Derived Brackets and the Gauge Algebra of Closed String Theory

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"Quantum Group Symposium at Group21"
(Proceedings of the Quantum Group Symposium
at the XXI International Colloquium on Group Theoretical Methods in Physics, Goslar, 1996),
eds. H.-D. Doebner and V.K. Dobrev (Heron Press, Sofia, 1997)
pp. 53-61

Abstract. We review the graded Lie algebra structures that appear in field theories, namely graded Poisson algebras, Gerstenhaber algebras and BV-algebras, and their generalizations to the non-commutative setting. We explain the construction of a derived bracket on a differential Lie algebra, in the category of Loday (Leibniz) algebras, which are generalizations of Lie algebras where the bracket is not skew-symmetric, and the numerous applications of that construction. We show that the Lie bracket of infinitesimal gauge transformations defined by Sen and Zwiebach in closed string field theory is a derived bracket.

1 Introduction

The role played in field theory by graded Poisson algebras and their odd version, called Gerstenhaber algebras (G-algebras), is well-known, so much so that the term "antibracket" has been introduced in physics to designate the graded bracket in the case of the Gerstenhaber algebras. The Batalin-Vilkovisky algebras (BV-algebras), a particular case of the Gerstenhaber algebras, figure in the theory of BRST cohomology. See for example [14] [10] [6] [7] and references therein. Moreover, it has recently become clear that the noncommutative version of Lie algebras, introduced by J.-L. Loday under the name of Leibniz algebras [11] [12], or rather the even and odd graded versions of these algebras, also play a role in string theory [1]. In [7], we introduced a very general construction that produces a Leibniz algebra, which we call the derived algebra, from a differential Lie algebra, and more generally, also from a differential Leibniz algebra - but not a new Lie algebra from a differential Lie algebra! - whence the necessity to leave the category of Lie algebras to carry out this construction. In [7], we give many instances of derived brackets, in differential geometry and its noncommutative variants, in Poisson calculus, and in the theory of generalized BV-algebras. We also explain why we have chosen to call these algebras Loday algebras rather than Leibniz algebras, the term used in the earlier literature. In this lecture, we briefly recall the main definitions and theorems, together with a few examples, and then we show how derived brackets arise in the definition and properties of the Lie algebra of infinitesimal gauge transformations in closed string field theory, as described by Sen and Zwiebach [13].

2 Graded Poisson algebras, Gerstenhaber algebras and BV-algebras.

Both graded Poisson algebras and Gerstenhaber algebras are graded versions of the familiar Poisson algebras (such as the algebra of functions on a Poisson manifold), in the even and odd case, respectively. They are associative, graded commutative algebras with a Lie bracket of degree m, even in the case of graded Poisson algebras (0 for ordinary Poisson algebras), odd in the case of Gerstenhaber algebras, satisfying the graded Leibniz rule :

$$[a, bc] = [a, b]c + (-1)^{(|a|+m)|b|} b[a, c] \ . \tag{2.1}$$

2.1 Remark.

When the underlying associative algebra is a chain complex, graded Poisson algebras (resp., Gerstenhaber algebras) are n-algebras or algebras of degree

$(n-1)$, in the sense of Getzler and Jones [5], with $(n-1)$ even (resp., odd).

2.2 Examples.

Graded Poisson algebras arise as algebras of functions on even Poisson supermanifolds, *e.g.*, the exterior algebra $\Lambda(F \oplus F^*)$, where F is any finite-dimensional vector space with dual F^*. In this case, the bracket, which we call the big bracket, is the unique extension of the duality pairing to a graded Poisson bracket of degree -2.

The basic example of a Gerstenhaber algebra is the Schouten algebra of a smooth manifold, the exterior algebra of multivectors (fields of contravariant, skewsymmetric tensors), equipped with the Schouten bracket. Other examples of Gerstenhaber algebras are the exterior algebra of a Lie algebra, the Hochschild cohomology of an associative algebra (of which the Schouten algebra of a smooth manifold is a particular case, by the Hochschild-Kostant-Rosenberg theorem), the Koszul algebra of forms on a Poisson manifold, the algebra of functions on a supermanifold with an odd Poisson structure (of which both the Schouten and Koszul algebras are particular cases).

2.3 BV-algebras.

BV-algebras are a particular instance of Gerstenhaber algebras, in which the bracket measures the defect in the derivation property of an odd, square 0, linear operator, which is said to generate the bracket. Let a denote both an element and left multiplication by this element in an associative, graded commutative algebra A with unit. Let Δ be an endomorphism of A, which vanishes on the unit. Define

$$\Phi^1_\Delta = \Delta \ , \tag{2.2}$$

$$\Phi^2_\Delta(a) = [\Phi^1_\Delta, a] - \Phi^1_\Delta(a), \tag{2.3}$$

$$\Phi^3_\Delta(a,b) = [\Phi^2_\Delta(a), b] - \Phi^2_\Delta(a)(b) \ , \tag{2.4}$$

for a, b in A. The operator Δ is a derivation if and only if Φ^2_Δ vanishes. By definition, the operator Δ is called a second-order derivation if Φ^3_Δ vanishes. In [9], Koszul proved the following fundamental lemma.

Koszul's lemma. Let $\Delta : A \to A$ be an odd, square 0, second-order derivation on an associative, graded commutative algebra with unit. Then

$$[a,b] = (-1)^{|a|}(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)) \tag{2.5}$$

defines a Gerstenhaber bracket, and Δ is a derivation of the Gerstenhaber algebra thus defined.

Definition. A Batalin-Vilkovisky or BV-algebra (A, Δ) is an associative, graded commutative algebra A, with unit, where Δ is an odd, square 0, second-order derivation.

By Koszul's lemma, any BV-algebra is a differential Gerstenhaber algebra.

2.4 Remark.

It is easy to establish a table of correspondences between the geometry of supermanifolds and the algebraic structures on their spaces of functions.

3 Towards a noncommutative generalization : Loday algebras and generalized BV-algebras.

In his book [11], Loday introduced a generalization of Lie algebras by relaxing the skew-symmetry axiom and formulating the Jacobi identity as the derivation property of the adjoint action, and he subsequently [12] developed the corresponding homology and cohomology theories. Here we formulate the definition of the graded version of these algebras, which we call Loday algebras.

Definition. A Loday algebra of degree m is a graded linear space A with a bilinear bracket satisfying

$$[a,[b,c]] = [[a,b],c] + (-1)^{(|a|+m)(|b|+m)}[b,[a,c]] \ , \qquad (3.1)$$

for all a of degree $|a|$, b of degree $|b|$, and c in A.

When moreover A has a multiplication (not necessarily associative nor graded commutative), we obtain the Loday-Poisson algebras (for m even) and the Loday-Gerstenhaber algebras (for m odd) by imposing the graded Leibniz rule (2.1). Generalizing the notion of second-order derivations by means of conditions $\Delta(1) = 0$ and $\Phi^3_\Delta = 0$, Koszul's lemma can be generalized as follows [1].

Akman's lemma. Let $\Delta : A \to A$ be an odd, square 0, second-order derivation on an algebra with unit. Then (2.5) defines a Loday-Gerstenhaber bracket, and Δ is a derivation of the Loday-Gerstenhaber algebra thus defined.

Thus it is natural to define a generalized BV-algebra as an algebra with unit, and an odd, square 0, second-order derivation. By the preceding lemma, any generalized BV-algebra is a Loday-Gerstenhaber algebra.

As an example, the u_1 mode of a vertex operator is a second-order derivation in a vertex operator algebra, and therefore defines a generalized BV-algebra structure whenever it is odd and of square 0. In particular, the coefficient b_0 of the z^{-2} term in the anti-ghost operator of a topological chiral algebra defines a generalized BV-algebra structure on the BRST complex itself [1].

4 Derived brackets.

There is a general construction, which produces a Loday algebra of odd (resp., even) degree from a differential Loday algebra of even (resp., odd) degree, *i.e.*, a Loday algebra with an odd, square 0 derivation of the bracket. In particular, one obtains by this method Loday-Gerstenhaber algebras from differential Loday-Poisson algebras, and vice-versa.

4.1 The main results.

We review the main results from [7].

Theorem 1 *Let $(A, [\ ,\ ], d)$ be a differential Loday algebra of degree m. Then*

$$[a, b]_d = (-1)^{|a|+m+1}[da, b] \ , \tag{4.1}$$

is a Loday bracket of degree $m + |d|$, and d is a derivation of $[\ ,\ ]_d$.

We call $[\ ,\ ]_d$ the derived bracket defined by d, and we speak of the derived Loday algebra defined by d. We stress the fact that when $[\ ,\ ]$ is a Lie bracket, $[\ ,\ ]_d$ is not in general a Lie bracket.

The mapping d satisfies $d([a, b]_d) = [da, db]$, so that it is morphism of Loday algebras.

Theorem 2 *The derived Loday algebra of a Loday-Poisson (resp., Loday-Gerstenhaber) algebra is a Loday-Gerstenhaber (resp., Loday-Poisson) algebra.*

Theorem 3 *If u is a morphism of differential Loday algebras from $(A[\ ,\ ],d)$ to $(A',[\ ,\ ]',d')$, then it is a morphism of derived brackets,*

$$u([a,b]_d) = [ua, ub]'_{d'} \ . \tag{4.2}$$

4.2 The case of an interior derivation.

Any element $a_0 \in A$, such that $|a_0| + m$ is odd, and $[a_0, a_0] = 0$, defines an odd derivation $[a_0, .]$, of square 0. We shall denote the derived Loday bracket defined by the interior derivation $[a_0, .]$ by $[\ ,\ ]_{a_0}$. If $[\ ,\ ]$ is skew-symmetric, then

$$[a,b]_{a_0} = [[a, a_0], b] \ . \tag{4.3}$$

4.3 The Lie algebra of co-exact elements on $A/\mathrm{Im}\, d$.

Assume that $[\ ,\ ]$ is a Lie bracket on A. Then, on $A/\mathrm{Im}\, d$, the derived Loday bracket $[\ ,\ ]_d$ induces a Lie bracket of opposite parity to that of $[\ ,\ ]$, and d is a morphism of Lie algebras.

4.4 Examples from the Poisson calculus.

Let (M, P) be a Poisson manifold, where P denotes the Poisson bivector. We shall show that the Poisson bracket of functions is the restriction of both a Loday-Poisson bracket defined on the whole algebra of differential forms on M, and a Loday-Poisson bracket defined on the whole algebra of multivectors on M. On the exterior algebra $\Omega(M)$ of differential forms both the Koszul bracket, $[\ ,\ ]_P$, which is a Gerstenhaber bracket, and the de Rham differential, d, which is a derivation of the Koszul bracket, are defined. Therefore, by Theorems 1 and 2, there exists a derived bracket on $\Omega(M)$ and it is a Loday-Poisson bracket,

$$[\alpha, \beta]_{P,d} = (-1)^{|\alpha|}[d\alpha, \beta]_P \ . \tag{4.4}$$

This bracket induces a Lie bracket on the space of co-exact forms (differential forms modulo exact forms) which extends the Poisson bracket of functions.

Dually, on the exterior algebra $V(M)$ of multivectors, both the Schouten bracket, $[\ ,\ ]$, which is a Gerstenhaber bracket, and the Lichnerowicz-Poisson differential, $d_P = [P, .]$, which is a derivation of the Schouten bracket, are defined. Therefore, there exists a derived bracket on $V(M)$ which is a Loday-Poisson bracket,

$$[Q, Q']_{d_P} = (-1)^{|Q|}[d_P Q, Q'] \ . \tag{4.5}$$

This bracket induces a Lie bracket on the space of co-exact multivectors (multivectors modulo d_P-exact multivectors) which also extends the Poisson bracket of functions. Thus the general construction of derived brackets gives a straightforward definition and interpretation of the various brackets on forms and multivectors extending the Poisson bracket of functions. See our detailed discussion in [7], and, in particular, the comparison of the results of [2].

Moreover, there exists an extension from functions to differential forms of the usual Hamiltonian mapping, defined by $\alpha \in \Omega(M) \to X^P_\alpha \in \mathrm{End}(\Omega(M))$, where

$$X^P_\alpha(\beta) = [\alpha, \beta]_{P,d} \ , \tag{4.6}$$

which is a morphism from the Loday algebra $(\Omega(M), [\ ,\ ]_{P,d})$ to the Lie algebra $\mathrm{End}(\Omega(M))$ with the graded commutator. In fact, if α is a 0-form, the restriction of X^P_α to 0-forms is just the Poisson bracket with α.

4.5 Even (resp., odd) Lie brackets on Abelian subalgebras of odd (resp., even) Lie algebras.

If $(A, [\ ,\ ])$ is a Lie algebra, on any Abelian subalgebra $\mathfrak{a}$ such that $[\mathfrak{a}, d\mathfrak{a}] \subset \mathfrak{a}$, the restriction of the derived bracket $[\ ,\ ]_d$ is a Lie bracket of opposite parity, satisfying

$$[a, b]_d = [a, db] \tag{4.7}$$

for a, b in $\mathfrak{a}$.

The algebraic Schouten bracket as a derived bracket of the big bracket. We know that on $\wedge(F \oplus F^*)$ we can define the big bracket $[\ ,\ ]$, which is a Poisson bracket of degree -2. Let $\mu \in \wedge^2 F^* \otimes F$ be a Lie algebra structure on F. It satisfies $[\mu, \mu] = 0$, and we can consider the derived bracket on $\wedge(F \oplus F^*)$ defined by the interior derivation, $[\mu, .]$. The restriction of this derived bracket to the Abelian subalgebra $\wedge F$ is the algebraic Schouten bracket which, for $a, b \in \wedge F$, satisfies $[a, b]_\mu = [a, [\mu, b]]$. It is also the unique extension of the Lie bracket defined on F by μ to a Gerstenhaber bracket of degree -1 on $\wedge F$.

The Schouten bracket of multivectors and the Lie bracket of vector fields as derived brackets of a graded commutator. Let M be a smooth manifold. We consider the algebra $\Omega(M)$ of differential forms on M, and the de Rham differential $d \in \mathrm{End}(\Omega(M))$. In $\mathrm{End}(\Omega(M))$, equipped with the graded commutator $[\ ,\ ]$, we can consider the interior derivation $[d, .]$ and the corresponding derived bracket $[\ ,\ ]_d$. If a, b are multivectors, interior products i_a, i_b by a, b commute in $\mathrm{End}(\Omega(M))$. When we compute the derived bracket of i_a, i_b, we obtain

$$[i_a, i_b]_d = [[i_a, d], i_b] = [L_a, i_b] = i_{[a,b]}, \tag{4.8}$$

where $L_a = [i_a, d]$ is the Lie operator with respect to a, and the bracket in the last term is the Schouten bracket. Thus, if we identify $a \in V(M)$ with $i_a \in \mathrm{End}(\Omega(M))$, we see that the Schouten bracket of multivectors is a derived bracket. In particular, if, $a, b \in V^1(M)$ are vector fields, then

$$[i_a, i_b]_d = i_{[a,b]}\ , \tag{4.9}$$

where $[a, b]$ is the Lie bracket of vector fields. Similarly, on a Poisson manifold (M, P), $[i_\alpha, i_\beta]_{d_P} = i_{[\alpha,\beta]_P}$, for $\alpha, \beta \in \Omega(M)$.

4.6 Generalizations to "noncommutative geometry".

In the spirit of [4], one can use the preceding facts to build non-commutative analogues of the calculus on manifolds. The basic datum is the graded Lie algebra of "vector fields" $\mathfrak{g}$, and a $\mathfrak{g}$-module $\mathfrak{m}$ which generalizes the commutative algebra of functions. One can then reconstruct the "differential forms" and the "de Rham differential", d, entirely from this structure. The Cartan formula is used to define d,

$$d\alpha(x_0, x_1, \cdots, x_n) = \Sigma(-1)^i x_i.\alpha(x_0, \cdots, \hat{x}_i, \cdots, x_n)$$
$$+\Sigma(-1)^{i+j}\alpha([x_i, x_j], x_0, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n)\ .$$

In the same way, one can reconstruct the "Poisson calculus" from $P \in \wedge^2\mathfrak{g}$. Other generalizations make use of Lie algebroids, and Lie-Rinehart pairs. (See *e.g.*, [8].)

4.7 The BV-bracket as a derived bracket of a graded commutator.

Now let A be a graded associative algebra with unit, and let Δ be an endomorphism of A, of degree -1, square 0 and such that $\Delta(1) = 0$. Then, we can consider the derived bracket $[\ ,\ ]_\Delta$ on End A, with respect to the interior derivation $[\Delta, .]$. For $u, v \in \mathrm{End}A$,

$$[u, v]_\Delta = (-1)^{|u|+1}[[\Delta, u], v] \ . \tag{4.10}$$

The derived bracket is a Loday-Gerstenhaber bracket of degree -1 on the graded associative algebra End A. We can imbed A itself in End A by associating to $a \in A$ the left multiplication by a, and consider the restriction of $[\ ,\ ]_\Delta$ to A. We claim that if $\Phi^3_\Delta = 0$, *i.e.*, if Δ is a second-order derivation, then $[\ ,\ ]_\Delta$ restricts to a Loday-Gerstenhaber bracket on A, given by

$$[a, b]_\Delta = (-1)^{|a|}(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)) \ ,$$

for $a, b \in A$, which is the generalized BV-bracket generated by Δ. If A is graded commutative, A is an Abelian subalgebra of End A, and we infer that $[\ ,\ ]_\Delta$ restricts to a Gerstenhaber bracket on A, which it is the BV-bracket generated by Δ. (We owe these remarks to I. Krasilsh'chik, unpublished.) Now Koszul's and Akman's lemmas follow without computations from these remarks and our general results on derived brackets.

As an example, we obtain the formula (cf., [3]) in the Poisson calculus,

$$e_{[\alpha,\beta]_P} = [e_\alpha, e_\beta]_{\partial_P} = [[e_\alpha, \partial_P], e_\beta] \ , \tag{4.11}$$

where e_α is left exterior multiplication by the form α, and $\partial_P = [i_P, d]$ is the Poisson homology operator which generates the Koszul bracket. Similarly,

$$e_{[\alpha,\beta]_{P,d}} = [e_\alpha, e_\beta]_{d,\partial_P} \ , \tag{4.12}$$

where $[u, v]_d = [[u, d], v]$ and $[u, v]_{d,\partial_P} = [[u, \partial_P]_d, v]_d$.

5 An application to the gauge Lie algebra of closed string field theory.

In this section, we wish to apply the theory of derived brackets to an example, that of the Lie bracket of infinitesimal gauge transformations, defined by Sen and Zwiebach in [13]. Although Sen and Zwiebach do not consider the general theory of derived brackets at all, their formulas are absolutely identical to a special case of ours, and it is striking that the proof of their result on the background independence of the gauge Lie algebra of closed string field theory is easily obtained as a corollary of the preceding theorems. We first recall the main statements from [13].

(i) There is a BV-algebra structure defined by Δ, an operator on the algebra A of string functionals, equipped with the "dot product", which is background dependent.

(ii) There is a background-independent BV-algebra structure on the free associative algebra A_{RS} generated by the space of orientable subspaces of moduli spaces of Riemann surfaces with punctures, defined by the operator of "twist-sewing", also denoted by Δ.

(iii) The choice of a background conformal field theory yields a morphism, f, from the latter BV-algebra to the former,

$$f : A_{RS} \to A \ .$$

(iv) The boundary operator d on A_{RS} is an odd derivation of the dot product and of the BV-bracket, and satisfies $[d, \Delta] = 0$ and $d^2 = 0$.

Thus $(A_{RS} \, , \Delta \, , d)$ is a differential BV-algebra. It follows that $d + \hbar\Delta = \delta_\hbar$, where $\hbar$ is a formal parameter, is a derivation of square 0. To conform to the notation of [13], we use $\{ \ , \ \}$ to denote the BV-brackets in this section.

For each choice of string vertices, $V_{g,n}$, setting $V = \sum_{g,n} \hbar^g V_{g,n}$, the recursion relations that have to be satisfied can be written compactly as

$$\delta_\hbar V + \frac{1}{2}\{V, V\} = 0 \ , \tag{5.1}$$

i.e.,

$$\delta_\hbar(\exp \frac{V}{\hbar}) = 0 \ ,$$

so that each choice of string vertices defines a cohomology class of $\delta_\hbar$. In the following table, we review the definition of the Lie bracket of infinitesimal gauge transformations in the algebra of string functionals A, and in the algebra of Riemann surfaces A_{RS}. In both cases, it is the Lie bracket obtained from a derived Loday bracket by passing to the quotient modulo exact elements, (a) with respect to the derivation Δ_S, where S is the master action, in the algebra of string functionals A, and $\Delta_S = \Delta + \frac{1}{\hbar}\{S, .\}$, and (b) with respect to the derivation $\delta_V = \delta_\hbar + \{V, .\}$, in the algebra of Riemann surfaces A_{RS}.

String functionals	Riemann surfaces
BV-algebra A S master action $\Delta S + \frac{1}{2\hbar}\{S, S\} = 0$	BV-algebra A_{RS} V string vertices $\delta_\hbar V + \frac{1}{2}\{V, V\} = 0$
$\Delta_S = \Delta + \frac{1}{\hbar}\{S, .\}$ $(\Delta_S)^2 = 0$	$\delta_V = \delta_\hbar + \{V, .\}$ $(\delta_V)^2 = 0$
gauge parameters : equivalence classes of string functionals mod. Im Δ_S	gauge transformations : equivalence classes of moduli spaces mod. Im δ_V
derived bracket $\{\ ,\ \}_{\Delta_S}$ $\rightarrow$ Lie bracket on $A/\mathrm{Im}\ \Delta_S$	derived bracket $\{\ ,\ \}_{\delta_V}$ $\rightarrow$ Lie bracket on $A_{RS}/\mathrm{Im}\ \delta_V$
$\{\Lambda_1, \Lambda_2\}_{\Delta_S} = \{\Lambda_1, \Delta_S \Lambda_2\}$ $= (-1)^{\lvert\Lambda_1\rvert}\{\Delta_S \Lambda_1, \Lambda_2\}$ $= \frac{1}{2}(\{\Lambda_1, \Delta_S \Lambda_2\}$ $-(-1)^{\lvert\Lambda_1\rvert\lvert\Lambda_2\rvert}\{\Lambda_2, \Delta_S \Lambda_1\})$	$\{X_1, X_2\}_{\delta_V} = \{X_1, \delta_V X_2\}$ $= (-1)^{\lvert X_1\rvert}\{\delta_V X_1, X_2\}$ $= \frac{1}{2}(\{X_1, \delta_V X_2\}$ $-(-1)^{\lvert X_1\rvert\lvert X_2\rvert}\{X_2, \delta_V X_1\})$

The choice of a background conformal field theory defines $f : A_{RS} \rightarrow A$, which is a morphism of BV-algebras, and intertwines δ_V and Δ_S. It follows from Theorem 3 that f is a morphism of derived Loday algebras,

$$f\left(\{X_1, X_2\}_{\delta_V}\right) = \{f(X_1), f(X_2)\}_{\Delta_S} \ . \tag{5.2}$$

What is the behaviour of the Lie algebra of infinitesimal gauge transformations under a change of string vertices? We assume that we replace V by $V' = V + \delta_V W$, such that $(\delta_{V'})^2 = 0$. Then V and V' yield isomorphic derived Loday brackets on A_{RS} and hence isomorphic Lie algebras of gauge transformations on $A_{RS}/\mathrm{Im}\ \delta_V$ and $A_{RS}/\mathrm{Im}\ \delta_{V'}$. In fact, $\delta_{V'} = \delta_V + \{\delta_V W, .\}$. Then $\exp\{W, .\}$ intertwines δ_V and $\delta_{V'}$, and is an isomorphism from $\{\ ,\ \}_{\delta_{V'}}$ to $\{\ ,\ \}_{\delta_V}$.

Thus the Lie algebra of gauge transformations $A_{RS}/\mathrm{Im}\ \delta_V$ is "universal", *i.e.*, it does not depend upon the choice of string vertices.

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