MANIN PAIRS AND MOMENT MAPS

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Abstract. A Lie group G in a group pair (D, G), integrating the Lie algebra $\mathfrak{g}$ in a Manin pair $(\mathfrak{d}, \mathfrak{g})$, has a quasi-Poisson structure. We define the quasi-Poisson actions of such Lie groups G, and show that they generalize the Poisson actions of Poisson Lie groups. We define and study the moment maps for those quasi-Poisson actions which are hamiltonian. These moment maps take values in the homogeneous space D/G. We prove an analogue of the hamiltonian reduction theorem for quasi-Poisson group actions, and we study the symplectic leaves of the orbit spaces of hamiltonian quasi-Poisson spaces.

1. Introduction

The purpose of this article is to provide a framework for Lie-group valued moment map theories. In the usual theory (see, *e.g.*, [10]), the moment map corresponding to an action of a Lie group G on a symplectic manifold (M, ω) takes values in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. In the case of $G = S^1$, an abelian Lie-group valued moment map taking values in S^1 instead of $\mathfrak{u}(1)^* = \mathbb{R}$ was introduced in [18] by McDuff. The first nonabelian Lie-group valued moment map theory to be proposed was that of Lu and Weinstein [16], [17]. In their approach, G is a Poisson Lie group, M has a symplectic form ω (or, more generally, a Poisson bivector) which is not G-invariant, and the moment map takes values in the Poisson Lie group dual to G. When G is a compact semi-simple Lie group, the target space is the symmetric space $G^{\mathbb{C}}/G$ which is equipped with the structure of a Lie group. Recently, another nonabelian moment map theory where the moment map takes values in the same compact simple Lie group G as the one which acts on the manifold has been developed [3]. In this theory, the 2-form ω is G-invariant but not closed.

These various formulations of the moment map theories share many features. For instance, they all have a well defined notion of a hamiltonian reduction [16], [3], convexity properties [8], [1], [19] and localization formulas [9], [24], [4]. We think that these common features justify a search for a unified formulation.

Our proposal of a general moment map theory is based on the notions of a Manin pair and of a Manin quasi-triple, and our main technical tool is the theory of quasi-Poisson Lie groups developed in [11] and [12]. We introduce the notion of a quasi-Poisson space as a space with a G-action and a bivector P such that the Schouten bracket $[P, P]$ is expressed as a certain trilinear combination of the vector fields generating the G-action. These spaces are the natural objects upon which quasi-Poisson Lie groups act. We define and study the moment maps for the actions on a quasi-Poisson space of a Lie group G whose Lie algebra, $\mathfrak{g}$, belongs to a Manin pair $(\mathfrak{d}, \mathfrak{g})$. Our formulation is close in spirit to that of Lu and Weinstein, which is based on the notions of a Manin triple and a Poisson Lie group. While Manin triples and Poisson Lie groups had been introduced as the classical limits of quantum groups

[6], which are Hopf algebra deformations of the enveloping algebras of Lie algebras, the generalized objects that we consider here are the classical limits of deformations which are only quasi-Hopf algebras [7]. The present generalization to Manin quasi-triples and quasi-Poisson Lie groups is necessary in order to provide a conceptual explanation for the theory of group valued moment maps introduced and developed in [3] and [4].

Section 2 includes some basic information on Manin pairs and Manin quasi-triples. In Section 3 we introduce *group pairs* which integrate Manin pairs. (Group pairs with a symmetric structure were also studied in [15].) The definition of quasi-Poisson actions is presented in Section 4. If (D, G) is a group pair, then the dressing action of G on D/G is quasi-Poisson. We show that a given action remains quasi-Poisson when both the quasi-Poisson Lie group and the quasi-Poisson space on which it acts are modified by a twist. Thus the notion of a quasi-Poisson action is related to that of a Manin pair, rather than to a particular Manin quasi-triple. We prove that if (M, P) is a quasi-Poisson space with a bivector P acted upon by a quasi-Poisson Lie group, the bivector P induces a genuine Poisson bracket on the space of G-invariant smooth functions on M. The definition of generalized moment maps is given in Section 5. In our setting, the moment maps are always assumed to be equivariant, a property which the usual moment maps may or may not have. We then prove that moment maps are bivector maps. This generalizes the theorem stating that the moment map is equivariant if and only if it is a Poisson map. The dressing action of G on D/G has the identity of D/G as a moment map. We call hamiltonian those quasi-Poisson spaces that admit a moment map. We show that there is a well-defined integrable distribution on a hamiltonian quasi-Poisson space which, under an additional assumption, gives rise to a generalized foliation by nondegenerate hamiltonian quasi-Poisson spaces that contain the G-orbits. We also prove that, in a hamiltonian quasi-Poisson space with a nondegenerate bivector P, the symplectic leaves in the orbit space are connected components of the projection of the level sets of the moment map. This is a generalization of the usual symplectic reduction theorem.

The fundamental example of a Manin pair is that of a Lie algebra $\mathfrak{g}$ with an invariant scalar product, embedded diagonally in the double $\mathfrak{g} \oplus \mathfrak{g}$, equipped with the corresponding hyperbolic metric (see Example 2.1.5). For the corresponding group pair $(G \times G, G)$, the target of the moment map $(G \times G)/G$ can be identified with G itself. It can be shown that any hamiltonian G-space with group-valued moment map in the sense of [3] and [4] is also a hamiltonian quasi-Poisson space associated to this group pair, in the sense of Definition 5.1.1. This can be easily shown (see Section 5.2) in the case of abelian Lie groups, and will be proved in the general case in a subsequent paper [2].

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2. Manin pairs and r-matrices

In this section we introduce the notions of a Manin pair and of a classical r-matrix which provide the main building blocks of the generalized moment map theory. All

vector spaces are over $\mathbb{R}$ or $\mathbb{C}$ and, for simplicity, we assume that they are finite-dimensional. We use the Einstein summation convention.

2.1. **Manin pairs and Manin quasi-triples.** We consider a finite-dimensional vector space V with a nondegenerate symmetric bilinear form $(\ |\ )$. An isotropic subspace $W \subset V$ is called maximal if it is not strictly contained in another isotropic subspace of V. Using Witt's theorem and its corollaries [14] it is easy to prove that, if V is a real vector space of signature (n, n) or a complex vector space of dimension $2n$, the dimension of any maximal isotropic subspace, W, of V is equal to n, and that an isotropic subspace is maximal if and only if it is equal to its own orthogonal. One can also show that any maximal isotropic subspace $W \subset V$ has isotropic complements, and that they are maximal. A choice of such a maximal isotropic complement, $W' \subset V, V = W \oplus W'$, determines an isomorphism between the space W^* dual to W and the space W'.

If, in addition, we assume that the spaces V and W possess a Lie-algebra structure, we arrive at the definition of a Manin pair.

Definition 2.1.1. A *Manin pair* is a pair, $(\mathfrak{d}, \mathfrak{g})$, where $\mathfrak{d}$ is a Lie algebra of even dimension $2n$, with an invariant, nondegenerate symmetric bilinear form, of signature (n, n) in the real case, and $\mathfrak{g}$ is both a maximal isotropic subspace and a Lie subalgebra of $\mathfrak{d}$.

Definition 2.1.2. A *Manin quasi-triple* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin pair $(\mathfrak{d}, \mathfrak{g})$ with an isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$.

Thus, in a Manin quasi-triple, $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$, $\mathfrak{g}$ is a maximal isotropic Lie subalgebra and $\mathfrak{h}$ is a maximal isotropic linear subspace of $\mathfrak{d}$. We shall denote by $f : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ the Lie-algebra structure of $\mathfrak{g}$ which is the restriction of that of $\mathfrak{d}$. We shall denote by $j : \mathfrak{g}^* \to \mathfrak{h}$ the isomorphism of vector spaces defined by the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$, which satisfies

$$(j(\xi)|x) = < \xi, x > ,$$

for each $\xi \in \mathfrak{g}^*$, and $x \in \mathfrak{g}$, and we shall denote the projections from $\mathfrak{d}$ to $\mathfrak{g}$ and $\mathfrak{h}$ by $p_{\mathfrak{g}}$ and $p_{\mathfrak{h}}$ respectively. On $\mathfrak{g}$, we introduce the cobracket, $F : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$. It is the transpose of the map from $\bigwedge^2 \mathfrak{g}^*$ to $\mathfrak{g}^*$, which we denote by the same letter, defined by

$$F(\xi, \eta) = j^{-1} p_{\mathfrak{h}}[j(\xi), j(\eta)],$$

for $\xi, \eta \in \mathfrak{g}^*$. We also introduce an element $\varphi \in \bigwedge^3 \mathfrak{g}$ which is defined by the map from $\bigwedge^2 \mathfrak{g}^*$ to $\mathfrak{g}$, denoted again by the same letter,

$$\varphi(\xi, \eta) = p_{\mathfrak{g}}[j(\xi), j(\eta)].$$

The Lie algebra $\mathfrak{g}$, with cobracket F and element $\varphi \in \bigwedge^3 \mathfrak{g}$, is called a *Lie quasi-bialgebra* [7]. (The element φ is the classical limit of the co-associator of a quasi-Hopf algebra.) Conversely, from a Lie quasi-bialgebra $(\mathfrak{g}, F, \varphi)$, we obtain a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$, where $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ with its canonical scalar product induced by the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$,

$$((x, \xi)|(y, \eta)) =< x, \eta > + < y, \xi >,$$

the Lie bracket on $\mathfrak{d}$ being defined as follows [7],

$$[x,y] = f(x,y), \quad [x,\xi] = \mathrm{ad}^*_x \xi - \mathrm{ad}^*_\xi x, \quad [\xi,\eta] = F(\xi,\eta) + \varphi(\xi,\eta), \tag{2.1.3}$$

where $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. Sometimes we refer to the bialgebra data F and φ corresponding to the complement $\mathfrak{h} \subset \mathfrak{d}$ as $F_\mathfrak{h}$ and $\varphi_\mathfrak{h}$.

Example 2.1.4. For any Lie algebra $\mathfrak{g}$, the choice $F = 0$ and $\varphi = 0$ defines the Manin pair $(\mathfrak{d}, \mathfrak{g})$, where $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ with the Lie bracket

$$[x,y] = f(x,y), \quad [x,\xi] = \mathrm{ad}^*_x \xi, \quad [\xi,\eta] = 0.$$

We call this Manin pair the *standard Manin pair* associated to $\mathfrak{g}$. In the standard Manin pair, $\mathfrak{g}$ possesses a canonical complement, $\mathfrak{h} = \mathfrak{g}^*$, defining a Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$.

Example 2.1.5. If a Lie algebra $\mathfrak{g}$ possesses an invariant, nondegenerate symmetric bilinear form K, one can construct another Manin pair with $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, the direct sum of two copies of the Lie algebra $\mathfrak{g}$. (See [20].) The scalar product on $\mathfrak{d}$ is defined as the difference of the bilinear forms on the two copies of $\mathfrak{g}$,

$$((x_1,x_2)|(y_1,y_2)) = K(x_1,y_1) - K(x_2,y_2)\ , \tag{2.1.6}$$

and $\mathfrak{g}$ is embedded into $\mathfrak{d}$ by the diagonal embedding $\Delta : x \mapsto (x,x)$. A possible choice of an isotropic complement to $\mathfrak{g}$ is given by $\mathfrak{g}_- = \frac{1}{2}\Delta_-(\mathfrak{g})$, where $\Delta_- : x \mapsto (x,-x)$ is the anti-diagonal embedding. In general, the isotropic subspace $\mathfrak{g}_- \subset \mathfrak{d}$ is not a Lie subalgebra, and $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}_-)$ is only a Manin quasi-triple. The map F vanishes because $[\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}$. If we identify elements in $\mathfrak{g}$ with elements in $\mathfrak{g}_-$ by $\frac{1}{2}\Delta_-$, and then with elements in $\mathfrak{g}^*$ by the isomorphism j^{-1} of this Manin quasi-triple, the element $\varphi \in \bigwedge^3 \mathfrak{g}$ can be identified with the trilinear form on $\mathfrak{g}$,

$$(x,y,z) \mapsto \frac{1}{4} K(z,[x,y]),$$

which is actually anti-symmetric and $\mathrm{ad}(\mathfrak{g})$-invariant. Let $(e_i), i = 1,\dots,n$, be a basis of $\mathfrak{g}$, and let (K^{ij}) be the matrix inverse to the matrix (K_{ij}), where $K_{ij} = K(e_i,e_j)$. Let us denote the structure constants of $\mathfrak{g}$ in this basis by f^k_{ij}. Then

$$\varphi = \frac{1}{4} K^{il} K^{jm} f^k_{lm} e_i \otimes e_j \otimes e_k = \frac{1}{24} K^{il} K^{jm} f^k_{lm} e_i \wedge e_j \wedge e_k.$$

If, in particular, we choose an orthogonal basis such that $K(e_i,e_j) = \frac{1}{2}\delta_{ij}$, then

$$\varphi = \sum_{ijk} f^i_{jk} e_i \otimes e_j \otimes e_k = \frac{1}{6} \sum_{ijk} f^i_{jk} e_i \wedge e_j \wedge e_k.$$

2.2. **Twisting.** All Manin quasi-triples corresponding to the same Manin pair $(\mathfrak{d}, \mathfrak{g})$ differ by a twist $t \in \bigwedge^2 \mathfrak{g}$, defined as follows. An isotropic complement to $\mathfrak{g}$ in $\mathfrak{d}$ always exists and is by no means unique. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two isotropic complements of $\mathfrak{g}$ in $\mathfrak{d}$, and let $1_\mathfrak{d} = p_\mathfrak{g} + p_\mathfrak{h}$ and $1_\mathfrak{d} = p'_\mathfrak{g} + p'_{\mathfrak{h}'}$ be the decompositions of the identity map of $\mathfrak{d}$ into the sum of the projections defined by the direct decompositions $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}'$. Then $1_\mathfrak{h} = \kappa + \lambda$, where κ and λ are the restrictions to $\mathfrak{h}$ of the projections $p'_\mathfrak{g}$ and $p'_{\mathfrak{h}'}$.

Let $j : \mathfrak{g}^* \to \mathfrak{h}$ and $j' : \mathfrak{g}^* \to \mathfrak{h}'$ be the isomorphisms of vector spaces defined by these direct decompositions. We consider the linear map from $\mathfrak{g}^*$ to $\mathfrak{d}$, called the *twist* from $\mathfrak{h}$ to $\mathfrak{h}'$,

$$t = j' - j \ . \tag{2.2.1}$$

We first remark that t takes values in $\mathfrak{g}$. In fact, $j' = \lambda \circ j$ and $t = -\kappa \circ j$. It is easy to show that t is anti-symmetric. In fact, for $\xi, \eta \in \mathfrak{g}^*$,

$$< t(\xi), \eta > + < \xi, t(\eta) > = (t(\xi)|j(\eta)) + (j'(\xi)|t(\eta)) = (j(\xi)|j(\eta)) = 0,$$

where we have used the isotropy of $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{h}'$. Hence, the map t defines an element in $\bigwedge^2 \mathfrak{g}$ which we denote by the same letter. By convention, $t(\xi, \eta) =< t(\xi), \eta >$.

2.3. **The canonical r-matrix.** Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a Manin quasi-triple. We identify $\mathfrak{d}$ with $\mathfrak{g} \oplus \mathfrak{g}^*$ using the isomorphism j^{-1} of $\mathfrak{h}$ onto $\mathfrak{g}^*$. The map $r_{\mathfrak{d}} : \mathfrak{d}^* \to \mathfrak{d}$ defined by $r_{\mathfrak{d}} : (\xi, x) \mapsto (0, \xi)$ for $x \in \mathfrak{g}, \xi \in \mathfrak{g}^*$ defines an element $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$, called the *canonical r-matrix*. Let $(e_i), i = 1, \ldots, n$, be a basis of $\mathfrak{g}$ and $(\varepsilon^i), i = 1, \ldots, n$, be the dual basis in $\mathfrak{g}^*$. Then

$$r_{\mathfrak{d}} = \sum_{i=1}^{n} e_i \otimes \varepsilon^i. \tag{2.3.1}$$

The symmetric part of $r_{\mathfrak{d}}$ coincides with the scalar product of $\mathfrak{d}$ up to a factor of 2, and therefore it is $\mathrm{ad}(\mathfrak{d})$-invariant. The element $r_{\mathfrak{d}}$ satisfies a relation that generalizes the classical Yang-Baxter equation. To derive that relation we introduce the notion of a Drinfeld bracket for elements of $\mathfrak{a} \otimes \mathfrak{a}$, for any Lie algebra $\mathfrak{a}$.

Definition 2.3.2. Let $\mathfrak{a}$ be a Lie algebra with a basis $(e_\alpha), \alpha = 1, \ldots, N$, and let $r = \sum_{\alpha\beta} r^{\alpha\beta} e_\alpha \otimes e_\beta$ be an element of $\mathfrak{a} \otimes \mathfrak{a}$. The *Drinfeld bracket* of r is the element in $\mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}$ defined as follows,

$$\langle r, r \rangle = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \tag{2.3.3}$$

where $r_{12} = r \otimes 1, r_{13} = \sum_{\alpha\beta} r^{\alpha\beta} e_\alpha \otimes 1 \otimes e_\beta, r_{23} = 1 \otimes r$, and 1 is the unit of the universal enveloping algebra $U(\mathfrak{a})$.

If r is anti-symmetric, then

$$\langle r, r \rangle = -\frac{1}{2}[r, r] \ , \tag{2.3.4}$$

where $[\ ,\ ]$ is the algebraic Schouten bracket of $\bigwedge \mathfrak{a}$. (See, *e.g.*, [12] or [13].) If r is symmetric and $\mathrm{ad}(\mathfrak{a})$-invariant, then

$$\langle r, r \rangle = [r_{12}, r_{13}] \ , \tag{2.3.5}$$

and $\langle r, r \rangle$ is the $\mathrm{ad}(\mathfrak{a})$-invariant element in $\bigwedge^3 \mathfrak{a}$ with components $r^{\kappa\beta} r^{\lambda\gamma} f^{\alpha}_{\kappa\lambda}$. For any $r \in \mathfrak{a} \otimes \mathfrak{a}$ with $\mathrm{ad}(\mathfrak{a})$-invariant symmetric part s,

$$\langle r, r \rangle = \langle a, a \rangle + \langle s, s \rangle \ , \tag{2.3.6}$$

where a is the anti-symmetric part of r. Therefore, when the symmetric part of r is invariant, $\langle r, r \rangle$ is in $\bigwedge^3 \mathfrak{a}$.

Proposition 2.3.7. [5] *The canonical r-matrix, $r_{\mathfrak{d}}$, associated to the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ satisfies*

$$\langle r_{\mathfrak{d}}, r_{\mathfrak{d}} \rangle = \varphi, \tag{2.3.8}$$

where the element $\varphi \in \bigwedge^3 \mathfrak{g}$ is considered as an element in $\bigwedge^3 \mathfrak{d}$.

Proof. In the basis (e_i, ε^i), the commutation relations of $\mathfrak{d}$ have the form,

$$[e_i, e_j] = f_{ij}^k e_k \ , \ [e_i, \varepsilon^j] = -f_{ik}^j \varepsilon^k + F_i^{jk} e_k \ , \ [\varepsilon^i, \varepsilon^j] = F_k^{ij} \varepsilon^k + \varphi^{ijk} e_k \ .$$

The computation of the three terms entering the Drinfeld bracket yields

$$\begin{aligned}
[r_{12}, r_{13}] &= [e_i \otimes \varepsilon^i \otimes 1, e_j \otimes 1 \otimes \varepsilon^j] = f_{ij}^k e_k \otimes \varepsilon^i \otimes \varepsilon^j; \\
[r_{12}, r_{23}] &= [e_i \otimes \varepsilon^i \otimes 1, 1 \otimes e_j \otimes \varepsilon^j] = f_{jk}^i e_i \otimes \varepsilon^k \otimes \varepsilon^j - F_j^{ik} e_i \otimes e_k \otimes \varepsilon^j; \\
[r_{13}, r_{23}] &= [e_i \otimes 1 \otimes \varepsilon^i, 1 \otimes e_j \otimes \varepsilon^j] = F_k^{ij} e_i \otimes e_j \otimes \varepsilon^k + \varphi^{ijk} e_i \otimes e_j \otimes e_k,
\end{aligned}$$

whence $\langle r_{\mathfrak{d}}, r_{\mathfrak{d}} \rangle = \varphi^{ijk} e_i \otimes e_j \otimes e_k = \varphi$. □

We now list properties of the anti-symmetric part of the canonical r-matrix, $r_{\mathfrak{d}}$.

Proposition 2.3.9. *Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a Manin quasi-triple, where we identify $\mathfrak{h}$ with $\mathfrak{g}^*$. Let $a_{\mathfrak{d}} \in \bigwedge^2 \mathfrak{d}$ be the anti-symmetric part of $r_{\mathfrak{d}}$. For any $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$,*

$$[x, a_{\mathfrak{d}}] = F(x), \quad [\xi, a_{\mathfrak{d}}] = -f(\xi) + \varphi(\xi), \quad [a_{\mathfrak{d}}, \varphi] = 0. \tag{2.3.10}$$

Proof. The proof of the first two equalities is by computation, using the formula $a_{\mathfrak{d}} = \frac{1}{2} \sum_{i=1}^n e_i \wedge \varepsilon^i$ and the derivation property of the algebraic Schouten bracket. To prove the third equality, we use formulas (2.3.6), (2.3.8) and (2.3.4) to obtain

$$[a_{\mathfrak{d}}, \varphi] = -\frac{1}{2}[a_{\mathfrak{d}}, [a_{\mathfrak{d}}, a_{\mathfrak{d}}]] + [a_{\mathfrak{d}}, \langle s_{\mathfrak{d}}, s_{\mathfrak{d}} \rangle].$$

The first term vanishes by the graded Jacobi identity and the second term vanishes because $\langle s_{\mathfrak{d}}, s_{\mathfrak{d}} \rangle$ is $\mathrm{ad}(\mathfrak{d})$-invariant. □

Under a twist t, the canonical r-matrix $r_{\mathfrak{d}}$ is modified to

$$r'_{\mathfrak{d}} = r_{\mathfrak{d}} + t. \tag{2.3.11}$$

Indeed, any element of $\mathfrak{d}$ can be decomposed in two ways, as $x + j(\xi)$ and as $x' + j'(\xi')$, where $x, x' \in \mathfrak{g}$, $\xi, \xi' \in \mathfrak{g}^*$. Then, by definition, $t(\xi) = j'(\xi') - j(\xi) \in \mathfrak{g}$, while

$$(r'_{\mathfrak{d}} - r_{\mathfrak{d}})(j(\xi) + x) = r'_{\mathfrak{d}}(j'(\xi') + x') - r_{\mathfrak{d}}(j(\xi) + x) = j'(\xi') - j(\xi) = t(\xi).$$

If $t = \sum_{ij} t^{ij} e_i \otimes e_j = \frac{1}{2} \sum_{ij} t^{ij} e_i \wedge e_j$, then after twisting by t, the dual basis of $\mathfrak{g}^*$ becomes

$$\varepsilon'^i = \varepsilon^i + t^{ij} e_j \ . \tag{2.3.12}$$

Under the twist t relating isotropic complements $\mathfrak{h}$ and $\mathfrak{h}'$, the cobracket F of the Lie quasi-bialgebra and the element φ are modified as follows [7] [11]:

$$F_{\mathfrak{h}'} = F_{\mathfrak{h}} + F_1, \tag{2.3.13}$$

$$\varphi_{\mathfrak{h}'} = \varphi_{\mathfrak{h}} - \langle t, t \rangle + \varphi_1 \ , \tag{2.3.14}$$

where $F_1(x) = \mathrm{ad}_x\, t$ and $\varphi_1(\xi) = \overline{\mathrm{ad}}^*_\xi t$. Here ad denotes the adjoint action of $\mathfrak{g}$ on $\bigwedge^2 \mathfrak{g}$, while $\overline{\mathrm{ad}}^*_\xi t$ denotes the projection of $\mathrm{ad}^*_\xi\, t$ onto $\bigwedge^2 \mathfrak{g}$, where $\mathfrak{g}^* \subset \mathfrak{d}^*$ acts on $\bigwedge^2 \mathfrak{g} \subset \bigwedge^2 \mathfrak{d}$ by the coadjoint action. In fact,

$$\varphi_1^{ijk} = F_l^{jk} t^{il} - F_l^{ik} t^{jl} \ . \tag{2.3.15}$$

3. Group pairs and quasi-Poisson structures

In this section, we study the global objects corresponding to Manin pairs and Manin quasi-triples. In the rest of this paper, we shall assume that the base field is $\mathbb{R}$, and all manifolds and maps will be assumed to be smooth. We shall often abbreviate "fields of multivectors" to "multivectors". A *bivector map* between (M_1, P_{M_1}) and (M_2, P_{M_2}), where P_{M_i} is a bivector on manifold M_i $(i = 1, 2)$, is a map u from M_1 to M_2 such that

$$u_* P_{M_1} = P_{M_2} \ .$$

(If P_{M_1} and P_{M_2} are Poisson bivectors, such a map is called a *Poisson map.*)

3.1. Group pairs and quasi-triples.

We first introduce the objects which integrate Manin pairs and Manin quasi-triples.

Definition 3.1.1. A *group pair* is a pair (D, G), where D is a connected Lie group with a bi-invariant scalar product and G is a connected, closed Lie subgroup of D, such that the Lie algebras, $\mathfrak{d}$ and $\mathfrak{g}$, of D and G form a Manin pair.

It is evident from this definition that, given a finite-dimensional Manin pair $(\mathfrak{d}, \mathfrak{g})$ it can be integrated into a unique group pair, where D is simply connected, provided $\mathfrak{g}$ is the Lie algebra of a closed Lie subgroup of D.

Example 3.1.2. A group pair corresponding to the standard Manin pair $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is (T^*G, G). Here T^*G is the cotangent bundle of the connected, simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$, equipped with the group structure of a semi-direct product, upon identification with $G \times \mathfrak{g}^*$ by left translations. The group G is embedded into T^*G as the zero section.

Example 3.1.3. For a Lie algebra $\mathfrak{g}$ with an invariant, nondegenerate symmetric bilinear form, the Manin pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$ of Example 2.1.5 can be integrated to a group pair $(G \times G, G)$. The group G is embedded into $D = G \times G$ as the diagonal.

Definition 3.1.4. A *quasi-triple* $(D, G, \mathfrak{h})$ is a group pair (D, G), where an isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ has been chosen.

Note that a quasi-triple is not a triple of Lie groups. In general, the subspace $\mathfrak{h} \subset \mathfrak{d}$ is not a Lie subalgebra and cannot be integrated into a Lie subgroup. If $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, integrating $\mathfrak{h}$ to a Lie group H yields a triple of groups (D, G, H). (See [17].)

We shall now study the bivectors on D and G which generalize the multiplicative bivectors obtained when $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, and D and G are Poisson Lie groups [6] [20].

3.2. **The quasi-Poisson structure of D.** Let $(D, G, \mathfrak{h})$ be a quasi-triple. We have recalled in Section 2 the definition of the canonical r-matrix of $\mathfrak{d}$, $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$. Thus $r_{\mathfrak{d}}$ defines a contravariant 2-tensor on D,

$$P_D = r_{\mathfrak{d}}^{\lambda} - r_{\mathfrak{d}}^{\rho}, \tag{3.2.1}$$

where $r_{\mathfrak{d}}^{\lambda}$ (resp., $r_{\mathfrak{d}}^{\rho}$) denotes the left- (resp., right-) invariant 2-tensor on the Lie group D with value $r_{\mathfrak{d}}$ at the identity. Sometimes we denote the bivector P_D corresponding to the complement $\mathfrak{h} \subset \mathfrak{d}$ by $P_D^{\mathfrak{h}}$ to render the dependence on $\mathfrak{h}$ explicit.

Actually, P_D is a bivector because the symmetric part of $r_{\mathfrak{d}}$, being $\mathrm{ad}(\mathfrak{d})$-invariant, is invariant with respect to the adjoint action of D and cancels in (3.2.1).

At the identity of D, P_D vanishes because there $r_{\mathfrak{d}}^{\lambda}$ coincides with $r_{\mathfrak{d}}^{\rho}$. The following multiplicativity property of P_D is obvious from the definition.

Proposition 3.2.2. *The bivector P_D on the Lie group D is multiplicative with respect to the multiplication of D, i.e., the multiplication map $m : D \times D \to D$ is a bivector map from $D \times D$ with the product bivector to (D, P_D).*

In general, the Schouten bracket $[P_D, P_D]$ does not vanish.

Proposition 3.2.3. *The Schouten bracket of the bivector P_D is given by*

$$\frac{1}{2}[P_D, P_D] = \varphi^{\rho} - \varphi^{\lambda}. \tag{3.2.4}$$

Proof. Since $P_D = a_{\mathfrak{d}}^{\lambda} - a_{\mathfrak{d}}^{\rho}$, and since left- and right-invariant vector fields commute with each other, we obtain

$$[P_D, P_D] = [a_{\mathfrak{d}}^{\lambda} - a_{\mathfrak{d}}^{\rho}, a_{\mathfrak{d}}^{\lambda} - a_{\mathfrak{d}}^{\rho}] = [a_{\mathfrak{d}}^{\lambda}, a_{\mathfrak{d}}^{\lambda}] + [a_{\mathfrak{d}}^{\rho}, a_{\mathfrak{d}}^{\rho}] = [a_{\mathfrak{d}}, a_{\mathfrak{d}}]^{\lambda} - [a_{\mathfrak{d}}, a_{\mathfrak{d}}]^{\rho} \ .$$

Using (2.3.4), (2.3.6) and the fact that $\langle s_{\mathfrak{d}}, s_{\mathfrak{d}} \rangle$ is $\mathrm{ad}(\mathfrak{d})$-invariant, the right-hand side is $2(\langle r_{\mathfrak{d}}, r_{\mathfrak{d}} \rangle^{\rho} - \langle r_{\mathfrak{d}}, r_{\mathfrak{d}} \rangle^{\lambda})$. Since $\langle r_{\mathfrak{d}}, r_{\mathfrak{d}} \rangle = \varphi$, we arrive at formula (3.2.4). □

We collect some properties of the bivector P_D in the following proposition.

Proposition 3.2.5. *The bivector P_D satisfies*

$$\mathcal{L}_{x^{\lambda}} P_D = F(x)^{\lambda}, \ \mathcal{L}_{x^{\rho}} P_D = F(x)^{\rho} \ , \tag{3.2.6}$$

for $x \in \mathfrak{g}$, and

$$\mathcal{L}_{\xi^{\lambda}} P_D = -f(\xi)^{\lambda} + \varphi(\xi)^{\lambda}, \ \mathcal{L}_{\xi^{\rho}} P_D = -f(\xi)^{\rho} + \varphi(\xi)^{\rho} \ , \tag{3.2.7}$$

for $\xi \in \mathfrak{h}$, and

$$[P_D, \varphi^{\lambda}] = 0, \ [P_D, \varphi^{\rho}] = 0. \tag{3.2.8}$$

Proof. Here and below $\mathfrak{h}$ is identified with $\mathfrak{g}^*$. By Proposition 2.3.9,

$$[x^{\lambda}, a_{\mathfrak{d}}^{\lambda}] = [x, a_{\mathfrak{d}}]^{\lambda} = F(x)^{\lambda},$$
$$[\xi^{\lambda}, a_{\mathfrak{d}}^{\lambda}] = [\xi, a_{\mathfrak{d}}]^{\lambda} = -f(\xi)^{\lambda} + \varphi(\xi)^{\lambda},$$
$$[a_{\mathfrak{d}}^{\lambda}, \varphi^{\lambda}] = [a_{\mathfrak{d}}, \varphi]^{\lambda} = 0.$$

To conclude we use the fact that $P_D = a_{\mathfrak{d}}^{\lambda} - a_{\mathfrak{d}}^{\rho}$ and, again, the fact that left- and right-invariant vector fields commute. The proof for right-invariant vector fields is similar. □

In fact, the bivector P_D defines a quasi-Poisson structure on the Lie group D (see the definition in Section 3.3).

Modifying the chosen complement $\mathfrak{h}$ of $\mathfrak{g}$ by a twist $t \in \bigwedge^2 \mathfrak{g}$ leads to modifying the bivector P_D in the following simple way,

$$P_D^{\mathfrak{h}'} = P_D^{\mathfrak{h}} + t^\lambda - t^\rho \ , \tag{3.2.9}$$

since $r_{\mathfrak{d}}$ is modified according to (2.3.11).

3.3. **The quasi-Poisson structure of G.** The bivector P_D has a natural restriction to the subgroup $G \subset D$. An isotropic complement $\mathfrak{h}$ such that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ being chosen, to any $g \in G$, one can associate another splitting of $\mathfrak{d}$, namely $\mathfrak{d} = \mathfrak{g} \oplus \mathrm{Ad}_g\, \mathfrak{h}$. Here we use the fact that the Lie subalgebra $\mathfrak{g}$ is stable with respect to Ad_g. The identification j' of $\mathfrak{g}^*$ with $\mathfrak{h}' = \mathrm{Ad}_g\, \mathfrak{h}$ is $j' = \mathrm{Ad}_g \circ j \circ {}^t\mathrm{Ad}_g$, and therefore $r_{\mathfrak{d}}$ is modified to

$$r'_{\mathfrak{d}} = \mathrm{Ad}_g\, r_{\mathfrak{d}} \ , \tag{3.3.1}$$

where Ad denotes the adjoint action of D on the tensor product $\mathfrak{d} \otimes \mathfrak{d}$. We denote the corresponding twist by t_g. According to (2.3.11) we obtain

$$t_g = \mathrm{Ad}_g\, r_{\mathfrak{d}} - r_{\mathfrak{d}} \ . \tag{3.3.2}$$

For each $g \in G$, t_g is an element of $\bigwedge^2 \mathfrak{g}$. Thus, we can define a bivector P_G on G by right translation of t_g, and then

$$P_G = r_{\mathfrak{d}}^\lambda - r_{\mathfrak{d}}^\rho. \tag{3.3.3}$$

Therefore, the embedding of (G, P_G) into (D, P_D) is a bivector map. Actually, this requirement determines P_G uniquely.

It is clear that P_G inherits the multiplicativity property of P_D and the Schouten bracket of P_G is given by the same formula,

$$\frac{1}{2}[P_G, P_G] = \varphi^\rho - \varphi^\lambda. \tag{3.3.4}$$

Note that both φ_g^ρ and φ_g^λ are already elements of $\bigwedge^3 T_g G$, whereas for a canonical r-matrix it is only the difference $(r_{\mathfrak{d}}^\lambda - r_{\mathfrak{d}}^\rho)_g$ which lies in $\bigwedge^2 T_g G$. Moreover,

$$[P_G, \varphi^\lambda] = 0, \quad [P_G, \varphi^\rho] = 0 \ . \tag{3.3.5}$$

We now recall from [11] and [12] the definition of quasi-Poisson Lie groups, which we shall use in the study of quasi-Poisson actions in Section 4.

Definition 3.3.6. A *quasi-Poisson structure* on a Lie group G is defined by a multiplicative bivector P_G and an element φ in $\bigwedge^3 \mathfrak{g}$ such that $\frac{1}{2}[P_G, P_G] = \varphi^\rho - \varphi^\lambda$ and $[P_G, \varphi^\lambda] = 0$.

It also follows from the definition that $[P_G, \varphi^\rho] = 0$, because, by the graded Jacobi identity, $[P_G, \varphi^\rho - \varphi^\lambda] = 0$. The identity $[P_G, \varphi^\lambda] = 0$ is the classical limit of the pentagon identity for quasi-Hopf algebras [7]. For a connected Lie group G, the multiplicativity condition is equivalent to its infinitesimal version,

$$\mathcal{L}_{x^\lambda} P_G = F(x)^\lambda, \tag{3.3.7}$$

for each $x \in \mathfrak{g}$.

Equations (3.3.4) and (3.3.5) show that the Lie group G, equipped with the bivector P_G and the element φ in $\bigwedge^3 \mathfrak{g}$, is a quasi-Poisson Lie group, in the sense of the preceding definition, and so is the Lie group D, with the bivector P_D and the same element φ in $\bigwedge^3 \mathfrak{g}$, considered as an element in $\bigwedge^3 \mathfrak{d}$. This last fact follows from the properties of P_D stated in Propositions 3.2.3 and 3.2.5.

We sometimes refer to the bivector P_G as $P_G^{\mathfrak{h}}$ to emphasize the dependence on the choice of a complement $\mathfrak{h}$. For the bivector $P_G^{\mathfrak{h}}$ corresponding to the complement $\mathfrak{h}$, all the properties required in the definition of a quasi-Poisson Lie group follow from the analogous properties of $P_D^{\mathfrak{h}}$, considered in the previous section. Hence, $(G, P_G^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is a quasi-Poisson Lie group integrating the Lie quasi-bialgebra structure of $\mathfrak{g}$, defined by the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. We sometimes refer to $(G, P_G^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ as $G_D^{\mathfrak{h}}$.

The bivector $P_G^{\mathfrak{h}}$ vanishes if the complement $\mathfrak{h} \subset \mathfrak{d}$ is $\mathrm{ad}(\mathfrak{g})$-invariant, $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, because from $\mathrm{Ad}_g\, \mathfrak{h} = \mathfrak{h}$ it follows that $t_g = \mathrm{Ad}_g\, r_{\mathfrak{d}} - r_{\mathfrak{d}} = 0$. This simple observation implies that, in the standard quasi-triple $(T^*G, G, \mathfrak{g}^*)$, the bivector $P_G = P_G^{\mathfrak{g}^*}$ on the group G vanishes, and also in the quasi-triple $(G \times G, G, \mathfrak{g}_-)$, the bivector $P_G = P_G^{\mathfrak{g}_-}$ vanishes.

Under a twist t, the bivector $P_G^{\mathfrak{h}}$ is modified in the same way as the bivector $P_D^{\mathfrak{h}}$,

$$P_G^{\mathfrak{h}'} = P_G^{\mathfrak{h}} + t^\lambda - t^\rho. \tag{3.3.8}$$

3.4. **The dressing action of D on D/G.** To any group pair (D, G) one can associate the quotient space D/G, which we shall denote by S. The space S will be the target of the generalized moment maps in Section 5.

The action of D on itself by left multiplication induces an action of D on S. Because this action generalizes the dressing of group-valued solutions of field equations, it is called the 'dressing action' [20] [21]. We denote the corresponding infinitesimal action by $X \mapsto X_S$, for $X \in \mathfrak{d}$. By definition, X_S is the projection onto S of the opposite of the right-invariant vector field on D with value X at the identity, e.

Definition 3.4.1. An isotropic complement $\mathfrak{h}$ to $\mathfrak{g}$ in $\mathfrak{d}$ is called *admissible* at the point $s \in D/G$ if the infinitesimal dressing action restricted to $\mathfrak{h}$ defines an isomorphism from $\mathfrak{h}$ onto $T_s(D/G)$.

It is clear that any isotropic complement $\mathfrak{h}$ to $\mathfrak{g}$ is admissible in some open neighborhood of $eG \in D/G$. If the complement $\mathfrak{h}$ is admissible at a point $s \in D/G$, it is also admissible in some open neighborhood U of s. If there exists an $\mathfrak{h}$ which is admissible everywhere on D/G, we call the corresponding quasi-triple $(D, G, \mathfrak{h})$ *complete*.

If we identitfy the tangent spaces to D with $\mathfrak{d}$ by means of right translations, then the tangent space to D/G at $s \in D/G$ is identified with $\mathfrak{d}/\mathrm{Ad}_s\, \mathfrak{g}$, and, for $X \in \mathfrak{d}$, $X_S(s)$ is identified with the class of X in $\mathfrak{d}/\mathrm{Ad}_s\, \mathfrak{g}$. If $\mathfrak{h}$ is admissible, then $\mathfrak{h}$ is isomorphic to $T_s(D/G)$, and we obtain both decompositions,

$$\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h} = \mathrm{Ad}_s\, \mathfrak{g} \oplus \mathfrak{h}\ .$$

If x is in $\mathfrak{g}$, then $x_S(s)$ is identified with the element $\theta_s(x) = (p_{\mathrm{Ad}_s\, \mathfrak{g}} - p_{\mathfrak{g}})(x) \in \mathfrak{h}$, the difference of the projections of x onto $\mathrm{Ad}_s\, \mathfrak{g}$ and $\mathfrak{g}$, parallel to $\mathfrak{h}$. Thus $x_S(s) =$

$(\theta_s(x))_S(s)$. So, if this map θ_s from $\mathfrak{g}$ to $\mathfrak{h}$ is composed with the isomorphism j^{-1} from $\mathfrak{h}$ to $\mathfrak{g}^*$ defined in Section 2.1, we obtain $\tau_s = j^{-1} \circ \theta_s$ from $\mathfrak{g}$ to $\mathfrak{g}^*$ satisfying

$$x_S(s) = ((j \circ \tau_s)(x))_S(s) \tag{3.4.2}$$

and, in particular,

$$(e_i)_S(s) = (\tau_s)_{ik}(j\varepsilon^k)_S(s). \tag{3.4.3}$$

Because $\mathfrak{g}, \mathfrak{h}$ and $\mathrm{Ad}_s\,\mathfrak{g}$ are isotropic, τ_s is anti-symmetric and defines an element in $\bigwedge^2 \mathfrak{g}^*$, which we denote by the same letter.

Proposition 3.4.4. *At any point $s \in D/G$ there exists an admissible complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$.*

Proof. Maximal isotropic subspaces in $\mathfrak{d}$ form a Grassmannian which we denote by $\mathcal{G}(\mathfrak{d})$. Since D is a connected Lie group, the subspaces $\mathfrak{g}$ and $\mathrm{Ad}_s\,\mathfrak{g}$ belong to the same connected component of $\mathcal{G}(\mathfrak{d})$. Let $\mathfrak{h}'$ be an isotropic complement of $\mathfrak{g}$ in $\mathfrak{d}$. The Grassmannian $\mathcal{G}(\mathfrak{d})$ being an algebraic variety, the set of isotropic complements $\mathfrak{h}$ of $\mathfrak{g}$ in the connected component of $\mathfrak{h}'$ is a Zariski open set. Since $\mathrm{Ad}_s\,\mathfrak{g}$ is in the same connected component of $\mathcal{G}(\mathfrak{d})$ as $\mathfrak{g}$, the set of isotropic complements to $\mathrm{Ad}_s\,\mathfrak{g}$ in the connected component of $\mathfrak{h}'$ is also a Zariski open set. An intersection of two nonempty Zariski open sets being nonempty, one can always find an $\mathfrak{h}$ which is an isotropic complement of both $\mathfrak{g}$ and $\mathrm{Ad}_s\,\mathfrak{g}$. Any such subspace is admissible at the point s. □

The choice of an admissible isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ at a point s gives rise to an additional structure on D/G. The space $\mathfrak{h}$ being isomorphic to $T_s(D/G)$, we can define a map from $\mathfrak{g}$ to the 1-forms on D/G at the point s, $x \mapsto \hat{x}_{\mathfrak{h}}(s)$, as follows,

$$< \hat{x}_{\mathfrak{h}}(s), \xi_S(s) > = -(x \mid \xi),$$

for each $\xi \in \mathfrak{h}$, where in the left-hand side we have used the canonical pairing between 1-forms and vectors. The map $\mathfrak{h} \to T_s(D/G)$ being an isomorphism of linear spaces, so is the map $\mathfrak{g} \to T_s^*(D/G)$. In other words, the forms $\hat{x}_{\mathfrak{h}}(s)$ span the cotangent space to D/G at s. Thus if $\mathfrak{h}$ is admissible in an open neighborhood U of s, we define the map $x \mapsto \hat{x}_{\mathfrak{h}}$ from $\mathfrak{g}$ to the 1-forms on $U \subset D/G$, such that

$$< \hat{x}_{\mathfrak{h}}, \xi_S > = -(x \mid \xi)\ . \tag{3.4.5}$$

When the quasi-triple $(D, G, \mathfrak{h})$ is complete, the forms $\hat{x}_{\mathfrak{h}}$ are globally defined on D/G. When $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, integrated to Lie groups D, G, H, then ξ_S is identified with the opposite of the right-invariant vector field on H with value ξ at the identity, and therefore $\hat{x}_{\mathfrak{h}}$ is the right-invariant 1-form on H with value $x \in \mathfrak{g} \simeq \mathfrak{h}^*$ at the identity.

3.5. **Properties of the forms $\hat{x}_{\mathfrak{h}}$ and examples.** The forms $\hat{x}_{\mathfrak{h}}$ will play an essential role in the theory of the moment map that we shall develop in Section 5. In this Section we study the properties of these forms and give examples.

We study the effect of a twist t on the map $x \mapsto \hat{x}_{\mathfrak{h}}$. Let us choose two complements $\mathfrak{h}$ and $\mathfrak{h}'$ admissible at $s \in D/G$. We first compare the dressing vector fields ξ_S and ξ'_S on D/G defined by $\xi_0 \in \mathfrak{g}^*$ corresponding to the splittings $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}'$, respectively, where the twist from $\mathfrak{h}$ to $\mathfrak{h}'$ is $t : \mathfrak{g}^* \to \mathfrak{g}$. Then $\xi_S = (j\xi_0)_S$,

$\xi'_S = (j'\xi_0)_S$. It follows from (3.4.2) that $(t\xi_0)_S(s) = ((j\circ\tau_s)(t\xi_0))_S(s)$, and therefore $\xi'_S(s) = (j(\sigma_s\xi_0))_S(s)$, where $\sigma_s = 1_{\mathfrak{g}^*} + \tau_s \circ t$. Thus

$$< x, \xi_0 > = < \hat{x}_{\mathfrak{h}}(s), (j\xi_0)_S(s) > = < \hat{x}_{\mathfrak{h}'}(s), (j(\sigma_s\xi_0))_S(s) > .$$

The pairing on the left hand side being nondegenerate, σ_s is invertible. Then

$$< x, \xi_0 > = < \ {}^t\sigma_s^{-1}x \ , \ \sigma_s\xi_0 > = < (\widehat{{}^t\sigma_s^{-1}x})_{\mathfrak{h}}(s), (j(\sigma_s\xi_0))_S(s) > \ .$$

Therefore,

$$\hat{x}_{\mathfrak{h}'}(s) = \widehat{(\nu_s x)}_{\mathfrak{h}}(s) \ , \tag{3.5.1}$$

where $\nu_s = \ {}^t\sigma_s^{-1} = (1_{\mathfrak{g}} + t\circ\tau_s)^{-1}$, because t and τ_s are anti-symmetric.

Example 3.5.2. For the standard group pair (T^*G, G), the space $S = T^*G/G$ coincides with $\mathfrak{g}^*$, the dual space to the Lie algebra $\mathfrak{g}$. Using dual bases (e_i) and (ε^i) of $\mathfrak{g}$ and $\mathfrak{g}^*$ one can introduce linear coordinates (ξ_i) on $\mathfrak{g}^*$. The vector fields generating the dressing action are as follows, $\varepsilon^i_S = -\partial/\partial\xi_i$, for the action of $\mathfrak{g}^*$, and $(e_i)_S(\xi) = -\operatorname{ad}^*_{e_i}\xi = f^k_{ij}\xi_k\partial/\partial\xi_j$, for the action of $\mathfrak{g}$. The quasi-triple $(T^*G, G, \mathfrak{g}^*)$ is complete, and the 1-forms corresponding to the elements $e_i \in \mathfrak{g}$ correspond to the differentials of the linear coordinates, $\hat{e}_i = d\xi_i$.

Example 3.5.3. For a Lie algebra $\mathfrak{g}$ with an invariant, nondegenerate symmetric bilinear form, the group pair $(G\times G, G)$ of Example 3.1.3 defines the space $S = (G\times G)/G \cong G$. One can express the dressing vector fields as $(x,0)_S = -x^\rho$ and $(0,x)_S = x^\lambda$, where x^ρ and x^λ are the right- and left-invariant vector fields on the group G defined by the element $x \in \mathfrak{g}$, so that G acts on $S \cong G$ by the adjoint action. The quasi-triple $(G\times G, G, \mathfrak{g}_-)$ is not necessarily complete. The 1-forms $\hat{x}$ are defined by the equation $< \hat{x}, \frac{1}{2}(y^\lambda + y^\rho) > = ((x,x)|(y,-y))$. Using the $\operatorname{Ad}(G)$-invariance of K, we obtain

$$< \hat{x}, y^\lambda > (g) = 2K(x, (1_{\mathfrak{g}} + \operatorname{Ad}_g^{-1})^{-1}y) = 2K((1_{\mathfrak{g}} + \operatorname{Ad}_g)^{-1}x, y) \ .$$

Therefore

$$\hat{x}_g = 2(K^\sharp \circ (1_{\mathfrak{g}} + \operatorname{Ad}_g)^{-1}x)^\lambda_g \ , \tag{3.5.4}$$

where $K^\sharp(x)(y) = K(x,y)$. It is clear that the 1-forms $\hat{x}$ are well defined if -1 is not an eigenvalue of the operator Ad_g.

At the points $g \in G$ such that Ad_g has eigenvalue -1, the complement $\mathfrak{g}_-$, defined by $\mathfrak{g}_- = \{\frac{1}{2}(x,-x)|x \in \mathfrak{g}\}$, is not admissible. Let us assume that G is a compact simple Lie group. Without loss of generality, we can assume that g belongs to a maximal torus T. The eigenspace V corresponding to the eigenvalue -1 of Ad_g in $\mathfrak{g}$ is then even-dimensional, and it splits into the orthogonal direct sum,

$$V = \oplus_{\alpha\in\Gamma} V_\alpha,$$

where each V_α is two-dimensional, and spanned by the linear combination of the root vectors $e_\alpha, e_{-\alpha}$,

$$a_\alpha = \frac{e_\alpha + e_{-\alpha}}{\sqrt{2}}, \ b_\alpha = \frac{e_\alpha - e_{-\alpha}}{i\sqrt{2}} \ ,$$

the sum being taken over a subset Γ of the set of positive roots. If $K(e_\alpha, e_{-\alpha}) = 1$, then (a_α, b_α) is an orthonormal basis of V_α. Let us twist the complement $\mathfrak{g}_-$ by $t = \frac{\varepsilon}{2}\sum_{\alpha\in\Gamma} a_\alpha \wedge b_\alpha$, where ε is a nonzero real number. After the twist, the

new complement is the direct sum of the orthogonal of V and of the subsbace of $\oplus_\alpha(V_\alpha \oplus V_\alpha)$ spanned by the vectors,

$$\frac{1}{2}(a_\alpha, -a_\alpha) + \varepsilon(b_\alpha, b_\alpha),\ \frac{1}{2}(b_\alpha, -b_\alpha) - \varepsilon(a_\alpha, a_\alpha).$$

Let us compute the 1-forms $\hat{x}'$ defined by the choice of the admissible complement thus obtained by twisting. If x is in the orthogonal complement to the eigenspace V, then $\hat{x}'$ coincides with the 1-forms $\hat{x}$ of formula (3.5.4). In order to determine the 1-forms corresponding to a basis of V, it is sufficient to deal with one of the V_α's, and we shall omit the index α. The dressing vector field on G corresponding to (x, x) is $x^\lambda - x^\rho$, and that corresponding to $(x, -x)$ is $-(x^\lambda + x^\rho)$. Assuming that x is in V, then $\mathrm{Ad}_g\, x = -x$, and therefore $(x, x)_G = 2x^\lambda$ and $(x, -x)_G = 0$. Thus, the dressing vector fields corresponding to this basis of V are $2\varepsilon b^\lambda$ and $-2\varepsilon a^\lambda$, respectively. The dual 1-forms are defined by

$$< \hat{a}', 2\varepsilon b^\lambda > = -\left((a, a)|(\frac{1}{2}a + \varepsilon b, -\frac{1}{2}a + \varepsilon b)\right) = -K(a, a)\ ,$$

$$< \hat{a}', 2\varepsilon a^\lambda > = \left((a, a)|(\frac{1}{2}b - \varepsilon a, -\frac{1}{2}b - \varepsilon a)\right) = K(a, b)$$

and similarly for b,

$$< \hat{b}', 2\varepsilon b^\lambda > = -K(a, b)\ ,$$
$$< \hat{b}', 2\varepsilon a^\lambda > = K(b, b)\ .$$

Therefore

$$\hat{a}' = -\frac{1}{2\varepsilon}K(b, \theta),\ \hat{b}' = \frac{1}{2\varepsilon}K(a, \theta),$$

where θ is the left-invariant Maurer-Cartan form.

3.6. **The bivector on D/G.** If we choose a quasi-triple $(D, G, \mathfrak{h})$ corresponding to the group pair (D, G), a bivector is defined on the space $S = D/G$ introduced in the previous subsection. Since the bivector $P_D = r^\lambda_{\mathfrak{d}} - r^\rho_{\mathfrak{d}}$ is projectable by the projection of D onto D/G, it defines a bivector P_S on $S = D/G$. Because all left-invariant vector fields generated by $\mathfrak{g}$ are projected to zero, the projection of $r^\lambda_{\mathfrak{d}}$ vanishes, and therefore

$$P_S = -(r_{\mathfrak{d}})_S, \tag{3.6.1}$$

The notation $(r_{\mathfrak{d}})_S$ refers to the homomorphism from $\mathfrak{d}$ to the vector fields on D/G induced by the action of D on D/G, extended to a homomorphism from the tensor algebra of $\mathfrak{d}$ to the algebra of contravariant tensors on $S = D/G$. Note that although $r_{\mathfrak{d}} \in \mathfrak{d} \otimes \mathfrak{d}$ is not anti-symmetric, after projection it defines a bivector on S. The bivector P_S corresponding to a complement $\mathfrak{h}$ will often be referred to as $P^{\mathfrak{h}}_S$.

The properties of P_S follow from those of P_D. Since the Schouten bracket of P_S is the projection of that of P_D, from (3.2.4) we obtain,

$$\frac{1}{2}[P_S, P_S] = \varphi_S. \tag{3.6.2}$$

Clearly, the action of D on D/G is a bivector map from $(D, P_D) \times (S, P_S)$ to (S, P_S). It follows from (2.3.10) that

$$\mathcal{L}_{x_S} P_S = -F(x)_S\ ,\ \mathcal{L}_{\xi_S} P_S = f(\xi)_S - \varphi(\xi)_S\ , \tag{3.6.3}$$

for all $x \in \mathfrak{g}, \xi \in \mathfrak{h}$.

Under a twist t of the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ into $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}')$, the bivector $P_S^{\mathfrak{h}}$ is modified to $P_S^{\mathfrak{h}'} = P_S^{\mathfrak{h}} - t_S$.

If the isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Lie subalgebra of $\mathfrak{d}$, all the bivectors $P_D^{\mathfrak{h}}$, $P_G^{\mathfrak{h}}$ and $P_S^{\mathfrak{h}}$ have vanishing Schouten brackets, and define Poisson brackets satisfying the Jacobi identity on the corresponding spaces.

The bivector P_S has an interesting characteristic property which plays the key role in the moment map theory. Let $(P_S^{\mathfrak{h}})^\sharp$ be the map from 1-forms to vectors defined by $< (P_S^{\mathfrak{h}})^\sharp \alpha, \beta > = P_S^{\mathfrak{h}}(\alpha, \beta)$, for any 1-forms α, β on S.

Proposition 3.6.4. *Let $(D, G, \mathfrak{h})$ be a quasi-triple such that $\mathfrak{h}$ is admissible on an open neighborhood U of $s \in D/G$, and let $P_S^{\mathfrak{h}}$ be the corresponding bivector on $S = D/G$. Then, for any $x \in \mathfrak{g}$,*

$$(P_S^{\mathfrak{h}})^\sharp(\hat{x}_{\mathfrak{h}}) = x_S \ , \tag{3.6.5}$$

holds on U, where $\hat{x}_{\mathfrak{h}}$ is defined by (3.4.5)*. This property uniquely characterizes the bivector $P_S^{\mathfrak{h}}$ in the neighborhood U.*

Proof. We choose an isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$, admissible in an open neighborhood U of $s \in D/G$. Let x be an element in $\mathfrak{g}$. We apply the map $(P_S^{\mathfrak{h}})^\sharp$, where $P_S^{\mathfrak{h}} = -(e_i)_S \otimes (\varepsilon^i)_S$, to the 1-form $\hat{x}_{\mathfrak{h}}$. By definition, we obtain,

$$(P_S^{\mathfrak{h}})^\sharp(\hat{x}_{\mathfrak{h}}) = (x|\varepsilon^i)(e_i)_S = x_S \ , \tag{3.6.6}$$

which proves formula (3.6.5). The complement $\mathfrak{h}$ being admissible, the 1-forms $(\hat{e}_i)_{\mathfrak{h}}$ form a basis of the cotangent space to D/G at each point in U. Hence, formula (3.6.5) gives a characterization of the bivector $P_S^{\mathfrak{h}}$ on U. □

Example 3.6.7. The standard Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ is in fact a Manin triple, so the bivector P_S has a vanishing Schouten bracket and defines a Poisson bracket on $\mathfrak{g}^*$. Actually, the induced bivector on $S = \mathfrak{g}^*$ coincides with the Kirillov-Kostant-Souriau bivector,

$$P_S = -(r_{\mathfrak{d}})_S = -(e_i)_S \otimes (\varepsilon^i)_S = \frac{1}{2} \sum_{ijk} f_{ij}^k \xi_k \, \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j}. \tag{3.6.8}$$

Example 3.6.9. For the quasi-triple $(G \times G, G, \mathfrak{g}_-)$, the bivector P_S has the form

$$P_S = -(r_{\mathfrak{d}})_S = -\sum_i (\Delta e_i)_S \otimes (\Delta_- e_i)_S = \frac{1}{2} \sum_i e_i^\lambda \wedge e_i^\rho. \tag{3.6.10}$$

Here we used the fact that $\sum_i e_i e_i \in U(\mathfrak{g})$ is a Casimir element, and therefore $\sum_i e_i^\lambda \otimes e_i^\lambda = \sum_i e_i^\rho \otimes e_i^\rho$. Usually, the Schouten bracket of the bivector (3.6.10) is nonvanishing. A notable exception is the group $G = SU(2)$, where φ_S vanishes although $\varphi \neq 0$ [23].

4. Quasi-Poisson actions

We shall now introduce quasi-Poisson actions in general, and show that the actions that we have described in Section 3 are examples of quasi-Poisson actions arising from Manin pairs.

4.1. **The definition of quasi-Poisson actions.** Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let M be a manifold on which the Lie group G acts. We shall denote by x_M the vector field on M corresponding to $x \in \mathfrak{g}$, and, more generally, the multivector field on M corresponding to $x \in \bigwedge \mathfrak{g}$. By convention, x_M satisfies

$$(x_M.f)(m) = \frac{d}{dt} f(\exp(-tx).m)|_{t=0} \ , \tag{4.1.1}$$

for $x \in \mathfrak{g}, m \in M$ and $f \in C^\infty(M)$. We first recall from [17] the following characterization of Poisson actions of connected Poisson Lie groups on Poisson manifolds.

Proposition 4.1.2. *If the action of a Poisson Lie group (G, P_G) on a Poisson manifold (M, P_M) is a Poisson action, then, for each $x \in \mathfrak{g}$,*

$$\mathcal{L}_{x_M} P_M = -F(x)_M. \tag{4.1.3}$$

The converse holds if G is connected.

Thus, in this case $[P_M, P_M] = 0$ (since P_M is Poisson) and (4.1.3) holds.

Remark 4.1.4. There is a simple interpretation of the above characterization of Poisson actions. Equation (4.1.3) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \bigwedge \mathfrak{g} & \xrightarrow{d_F} & \bigwedge \mathfrak{g} \\ \downarrow & & \downarrow \\ \bigwedge \mathcal{X}(M) & \xrightarrow{d_{P_M}} & \bigwedge \mathcal{X}(M) \end{array}$$

where the vertical arrows are induced by the infinitesimal action $x \in \mathfrak{g} \mapsto x_M \in \mathcal{X}(M)$, the linear space of vector fields on M. The map d_F is, up to a sign, the Chevalley-Eilenberg cohomology operator of the Lie algebra $\mathfrak{g}^*$ with bracket F, with values in the trivial $\mathfrak{g}^*$-module. More precisely, $d_F = [F, \ . \ ]$, where $[\ ,\ ]$ is the "big bracket" on $\bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*)$. (See [12].) The map $d_{P_M} = [P_M, \ . \ ]$, where $[\ ,\ ]$ is the Schouten bracket of multivectors, is the Lichnerowicz-Poisson cohomology operator on multivectors on M. In other words, the action $x \in \mathfrak{g} \mapsto x_M \in \mathcal{X}(M)$ is the infinitesimal of a Poisson action if and only if it induces a morphism from the complex $(\bigwedge \mathfrak{g}, d_F)$ to the complex $(\bigwedge \mathcal{X}(M), d_{P_M})$.

The following definition generalizes the notion of a Poisson action.

Definition 4.1.5. Let (G, P_G, φ) be a connected quasi-Poisson Lie group acting on a manifold M with a bivector P_M. The action of G on M is said to be a *quasi-Poisson action* if and only if

$$\frac{1}{2}[P_M, P_M] = \varphi_M, \tag{4.1.6}$$

$$\mathcal{L}_{x_M} P_M = -F(x)_M, \tag{4.1.7}$$

for each $x \in \mathfrak{g}$.

Let $(D, G, \mathfrak{h})$ be a quasi-triple. We consider G with the quasi-Poisson structure defined in Section 3.3, and the space $S = D/G$ with the bivector P_S defined in Section 3.5. Then the dressing action of G on D/G is quasi-Poisson. In fact, properties (4.1.6) and (4.1.7) were proved in Section 3.5. The action of G obtained

by restriction to any G-invariant embedded submanifold of D/G is also a quasi-Poisson action. In particular, those G-orbits that are embedded submanifolds of D/G are *quasi-Poisson spaces*, in the sense that they are manifolds with a bivector on which a quasi-Poisson Lie group acts by a quasi-Poisson action.

Under a twist t, the quasi-Poisson Lie group $(G, P_G^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is modified to $(G, P_G^{\mathfrak{h}'}, \varphi_{\mathfrak{h}'})$, where $P_G^{\mathfrak{h}'}, \varphi_{\mathfrak{h}'}$ are given by (3.3.8) and (2.3.14), and $(M, P_M^{\mathfrak{h}})$ is modified to $(M, P_M^{\mathfrak{h}'})$, where

$$P_M^{\mathfrak{h}'} = P_M^{\mathfrak{h}} - t_M \ . \tag{4.1.8}$$

A simple computation in terms of Schouten brackets shows that after the twist the action remains quasi-Poisson,

$$\mathcal{L}_{x_M} P_M^{\mathfrak{h}'} = \mathcal{L}_{x_M} P_M^{\mathfrak{h}} - [x_M, t_M] = -(F_{\mathfrak{h}}(x) + \mathrm{ad}_x\, t)_M = -F_{\mathfrak{h}'}(x)_M,$$
$$\frac{1}{2}[P_M^{\mathfrak{h}'}, P_M^{\mathfrak{h}'}] = \frac{1}{2}[P_M^{\mathfrak{h}}, P_M^{\mathfrak{h}}] + \frac{1}{2}[t_M, t_M] - [P_M^{\mathfrak{h}}, t_M] = (\varphi_{\mathfrak{h}} - \langle t, t\rangle + \varphi_1)_M = (\varphi_{\mathfrak{h}'})_M,$$

where we have used the transformation rules and the notations of (2.3.13) and (2.3.14).

The last calculation shows that one can consider a family of quasi-Poisson Lie groups $G_D^{\mathfrak{h}}$ acting on a family of quasi-Poisson spaces $(M, P_M^{\mathfrak{h}})$, where $P_M^{\mathfrak{h}'} = P_M^{\mathfrak{h}} - t_M$ when the complements $\mathfrak{h}$ and $\mathfrak{h}'$ are related by a twist t. We have just shown that when the action of $G_D^{\mathfrak{h}}$ on $(M, P_M^{\mathfrak{h}})$ is quasi-Poisson for an isotropic complement $\mathfrak{h}$ it is also quasi-Poisson for any $\mathfrak{h}'$. In the moment map theory which we shall present in the next section, it is more convenient to consider families $G_D^{\mathfrak{h}}$ acting on $(M, P_M^{\mathfrak{h}})$ than individual quasi-Poisson Lie groups acting on individual quasi-Poisson spaces. It would be interesting to find a geometric framework for this construction which does not explicitly refer to the choice of an isotropic complement.

Remark 4.1.9. In the case of quasi-Poisson actions, the operators $d_F = \big[F,\ .\ \big]$ and $d_{P_M} = [P_M,\ .\ ]$ can still be defined, but their squares do not vanish. In fact, $(d_F)^2 = [\varphi,\ .\ ]$ and $(d_{P_M})^2 = [\varphi_M,\ .\ ]$. In the first formula the bracket is the algebraic Schouten bracket on $\bigwedge \mathfrak{g}$ and, in the second, it is the Schouten bracket of multivectors on M. Formula (4.1.7) can still be interpreted as the commutattivity of the diagram of Remark 4.1.4, defined by the map from $\bigwedge \mathfrak{g}$ to $\bigwedge \mathcal{X}(M)$ induced by the action of G on M, and by the operators d_F and d_{P_M}. It follows that the squares of these operators commute with the induced map, and this implies that the 3-vector $\frac{1}{2}[P, P] - \varphi_M$ has vanishing Schouten bracket with any multivector in the image of the induced map. Condition (4.1.6) expresses the fact that this 3-vector actually vanishes.

4.2. **Properties of quasi-Poisson actions.** First, we characterize the quasi-Poisson actions as bivector maps.

Proposition 4.2.1. *Let $\rho\ : G \times M \to M$ be an action of a connected quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ on a manifold M equipped with a bivector $P_M^{\mathfrak{h}}$ which satisfies the property $\frac{1}{2}[P_M^{\mathfrak{h}}, P_M^{\mathfrak{h}}] = (\varphi_{\mathfrak{h}})_M$. Then ρ is a quasi-Poisson action if and only if ρ maps the bivector $P_G^{\mathfrak{h}} + P_M^{\mathfrak{h}}$ on $G \times M$ to $P_M^{\mathfrak{h}}$.*

Proof. The proof is identical to that in the case of Poisson actions. See [17] [13]. □

Next, we introduce the notion of quasi-Poisson reduction, similar to the usual Poisson reduction, which yields a genuine Poisson structure on the space of orbits.

Theorem 4.2.2. *Let $G_D^{\mathfrak{h}}$ be a connected quasi-Poisson Lie group acting on a manifold $(M, P_M^{\mathfrak{h}})$ by a quasi-Poisson action. Then the bivector $P_M^{\mathfrak{h}}$ defines a Poisson bracket on the space $C^\infty(M)^G$ of smooth G-invariant functions on M. This Poisson bracket is independent of the choice of $\mathfrak{h}$.*

Proof. First, we show that the bracket of G-invariant functions, f_1, f_2, is G-invariant. Indeed,

$$(\mathcal{L}_{x_M} P_M^{\mathfrak{h}})(df_1, df_2) = -F(x)_M(df_1, df_2) = 0 \ ,$$

for any $x \in \mathfrak{g}$, because $F(x) \in \bigwedge^2 \mathfrak{g}$ and, hence, $F(x)_M$ annihilates $df_1 \wedge df_2$. Next, we observe that the bracket defined by $P_M^{\mathfrak{h}}$ on invariant functions is a Poisson bracket,

$$\frac{1}{2}[P_M^{\mathfrak{h}}, P_M^{\mathfrak{h}}](df_1, df_2, df_3) = \varphi_M(df_1, df_2, df_3) = 0 \ ,$$

because $\varphi \in \bigwedge^3 \mathfrak{g}$ and f_1, f_2 and f_3 are G-invariant. Finally, if one modifies the complement $\mathfrak{h}$ to $\mathfrak{h}'$, the bivector on M is modified by a twist, $P_M^{\mathfrak{h}'} = P_M^{\mathfrak{h}} - t_M$, and the Poisson bracket of invariant functions is unchanged,

$$P_M^{\mathfrak{h}'}(df_1, df_2) = P_M^{\mathfrak{h}}(df_1, df_2) - t_M(df_1, df_2) = P_M^{\mathfrak{h}}(df_1, df_2)$$

because t_M annihilates invariant functions. □

Let us introduce the projection $p : M \to M/G$ onto the space of G-orbits on M. If the G-action is free and proper in a neighborhood U of $x \in M$, the space M/G is smooth near $p(x)$, and P_M defines a Poisson structure on $p(U)$.

5. Generalized moment maps

In this section we define moment maps for quasi-Poisson actions. For the actions of a quasi-Poisson Lie group $G_D^{\mathfrak{h}}$, the space $S = D/G$ is the target of the moment maps. We always assume that a moment map $\mu : M \to D/G$ is equivariant with respect to the G-action on M and the dressing action on D/G. As we show in Proposition 5.1.5, the equivariance condition ensures that the definition is independent of the choice of an admissible complement $\mathfrak{h} \subset \mathfrak{d}$. The following definition of a moment map for a quasi-Poisson action of a quasi-Poisson Lie group on a manifold with a bivector is a generalization of the notion of equivariant moment map for the Poisson action of a Poisson Lie group on a Poisson manifold, defined in [16], which itself generalizes the usual notion of equivariant moment map for a hamiltonian action of a Lie group on a Poisson manifold.

5.1. **Definition of a moment map.** Let $G_D^{\mathfrak{h}}$ be a connected quasi-Poisson Lie group, and let $x \mapsto x_M$ be the action of $\mathfrak{g}$ on M by the infinitesimal generators, defined by (4.1.1), of a quasi-Poisson action of $G_D^{\mathfrak{h}}$ on $(M, P_M^{\mathfrak{h}})$. We define the map $(P_M^{\mathfrak{h}})^\sharp$ from 1-forms on M to vector fields on M by $< (P_M^{\mathfrak{h}})^\sharp \alpha, \beta > = P_M^{\mathfrak{h}}(\alpha, \beta)$.

Definition 5.1.1. A map μ from M to D/G, equivariant with respect to the action of G on M and to the dressing action of G on D/G, is called a *moment map* for the action of $G_D^{\mathfrak{h}}$ on $(M, P_M^{\mathfrak{h}})$ if, on any open subset $\Omega \subset M$,

$$(P_M^{\mathfrak{h}})^\sharp(\mu^* \hat{x}_{\mathfrak{h}}) = x_M \ , \tag{5.1.2}$$

for any $\mathfrak{h}$ admissible on $\mu(\Omega)$. The action of $G_D^{\mathfrak{h}}$ on $(M, P_M^{\mathfrak{h}})$ is called *hamiltonian* if it admits a moment map. A *hamiltonian quasi-Poisson space* is a manifold with a bivector on which a quasi-Poisson Lie group acts by a hamiltonian action.

We observe that, in this generalized situation, the target of the moment map is a hamiltonian quasi-Poisson space. (The term "quasi-hamiltonian" was used in [3] but in the context of group valued moment maps.) From (3.6.5), it follows immediately that the dressing action of G on D/G is hamiltonian and has the identity of D/G as a moment map. Moreover, let N be any G-invariant embedded submanifold of $S = D/G$. There is a unique bivector P_N on N such that the embedding of N into S is a bivector map. Then the hamiltonian dressing action of G on S restricts to a hamiltonian action of G on N, and has the embedding of N into $S = D/G$ as a moment map. In particular, the orbits of the dressing action of G on D/G which are embedded submanifolds are hamiltonian quasi-Poisson spaces.

Example 5.1.3. If the Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, then $\varphi = 0$, so (G, P_G) is a Poisson Lie group and (M, P_M) is a Poisson manifold. In this case, the preceding definition of an equivariant moment map reduces to that given by Lu in [16], that is, the infinitesimal generator of the group action x_M is the image under the map $P_M^\sharp$ of the pull-back by the moment map of the right-invariant 1-form with value x at the identity in the dual group D/G of G.

In the particular case of the standard quasi-triple, $(T^*G, G, \mathfrak{g}^*)$, which corresponds to the Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$, with $F = 0$ and $\varphi = 0$, the Poisson bivector P_G vanishes and the dual group is the abelian group $\mathfrak{g}^*$. The moment map then takes values in the vector space $\mathfrak{g}^*$. For any $x \in \mathfrak{g}$, the right-invariant 1-form x^ρ is the constant form x on $\mathfrak{g}^*$, and its pull-back by the moment map is $d(\mu(x))$, so we recover the usual definition of the moment map for a hamiltonian action. In this case, the orbits of the dressing action of G on $\mathfrak{g}^*$ are the coadjoint orbits familiar from the usual moment map theory.

Example 5.1.4. In the case of $(G \times G, G, \mathfrak{g}_-)$, the quotient space $D/G = G$ is diffeomorphic to the group G, and the dressing action is the action by conjugation. Hence, the conjugacy classes in G are hamiltonian quasi-Poisson spaces, and the inclusion is a moment map.

In fact, we do not need to impose the moment map condition (5.1.2) for all admissible complements because conditions (5.1.2) for different admissible complements are equivalent.

Proposition 5.1.5. *Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two complements admissible at a point $s \in D/G$, and let $m \in M$ be such that $\mu(m) = s$. Then, at the point m, conditions* (5.1.2) *for $\mathfrak{h}$ and $\mathfrak{h}'$ are equivalent, namely $(P_M^{\mathfrak{h}})^\sharp(\mu^*\hat{x}_{\mathfrak{h}}) = (P_M^{\mathfrak{h}'})^\sharp(\mu^*\hat{x}_{\mathfrak{h}'})$.*

Proof. To prove the proposition, we use both (3.5.1) and the above definition of the moment map and its equivariance. Thus, at a point m such that $\mu(m) = s$,

$$\begin{aligned}(P_M^{\mathfrak{h}'})^\sharp(\mu^*\hat{x}_{\mathfrak{h}'}) = (P_M^{\mathfrak{h}} - t_M)^\sharp(\mu^*\hat{x}_{\mathfrak{h}'}) &= (P_M^{\mathfrak{h}})^\sharp(\mu^*\widehat{\nu x}_{\mathfrak{h}}) - t_M(\mu^*\widehat{\nu x}_{\mathfrak{h}}) \\ &= (\nu x)_M + ((t \circ \tau_s)(\nu x))_M = ((1_{\mathfrak{g}} + t \circ \tau_s)(\nu x))_M = x_M.\end{aligned}$$

Here we have used the equivariance property of the moment map and formula (3.4.3), which imply that, for any y in $\mathfrak{g}$,

$$< \mu^* \hat{y}_{\mathfrak{h}}, (e_i)_M > (m) = < \hat{y}_{\mathfrak{h}}, (e_i)_S > (s) = (\tau_s)_{ik} y^k,$$

when $\mu(m) = s$, and hence

$$t_M(\mu^* \hat{y}_{\mathfrak{h}})(m) = -((t \circ \tau_s)y)_M(m) \ . \tag{5.1.6}$$

The proposition is therefore proved. □

5.2. **Torus-valued moment maps.** Let us consider the Manin pair $(\mathfrak{d}, \mathfrak{g})$, where $\mathfrak{d} = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ and $\mathfrak{g} = \mathfrak{u}(1)$. Here $\mathfrak{d}$ is the direct sum of two copies of $\mathfrak{g}$, and this is a particular case of Example 2.1.5. There are two group pairs with a compact subgroup corresponding to this Manin pair, $(S^1 \times \mathbb{R}, S^1)$ and $(S^1 \times S^1, S^1)$. The first group pair corresponds to the usual moment map theory. Let us show that the notion of a moment map for the second group pair extends that of an S^1-valued moment map in the sense of McDuff [18] [9] [24] to the case of a manifold with a possibly degenerate Poisson bivector. A symplectic action of S^1 on a symplectic manifold (M, ω_M) is also a quasi-Poisson action of the quasi-Poisson Lie group S^1 defined by the quasi-triple $(S^1 \times S^1, S^1, \mathfrak{g}_-)$, where $\mathfrak{g}_- = \frac{1}{2}\Delta_-(\mathfrak{u}(1))$ (see 2.1.5), because in this case $F = 0$ and $\varphi = 0$. Here $S = (S^1 \times S^1)/S^1 \simeq S^1$. Since S^1 is abelian, we see from (3.5.4) that $\hat{X} = -d\alpha$, where X is the generator of $\mathfrak{u}(1)$ and α is the parameter on S^1. A moment map for this quasi-Poisson action is a map, μ, from M to S^1, satisfying $P_M^\sharp(\mu^* d\alpha) = X_M$. When P_M is nondegenerate with inverse ω_M, this condition is equivalent to $i_{X_M}\omega_M = \mu^* d\alpha$, where i denotes the interior product, which is the defining property of an S^1-valued moment map.

More generally, for $r > 1$, T^r-valued moment maps (see [9]) for symplectic actions of an r-dimensional torus $T^r = S^1 \times \ldots \times S^1$ are also moment maps for the quasi-Poisson action of the quasi-Poisson Lie group T^r defined by the quasi-triple $(T^r \times T^r, T^r, \mathfrak{g}_-)$, where $\mathfrak{g}$ is the Lie algebra $\mathfrak{t}^r = \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1)$, since in this case also $F = 0$ and $\varphi = 0$, and the preceding relation is valid for each copy of S^1.

When G is a compact, connected abelian Lie group, quasi-hamiltonian G-spaces (M, ω_M) in the sense of [3] are necessarily symplectic. When M is equipped with the nondegenerate bivector P_M defined by ω_M, it is also a hamiltonian quasi-Poisson space for the quasi-Poisson Lie group G defined by the quasi-triple $(G \times G, G, \mathfrak{g}_-)$, because in this case, (5.1.2) coincides with the defining property of the group-valued moment map of [3]. For nonabelian compact Lie groups, it can be shown [2] that the moment map theory developed here also coincides with the moment map theory of [3].

5.3. **The standard quasi-triple and group-valued moment maps.** We now summarize the case of the hamiltonian quasi-Poisson spaces, already considered in Examples 2.1.5, 3.1.3, 3.5.3, 3.6.9 and 5.1.4.

If G is a connected Lie group with a bi-invariant scalar product, then there is a well-defined Ad-invariant element φ in $\bigwedge^3 \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. If $(K^\sharp)^{-1}$ denotes the isomorphism between $\mathfrak{g}^*$ and $\mathfrak{g}$ defined by the scalar product K on $\mathfrak{g}$, then φ satisfies

$$\varphi(\xi, \eta, \zeta) = \frac{1}{4} \ < \xi, [(K^\sharp)^{-1}\eta, (K^\sharp)^{-1}\zeta] > \ .$$

The Lie group G diagonally embedded in $G\times G$ defines a group pair and an associated quasi-triple is $(\mathfrak{g}\oplus\mathfrak{g}, \Delta(\mathfrak{g}), \frac{1}{2}\Delta_-(\mathfrak{g}))$, with the scalar product on $\mathfrak{g}\oplus\mathfrak{g}$ defined by (2.1.6). In this quasi-triple, $F=0$ and the element φ in $\bigwedge^3\mathfrak{g}$ is the one defined above.

Let M be a manifold on which the Lie group G acts and let P be a bivector field on M. Then (M,P) is a *quasi-Poisson space* if P is G-invariant and

$$\frac{1}{2}[P,P]=\varphi_M\ , \tag{5.3.1}$$

where φ_M is the field of trivectors on M induced from φ by the infinitesimal action of $\mathfrak{g}$ on M.

If one identifies G with $S=(G\times G)/G$, then the dressing action of G on S is identified with the adjoint action of G on itself. The 1-forms $\hat{x}$ on G defined by the choice of the above quasi-triple are such that

$$\hat{x}_g=2K((1_{\mathfrak{g}}+\mathrm{Ad}_g)^{-1}x,\theta_g)\ , \tag{5.3.2}$$

for $g\in G$, where θ is the left Maurer-Cartan form on G.

According to Definition 5.1.1, (M,P) is called a *hamiltonian quasi-Poisson space* if it is a quasi-Poisson space and if, moreover, there exists a moment map for the quasi-Poisson action of G on M.

Proposition 5.3.3. *Let (M,P) be a manifold with a bivector on which the compact simple Lie group G acts and which is a quasi-Poisson space. Then (M,P) is a hamiltonian quasi-Poisson space if and only if there exists a map $\mu: M\to G$ which is equivariant with respect to the given action of G on M and the adjoint action of G on itself, and which satisfies*

$$P^\sharp(\mu^*K(x,\theta))=\frac{1}{2}((1_{\mathfrak{g}}+\mathrm{Ad}_\mu)x)_M\ , \tag{5.3.4}$$

for all $x\in\mathfrak{g}$.

Proof. At each point $m\in M$, let us apply the construction of Example 3.5.3, with $g=\mu(m)$, to obtain an admissible complement by means of which we formulate the definition of the moment map. The bivector on M, after the twist $t=\frac{\varepsilon}{2}\sum_{\alpha\in\Gamma}a_\alpha\wedge b_\alpha$, is

$$P'=P-\frac{\varepsilon}{2}\sum_{\alpha\in\Gamma}(a_\alpha)_M\wedge(b_\alpha)_M\ .$$

If x is in the orthogonal complement of the kernel of $1_{\mathfrak{g}}+\mathrm{Ad}_{\mu(m)}$, then $\hat{x}'=\hat{x}$, and $P'(\mu^*\hat{x})=P(\mu^*\hat{x})$. Taking into account (5.3.2), we see that the condition (5.3.4) is then equivalent to the definition $P^\sharp_m(\mu^*(\hat{x}_{\mu(m)}))=x_M(m)$. For $a_\alpha,b_\alpha\in V_\alpha$, the moment map conditions are $P'^\sharp_M(\mu^*\hat{a}'_\alpha)=(a_\alpha)_M$, $P'^\sharp_M(\mu^*\hat{b}'_\alpha)=(b_\alpha)_M$. We replace $\hat{a}'_\alpha,\hat{b}'_\alpha$ by their values found in Example 3.5.3. Using the equivariance of the moment map, and the fact that $x^\rho=-x^\lambda$, for $x\in V$, we find, for all $\alpha,\beta\in\Gamma$,

$$\frac{\varepsilon}{2}\sum_{\alpha\in\Gamma}((a_\alpha)_M\wedge(b_\alpha)_M)(\mu^*\hat{a}'_\beta)=-(a_\beta)_M,$$

$$\frac{\varepsilon}{2}\sum_{\alpha\in\Gamma}((a_\alpha)_M\wedge(b_\alpha)_M)(\mu^*\hat{b}'_\beta)=-(b_\beta)_M,$$

Thus the moment map condition, for $x \in V$, reduces to

$$P^\sharp(\mu^* K(x,\theta)) = 0 \ ,$$

and therefore it coincides with (5.3.4). □

5.4. **Generalized foliations of hamiltonian quasi-Poisson spaces.** We now consider a hamiltonian quasi-Poisson G-space $(M, P_M^{\mathfrak{h}})$, as in Definition 5.1.1, where $G = G_D^{\mathfrak{h}}$ is a quasi-Poisson Lie group defined by a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. We wish to show that, under an additional assumption, there is an integrable generalized distribution on M defined by the image of any bivector $P_M^{\mathfrak{h}}$, where $\mathfrak{h}$ is admissible.

It follows from the existence of a moment map for the action of G on M that, at each point in M where $\mathfrak{h}$ is an admissible complement, the image of $(P_M^{\mathfrak{h}})^\sharp$ contains the tangent space to the G-orbit through this point.

We further observe that, under a change of admissible complement, when $P_M^{\mathfrak{h}}$ is modified to $P_M^{\mathfrak{h}'} = P_M^{\mathfrak{h}} - t_M$ (see (4.1.8)), the image of $(P_M^{\mathfrak{h}'})^\sharp$ coincides with that of $(P_M^{\mathfrak{h}})^\sharp$. In fact, since by the moment map property the image of $t_M^\sharp$ is contained in the tangent space to the G-orbit, then $\operatorname{Im} t_M^\sharp \subset \operatorname{Im}(P_M^{\mathfrak{h}})^\sharp$, and therefore $\operatorname{Im}(P_M^{\mathfrak{h}'})^\sharp \subset \operatorname{Im}(P_M^{\mathfrak{h}})^\sharp$. By symmetry, the two images coincide. We denote by $\mathcal{D}$ the generalized distribution thus defined on M.

Proposition 5.4.1. *The distribution $\mathcal{D}$ on the quasi-Poisson space M satisfies the Frobenius property, $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.*

Proof. Let $P = P_M^{\mathfrak{h}}$ for an admissible $\mathfrak{h}$. Let f and g be arbitrary functions on M, and let $[P,P]^\sharp(df, dg)$ denote the vector field on M such that

$$< [P,P]^\sharp(df,dg), \alpha >= [P,P](df, dg, \alpha) \ ,$$

for any 1-form α on M. Then

$$[P^\sharp(df), P^\sharp(dg)] - P^\sharp d\{f,g\} = -\frac{1}{2}[P,P]^\sharp(df,dg) \ .$$

Since $\frac{1}{2}[P,P] = \varphi_M$, and since the tangent space to the G-orbit at each point is contained in the distribution $\operatorname{Im} P^\sharp = \mathcal{D}$, we conclude that $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. □

To conclude that the distribution $\mathcal{D}$ is completely integrable, it is enough, by the Stefan-Sussmann theorem (see *e.g.*, [22]), to assume that the rank of $\mathcal{D}$ is constant along the trajectory of any "hamiltonian vector field", $P^\sharp(df), f \in C^\infty(M)$. If μ_t is the flow of $P^\sharp(df)$ in the neighbourhood of $m \in M$, then the dimension of the image of $P^\sharp$ at $\mu_t m$ is at least equal to that of the image of $P^\sharp$ at m. Reversing the argument, we see that the dimensions of the images at both points are equal.

Under this assumption, there is a well-defined generalized foliation (in the sense of Stefan and Sussmann) on M, whose leaves are nondegenerate hamiltonian quasi-Poisson spaces containing the G-orbits. To see that each leaf satisfies the nondegeneracy property, we observe that, by the skew-symmetry of P, the kernel of $P_m^\sharp$ is the orthogonal of $\operatorname{Im} P_m^\sharp$. Therefore, $P_m^\sharp$ factorizes through the dual of $\operatorname{Im} P_m^\sharp$ and, by dimension counting, is an isomorphism onto $\operatorname{Im} P_m^\sharp$.

Example 5.4.2. Let the quasi-Poisson space M be $D/G = S$. Then the leaves of the generalized foliation $\mathcal{D}$ are the orbits of the dressing action of G on S.

Let, in particular, the quasi-Poisson space M be $(G \times G)/G \cong G$ as in Section 5.3. In this case, the orbits of the dressing action are the conjugacy classes of G, each of which is a nondegenerate hamiltonian quasi-Poisson space.

5.5. **Properties of moment maps.** The generalized moment maps introduced in the previous section possess properties which resemble the properties of the usual moment maps.

Theorem 5.5.1. *Let $(M, P_M^{\mathfrak{h}}, \mu)$ be a hamiltonian quasi-Poisson space acted upon by a connected quasi-Poisson Lie group $G_D^{\mathfrak{h}}$. Then, on any open set $\Omega \subset M$ such that $\mathfrak{h}$ is admissible on $\mu(\Omega)$, the moment map is a bivector map from $(M, P_M^{\mathfrak{h}})$ to $(S, P_S^{\mathfrak{h}})$,*

$$\mu_* P_M^{\mathfrak{h}} = P_S^{\mathfrak{h}} \ . \tag{5.5.2}$$

Proof. By the definition of the moment map, for all $x \in \mathfrak{g}$,

$$\mu_* x_M = \mu_* (P_M^{\mathfrak{h}})^\sharp (\mu^* \hat{x}_{\mathfrak{h}}),$$

while, by the characteristic property of P_S, $x_S = (P_S^{\mathfrak{h}})^\sharp(\hat{x}_{\mathfrak{h}})$. Thus, we see that $\mu_* P_M^{\mathfrak{h}} = P_S^{\mathfrak{h}}$ follows from the equivariance condition, $\mu_* x_M = x_S$, and from the fact that the 1-forms $\hat{x}_{\mathfrak{h}}$ span the cotangent space to S. □

Sometimes it is convenient to require that the bivector $P_M^{\mathfrak{h}}$ be nondegenerate, *i.e.*, that the map $(P_M^{\mathfrak{h}})^\sharp$ be an isomorphism. We next show that for hamiltonian quasi-Poisson spaces this condition is independent of the particular choice of a complement.

Proposition 5.5.3. *Let $(M, P_M^{\mathfrak{h}}, \mu)$ be a hamiltonian quasi-Poisson space acted upon by $G_D^{\mathfrak{h}}$. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two isotropic complements of $\mathfrak{g}$ admissible on $\mu(\Omega)$, where Ω is an open subset of M. Then $P_M^{\mathfrak{h}}$ is nondegenerate on Ω if and only if $P_M^{\mathfrak{h}'}$ is nondegenerate on Ω.*

Proof. We assume that $P_M^{\mathfrak{h}}$ is nondegenerate and we let α be a 1-form in the kernel of $P_M^{\mathfrak{h}'}$. Then,

$$(P_M^{\mathfrak{h}})^\sharp(\alpha) = t_M \alpha.$$

Since $\mathfrak{h}$ is admissible, there exists $x \in \mathfrak{g}$, such that $t_M \alpha = x_M$. (If $\alpha = \alpha_i (\varepsilon^i)_M$, we can set $x = t^{ik} \alpha_i e_k$.) By the nondegeneracy of $P_M^{\mathfrak{h}}$ and the definition of the moment map, $\alpha = \mu^* \hat{x}_{\mathfrak{h}}$. Applying (5.1.6), we obtain $x_M(m) = -((t \circ \tau_s)x)_M(m)$, where $s = \mu(m)$. The equivariance of the moment map implies that $((1_{\mathfrak{g}} + t \circ \tau_s)x)_S = 0$, which in turn implies that $\tau_s(1_{\mathfrak{g}} + t \circ \tau_s)x = 0$. If both complements $\mathfrak{h}$ and $\mathfrak{h}'$ are admissible at s, the operator $1_{\mathfrak{g}} + t \circ \tau_s$ is invertible, and therefore from $(t \circ \tau_s)(1_{\mathfrak{g}} + t \circ \tau_s)x = 0$, we obtain $(t \circ \tau_s)x = 0$. Since $x_M = -((t \circ \tau_s)x)_M$, the vector field x_M vanishes. Therefore $\alpha = 0$, and $P_M^{\mathfrak{h}'}$ is nondegenerate. □

Because of the preceding proposition, it is justified to call a family of bivectors $P_M^{\mathfrak{h}}$ nondegenerate if $P_M^{\mathfrak{h}}$ is nondegenerate for an admissible $\mathfrak{h}$.

Example 5.5.4. In the case of the standard quasi-triple studied in Section 5.3, it follows from (5.3.4) that (M, P, μ) is a nondegenerate hamiltonian quasi-Poisson space if and only if, for each $m \in M$,

$$\ker(P_m^\sharp) = \{\mu^* K(x, \theta) | x \in \ker(1_{\mathfrak{g}} + \mathrm{Ad}_{\mu(m)})\} \ .$$

Finally, we establish a relation between the Poisson reduction of Theorem 4.2.2 and moment maps.

Theorem 5.5.5. *Let $(M, P_M^{\mathfrak{h}}, \mu)$ be a hamiltonian quasi-Poisson space such that the bivector $P_M^{\mathfrak{h}}$ is everywhere nondegenerate. Assume that M/G is a smooth manifold in a neighbourhood U of $p(x_0)$, where $x_0 \in M$. Let $x \in M$ be such that $p(x) \in U$ and $s = \mu(x) \in D/G$ is a regular value of the moment map, μ. Then the symplectic leaf through $p(x)$ in the Poisson manifold U is the connected component of the intersection with U of the projection of the manifold $\mu^{-1}(s)$.*

Proof. The proof is analogous to that in the case of Poisson actions. See, *e.g.*, [16] .

Let $x \in M$ be as stated and let $y = p(x)$ be its projection in $U \subset M/G$. Let $L \subset U \subset M/G$ be the symplectic leaf passing through y in the Poisson manifold U. We choose any complement $\mathfrak{h}$ admissible at $s = \mu(x)$. We need to prove that the projection of the tangent space to the level submanifold, $p_* T_x \mu^{-1}(s)$, coincides with the tangent space to the symplectic leaf, $T_y L$.

We denote the Poisson bivector which is locally defined on M/G near y by Q. Let v be a vector in $T_y L$. By definition, $v = Q^\sharp(\alpha)$, where α is a 1-form in $T_y^*(M/G)$, and we can also represent v as $v = p_*(P_M^{\mathfrak{h}})^\sharp(p^*\alpha)$.

The vectors u that are tangent to the level submanifold $\mu^{-1}(s) \subset M$ are characterized by the property $\langle u, \mu^* \hat{x}_{\mathfrak{h}} \rangle = 0$ for any $x \in \mathfrak{g}$. The bivector $P_M^{\mathfrak{h}}$ being nondegenerate, any vector u tangent to M at x can be represented as $u = (P_M^{\mathfrak{h}})^\sharp(\beta)$ for some $\beta \in T_x^* M$. Then u is tangent to $\mu^{-1}(s)$ if and only if $< \beta, x_M > = 0$, for all $x \in \mathfrak{g}$, in other words, the 1-form β is the inverse image of a 1-form $\alpha \in T_y^*(M/G)$, $\beta = p^*\alpha$.

We conclude that $T_y L = p_*(P_M^{\mathfrak{h}})^\sharp(p^* T_y^*(M/G)) = p_* T_x \mu^{-1}(s)$ which proves the theorem. □

Thus, we have extended several of the basic properties of hamiltonian group actions on Poisson manifolds to the case of hamiltonian quasi-Poisson actions.

References

[1] A. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Diff. Geom. **45** (1997) 241-256.

[2] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, *Quasi-Poisson manifolds*, submitted for publication.

[3] A. Alekseev, A. Malkin, E. Meinrenken, *Lie group valued moment maps*, J. Diff. Geom. **48** (1998) 445-495.

[4] A. Alekseev, E. Meinrenken, C. Woodward, *Duistermaat-Heckman distributions for group valued moment maps*, preprint, math.DG/9903087.

[5] M. Bangoura, Y. Kosmann-Schwarzbach, *The double of a Jacobian quasi-bialgebra*, Lett. Math. Phys. **28** (1993) 13-29.

[6] V.G. Drinfeld, *Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Soviet Math. Dokl. **27** (1983) 68-71.

[7] V.G. Drinfeld, *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990) 1419-1457.

[8] H. Flaschka, T. Ratiu, *A convexity theorem for Poisson action of compact Lie groups*, Ann. Sci. École Norm. Sup. **29** (1996) 787-809.

[9] V. L. Ginzburg, *Some remarks on symplectic actions of compact groups*, Math. Z. **210** (1992) 625-640.

[10] V. Guillemin, S. Sternberg, *Symplectic Techniques in Physics*, Cambridge Univ. Press, 1984.

[11] Y. Kosmann-Schwarzbach, *Quasi-bigèbres de Lie et groupes de Lie quasi-Poisson*, C.R. Acad. Sci. Paris **312**, Série I (1991) 391-394.
[12] Y. Kosmann-Schwarzbach, *Jacobian quasi-bialgebras and quasi-Poisson Lie groups*, Contemporary Math. **132** (1992) 459-489.
[13] Y. Kosmann-Schwarzbach, *Lie bialgebras, Poisson Lie groups and dressing transformations*, Lecture Notes Phys. **495**, Springer-Verlag, 1997, 104-170.
[14] S. Lang, *Algebra*, Addison-Wesley, 1965.
[15] M. Leingang, *Symmetric pairs and moment spaces*, preprint, math.SG/9810064.
[16] J.-H. Lu, *Momentum mappings and reduction of Poisson actions*, in Symplectic Geometry, Groupoids and Integrable Systems, P. Dazord and A. Weinstein, eds., Springer-Verlag, 1991, 209-226.
[17] J.-H. Lu, A. Weinstein, *Poisson-Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom. **31** (1990) 501-526.
[18] D. McDuff, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. **5** (1988) 149-160.
[19] E. Meinrenken, C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, J. Diff. Geom. **50** (1998) 417-470.
[20] M.A. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions*, Publ. R.I.M.S. **21** (1985) 1237-1260.
[21] M.A. Semenov-Tian-Shansky, *Poisson Lie groups, quantum duality and the quantum double*, Contemporary Math. **175** (1994) 219-248.
[22] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Birkhäuser, 1994.
[23] A. Volkov, private communication.
[24] J. Weitsman, *A Duistermaat-Heckman formula for symplectic circle actions*, Internat. Math. Res. Notices **12** (1993) 309-312 [Duke Math. J. **72** (3)].

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