

# Modular vector fields and Batalin-Vilkovisky algebras\*

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**Abstract.** We show that a modular class arises from the existence of two generating operators for a Batalin-Vilkovisky algebra. In particular, for every triangular Lie bialgebroid  $(A, P)$ , such that its top exterior power is a trivial line bundle, there is a section of the vector bundle  $A$  whose  $d_P$ -cohomology class is well-defined. We give simple proofs of its properties. The modular class of an orientable Poisson manifold is an example. We analyse the relationships between generating operators of the Gerstenhaber algebra of a Lie algebroid, right actions on the elements of degree 0, and left actions on the elements of top degree. We show that the modular class of a triangular Lie bialgebroid coincides with the characteristic class of a Lie algebroid with representation on a line bundle.

## 1 Introduction.

We present the definition and properties of the modular class of a triangular Lie bialgebroid such that its top exterior power is a trivial line bundle. This is only a slight generalization of the theory of modular vector fields of orientable Poisson manifolds. However, we believe that it is important to show that the existence of a modular field is due to the existence of two different generating operators for a Batalin-Vilkovisky algebra. The main features of the construction are valid in the more general algebraic framework of Lie-Rinehart algebras and their associated Gerstenhaber algebras [9] [10]. The modular field is then an element of the dual over the base ring of the Lie-Rinehart algebra. To simplify, we restrict most of our remarks to the case of Gerstenhaber algebras associated to Lie algebroids.

The *modular vector field* of an orientable Poisson manifold was defined, under the name of a “curl” (“*rotationnel*”) by Dufour and Haraki [5] who showed that

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it preserves the Poisson structure, and used it to classify the quadratic Poisson structures on a 3-dimensional vector space. It was also used by Liu and Xu in [16], and by Grabowski, Marmo and Perelomov in [8]. In [24], Weinstein showed in what sense the modular vector field is the infinitesimal generator of the analogue in Poisson geometry of the modular automorphism group of a von Neumann algebra, and he introduced the *modular class*. See also [25]. In [6], Evens, Lu and Weinstein defined the modular class of a Lie algebroid and proved that the modular class of a Poisson manifold  $(M, P)$  is one-half the modular class of  $T^*M$ , the cotangent bundle of  $M$  with the Lie algebroid structure defined by  $P$ , according to formula (3) below. In [26], Xu related the study of modular vector fields on Poisson manifolds to the notion of a Gerstenhaber algebra, more precisely to that of a Batalin-Vilkovisky algebra (BV-algebra). A full algebraic theory was developed by Huebschmann in [10] [11] and [12].

Here, we show (Section 4) that the existence of a closed (in the Lie algebroid cohomology) section of the dual  $E^*$  of a Lie algebroid  $E$  follows from the existence of two distinct generating operators of square zero for the Gerstenhaber bracket on the sections of  $\bigwedge E$ . When  $E = A^*$ , where  $(A, P)$  is a triangular Lie bialgebroid such that its top exterior power is a trivial line bundle, we obtain (Section 5) the definition of its modular field, which is a section of the Lie algebroid  $A$ . In the particular case of the triangular Lie bialgebroid  $(TM, P)$  of an orientable Poisson manifold  $(M, P)$ , we recover the modular vector field of the Poisson manifold, a section of  $TM$ . Section 6 deals with the notion of “Laplacian”.

We then show how the notions of representations and right actions connect our results with those in the literature. We show (Section 7) that it is possible to modify the right action of the Lie algebra of the elements of degree 1 on the elements of degree 0 in a Batalin-Vilkovisky algebra by the addition of the cocycle defined by the restriction of the generating operator of the bracket, and we recover the right action of differential 1-forms on functions on a Poisson manifold discovered by Huebschmann in [10]. More precisely, as shown in [10], there is a one-to-one correspondence between generating operators and right actions on the elements of degree 0. In Section 8, we explain the one-to-one correspondence between generating operators of the Gerstenhaber algebra of a Lie algebroid,  $E$ , and left actions on the elements of top degree. If, in particular, the top exterior power of the Lie algebroid is a trivial line bundle, the choice of a nowhere vanishing section  $\lambda$  of this bundle uniquely determines both a left action  $\nabla^\lambda$  of the sections of the Lie algebroid on the line bundle and a generator  $\partial_\lambda$  of the Gerstenhaber algebra of the Lie algebroid. In fact,  $\nabla^\lambda \lambda = 0$  and  $\partial_\lambda$  is nothing but the Lie algebroid differential transported to the sections of the exterior powers of the Lie algebroid itself by means of the isomorphism defined by  $\lambda$  between sections of  $\bigwedge E$  and sections of  $\bigwedge E^*$ . We show (Section 9) that the modular class considered here coincides with the characteristic class of a Lie algebroid with a representation on its top exterior bundle in the sense of Evens, Lu and Weinstein [6], and therefore, when the triangular Lie bialgebroid is the

one associated to a Poisson manifold  $(M, P)$ , it is also one-half the modular class of the Lie algebroid  $T^*M$ , which is itself one-half the modular class, in the sense of Huebschmann [10], of the associated Lie-Rinehart algebra.

## 2 Gerstenhaber algebras, Batalin-Vilkovisky algebras and Lie algebroid cohomology.

We first recall some definitions. A *Gerstenhaber algebra* is a graded commutative, associative algebra,  $\mathcal{A}$ , equipped with a bracket,  $[ \ , \ ]$ , of degree  $-1$ , which makes  $\mathcal{A}[1]$  a graded Lie algebra, where  $\mathcal{A}[1]^k = \mathcal{A}^{k+1}$ , and such that, for  $a \in \mathcal{A}[1]^k$ ,  $[a, \cdot]$  is a derivation of degree  $k$  of  $\mathcal{A}$ .

A *Batalin-Vilkovisky algebra* (hereafter, *BV-algebra*) is a graded commutative, associative algebra,  $\mathcal{A}$ , equipped with an operator  $\partial$  of degree  $-1$  and of square 0 such that

$$[a, b] = (-1)^{|a|}(\partial(ab) - (\partial a)b - (-1)^{|a|}a(\partial b)), \quad (1)$$

for  $a \in \mathcal{A}^{|a|}, b \in \mathcal{A}^{|b|}$ , defines a Gerstenhaber algebra structure on  $\mathcal{A}$  (see [15]). In this case we say that  $\partial$  *generates* the bracket  $[ \ , \ ]$  of  $\mathcal{A}$ , or that  $\partial$  is a *generating operator* for the bracket. BV-algebras are also called exact Gerstenhaber algebras [13] because, in the preceding fundamental formula, the bracket, seen as a 2-cochain on  $\mathcal{A}$  with coefficients in  $\mathcal{A}$ , is the graded Hochschild coboundary of  $\partial$  with respect to the associative multiplication. It follows from the definition of  $\partial$  that it is a derivation of the bracket of  $\mathcal{A}$ , thus  $(\mathcal{A}[1], \partial)$  is a differential graded Lie algebra. See [15] and, for a recent reference, [19].

One can also introduce a more general notion of a generating operator of a Gerstenhaber algebra, one that satisfies (1) but which is not necessarily of square 0. But in this paper we shall only consider generating operators of Gerstenhaber algebras of square zero, which we choose to call “generators”, although they are more often called “exact generators”. Thus, we adhere here to the following

CONVENTION. The generating operators of Gerstenhaber algebras are assumed to be of square 0.

A *strong differential Gerstenhaber algebra* is a Gerstenhaber algebra with a differential, *i. e.*, a derivation of degree 1 and of square 0 of the associative algebra, which differential is also a derivation of its bracket. A *strong differential BV-algebra* is a BV-algebra with a differential that is also a derivation of the bracket.

Any Lie-Rinehart algebra [9] defines a Gerstenhaber algebra, and conversely. If  $\mathcal{A}$  is a Gerstenhaber algebra over a field  $k$ , then  $(\mathcal{A}^0, \mathcal{A}^1)$  is a Lie-Rinehart algebra, more precisely,  $\mathcal{A}^1$  is a  $(k, \mathcal{A}^0)$ -algebra, namely,  $\mathcal{A}^0$  is a commutative  $k$ -algebra and  $\mathcal{A}^1$  is an  $\mathcal{A}^0$ -module and a  $k$ -Lie algebra which acts on  $\mathcal{A}^0$  by derivations, such that  $(fa).g = f(a.g)$  and  $[a, fb] = f[a, b] + (a.f)b$ , for  $f \in \mathcal{A}^0$  and  $a, b \in \mathcal{A}^1$ . Conversely, if  $\mathcal{A}^1$  is a  $(k, \mathcal{A}^0)$ -algebra, then  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$  is a

Gerstenhaber algebra over  $k$ . (See [7].) If  $\mathcal{A}^1$  is a  $(k, \mathcal{A}^0)$ -algebra, and if  $\mathcal{A}^1$  is a projective  $\mathcal{A}^0$ -module, the *cohomology* of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$  is the cohomology of the complex  $(\text{Alt}_{\mathcal{A}^0}(\mathcal{A}^1, \mathcal{A}^0), d)$ , where  $d$  is defined by a formula [21] generalizing both that of the de Rham differential and that of the Chevalley-Eilenberg differential, using the Lie bracket of  $\mathcal{A}^1$  and the left action of  $\mathcal{A}^1$  on  $\mathcal{A}^0$ . More generally, one can define the cohomology of  $\mathcal{A}^1$  with coefficients in a left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module, and the homology of  $\mathcal{A}^1$  with coefficients in a right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module. See [9]. We recall that an  $\mathcal{A}^0$ -module,  $M$  (resp.,  $N$ ), which is also a left (resp., right)  $\mathcal{A}^1$ -module is called a left (resp., right)  $(\mathcal{A}^0, \mathcal{A}^1)$ -module if, for each  $m \in M$  (resp.,  $n \in N$ ),  $(fa).m = f(a.m)$  and  $a.(fm) = f(a.m) + (a.f)m$  (resp.,  $n.(fa) = (nf).a$  and  $(n.a)f = n.(fa) + n(a.f)$ ).

If  $E$  is a *Lie algebroid* with base  $M$  and anchor  $\rho$  (see [17] [23]), the space of sections of  $E$  is an  $(\mathbb{R}, C^\infty(M))$ -algebra, and there are related structures naturally defined on the sections of the exterior bundle of  $E$  and its dual [14] [18].

(i) The space of sections of  $\wedge E$  (the Whitney sum of the exterior powers of the vector bundle  $E$ ) is a Gerstenhaber algebra. The Gerstenhaber bracket on  $\Gamma(\wedge E)$ , denoted by  $[\cdot, \cdot]_E$  and often called the *Schouten bracket*, extends the Lie bracket of  $\Gamma(E) = \Gamma(\wedge^1 E)$  to  $\Gamma(\wedge E)$ , and is such that  $[a, f]_E = \rho(a)f$ , for  $f \in C^\infty(M)$ ,  $a \in \Gamma(E)$ . The *Lie derivation* by  $a \in \Gamma(E)$  is the derivation  $\mathcal{L}_a^E = [a, \cdot]_E$  of  $\Gamma(\wedge E)$ .

(ii) The space of sections of  $\wedge E^*$ , where  $E^*$  is the dual vector bundle of  $E$ , is a differential graded commutative, associative algebra. The differential on  $\Gamma(\wedge E^*)$ , denoted by  $d_E$ , is called the *de Rham differential*. The cohomology of the complex  $(\Gamma(\wedge E^*), d_E)$ , called the *Lie algebroid cohomology* of  $E$  with coefficients in  $C^\infty(M)$ , is the cohomology of the Lie-Rinehart algebra  $\mathcal{A}^1 = \Gamma(E)$  with coefficients in  $\mathcal{A}^0 = C^\infty(M)$ . For  $a \in \Gamma(E)$ , the Lie derivation of  $\Gamma(\wedge E^*)$  is the operator  $\mathcal{L}_a = [i_a, d_E]$ , where the bracket is the graded commutator of operators.

We denote the interior product of a section  $\alpha$  of  $\wedge E^*$  by a section  $a$  of  $\wedge E$  by  $i_a \alpha$ . Our convention is that  $i_a = i_{a_1} i_{a_2} \dots i_{a_{k-1}} i_{a_k}$ , for a decomposable element  $a = a_1 \wedge a_2 \wedge \dots \wedge a_k$ , where  $a_j \in \Gamma(E)$ ,  $j = 1, 2, \dots, k$ . Under this convention (see, e. g., [4]), the Schouten bracket of multivectors  $a, b \in \Gamma(\wedge E)$  satisfies

$$i_{[a,b]_E} = [\mathcal{L}_a, i_b] = [[i_a, d_E], i_b]. \quad (2)$$

*Remark.* Many authors, including those of [6], [26] and [20], adopt a different sign convention for the interior product, namely they define the interior product by a decomposable element  $a = a_1 \wedge a_2 \wedge \dots \wedge a_k$  as  $i_{a_k} i_{a_{k-1}} \dots i_{a_2} i_{a_1}$ . We shall denote this other interior product by  $\tilde{i}_a$ . The two operators are related by  $\tilde{i}_a = (-1)^{\frac{|a|(|a|-1)}{2}} i_a$ . In particular, if  $P$  is a bivector, then  $\tilde{i}_P = -i_P$ . This fact explains the sign differences between some of their formulae and ours.

When  $E$  is a vector bundle of rank  $n$ , we denote  $\bigwedge^n E$  by  $\bigwedge^{\text{top}} E$ .

We shall use the notation  $\partial$ , with or without a subscript, for operators of degree  $-1$ , and the notation  $d$ , with or without a subscript, for operators of degree 1.

### 3 Triangular Lie bialgebroids.

We now recall the definition and main properties of triangular Lie bialgebroids.

A *Lie bialgebroid* is a pair  $(A, A^*)$  of Lie algebroids in duality such that  $d_{A^*}$  is a derivation of  $[\ , \ ]_A$ . Then [18] [13]  $d_A$  is also a derivation of  $[\ , \ ]_{A^*}$ .

In particular, a *triangular Lie bialgebroid* is a pair  $(A, P)$ , where  $A$  is a Lie algebroid and  $P$  is a section of  $\bigwedge^2 A$  such that  $[P, P]_A = 0$ . Then

$$[\alpha, \beta]_P = \mathcal{L}_{P^\sharp \alpha} \beta - \mathcal{L}_{P^\sharp \beta} \alpha - d_A(P(\alpha, \beta)) , \quad (3)$$

for  $\alpha, \beta \in \Gamma(A^*)$ , defines a Lie algebroid structure on  $A^*$ , with anchor  $\rho_* = \rho \circ P^\sharp$ , where  $P^\sharp(\alpha)(\beta) = P(\alpha, \beta)$ , and  $\rho$  is the anchor of  $A$ . Furthermore,  $(A, A^*)$  is a Lie bialgebroid. We use the same notation for the extension of  $[\ , \ ]_P$  to the Gerstenhaber bracket on the algebra  $\Gamma(\bigwedge A^*)$ . In this case, the de Rham differential on  $\Gamma(\bigwedge A)$  is denoted by  $d_P$ . It was proved in [1] and [14] that

$$d_P = [P, \cdot]_A . \quad (4)$$

From this property it follows that  $(\Gamma(\bigwedge A), [\ , \ ]_A, d_P)$  is a strong differential Gerstenhaber algebra.

We also know ([15] [13]) that  $\Gamma(\bigwedge A^*)$  has a BV-algebra structure, since its Gerstenhaber bracket,  $[\ , \ ]_P$ , is generated by the generalized Poisson homology operator,

$$\partial_P = [d_A, i_P] , \quad (5)$$

where  $d_A$  is the de Rham differential on  $\Gamma(\bigwedge A^*)$  coming from the Lie algebroid structure of  $A$ , and  $i_P$  denotes the interior product by  $P$ . In this formula, the bracket is the graded commutator of operators. This operator was introduced by Koszul [15] and studied by Brylinski in [2]. It is sometimes called the Koszul-Brylinski operator.

It follows from the general result on BV-algebras recalled in Section 2 that the operator  $\partial_P$  is a derivation of the bracket  $[\ , \ ]_P$ . Moreover, the de Rham differential,  $d_A$ , is a derivation of the bracket  $[\ , \ ]_P$ , so that  $(\Gamma(\bigwedge A^*), \partial_P, d_A)$  is a strong differential BV-algebra.

The main example of a triangular Lie bialgebroid is a pair  $(TM, P)$ , where  $TM$  is the tangent bundle of a manifold  $M$  and  $P$  is a Poisson bivector on  $M$ . In this case,  $[\ , \ ]_P$  is a Lie bracket on the vector space of differential 1-forms on  $M$ , and its extension to  $\Gamma(\bigwedge T^*M)$  is a Gerstenhaber bracket on the algebra of differential forms on  $M$ , which was defined by Koszul [15].

*Remark.* The operator  $[i_P, d]$  introduced in [15], where it is denoted by  $\nabla$ , is the opposite of the operator that we have denoted here by  $\partial_P$ . (Although the convention for the interior product is not explicitly stated in [15], it is clear from the context that it is the one that we have adopted here,  $i_P$ , and not  $\tilde{i}_P$ .) It generates the bracket denoted there by  $[ , ]$ , which satisfies  $[df, dg] = -d\{f, g\}$ , and is therefore the opposite of the bracket  $[ , ]_P$  defined by (3).

Just as Lie bialgebras in the sense of Drinfeld are examples of Lie algebroids (whose base is a point), triangular Lie bialgebras  $(\mathfrak{g}, r)$  are examples of triangular Lie bialgebroids, with base a point. Here,  $r$  is in  $\bigwedge^2 \mathfrak{g}$ . In this case,  $d_A$  is the Lie algebra coboundary operator  $d_{\mathfrak{g}}$  defined by the Lie algebra structure of  $\mathfrak{g}$ , and  $\partial_r = [d_{\mathfrak{g}}, i_r]$  generates the bracket of  $\mathfrak{g}^*$ . For  $\xi, \eta \in \mathfrak{g}^*$ ,

$$[\xi, \eta]_r = \text{ad}_{r\sharp\xi}^* \eta - \text{ad}_{r\sharp\eta}^* \xi = -(\partial_r(\xi \wedge \eta) - (\partial_r \xi) \wedge \eta + \xi \wedge (\partial_r \eta)) . \quad (6)$$

## 4 Generating operators of Gerstenhaber algebras and modular classes.

We first prove a generalization of formula (2.4) in [15].

**Proposition 1** *Let  $(\mathcal{A}, \partial)$  be a BV-algebra with the Gerstenhaber bracket  $[ , ]$  defined by (1), and let us assume that  $\mathcal{A} = \bigwedge_{\mathcal{A}^0} \mathcal{A}^1$ . If  $\xi$  is in  $\text{Hom}_{\mathcal{A}^0}(\bigwedge_{\mathcal{A}^0} \mathcal{A}^1, \mathcal{A}^0)$ , then*

$$[\partial, i_{\xi}] = -i_{d\xi} . \quad (7)$$

*Proof.* If  $\partial$  generates the bracket, it satisfies, for  $f \in \mathcal{A}^0, a \in \mathcal{A}^1$ ,  $\partial(fa) = f\partial a - [a, f]$ . Since  $i_{df}a = (df)(a) = [a, f]$ , (7) holds for  $\xi \in \mathcal{A}^0$ . Let  $\xi$  be in  $(\mathcal{A}^1)^* = \text{Hom}_{\mathcal{A}^0}(\mathcal{A}^1, \mathcal{A}^0)$ . Using (1), we can compute  $[\partial, i_{\xi}](a \wedge b)$  for  $a, b \in \mathcal{A}^1$  in terms of the bracket of  $\mathcal{A}$ , and we obtain

$$[\partial, i_{\xi}](a \wedge b) = [a, \langle \xi, b \rangle] - [b, \langle \xi, a \rangle] - \langle \xi, [a, b] \rangle = (d\xi)(a, b) = -i_{d\xi}(a \wedge b) , \quad (8)$$

by the definition of the cohomology operator  $d$  of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ . Since  $[\partial, i_{\xi}]$  is  $\mathcal{A}^0$ -linear and of degree  $-2$ , formula (7) holds for  $\xi \in \mathcal{A}^1$ . It is easy to see, using the relation  $i_{\xi \wedge \eta} = i_{\xi} i_{\eta}$  for  $\xi, \eta \in \mathcal{A}^1$ , that (7) holds for forms of all degrees.

If  $\partial'$  is another operator which generates the same bracket, then  $\partial' - \partial$  is  $\mathcal{A}^0$ -linear and therefore its restriction to  $\mathcal{A}^1$  defines an element  $\xi$  in  $(\mathcal{A}^1)^*$  such that, for any  $a \in \mathcal{A}^1$ ,  $(\partial' - \partial)a = \langle \xi, a \rangle$ . Since  $\partial' - \partial$  is a derivation, it follows that, for any  $a \in \mathcal{A}$ ,  $(\partial' - \partial)a = i_{\xi}a$ . So, any two generating operators of  $\mathcal{A}$  define an element  $\xi$  in  $(\mathcal{A}^1)^*$ . This element is in fact closed in the cohomology of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ , as we now show.

**Proposition 2** *Let  $\partial$  and  $\partial'$  be operators that each generates the Gerstenhaber algebra  $(\bigwedge_{\mathcal{A}^0} \mathcal{A}^1, [\ , \ ])$ . Then the element  $\xi$  of  $(\mathcal{A}^1)^*$  such that  $\partial' - \partial = i_\xi$  is a 1-cocycle of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ . Conversely, if  $\partial$  is a generating operator of  $\mathcal{A} = \bigwedge_{\mathcal{A}^0} \mathcal{A}^1$  and if  $\xi \in (\mathcal{A}^1)^*$  is a 1-cocycle of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ , then  $\partial + i_\xi$  is also a generating operator.*

Proof. We assume that  $\partial' - \partial = i_\xi$ . The graded commutator  $[\partial', \partial']$  is equal to  $[\partial, \partial] + 2[\partial, i_\xi]$ . Since by assumption,  $\partial^2 = \partial'^2 = 0$ , it follows from (7) that  $d\xi = 0$ . Conversely, if  $\partial$  is a generating operator of the bracket, and if  $\partial' = \partial + i_\xi$ , then, since  $i_\xi$  is a derivation,  $\partial'$  also satisfies (1). Furthermore,  $(\partial')^2 = 0$ , since  $\partial^2 = 0$  and  $\xi$  is a 1-cocycle, and therefore  $\partial'$  is a generating operator.

In Proposition 3.2 of [26], Xu proves this result for the case of the Gerstenhaber algebra of a Lie algebroid, making use of flat connections as in Section 8. See also Theorem 2.6 in [22], for the case where  $\mathcal{A}^1$  is the Lie algebra of derivations of a commutative ring and  $\xi$  is an exact 1-form,  $df$ .

In the rest of this paper, we shall consider those cases where  $\partial$  is a fixed generating operator, and where, by varying  $\partial'$ , we obtain equivalent 1-cocycles,  $\xi_{\partial'}$ . In those cases, the class of  $\xi_{\partial'}$  in the cohomology of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$  is well-defined, and we shall call it the *modular class* of the BV-algebra  $(\mathcal{A}, \partial)$ .

## 5 The modular class of a triangular Lie bialgebroid.

Let  $A$  be a Lie algebroid of rank  $n$ . We assume that there exists a nowhere vanishing element  $\mu$  in  $\Gamma(\bigwedge^{\text{top}} A^*)$ . Such a section defines an isomorphism  $*_\mu$  of  $\bigwedge A$  onto  $\bigwedge A^*$  such that, for each degree  $k$ ,  $0 \leq k \leq n$ ,  $*_\mu : \bigwedge^k A \rightarrow \bigwedge^{n-k} A^*$ . This isomorphism is defined by

$$*_\mu Q = i_Q \mu, \quad (9)$$

for  $Q \in \Gamma(\bigwedge^k A)$ ,  $k > 0$ , and  $*_\mu f = f\mu$  for  $f \in \Gamma(\bigwedge^0 A) = C^\infty(M)$ . (We denote by the same letter a morphism of vector bundles and the map on sections that it defines.)

Let us introduce the operator

$$\partial_\mu = - *_\mu^{-1} d_A *_\mu \quad (10)$$

on  $\Gamma(\bigwedge A)$ .

**Proposition 3** *The operator  $\partial_\mu$  is of degree  $-1$  and of square 0, it generates the Schouten bracket of  $\Gamma(\bigwedge A)$  and it is a derivation of the Schouten bracket.*

Proof. We first prove that

$$\mathcal{L}_X \mu = (-1)^k *_{\mu} (\partial_{\mu} X) , \quad (11)$$

for  $X \in \Gamma(\wedge^k A)$ . In fact, this equality follows from the fact that, on forms of top degree,  $\mathcal{L}_X = -(-1)^k d_A i_X$ , and from the definition of  $\partial_{\mu}$ .

Now, let  $X$  be in  $\Gamma(\wedge^k A)$  and  $Y$  in  $\Gamma(\wedge^l A)$ . We use the definition of  $\mathcal{L}_X^A$  and relation (2), as well as the relations  $i_{X \wedge Y} \mu = i_X i_Y \mu$ , for  $X, Y \in \Gamma(\wedge A)$ , and  $*_{\mu} e_X = i_X *_{\mu}$ , where  $e_X$  is the exterior product by  $X$ , to obtain first

$$\partial_{\mu}(X \wedge Y) = - *_{\mu}^{-1} d_A *_{\mu} e_X Y = - *_{\mu}^{-1} d_A i_X *_{\mu} Y ,$$

then

$$d_A i_X *_{\mu} Y = (-1)^{k-1} \mathcal{L}_X i_Y \mu + (-1)^k i_X d_A *_{\mu} Y ,$$

whence

$$\partial_{\mu}(X \wedge Y) = (-1)^k *_{\mu}^{-1} i_{[X, Y]_A} \mu + (-1)^{k+(k-1)l} *_{\mu}^{-1} i_Y \mathcal{L}_X \mu + (-1)^k e_X \partial_{\mu} Y .$$

Using relation (11), we obtain

$$\partial_{\mu}(X \wedge Y) = (-1)^k [X, Y]_A + (-1)^{(k-1)l} e_Y \partial_{\mu} X + (-1)^k e_X \partial_{\mu} Y ,$$

or

$$[X, Y]_A = (-1)^k (\partial_{\mu}(X \wedge Y) - (\partial_{\mu} X) \wedge Y - (-1)^k X \wedge (\partial_{\mu} Y)) ,$$

which proves the proposition.

Cf Lemma 4.6 of [6] (where we see that, taking into account the different conventions for the definition of the interior product, the operator  $b_{\mu_0}$  defined in formula (45) is equal to our  $\partial_{\mu_0}$ ) and Theorem 2.3 of [22] (where the sign conventions coincide with ours).

The operator  $\partial_{\mu}$ , called the *divergence* with respect to  $\mu$ , satisfies

$$\mathcal{L}_X \mu = -(\partial_{\mu} X) \mu , \quad (12)$$

for  $X \in \Gamma(A)$ . So, if  $A = TM$ , for a vector field  $X \in \Gamma(TM)$ ,  $\partial_{\mu} X$  is the opposite of the usual divergence,  $\text{div}_{\mu} X$ , which satisfies  $\mathcal{L}_X \mu = (\text{div}_{\mu} X) \mu$ . In fact,  $\partial_{\mu}$  is the unique generating operator of the Schouten bracket that satisfies (12). Explicitly (see [11], [22]), for  $X_j \in \Gamma(A), j = 1, 2, \dots, q$ ,

$$\begin{aligned} \partial_{\mu}(X_1 \wedge \dots \wedge X_q) &= \sum_{1 \leq j < k \leq q} (-1)^{j+k} [X_j, X_k]_A \wedge X_1 \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge \hat{X}_k \wedge \dots \wedge X_q \\ &+ \sum_{j=1}^q (-1)^{j-1} (\partial_{\mu} X_j) X_1 \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_q . \end{aligned}$$

Let us now assume that  $(A, P)$  is a triangular Lie bialgebroid. The operator  $\partial_P$  defined by formula (5) generates the bracket  $[ , ]_P$  of  $\Gamma(\wedge A^*)$ .

*Remark.* In the case of the generating operator,  $\partial_P = [d, i_P]$ , an alternate proof of relation (7),  $[\partial_P, i_Q] = -i_{d_P} Q$ , for  $Q \in \Gamma(\wedge A)$ , is obtained by setting  $E = A, a = P, b = Q$  in relation (2).

We shall now show that to any choice of a nonvanishing section  $\mu$  in  $\Gamma(\wedge^{\text{top}} A^*)$ , there corresponds another generating operator of this same bracket. Let us set

$$\partial_{P,\mu} = - *_{\mu} d_P *_{\mu}^{-1} . \quad (13)$$

By definition, the operator  $\partial_{P,\mu}$  on  $\Gamma(\wedge A^*)$  satisfies

$$\partial_{P,\mu}(i_Q \mu) = -i_{d_P} Q \mu , \quad (14)$$

for any  $Q \in \Gamma(\wedge A)$ .

**Proposition 4** *The operator  $\partial_{P,\mu}$  is of degree  $-1$  and of square 0, it generates the Gerstenhaber bracket  $[\ , ]_P$  of  $\Gamma(\wedge A^*)$ , and is a derivation of the bracket  $[\ , ]_P$ .*

Proof. We first choose a nowhere vanishing section  $\lambda$  of  $\Gamma(\wedge^{\text{top}} A)$ , and we introduce the operator  $\tilde{\partial}_{P,\lambda} = - *_{\lambda}^{-1} d_P *_{\lambda}$ , where  $*_{\lambda} \alpha = i_{\alpha} \lambda$ , for  $\alpha \in \Gamma(\wedge A^*)$ . Proceeding as in the proof of Proposition 3, we see that  $\tilde{\partial}_{P,\lambda}$  generates the bracket  $[\ , ]_P$ . Now, for  $\lambda \in \Gamma(\wedge^{\text{top}} A)$ ,  $\mu \in \Gamma(\wedge^{\text{top}} A^*)$ , and  $\alpha \in \Gamma(\wedge^k A^*)$ ,

$$*_{\mu}(i_{\alpha} \lambda) = (-1)^{k(n-k)} \langle \mu, \lambda \rangle \alpha . \quad (15)$$

Indeed, for  $X \in \Gamma(\wedge^k A)$ ,

$$\langle *_{\mu}(i_{\alpha} \lambda), X \rangle = \mu(i_{\alpha} \lambda \wedge X) = (-1)^{k(n-k)} \mu(X \wedge i_{\alpha} \lambda)$$

and

$$\mu(X \wedge i_{\alpha} \lambda) = \mu(e_X *_{\lambda} \alpha) = \mu(*_{\lambda} i_X \alpha) = \langle \alpha, X \rangle \mu(*_{\lambda} 1) = \langle \mu, \lambda \rangle \langle \alpha, X \rangle .$$

Choosing  $\lambda$  and  $\mu$  such that  $\langle \lambda, \mu \rangle = 1$ , we obtain that the operator  $\partial_{P,\mu}$  defined by (13) coincides with  $\tilde{\partial}_{P,\lambda}$ , thus proving that it generates the bracket  $[\ , ]_P$ . The rest of the proposition is clear.

*Remark.* We give an alternate proof of Proposition 4, based on Propositions 2 and 3. Since  $\partial_{\mu}$  generates the Schouten bracket, we obtain

$$[\partial_{\mu}, e_P] - d_P = e_{\partial_{\mu} P} , \quad (16)$$

whence

$$\partial_{P,\mu} - [d_A, i_P] = i_{\partial_{\mu} P} .$$

Since  $\partial_{\mu}$  is a derivation of the Schouten bracket,

$$d_P(\partial_{\mu} P) = [P, \partial_{\mu} P]_A = \frac{1}{2} \partial_{\mu} [P, P]_A = 0 ,$$

thus  $\partial_{\mu} P$  is  $d_P$ -closed. Since  $\partial_{\mu} P$  is a 1-cocycle and since  $\partial_P = [d_A, i_P]$  generates the bracket  $[\ , ]_P$ , so does  $\partial_{P,\mu} = \partial_P + i_{\partial_{\mu} P}$ .

**Definition 1** *The modular field of the triangular Lie bialgebroid  $(A, P)$ , associated with the nowhere vanishing section  $\mu$  of  $\bigwedge^{\text{top}} A^*$ , is the section  $X_\mu$  of  $A$  satisfying*

$$\partial_{P,\mu} - \partial_P = i_{X_\mu} . \quad (17)$$

We now derive the properties of the modular field.

**Proposition 5** *The modular field  $X_\mu$  associated with  $\mu$  satisfies the equivalent relations  $i_{X_\mu}\mu = -\partial_P\mu$ ,  $i_{X_\mu}\mu = d_A(i_P\mu)$  and*

$$X_\mu = - *_{\mu}^{-1} (\partial_P\mu) . \quad (18)$$

*Proof.* We must prove that  $i_{X_\mu}\mu = -\partial_P\mu$ . This equality follows from Definition 1 and the fact that  $\partial_{P,\mu}\mu = 0$ , since  $*_{\mu}^{-1}\mu = 1$ . The rest of the proposition follows from the definitions.

**Proposition 6** *The modular field  $X_\mu$  associated with  $\mu$  satisfies*

$$X_\mu = \partial_\mu P . \quad (19)$$

*Proof.* By definition,  $i_{\partial_\mu Q}\mu = -d_A(i_Q\mu) = (-1)^q [i_Q, d_A] \mu$ , for any section  $Q$  of  $\bigwedge^q A$ . If, in particular,  $Q = P$ , then  $i_{\partial_\mu P}\mu = -\partial_P\mu$ , or  $\partial_\mu P = - *_{\mu}^{-1} (\partial_P\mu)$ . The result now follows from Proposition 5.

This proposition means that the modular field associated with  $\mu$  is the divergence of  $P$  with respect to  $\mu$ . As an obvious consequence of the preceding result, we see that  $\partial_\mu X_\mu = 0$ . The main result of this section is

**Theorem 7** *The modular fields  $X_\mu$  of a triangular Lie bialgebroid  $(A, P)$  satisfy*

$$d_P X_\mu = 0 , \quad (20)$$

*and, for nowhere vanishing  $f \in C^\infty(M)$ ,*

$$X_{f\mu} = X_\mu + \frac{1}{f} d_P f . \quad (21)$$

*Proof.* Formula (20) is a special case of Proposition 2. We give a direct proof. By assumption,  $[P, P]_A = 0$ . Using the fact that the operator  $\partial_\mu$  is a derivation of the Schouten bracket, and the skew-symmetry of the Schouten bracket, it follows that  $[P, \partial_\mu P]_A = 0$ . By formula (4) and Proposition 6, we obtain  $d_P X_\mu = 0$ .

Let  $\mu' = f\mu$ . Then  $*_{\mu'} = f *_{\mu}$ , and using the relations  $i_{X \wedge Y}\mu = i_X i_Y \mu$ , for  $X, Y$  in  $\Gamma(A)$ , and  $*_{\mu} e_X = i_X *_{\mu}$ , we see that

$$\partial_{P,\mu'} \alpha = \partial_{P,\mu} \alpha + f^{-1} *_{\mu} (d_P f \wedge *_{\mu}^{-1} \alpha) = \partial_{P,\mu} \alpha + f^{-1} i_{d_P f} \alpha , \quad (22)$$

for any  $\alpha \in \Gamma(\bigwedge A^*)$ .

Formula (20) means that  $X_\mu$  leaves  $P$  invariant,

$$\mathcal{L}_{X_\mu} P = 0 . \quad (23)$$

**Corollary 8** For each nowhere vanishing section  $\mu$  of  $\bigwedge^{\text{top}} A^*$ , the modular field  $X_\mu$  is  $d_P$ -closed, and its class in the Lie algebroid cohomology of  $A$  with coefficients in  $C^\infty(M)$  is independent of  $\mu$ .

Proof. In fact,  $X_{f\mu} = X_\mu + d_P(\log|f|)$ .

**Definition 2** The class of the modular fields of a triangular Lie bialgebroid  $(A, P)$  is called the modular class of  $(A, P)$ . A triangular Lie bialgebroid is called unimodular if its modular class vanishes.

It follows from Proposition 5 that, when  $(A, P)$  is unimodular,  $i_P\mu$  is closed, and conversely. This is the answer to question 1 in [26].

**Proposition 9** The Lie derivation on  $\Gamma(\bigwedge A)$  with respect to the modular field  $X_\mu$  associated with  $\mu$  is the operator

$$\mathcal{L}_{X_\mu} = [X_\mu, \cdot]_A = [\partial_\mu, d_P], \quad (24)$$

where the first bracket is a Schouten bracket, while the second is a graded commutator. The Lie derivation on  $\Gamma(\bigwedge A^*)$  with respect to  $X_\mu$  is the operator

$$\mathcal{L}_{X_\mu} = [\partial_{P,\mu}, d_A]. \quad (25)$$

In particular, as derivations of  $C^\infty(M)$ ,

$$X_\mu = \partial_\mu \circ d_P = \partial_{P,\mu} \circ d_A. \quad (26)$$

If  $X_f^P = P^\sharp(d_A f) = -[P, f]_A = -d_P f$  is the Hamiltonian field with Hamiltonian  $f$  in  $C^\infty(M)$ , then

$$\mathcal{L}_{X_\mu} f = -\partial_\mu X_f^P. \quad (27)$$

Proof. Using the fact that  $\partial_\mu$  is a derivation of the Schouten bracket and formula (4), we find that, for any  $Q \in \Gamma(\bigwedge A)$ ,

$$\mathcal{L}_{\partial_\mu P} Q = [\partial_\mu P, Q]_A = \partial_\mu[P, Q]_A + [P, \partial_\mu Q]_A = (\partial_\mu d_P + d_P \partial_\mu)Q, \quad (28)$$

whence relation (24). To prove (25), we recall that  $\mathcal{L}_{X_\mu} = [i_{X_\mu}, d_A]$ . Since  $i_{X_\mu} = \partial_{P,\mu} - \partial_P$ , and since  $\partial_P$  commutes with  $d_A$ , relation (25) follows. If, in particular  $f \in \Gamma(\bigwedge^0 A)$ , then  $\partial_\mu f = \partial_{P,\mu} f = 0$ , and therefore we obtain (26), and (27) follows from the definitions.

**Corollary 10** The modular field  $X_\mu$ , associated with  $\mu$ , satisfies

$$\mathcal{L}_{X_\mu} \mu = 0. \quad (29)$$

Proof. From Proposition 5, we see that  $\mathcal{L}_{X_\mu} \mu = d_A(i_{X_\mu} \mu) = (d_A)^2(i_P \mu) = 0$ .

We collect various formulae in the following proposition.

**Proposition 11** For  $f \in C^\infty(M)$ ,  $[f, \mu]_P = \partial_{P, \mu}(f\mu) = -i_{d_P f} \mu$  and for  $\alpha \in \Gamma(A^*)$ ,

$$[\alpha, \mu]_P = (\partial_{P, \mu} \alpha) \mu . \quad (30)$$

In particular,  $[d_A f, \mu]_P = (X_\mu \cdot f) \mu$ .

Proof. These formulae follow from Proposition 4 and the fact that  $\partial_{P, \mu} \mu = 0$ . Moreover,  $\partial_{P, \mu} d_A f = \partial_P d_A f + i_{X_\mu} d_A f = X_\mu \cdot f$ , since  $\partial_P$  commutes with  $d_A$ .

If we now introduce the operator of degree 1 and of square 0 on  $\Gamma(\wedge A)$ ,

$$d_{P, \mu} = - *_{\mu}^{-1} \partial_P *_{\mu} , \quad (31)$$

using (5) and the relation  $i_P *_{\mu} = *_{\mu} e_P$ , where  $e_P$  denotes the exterior product by  $P$ , we see that

$$d_{P, \mu} = [\partial_{\mu}, e_P] . \quad (32)$$

Thus, equation (16) can be written as

$$d_{P, \mu} - d_P = e_{\partial_{\mu} P} . \quad (33)$$

By formula (19), this equation coincides with formula (43) of [6], where the operator  $d_{P, \mu}$  is denoted  $\delta'_{\pi, \mu_0}$ . This operator plays the role of a twisted cohomology operator of  $A^*$ .

**EXAMPLE 1. THE MODULAR CLASS OF A POISSON MANIFOLD.** Let  $(M, P)$  be an orientable Poisson manifold, where  $P$  denotes the Poisson bivector, and let  $\mu$  be a volume element on  $M$ , i. e., a nowhere vanishing section of  $\Gamma(\wedge^{\text{top}} T^* M)$ .

The triangular Lie bialgebroid  $(TM, P)$  of the Poisson manifold  $(M, P)$  was studied in [18]. If  $A = TM$ , and  $P \in \Gamma(\wedge^2 TM)$  is a Poisson structure on  $M$ , then the modular section is a vector field  $X_\mu$  on  $M$ , and, by formula (27),  $X_\mu \cdot f = -\partial_{\mu} X_f^P = \text{div}_{\mu} X_f^P$ . Therefore we recover Weinstein's definition [24], adopted in [6],

$$\mathcal{L}_{X_f^P} \mu = (X_\mu \cdot f) \mu . \quad (34)$$

**Proposition 12** The modular class of the triangular Lie bialgebroid  $(TM, P)$  of the Poisson manifold  $(M, P)$  is equal to the class of the modular vector field of  $(M, P)$ , defined by  $\mathcal{L}_{P^{\sharp} df} \mu = (X_\mu \cdot f) \mu$ .

**THE CASE OF A SYMPLECTIC MANIFOLD.** If, in particular, the bivector  $P$  on a manifold  $M$  of dimension  $n = 2m$  is of maximal rank, so that the Poisson structure,  $P$ , is actually associated with a symplectic structure  $\omega$ , then the modular class of  $(M, P)$  vanishes. In fact, taking the Liouville form  $\frac{\omega^m}{m!}$  as the volume form  $\mu$  on  $M$ , we see that any Hamiltonian vector field leaves  $\mu$

invariant and therefore, for any function  $f$ ,  $\partial_\mu X_f^P = 0$ . By (27), the modular vector field associated with  $\mu$  vanishes. So, when  $\mu$  is the Liouville form,

$$[d, i_P] = \partial_P = \partial_{P, \mu} . \quad (35)$$

Since, when  $(M, P)$  is symplectic, the modular vector field  $\partial_\mu P$  vanishes, formula (33) reduces to  $d_{P, \mu} = d_P$ . Using formula (32), we obtain

$$[\partial_\mu, e_P] = d_P = d_{P, \mu} . \quad (36)$$

In the symplectic case, there is a formula dual to (4),

$$d = [\omega, \cdot]_P . \quad (37)$$

(See [14].) In fact, this formula holds for any  $f \in C^\infty(M)$ , since it follows from the Leibniz rule that  $[\omega, f]_P = -i_{P^*df}\omega = df$ . Using the derivation property of the differential  $d$  with respect to the bracket  $[\cdot, \cdot]_P$  and the Leibniz rule again, we find  $[\omega, df]_P = 0$ , and for any  $g \in C^\infty(M)$ ,  $[\omega, gdf]_P = d(gdf)$ , and therefore (37) holds for any form.

Since  $\partial_P$  generates the bracket  $[\cdot, \cdot]_P$  and since  $\partial_P \omega = 0$ , we obtain, using (37),

$$[\partial_P, e_\omega] = d , \quad (38)$$

where  $e_\omega$  denotes the exterior product by  $\omega$ . This, in turn, implies, using the relation  $*_\mu i_\omega = e_\omega *_\mu$ , and (35),

$$[d_P, i_\omega] = \partial_\mu . \quad (39)$$

Thus, the operator  $[d_P, i_\omega]$  on fields of multivectors generates the Schouten bracket and coincides with  $\partial_\mu$  when  $\mu$  is the Liouville form.

*Remark.* It follows from relation (39) that, in the symplectic case, an alternate proof of relation (7),  $[\partial_\mu, i_\xi] = -i_{d\xi}$ , for any form  $\xi$ , is obtained by setting  $E = T^*M$ ,  $a = \omega$ ,  $b = \xi$  in relation (2).

**EXAMPLE 2. THE LINEAR CASE** (see [15]). If  $\mathfrak{g}$  is a finite-dimensional real Lie algebra, then  $M = \mathfrak{g}^*$  is a linear Poisson manifold, and conversely. The fields of multivectors on  $\mathfrak{g}^*$  are maps from  $\mathfrak{g}^*$  to  $\bigwedge \mathfrak{g}^*$ , and the linear fields of multivectors on  $\mathfrak{g}^*$  are vector-valued forms on  $\mathfrak{g}^*$ . Their Schouten bracket coincides with the Nijenhuis-Richardson bracket of vector-valued forms on the vector space  $\mathfrak{g}^*$ . If  $P$  is the linear Poisson structure on  $\mathfrak{g}^*$  defined by the Lie algebra structure of  $\mathfrak{g}$ , then the operator  $d_P$  is the Lie algebra cohomology operator on the cochains of  $\mathfrak{g}$  with coefficients in  $C^\infty(\mathfrak{g}^*)$ .

To compute the modular vector field  $X_\mu$  of  $(\mathfrak{g}^*, P)$  associated with the standard Lebesgue measure  $\mu$  on the vector space  $\mathfrak{g}^*$ , we choose a basis in  $\mathfrak{g}$ , and we

let  $(x^k)$  be the coordinates on  $\mathfrak{g}$  and  $(\xi_k)$  be the dual coordinates on  $\mathfrak{g}^*$ . Let  $C_{ij}^k$  be the structure constants of  $\mathfrak{g}$  in the chosen basis. Then  $P_{ij}(\xi) = C_{ij}^k \xi_k$ , and

$$(X_\mu)_i = (\partial_\mu P)_i = -\partial^j P_{ji} = C_{ij}^j. \quad (40)$$

This is a constant vector field on  $\mathfrak{g}^*$ , *i. e.*, an element of  $\mathfrak{g}^*$  which is equal to the linear 1-form on  $\mathfrak{g}$ ,  $tr(ad) : x \in \mathfrak{g} \mapsto tr(ad_x)$ , where  $tr$  denotes the trace. Thus,  $X_\mu$  is equal to the character of the adjoint representation of  $\mathfrak{g}$ , which is called the *infinitesimal modular character* of  $\mathfrak{g}$ .

**Proposition 13** *If  $M = \mathfrak{g}^*$ , the modular vector field associated with the standard Lebesgue measure is the infinitesimal modular character of  $\mathfrak{g}$ .*

Thus, the linear Poisson manifold  $\mathfrak{g}^*$  is unimodular if and only if the Lie algebra  $\mathfrak{g}$  is unimodular in the usual sense. This is one of the justifications for the use of the term “modular” in the context of Poisson geometry.

We now state the consequences of the preceding results for the homology-cohomology duality of Lie algebroids. See [26], and, for more general results concerning Lie-Rinehart algebras, see [10] and [11]. Let  $E$  be a Lie algebroid of rank  $n$ , such that there exists a nowhere vanishing section  $\mu$  of its top exterior power. Set  $\partial_{E,\mu} = -*\mu^{-1}d_E*\mu$ . Then the homology  $H_\bullet(E, \partial_{E,\mu})$  of the complex  $(\Gamma(\wedge E), \partial_{E,\mu})$  is isomorphic to the cohomology  $H^{n-\bullet}(E, d_E)$  of the complex  $(\Gamma(\wedge E^*), d_E)$ . In particular, if  $(A, P)$  is a triangular Lie bialgebroid, then for  $E = A^*$ ,  $d_E = d_{A^*} = d_P$  is the cohomology operator on  $\Gamma(\wedge E^*)$ , and therefore (i) the homology  $H_\bullet(A^*, \partial_{P,\mu})$  of the complex  $(\Gamma(\wedge A^*), \partial_{P,\mu})$  is isomorphic to the Poisson cohomology  $H^{n-\bullet}(A^*, d_P)$  of the complex  $(\Gamma(\wedge A), d_P)$ , and (ii) the Poisson homology  $H_\bullet(A^*, \partial_P)$  of the complex  $(\Gamma(\wedge A^*), \partial_P)$  is isomorphic to the twisted Poisson cohomology  $H^{n-\bullet}(A^*, d_P + e_{X_\mu})$  of the complex  $(\Gamma(\wedge A), d_P + e_{X_\mu})$ . This last fact follows from (33) and the definition of  $d_{P,\mu}$ . If the triangular Lie bialgebroid is unimodular, then  $\partial_{P,\mu} = \partial_P$ . So, both statements reduce to the fact that, in the unimodular case, the Poisson homology  $H_\bullet(A^*, \partial_P)$  is isomorphic to the Poisson cohomology  $H^{n-\bullet}(A^*, d_P)$ . Thus, in the case of a unimodular triangular Lie bialgebroid, in particular an orientable Poisson manifold with vanishing modular class, the Poisson homology is isomorphic to the Poisson cohomology.

## 6 The Laplacian of a strong differential BV-algebra. The case of a triangular Lie bialgebroid.

If  $(\mathcal{A}, \partial, d)$  is a strong differential BV-algebra, the operator,  $\Delta = [\partial, d]$ , on  $\mathcal{A}$  is called the *Laplacian* of  $(\mathcal{A}, \partial, d)$ . It is a derivation of degree 0 of both the bracket and the associative multiplication, and it is a differential operator of order 1 which vanishes on the unit of  $\mathcal{A}^0$ . If  $\mathcal{A}$  is the Gerstenhaber algebra of a

Lie algebroid  $E$  with base  $M$ , then  $\mathcal{A}^0 = C^\infty(M)$ , and the restriction of  $\Delta$  to  $\mathcal{A}^0$  is therefore a vector field on  $M$ .

Using the fact that  $\partial$  is a derivation of the Gerstenhaber bracket, we see that, if  $d$  is the interior derivation by an element  $P \in \mathcal{A}^2$ , then  $\Delta$  is the interior derivation by  $\partial P \in \mathcal{A}^1$ . This remark partly answers question 4 of [26].

If  $(A, P)$  is a triangular Lie bialgebroid, then  $(\Gamma(\wedge A^*), \partial_P, d_A)$  is a strong differential BV-algebra, but, in this case,  $\Delta = [\partial_P, d_A]$  vanishes identically. If moreover  $\mu$  is a nowhere vanishing section of  $\Gamma(\wedge^{\text{top}} A^*)$ , the operator  $\partial_\mu$  generates the Gerstenhaber bracket of  $\Gamma(\wedge A)$ , and  $d_P$  is an interior derivation of the Gerstenhaber bracket. Therefore the Laplacian of the strong differential BV-algebra,  $(\Gamma(\wedge A), \partial_\mu, d_P)$ , is

$$\Delta = [\partial_\mu, d_P] = [\partial_\mu P, \cdot]_A = \mathcal{L}_{\partial_\mu P}, \quad (41)$$

where the second bracket is the Gerstenhaber bracket of  $\Gamma(\wedge A)$ . In view of Proposition 6, this result coincides with the first statement of Proposition 9, which can be restated as:

**Proposition 14** *Let  $(A, P)$  be a triangular Lie bialgebroid, and let  $\mu$  be a nowhere vanishing section of  $\Gamma(\wedge^{\text{top}} A^*)$ . Then  $(\Gamma(\wedge A), \partial_\mu, d_P)$  is a strong differential BV-algebra, whose Laplacian is the Lie derivation with respect to the modular field of  $(A, P)$  associated with  $\mu$ .*

*Remark.* In [19], a strong differential BV-algebra which is unimodular is called a dGBV-algebra.

For a Poisson manifold  $(M, P)$  with a nowhere vanishing section  $\mu$  of  $\Gamma(\wedge^{\text{top}} T^*M)$ , the Laplacian is that of the strong differential BV-algebra  $(\Gamma(\wedge TM), \partial_\mu, d_P)$ , and it is equal to the derivation  $\mathcal{L}_{X_\mu}$  of  $\Gamma(\wedge TM)$ . This result was already in Koszul [15].

## 7 BV-algebras and modified actions.

The restriction of the generating operator of a BV-algebra to  $\mathcal{A}^1$  is an  $\mathcal{A}^0$ -valued derivation of the Lie algebra  $\mathcal{A}^1$ . Therefore

**Proposition 15** *Let  $(\mathcal{A}, \partial)$  be a BV-algebra. Then  $\partial|_{\mathcal{A}^1} : \mathcal{A}^1 \rightarrow \mathcal{A}^0$  is a 1-cocycle of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ , and the map  $(a, f) \in \mathcal{A}^1 \times \mathcal{A}^0 \mapsto [a, f] + \kappa(\partial a)f \in \mathcal{A}^0$ , for any scalar  $\kappa$ , is an action of the Lie algebra  $\mathcal{A}^1$  on  $\mathcal{A}^0$ .*

Thus to each generating operator of the Gerstenhaber bracket of  $\mathcal{A}$ , there corresponds a modified left  $\mathcal{A}^1$ -module structure on  $\mathcal{A}^0$ . However, it is easy to see that this action does not make  $\mathcal{A}^0$  a left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module in the sense of [9]. On the other hand, as has been shown by Huebschmann [11] [10],  $\mathcal{A}^0$  has a right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structure:

**Proposition 16** *Let  $(\mathcal{A}, \partial)$  be a BV-algebra. The map  $(f, a) \in \mathcal{A}^0 \times \mathcal{A}^1 \mapsto f.a \in \mathcal{A}^0$  defined by*

$$f.a = -[a, f] + (\partial a)f \quad (42)$$

*is a right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structure on  $\mathcal{A}^0$ , and  $f.a = \partial(fa)$ .*

Proof. We have to show that

$$(fg).a = f(g.a) - [a, f]g = g.(fa) , \quad (43)$$

for  $f, g \in \mathcal{A}^0$ ,  $a \in \mathcal{A}^1$ . In view of

$$\partial(fa) = f\partial a - [a, f] , \quad (44)$$

this is straightforward.

Conversely, if  $(f, a) \mapsto f.a$  is a right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structure on  $\mathcal{A}^0$ , then there exists a unique generating operator  $\partial$  of  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$  such that (42) holds. In particular, if we let 1 be the unity of  $\mathcal{A}^0$ , then  $\partial a = 1.a$ , for  $a \in \mathcal{A}^1$ . In [11], Huebschmann states and proves the following theorem. (He actually proves a more general result dealing with the generating operators which are not necessarily of square 0).

**Theorem 17** *There is a one-to-one correspondence between right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structures on  $\mathcal{A}^0$  and generating operators of  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$ .*

Proof. The proof consists of showing that any 1-cocycle  $\partial : \mathcal{A}^1 \rightarrow \mathcal{A}^0$  satisfying (44) can be uniquely extended to a generating operator of  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$ . This, in turn, is proved by means of the explicit formula given in Section 5.

In particular, if  $E$  is a Lie algebroid, there is a one-to-one correspondence between right  $(C^\infty(M), \Gamma(E))$ -module structures on  $C^\infty(M)$  and generating operators of  $\Gamma(\bigwedge E)$ .

For example, if  $M$  is a manifold and  $\mu$  is a volume element on  $M$ , then the modified right action of  $\Gamma(TM)$  on  $C^\infty(M)$  is  $X \in \Gamma(TM) \mapsto -X + (\partial_\mu X)Id \in \text{End}(C^\infty(M))$ , where  $Id$  is the identity endomorphism of  $C^\infty(M)$ . Thus, in any orientable manifold  $(M, \mu)$ , there is a right action of the Lie algebra of vector fields on the space of functions, defined by

$$f.X = -\mathcal{L}_X f + f\partial_\mu X . \quad (45)$$

If  $(A, P)$  is a triangular Lie bialgebroid, then  $\alpha \in \Gamma(A^*) \mapsto -P^\sharp \alpha + (\partial_P \alpha)Id \in \text{End}(C^\infty(M))$ , where  $\partial_P \alpha = i_P d_A \alpha$ , defines the modified right action of  $\Gamma(A^*)$  on  $C^\infty(M)$ . If, in particular,  $(M, P)$  is a Poisson manifold, then  $(\Gamma(\bigwedge T^*M), \partial_P)$  is a BV-algebra, and we obtain a modified right action of  $\Gamma(T^*M)$  on  $C^\infty(M)$  making  $C^\infty(M)$  a right  $(C^\infty(M), \Gamma(T^*M))$ -module. Thus, in any Poisson manifold  $(M, P)$ , there is a right action of the Lie algebra of differential 1-forms on the space of functions defined by

$$f.\alpha = -\mathcal{L}_{P^\sharp \alpha} f + f\partial_P \alpha . \quad (46)$$

In particular, for functions  $f, g, h \in C^\infty(M)$ ,  $f.dh$  is the Poisson bracket  $\{f, h\}$  and

$$f.(gdh) = \{fg, h\} . \quad (47)$$

Therefore, this is the structure defined in [10], formula (7.3). Another right action of differential 1-forms on functions is defined by

$$f.(gdh) = \{fg, h\} + fg(X_\mu.h) . \quad (48)$$

## 8 Representations and generating operators.

We now explain the one-to-one correspondence between generating operators and left structures on the top exterior power of projective, finite-rank Lie-Rinehart algebras. In [11], Huebschmann states the following theorem which follows from Theorem 17.

**Theorem 18** *Let  $\mathcal{A}^1$  be a  $(k, \mathcal{A}^0)$ -algebra which is a projective, finite rank  $\mathcal{A}^0$ -module. There is a one-to-one correspondence between generating operators for the Gerstenhaber algebra  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$  and left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structures on  $\bigwedge_{\mathcal{A}^0}^{\text{top}} \mathcal{A}^1$ .*

Recall that a *representation* of the Lie algebroid  $E$  with base  $M$  and anchor  $\rho$  on the vector bundle  $F$  is an  $\mathbb{R}$ -linear map  $\nabla$  from  $\Gamma(E) \times \Gamma(F)$  to  $\Gamma(F)$  which is  $C^\infty(M)$ -linear in the first argument and such that  $\nabla_a(fu) = f\nabla_a u + (\rho(a)f)u$ , and  $\nabla_{[a,b]} = [\nabla_a, \nabla_b]$ , for  $a, b \in \Gamma(E)$ ,  $f \in C^\infty(M)$ ,  $u \in \Gamma(F)$ .

Representations of  $E$  on  $F$  are called flat  $E$ -connections on  $F$  in [26]. They coincide with left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structures on  $\Gamma(F)$ , for  $\mathcal{A}^0 = C^\infty(M)$  and  $\mathcal{A}^1 = \Gamma(E)$ .

**Corollary 19** *Let  $E$  be a Lie algebroid. There is a one-to-one correspondence between generating operators for the Gerstenhaber algebra of  $E$ ,  $\Gamma(\bigwedge E)$ , and representations of  $E$  on  $\bigwedge^{\text{top}} E$ .*

Explicitly, if  $\partial$  is a generating operator of the Gerstenhaber bracket of  $\Gamma(\bigwedge E)$ , then  $\nabla$  defined by

$$\nabla_a \lambda = a.\lambda - (\partial a)\lambda , \quad (49)$$

for  $a \in \Gamma(E)$ ,  $\lambda \in \Gamma(\bigwedge^{\text{top}} E)$ , is a representation of  $E$  on  $\bigwedge^{\text{top}} E$ . Here  $a.\lambda = \mathcal{L}_a \lambda = [a, \lambda]_E$ .

Conversely, given a representation  $\nabla$  of  $E$  on  $\bigwedge^{\text{top}} E$ , there exists a unique generating operator,  $\partial$ , of the Gerstenhaber bracket satisfying the condition that, for any section  $\lambda$  of  $\bigwedge^{\text{top}} E$ , and for any section  $a$  of  $E$ ,

$$(\mathcal{L}_a - \nabla_a)\lambda = (\partial a)\lambda , \quad (50)$$

or equivalently,

$$\nabla_a \lambda = -a \wedge \partial \lambda . \quad (51)$$

Both conditions are equivalent since  $\mathcal{L}_a \lambda = [a, \lambda]_E = -\partial(a \wedge \lambda) + (\partial a) \lambda - a \wedge \partial \lambda$ .

This corollary states that to each representation of  $E$  on  $\bigwedge^{\text{top}} E$ , there corresponds a BV-algebra structure on the Gerstenhaber algebra of  $E$ , and conversely.

The analogue of this theorem is proved in [22] in the framework of schemes: there are mutually inverse bijections between the sets of ‘‘Calabi-Yau (CY) data on a scheme  $X$ ’’ and ‘‘Batalin-Vilkovisky (BV) data on  $X$ ’’. CY data correspond to representations of the tangent bundle of a complex manifold on the canonical bundle (holomorphic forms of top degree), while BV data correspond to generating operators of the Gerstenhaber algebra of holomorphic fields of multivectors. For the holomorphic case, see also [3].

*Remark.* The above corollary was proved by Koszul [15] when  $E = TM$ , and generalized to arbitrary Lie algebroids by Xu [26]. The results proved by Koszul and Xu are actually more general. The one-to-one correspondence of Corollary 19 is the restriction to the flat  $E$ -connections of a one-to-one correspondence between generating operators, not necessarily of square zero, of the Gerstenhaber algebra of  $E$  and  $E$ -connections on  $\bigwedge^{\text{top}} E$  which are not necessarily flat.

Combining the one-to-one correspondences in Theorems 17 and 19, one obtains a one-to-one correspondence between right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structures on  $\mathcal{A}^0$  and left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structures on  $\bigwedge_{\mathcal{A}^0}^{\text{top}} \mathcal{A}^1$ . Given a left structure  $\nabla$  on  $\bigwedge_{\mathcal{A}^0}^{\text{top}} \mathcal{A}^1$ , the corresponding right structure on  $\mathcal{A}^0$  is such that

$$(f.a)\lambda = f(a.\lambda) - \nabla_a(f\lambda) , \quad (52)$$

where  $a.\lambda = \mathcal{L}_a \lambda$ . Conversely, if the right structure on  $\mathcal{A}^0$  is given, then the left structure  $\nabla$  on  $\bigwedge_{\mathcal{A}^0}^{\text{top}} \mathcal{A}^1$  is such that

$$\nabla_a \lambda = a.\lambda - (1.a)\lambda . \quad (53)$$

In fact the correspondence established in Corollary 19 between  $\nabla$  and  $\partial$  is the composition of the map  $\nabla \mapsto \{\text{right structure on } \mathcal{A}^0\}$  which we have just described with the map  $\{\text{right structure on } \mathcal{A}^0\} \mapsto \partial$  of Theorem 17.

Huebschmann shows in [10] and [11] that, if  $\partial$  is the generating operator of  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$  associated to a left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structure  $\nabla$  on  $\bigwedge_{\mathcal{A}^0}^{\text{top}} \mathcal{A}^1$ , then the homology of  $\mathcal{A}^1$  with coefficients in the right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module  $\mathcal{A}^0$  defined by  $\partial$  and the cohomology of  $\mathcal{A}^1$  with coefficients in the left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module  $\bigwedge_{\mathcal{A}^0} \mathcal{A}^1$  defined by  $\nabla$  are isomorphic.

## 9 The modular class of a triangular Lie bialgebroid as the characteristic class of a Lie algebroid with a representation.

If  $\nabla$  and  $\nabla'$  are representations of a Lie algebroid  $E$  on  $\bigwedge^{\text{top}} E$ , there is a section  $\xi$  of  $E^*$  such that, for any  $\lambda \in \Gamma(\bigwedge^{\text{top}} E)$ , and for any section  $a$  of  $E$ ,

$$(\nabla_a - \nabla'_a)\lambda = \langle \xi, a \rangle \lambda . \quad (54)$$

If  $\partial$  and  $\partial'$  are the generating operators associated with  $\nabla$  and  $\nabla'$ , respectively, as in Corollary 19, then this relation is equivalent to

$$\partial' - \partial = i_\xi . \quad (55)$$

It follows from Proposition 2 that  $\xi$  is then  $d_E$ -closed in the Lie algebroid cohomology of  $E$  with coefficients in  $C^\infty(M)$ . Of course this 1-cocycle can be trivial, in which case, the representations  $\nabla$  and  $\nabla'$  are called *homotopic* [26], and their associated operators  $\partial$  and  $\partial'$  are also called homotopic.

Let the representation  $\nabla$  be fixed. If, for a class of representations  $\nabla'$ , the 1-cocycles,  $\xi$ , defined by (54), for various  $\nabla'$  or, equivalently, (55), for various  $\partial'$ , are equivalent, we can call their cohomology class *the modular class of the Lie algebroid with representation  $(E, \nabla)$* . The modular class vanishes if and only if all representations of  $E$  on  $\bigwedge^{\text{top}} E$  in the class  $\nabla'$  are homotopic to  $\nabla$ . If  $\partial$  is the generating operator of  $\Gamma(\bigwedge E)$  corresponding to  $\nabla$  by Corollary 19, we can also call this cohomology class *the modular class of the BV-algebra  $(\Gamma(\bigwedge E), \partial)$* .

The fundamental remark is the following. Assume that there is a well-defined generating operator  $\partial$  for the Gerstenhaber algebra of a Lie algebroid  $E$ . To any nowhere vanishing section,  $\lambda$ , of  $\bigwedge^{\text{top}} E$ , there corresponds an isomorphism  $*_\lambda$  from  $\bigwedge E^*$  to  $\bigwedge E$ , and a generating operator,  $\partial_\lambda$ , of the bracket of  $\Gamma(\bigwedge E)$ , defined by  $\partial_\lambda = - *_\lambda d_E *_\lambda^{-1}$ .

In this case, the correspondence of Corollary 19 is given as follows. (Cf [26].)

**Proposition 20** *The representation  $\nabla^\lambda$  of  $E$  on  $\bigwedge^{\text{top}} E$  associated with the generating operator of the bracket of  $\Gamma(\bigwedge E)$ ,  $\partial_\lambda = - *_\lambda d_E *_\lambda^{-1}$ , is defined by  $\nabla^\lambda \lambda = 0$ .*

This follows from (51) since, in this case,  $\partial_\lambda \lambda = 0$ . It follows from (55) and from a computation analogous to (22), that the class of the 1-cocycles  $\xi$  associated with the pairs  $(\nabla, \nabla^\lambda)$ , for various choices of  $\lambda$ , is independent of the choice of  $\lambda$ .

According to Evens, Lu and Weinstein [6], for any Lie algebroid  $E$  with a representation  $\nabla$  on a line bundle  $L$ , the *characteristic class* is the class of the section  $\hat{\xi}$  of  $E^*$  satisfying

$$\nabla_a \lambda = \langle \hat{\xi}, a \rangle \lambda , \quad (56)$$

for  $a \in \Gamma(E)$  and  $\lambda$  a nowhere vanishing section of  $\bigwedge^{\text{top}} E$ . (If  $\bigwedge^{\text{top}} E$  is not trivial, define the characteristic class as one-half of that of the square of this bundle with the associated representation.) The section  $\hat{\xi} \in \Gamma(E^*)$  depends on  $\nabla$  and  $\lambda$ , but it is  $d_E$ -closed and its  $d_E$ -cohomology class depends only on  $\nabla$ .

On the other hand, in view of the definition of  $\nabla^\lambda$ , the modular section  $\xi$  defining the modular class of the Lie algebroid with representation  $(E, \nabla)$  has to satisfy

$$(\nabla_a - \nabla_a^\lambda)\lambda = \langle \xi, a \rangle \lambda . \quad (57)$$

In view of the characterization of  $\nabla^\lambda$  given in Proposition 20, this condition coincides with (56), in other words,  $\xi = \hat{\xi}$ . Moreover, for the associated BV-algebra  $(\Gamma(\bigwedge E), \partial)$ , the modular class is defined as the class of  $\xi \in \Gamma(E^*)$  satisfying

$$\partial_\lambda - \partial = i_\xi , \quad (58)$$

where  $\partial$  is the fixed generating operator associated with  $\nabla$ , and  $\partial_\lambda$  is the generating operator corresponding to  $\nabla^\lambda$ . Therefore,

**Proposition 21** *The modular class of the Lie algebroid with representation  $(E, \nabla)$  coincides with the characteristic class defined in [6], and it is the class of the section  $\xi$  of  $\Gamma(E^*)$  satisfying (58).*

THE CASE OF A TRIANGULAR LIE BIALGEBROID. Let  $(A, P)$  be a triangular Lie bialgebroid. Then, there is a well-defined generating operator,  $\partial_P = [d_A, i_P]$ , for the Gerstenhaber bracket of  $\Gamma(\bigwedge A^*)$ . What is the corresponding representation  $\nabla^P$  of  $A^*$  on  $\bigwedge^{\text{top}} A^*$ ? According to formula (51), it is defined by

$$\nabla_\alpha^P \lambda = -\alpha \wedge \partial_P \lambda = -\alpha \wedge d_A(i_P \lambda) , \quad (59)$$

for  $\alpha$  a section of  $A^*$  and  $\lambda$  a section of  $\bigwedge^{\text{top}} A^*$ . Now let  $\nabla^\mu$  be the representation of  $A^*$  in  $\bigwedge^{\text{top}} A^*$  defined by  $\nabla^\mu \mu = 0$ , where  $\mu$  is a nowhere vanishing section of  $\Gamma(\bigwedge^{\text{top}} A^*)$ . Then the characteristic class of the Lie algebroid with representation in a line bundle  $(A^*, \nabla^P)$  is the class of the section  $\hat{X}_\mu$ , depending on  $\mu$ , defined by

$$\nabla_\alpha^P \mu = \langle \hat{X}_\mu, \alpha \rangle \mu , \quad (60)$$

for  $\alpha \in \Gamma(A^*)$ . In view of Proposition 20, and of the equivalence of (54) and (55), condition (60) is equivalent to

$$\partial_{P, \mu} - \partial_P = i_{\hat{X}_\mu} . \quad (61)$$

Therefore, the section  $\hat{X}_\mu$  is equal to the modular field  $X_\mu$  defined in Section 5, and  $\nabla_\alpha^P \mu = ((\partial_{P, \mu} - \partial_P)\alpha)\mu$ .

**Proposition 22** *If  $(A, P)$  is a triangular Lie bialgebroid, its modular class is equal to the characteristic class of the Lie algebroid with representation  $(A^*, \nabla^P)$ .*

THE CASE OF A POISSON MANIFOLD. We can apply the preceding analysis to the case of a Poisson manifold  $(M, P)$ .

In [6], it is proved that, if  $(M, P)$  is a Poisson manifold, there is a well defined representation of the Lie algebroid  $T^*M$  on  $\bigwedge^{\text{top}} T^*M$ . In our approach, this follows from the fact [15] [13] that there is a well defined generating operator for the Gerstenhaber bracket of  $\Gamma(\bigwedge T^*M)$ , the Koszul-Brylinski homology operator,  $\partial_P = [d, i_P]$ , combined with Corollary 19.

Applying our previous results to the triangular Lie bialgebroid  $(TM, P)$ , and combining them with Proposition 12, we obtain the following result.

**Proposition 23** *Let  $(M, P)$  be a Poisson manifold. The modular class of the triangular Lie bialgebroid  $(TM, P)$  coincides with the modular class of the Poisson manifold and with the characteristic class of the Lie algebroid with representation  $(T^*M, \nabla^P)$ .*

*Remark.* Formula (59) in the case of a Poisson manifold does coincide with formulae (36) of [6] and (18) of [26], although there is an apparent difference in the sign. This is due to the fact that the interior product used in [6] and [26] is the one that we have denoted by  $\tilde{i}$  (see the Remark in Section 2), which is such that  $\tilde{i}_P = -i_P$ . To summarize, in [6], the characteristic class of  $(T^*M, \nabla^P)$  is the class of  $\hat{X}_\mu$ , where

$$df \wedge d(\tilde{i}_P \mu) = \langle \hat{X}_\mu, df \rangle \mu .$$

The modular class of the Poisson manifold  $(M, P)$  is the class of  $\mathcal{X}_\mu$ , where by definition,

$$di_{P^\sharp} df \mu = \langle \mathcal{X}_\mu, df \rangle \mu ,$$

while, here, the modular class of the triangular Lie bialgebroid  $(TM, P)$  is the class of  $X_\mu$  where, by definition,

$$-df \wedge d(i_P \mu) = \langle X_\mu, df \rangle \mu .$$

Since  $i_{P^\sharp} df \mu = df \wedge i_P \mu = -df \wedge \tilde{i}_P \mu$ , we see that all three definitions agree.

In general, any Lie algebroid  $E$  has a well-defined representation  $\nabla^{Q_E}$  on the line bundle  $Q_E = \bigwedge^{\text{top}} E \otimes \bigwedge^{\text{top}} T^*M$ . If  $(M, P)$  is a Poisson manifold, the characteristic class of  $(T^*M, \nabla^P)$  is that of  $(T^*M, \nabla^{Q_{T^*M}})$  divided by 2. See [6].

If  $(A, P)$  is a triangular Lie bialgebroid, we have to compare its modular class, defined here, to the characteristic class of  $(Q_{A^*}, \nabla^{Q_{A^*}})$ , where  $Q_{A^*} = \bigwedge^{\text{top}} A^* \otimes \bigwedge^{\text{top}} T^*M$ , and

$$\nabla_\alpha^{Q_{A^*}} (\mu \otimes \nu) = \mathcal{L}_\alpha \mu \otimes \nu + \mu \otimes \mathcal{L}_{\rho(P^\sharp \alpha)} \nu . \quad (62)$$

The modular class of  $(A, P)$  is the class of the section  $X_\mu$  of  $A$  such that

$$\nabla_\alpha^P \mu = \langle X_\mu, \alpha \rangle \mu ,$$

where  $\nabla_\alpha^P \mu = \mathcal{L}_\alpha \mu - (\partial_P \mu) \alpha$ , while the characteristic class of  $(Q_{A^*}, \nabla^{Q_{A^*}})$  is the class of the section  $\tilde{X}_{\mu \otimes \nu}$  of  $A$  such that

$$\nabla_\alpha^{Q_{A^*}} (\mu \otimes \nu) = \langle \tilde{X}_{\mu \otimes \nu}, \alpha \rangle \mu \otimes \nu .$$

Here  $\mu$  is a section of  $\bigwedge^{\text{top}} A^*$  and  $\nu$  is a section of  $\bigwedge^{\text{top}} T^*M$ .

Let us assume that  $\alpha = d_A f$ ,  $f \in C^\infty(M)$ . To evaluate the first term in the right-hand side of (62), we use Proposition 11, and we obtain

$$\mathcal{L}_{d_A f} \mu = [d_A f, \mu]_P = \langle X_\mu, d_A f \rangle \mu = \langle \rho(X_\mu), df \rangle \mu ,$$

where  $d$  is the de Rham differential on the base manifold  $M$ .

The Lie bialgebroid structure of  $(A, A^*)$  defined by  $P$  induces a Poisson structure,  $P_M$ , on  $M$  [18] [13] satisfying

$$\langle P_M^\sharp(df), dg \rangle = \langle d_A f, d_{A^*} g \rangle ,$$

for  $f, g \in C^\infty(M)$ . Using  $d_{A^*} g = d_P g = -P^\sharp(d_A g)$ , the skew-symmetry of  $P^\sharp$  and the definition of  $d_A$ , we obtain  $P_M^\sharp(df) = \rho(P^\sharp(d_A f))$ . Therefore, we can evaluate the second term in the right-hand side of (62) by means of the modular vector field of the Poisson manifold  $(M, P_M)$  with volume form  $\nu$ , defined by

$$\mathcal{L}_{\rho(P^\sharp(d_A f))} \nu = \langle X_\nu^M, df \rangle \nu .$$

Therefore,

$$\langle \tilde{X}_{\mu \otimes \nu}, d_A f \rangle = \langle X_\mu, d_A f \rangle + \langle X_\nu^M, df \rangle \quad (63)$$

and

$$\rho(\tilde{X}_{\mu \otimes \nu}) = \rho(X_\mu) + X_\nu^M . \quad (64)$$

In the particular case of a Poisson manifold  $(M, P)$ , we consider the tangent bundle  $TM = A$ , with anchor the identity of  $TM$ . Then  $P_M = P$ , and  $X_\mu^M = X_\mu$ , and we recover the relation

$$\tilde{X}_{\mu \otimes \mu} = 2X_\mu . \quad (65)$$

To complete our discussion, we sketch the approach to the definition of the modular class of a Lie-Rinehart algebra given in [10]. There, Huebschmann defines the modular class of a Lie-Rinehart algebra  $(\mathcal{A}^0, \mathcal{A}^1)$  satisfying certain regularity conditions as the isomorphism class of the left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module,  $Q_{\mathcal{A}^1} = \text{Hom}_{\mathcal{A}^0}(\bigwedge_{\mathcal{A}^0}^{\text{top}}(\mathcal{A}^1)^*, \omega_{\mathcal{A}^0})$ , where  $\omega_{\mathcal{A}^0} = \text{Hom}(\bigwedge_{\mathcal{A}^0}^{\text{top}}(\text{Der} \mathcal{A}^0), \mathcal{A}^0)$ . He further shows that  $\bigwedge_{\mathcal{A}^0}^{\text{top}}(\mathcal{A}^1)^*$ , called the dualizing module of the Lie-Rinehart algebra  $(\mathcal{A}^0, \mathcal{A}^1)$ , as well as  $\omega_{\mathcal{A}^0}$  have a canonically defined structure of right  $(\mathcal{A}^0, \mathcal{A}^1)$ -module, and that  $Q_{\mathcal{A}^1}$  is therefore a left  $(\mathcal{A}^0, \mathcal{A}^1)$ -module. The module  $Q_{\mathcal{A}^1}$  generalizes the space of sections of the line bundle with a representation  $Q_E$ . In fact, if  $E$  is a Lie algebroid with base  $M$ , and if  $\mathcal{A}^0 = C^\infty(M)$ , and

$\mathcal{A}^1 = \Gamma(E)$ , then  $Q_{\mathcal{A}^1} = \text{Hom}_{C^\infty(M)}(\Gamma(\bigwedge^{\text{top}} E^*), \Gamma(\bigwedge^{\text{top}} T^*M)) = \Gamma(\bigwedge^{\text{top}} E \otimes \bigwedge^{\text{top}} T^*M) = \Gamma(Q_E)$ .

As a left  $\mathcal{A}^0$ -module, the tensor square of  $Q_{\mathcal{A}^1}$  is trivial, so its class is in the kernel of the forgetful map from the abelian group (introduced in [10]) of isomorphism classes of left  $(\mathcal{A}^0, \mathcal{A}^1)$ -modules which are projective and of rank one as  $\mathcal{A}^0$ -modules to the Picard group of isomorphism classes of projective rank one  $\mathcal{A}^0$ -modules. This kernel is precisely the space of derivations of  $\mathcal{A}^1$  with values in  $\mathcal{A}^0$  modulo the action of nowhere vanishing functions, and the isomorphism class of  $Q_{\mathcal{A}^1} \otimes Q_{\mathcal{A}^1}$  maps to a cohomology class of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ . In fact, if  $\mathcal{E}$  is a trivial  $\mathcal{A}^0$ -module of rank 1 with basis  $\lambda$ , setting  $\nabla_a \lambda = \langle \xi, a \rangle \lambda$ , for  $a \in \mathcal{E}$ , where  $\xi$  is a 1-cocycle of  $\mathcal{A}^1$  with coefficients in  $\mathcal{A}^0$ , defines an  $(\mathcal{A}^0, \mathcal{A}^1)$ -module structure on  $\mathcal{E}$ , and two such structures are isomorphic if and only if  $\xi$  is a coboundary. Taking  $\mathcal{A}^1 = \Gamma(E)$  and  $\mathcal{E} = \Gamma(Q_E \otimes Q_E)$  yields a cohomology class of  $E$  with values in  $C^\infty(M)$ . The modular class of the Lie algebroid  $E$  in the sense of Evens, Lu and Weinstein is this cohomology class divided by two.

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