

# Quantum homogeneous spaces and quasi-Hopf algebras

Benjamin Enriquez

*Département de mathématiques et applications (UMR 8553 du CNRS)*

*École normale supérieure, 45 rue d'Ulm, 75005 Paris, France*

e-mail: benjamin.enriquez@ens.fr

Yvette Kosmann-Schwarzbach

*Centre de mathématiques (UMR 7640 du CNRS)*

*École polytechnique, 91128 Palaiseau, France*

e-mail: yks@math.polytechnique.fr

**Abstract.** We propose a formulation of the quantization problem of Manin quadruples, and show that a solution to this problem yields a quantization of the corresponding Poisson homogeneous spaces. We then solve both quantization problems in an example related to quantum spheres.

**Keywords:** Manin triples, Manin pairs, Poisson-Lie groups, homogeneous spaces, Hopf algebras, quasi-Hopf algebras

**Mathematics Subject Classifications (2000):** 16B62, 53D17, 17B37, 20G42

*To the memory of Moshé Flato*

## Introduction

According to a theorem of Drinfeld, formal Poisson homogeneous spaces over a formal Poisson-Lie group  $G_+$  with Lie algebra  $\mathfrak{g}_+$  correspond bijectively to  $G_+$ -conjugation classes of Lagrangian (i.e., maximal isotropic) Lie subalgebras  $\mathfrak{h}$  of the double Lie algebra  $\mathfrak{g}$  of  $\mathfrak{g}_+$ . The formal Poisson homogeneous space is then  $G_+/(G_+ \cap H)$ , where  $H$  is the formal Lie group with Lie algebra  $\mathfrak{h}$ . The corresponding quantization problem is to deform the algebra of functions over the homogeneous space to an algebra-module over the quantized enveloping algebra of  $\mathfrak{g}_+$ .

In this paper, we show that there is a Poisson homogeneous structure on the formal homogeneous space  $G/H$ , such that the embedding of  $G_+/(G_+ \cap H)$  in  $G/H$  is Poisson, where  $G$  is the formal Lie group with Lie algebra  $\mathfrak{g}$ . It is therefore natural to seek a quantization of the function algebra of  $G_+/(G_+ \cap H)$  as a quotient of a quantization of  $G/H$ .

The data of  $(\mathfrak{g}, \mathfrak{h})$  and the  $r$ -matrix of  $\mathfrak{g}$  constitute an example of a quasitriangular Manin pair (see Section 1.1). We introduce the notion of the quantization of a quasitriangular Manin pair, which consists of a quasitriangular Hopf algebra quantizing the Lie bialgebra  $\mathfrak{g}$ , quasi-Hopf algebras quantizing the Manin pair

[105]

*G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 2, 105 – 121.*

© 2000 Kluwer Academic Publishers. Printed in the Netherlands.

$(\mathfrak{g}, \mathfrak{h})$ , and a twist element relating both structures. We then show (Theorem 2.1) that this data gives rise to a quantization of  $G/H$ .

The quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  formed by adjoining  $\mathfrak{h}$  to the Manin triple of  $\mathfrak{g}_+$  is called a Manin quadruple. The quantization of a Manin quadruple is the additional data of Hopf algebras quantizing the Manin triple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ , subject to compatibility conditions with the quantization of the underlying quasitriangular Manin pair. We show that any quantization of a given quadruple gives rise to a quantization of the corresponding homogeneous space (Theorem 3.1).

Finally, in Section 4.1, we explicitly solve the problem of quantizing a Manin quadruple in a situation related to quantum spheres.

## 1. Manin quadruples and Poisson homogeneous spaces

In this section, we define Lie-algebraic structures, i.e., quasitriangular Manin pairs and Manin quadruples, and the Poisson homogeneous spaces naturally associated to them.

### 1.1. Quasitriangular Manin pairs

Recall that a *quasitriangular Lie bialgebra* is a triple  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ , where  $\mathfrak{g}$  is a complex Lie algebra,  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$  is a nondegenerate invariant symmetric bilinear form on  $\mathfrak{g}$ , and  $r$  is an element of  $\mathfrak{g} \otimes \mathfrak{g}$  such that  $r + r^{(21)}$  is the symmetric element of  $\mathfrak{g} \otimes \mathfrak{g}$  defined by  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ , and  $r$  satisfies the classical Yang-Baxter equation,  $[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] + [r^{(13)}, r^{(23)}] = 0$ . (Such Lie bialgebras are also called factorizable.)

Assume that  $\mathfrak{g}$  is finite dimensional, and let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ . Then  $G$  is equipped with the Poisson-Lie bivector  $P_G = r^L - r^R$ , where, for any element  $a$  of  $\mathfrak{g} \otimes \mathfrak{g}$ , we denote the right- and left-invariant 2-tensors on  $G$  corresponding to  $a$  by  $a^L$  and  $a^R$ . If  $\mathfrak{g}$  is an arbitrary Lie algebra, the same statement holds for its formal Lie group.

We will call the pair  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$  of a quasitriangular Lie bialgebra  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$  and a Lagrangian Lie subalgebra  $\mathfrak{h}$  in  $\mathfrak{g}$  a *quasitriangular Manin pair*.

Assume that  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$  is a quasitriangular Manin pair, and that  $L$  is a Lagrangian complement of  $\mathfrak{h}$  in  $\mathfrak{g}$ . Let  $(\varepsilon^i)$  and  $(\varepsilon_i)$  be dual bases of  $\mathfrak{h}$  and  $L$ , and set  $r_{\mathfrak{h},L} = \sum_i \varepsilon^i \otimes \varepsilon_i$ . The restriction to  $L$  of the Lie bracket of  $\mathfrak{g}$  followed by the projection to the first factor in  $\mathfrak{g} = \mathfrak{h} \oplus L$  yields an element  $\varphi_{\mathfrak{h},L}$  of  $\wedge^3 \mathfrak{h}$ . Let us set  $f_{\mathfrak{h},L} = r_{\mathfrak{h},L} - r$ ; then  $f_{\mathfrak{h},L}$  belongs to  $\wedge^2 \mathfrak{g}$ .

Then the twist of the Lie bialgebra  $(\mathfrak{g}, \partial r)$  by  $f_{\mathfrak{h},L}$  is the quasitriangular Lie quasi-bialgebra  $(\mathfrak{g}, \partial r_{\mathfrak{h},L}, \varphi_{\mathfrak{h},L})$ . The cocycle  $\partial r_{\mathfrak{h},L}$  maps  $\mathfrak{h}$  to  $\wedge^2 \mathfrak{h}$ , so  $(\mathfrak{h}, (\partial r_{\mathfrak{h},L})|_{\mathfrak{h}}, \varphi_{\mathfrak{h},L})$  is a sub-Lie quasi-bialgebra of  $(\mathfrak{g}, \partial r_{\mathfrak{h},L}, \varphi_{\mathfrak{h},L})$ .

## 1.2. Manin quadruples

### 1.2.1. Definition

A *Manin quadruple* is a quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ , where  $\mathfrak{g}$  is a complex Lie algebra, equipped with a nondegenerate invariant symmetric bilinear form  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ , and  $\mathfrak{g}_+, \mathfrak{g}_-$  and  $\mathfrak{h}$  are three Lagrangian subalgebras of  $\mathfrak{g}$ , such that  $\mathfrak{g}$  is equal to the direct sum  $\mathfrak{g}_+ \oplus \mathfrak{g}_-$  (see [11]).

In particular,  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  is a Manin triple. This implies that for any Lie group  $G$  with subgroups  $G_+, G_-$  integrating  $\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-$ , we obtain a Poisson-Lie group structure  $P_G$  on  $G$ , such that  $G_+$  and  $G_-$  are Poisson-Lie subgroups of  $(G, P_G)$  (see [6]). We denote the corresponding Poisson structures on  $G_+$  and  $G_-$  by  $P_{G_+}$  and  $P_{G_-}$ .

Any Manin quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  gives rise to a quasitriangular Manin pair  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$ , where  $r = r_{\mathfrak{g}_+, \mathfrak{g}_-} = \sum_i e^i \otimes e_i$ , and  $(e^i), (e_i)$  are dual bases of  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$ .

### 1.2.2. Examples

In the case where  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  is the Manin triple associated with the Sklyanin structure on a semisimple Lie group  $G_+$ , the Manin quadruples were classified in [12]. (See also [4].) It was shown in [14] that the Lagrangian subalgebras  $\mathfrak{h}$  of  $\mathfrak{g}$  such that the intersection  $\mathfrak{g}_+ \cap \mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}_+$  correspond bijectively to the classical dynamical  $r$ -matrices for  $\mathfrak{g}_+$ . We will treat the quantization of this example in Section 4, in the case where  $\mathfrak{g}_+ = \mathfrak{sl}_2$ .

In [9], the following class of Manin quadruples was studied. Let  $\bar{\mathfrak{g}}$  be a semisimple Lie algebra with nondegenerate invariant symmetric bilinear form  $\langle \cdot, \cdot \rangle_{\bar{\mathfrak{g}}}$ , and Cartan decomposition  $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$ . Let  $\mathcal{K}$  be a commutative ring equipped with a nondegenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ . Assume that  $R \subset \mathcal{K}$  is a Lagrangian subring of  $\mathcal{K}$ , with Lagrangian complement  $\Lambda$ . Let us set  $\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathcal{K}$ , let us equip  $\mathfrak{g}$  with the bilinear form  $\langle \cdot, \cdot \rangle_{\bar{\mathfrak{g}}} \otimes \langle \cdot, \cdot \rangle_{\mathcal{K}}$  and let us set

$$\mathfrak{g}_+ = (\bar{\mathfrak{h}} \otimes R) \oplus (\bar{\mathfrak{n}}_+ \otimes \mathcal{K}), \quad \mathfrak{g}_- = (\bar{\mathfrak{h}} \otimes \Lambda) \oplus (\bar{\mathfrak{n}}_- \otimes \mathcal{K}), \quad \mathfrak{h} = \bar{\mathfrak{g}} \otimes R. \quad (1)$$

More generally, extensions of these Lie algebras, connected with the additional data of a derivation  $\partial$  of  $\mathcal{K}$  leaving  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$  invariant and preserving  $R$ , were considered in [9]. Examples of quadruples  $(\mathcal{K}, \partial, R, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ , where  $\mathcal{K}$  is an infinite-dimensional vector space, arise in the theory of complex curves.

## 1.3. Formal Poisson homogeneous spaces

In what follows, all homogeneous spaces will be formal, so if  $\mathfrak{a}$  is a Lie algebra and  $A$  is the associated formal group, the function ring of  $A$  is  $\mathcal{O}_A := (U\mathfrak{a})^*$ , and if  $\mathfrak{b}$  is a Lie subalgebra of  $\mathfrak{a}$ , and  $B$  is the associated formal group, the function ring of  $A/B$  is  $\mathcal{O}_{A/B} := (U\mathfrak{a}/(U\mathfrak{b}))^*$ .

We will need the following result on formal homogeneous spaces.

**Lemma 1.1** *Let  $\mathfrak{a}$  be a Lie algebra, let  $\mathfrak{a}_+$  and  $\mathfrak{b}$  be Lie subalgebras of  $\mathfrak{a}$ , and let  $A, A_+$ , and  $B$  be the associated formal groups. The restriction map*

$(U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b})^* \rightarrow (U\mathfrak{a}_+/(U\mathfrak{a}_+(\mathfrak{a}_+ \cap \mathfrak{b})))^*$  is a surjective morphism of algebras from  $\mathcal{O}_{A/B}$  to  $\mathcal{O}_{A_+/(A_+ \cap B)}$ .

*Proof.* Let  $L_+$  be a complement of  $\mathfrak{a}_+ \cap \mathfrak{b}$  in  $\mathfrak{a}_+$ , and let  $L$  be a complement of  $\mathfrak{b}$  in  $\mathfrak{a}$ , containing  $L_+$ . When  $V$  is a vector space, we denote by  $S(V)$  its symmetric algebra. The following diagram is commutative

$$\begin{array}{ccc} S(L_+) & \rightarrow & U\mathfrak{a}_+/(U\mathfrak{a}_+(\mathfrak{a}_+ \cap \mathfrak{b})) \\ \downarrow & & \downarrow \\ S(L) & \rightarrow & U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b} \end{array}$$

The horizontal maps are the linear isomorphisms obtained by symmetrization. Since the natural map from  $S(L_+)$  to  $S(L)$  is injective, so is the right-hand vertical map, and its dual is surjective. ■

A *Poisson homogeneous space*  $(X, P_X)$  for a Poisson-Lie group  $(\Gamma, P_\Gamma)$  is a Poisson formal manifold  $(X, P_X)$ , equipped with a transitive action of  $\Gamma$ , and such that the map  $\Gamma \times X \rightarrow X$  is Poisson. Then there is a Lie subgroup  $\Gamma'$  of  $\Gamma$  such that  $X = \Gamma/\Gamma'$ . Following [8], we observe that these two conditions are equivalent: a) the projection map  $\Gamma \rightarrow X$  is Poisson, b) the Poisson bivector  $P_X$  vanishes at one point of  $X$ . If either of these equivalent conditions is satisfied,  $(X, P_X)$  is said to be of *group type*. (See [11].)

#### 1.4. Poisson homogeneous space structure on $G/H$

Let  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$  be a quasitriangular Manin pair. Let  $H$  be the formal subgroup of  $G$  with Lie algebra  $\mathfrak{h}$ , and let  $P_{G/H}$  be the 2-tensor on  $G/H$  equal to the projection of  $r^L$ .

**Proposition 1.1** *Let  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$  be a quasitriangular Manin pair. Then  $P_{G/H}$  is a Poisson bivector on  $G/H$ , and  $G/H$  is a Poisson homogeneous space for  $(G, P_G)$ .*

*Proof.* The only nonobvious property is the antisymmetry of the bracket defined by  $r^L$ . Let  $t$  denote the symmetric element of  $\mathfrak{g} \otimes \mathfrak{g}$  defined by  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ . If  $f_1$  and  $f_2$  are right  $\mathfrak{h}$ -invariant functions on  $G$ ,  $\{f_1, f_2\} + \{f_2, f_1\} = t^L(df_1 \otimes df_2) = t^R(df_1 \otimes df_2)$ , by the invariance of  $t$ . Since  $\mathfrak{h}$  is Lagrangian,  $t$  belongs to  $\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$ . Since moreover  $f_1$  and  $f_2$  are  $\mathfrak{h}$ -invariant,  $\{f_1, f_2\} + \{f_2, f_1\}$  vanishes. ■

**Remark 1** In the case where  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$  corresponds to a Manin quadruple, this statement has been proved by Etingof and Kazhdan, who also constructed a quantization of this Poisson homogeneous space ([11]).

**Remark 2** For  $g$  in  $G$  and  $x$  in  $\mathfrak{g}$ , let us denote the adjoint action of  $g$  on  $x$  by  ${}^g x$ . In the case of a quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ , the Poisson homogeneous space  $(G/H, P_{G/H})$  is of group type if and only if there exists  $g$  in  $G$  such that  ${}^g \mathfrak{h}$  is graded for the Manin triple decomposition, i.e., such that  ${}^g \mathfrak{h} = ({}^g \mathfrak{h} \cap \mathfrak{g}_+) \oplus ({}^g \mathfrak{h} \cap \mathfrak{g}_-)$ .

**1.5. Poisson homogeneous space structure on  $G_+/(G_+ \cap H)$**

Let  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  be a Manin quadruple. The inclusion  $G_+ \subset G$  induces an inclusion map  $i : G_+/(G_+ \cap H) \rightarrow G/H$ . On the other hand, by Proposition 1.1,  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  defines a Poisson structure on  $G/H$ .

**Proposition 1.2** *There exists a unique Poisson structure  $P_{G_+/(G_+ \cap H)}$  on  $G_+/(G_+ \cap H)$  such that the inclusion  $i$  is Poisson. Then  $(G_+/(G_+ \cap H), P_{G_+/(G_+ \cap H)})$  is a Poisson homogeneous space for  $(G_+, P_{G_+})$ .*

*Proof.* Let  $\langle \cdot, \cdot \rangle_{\mathfrak{g} \otimes \mathfrak{g}}$  denote the bilinear form on  $\mathfrak{g} \otimes \mathfrak{g}$  defined as the tensor square of  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ . We must prove that for any  $g$  in  $G_+$ ,  $r_{\mathfrak{g}_+, \mathfrak{g}_-}$  belongs to  $\mathfrak{g}_+ \otimes \mathfrak{g}_+ + \mathfrak{g} \otimes {}^s\mathfrak{h} + {}^s\mathfrak{h} \otimes \mathfrak{g}$ . The annihilator of this space in  $\mathfrak{g} \otimes \mathfrak{g}$  with respect to  $\langle \cdot, \cdot \rangle_{\mathfrak{g} \otimes \mathfrak{g}}$  is  $(\mathfrak{g}_+ \cap {}^s\mathfrak{h}) \otimes {}^s\mathfrak{h} + {}^s\mathfrak{h} \otimes (\mathfrak{g}_+ \cap {}^s\mathfrak{h})$ . Let us show that for any  $x$  in this annihilator,  $\langle r_{\mathfrak{g}_+, \mathfrak{g}_-}, x \rangle_{\mathfrak{g} \otimes \mathfrak{g}}$  is zero. Assume that  $x = v \otimes w$ , where  $v \in {}^s\mathfrak{h}, w \in \mathfrak{g}_+ \cap {}^s\mathfrak{h}$ ; then

$$\langle r_{\mathfrak{g}_+, \mathfrak{g}_-}, x \rangle_{\mathfrak{g} \otimes \mathfrak{g}} = \left\langle \sum_i e^i \otimes e_i, v \otimes w \right\rangle_{\mathfrak{g} \otimes \mathfrak{g}} = \langle v, w \rangle_{\mathfrak{g}} = 0,$$

where the second equality follows from the facts that  $\mathfrak{g}_+$  is isotropic and that  $(e^i), (e_i)$  are dual bases, and the last equality follows from the isotropy of  ${}^s\mathfrak{h}$ . On the other hand, the isotropy of  $\mathfrak{g}_+$  implies that  $\langle r_{\mathfrak{g}_+, \mathfrak{g}_-}, (\mathfrak{g}_+ \cap {}^s\mathfrak{h}) \otimes {}^s\mathfrak{h} \rangle_{\mathfrak{g} \otimes \mathfrak{g}} = 0$ . Therefore  $r_{\mathfrak{g}_+, \mathfrak{g}_-}$  belongs to  $\mathfrak{g}_+ \otimes \mathfrak{g}_+ + \mathfrak{g} \otimes {}^s\mathfrak{h} + {}^s\mathfrak{h} \otimes \mathfrak{g}$ ; this implies the first part of the Proposition.

Let us prove that  $(G_+/(G_+ \cap H), P_{G_+/(G_+ \cap H)})$  is a Poisson homogeneous space for  $(G_+, P_{G_+})$ . We have a commutative diagram

$$\begin{array}{ccc} G_+ \times (G_+/(G_+ \cap H)) & \xrightarrow{a_{G_+}} & G_+/(G_+ \cap H) \\ i_G \times i \downarrow & & i \downarrow \\ G \times (G/H) & \xrightarrow{a_G} & G/H \end{array}$$

where  $i_G$  is the inclusion map of  $G_+$  in  $G$  and  $a_G$  (resp.,  $a_{G_+}$ ) is the action map of  $G$  on  $G/H$  (resp., of  $G_+$  on  $G_+/(G_+ \cap H)$ ). The maps  $i_G \times i, i$  and  $a_G$  are Poisson maps; since  $i$  induces an injection of tangent spaces, it follows that  $a_{G_+}$  is a Poisson map. ■

In [8], Drinfeld defined a Poisson bivector  $P'_{G_+/(G_+ \cap H)}$  on  $G_+/(G_+ \cap H)$ , which can be described as follows. When  $V$  is a Lagrangian subspace of  $\mathfrak{g}$ , identify  $(\mathfrak{g}_+ \cap V)^\perp$  with a subspace of  $\mathfrak{g}_-$ , and define  $\tilde{\xi}_V : (\mathfrak{g}_+ \cap V)^\perp \rightarrow \mathfrak{g}_+ / (\mathfrak{g}_+ \cap V)$  to be the linear map which, to any element  $a_-$  of  $(\mathfrak{g}_+ \cap V)^\perp \subset \mathfrak{g}_-$ , associates the class of an element  $a_+ \in \mathfrak{g}_+$  such that  $a_+ + a_-$  belongs to  $V$ . Then there is a unique element  $\xi_V \in (\mathfrak{g}_+ / (\mathfrak{g}_+ \cap V))^{\otimes 2}$ , such that  $(a_- \otimes id)(\xi_V) = \tilde{\xi}_V(a_-)$ , for any  $a_- \in (\mathfrak{g}_+ \cap V)^\perp$ .

For any  $g$  in  $G_+$ , identify the tangent space of  $G_+/(G_+ \cap H)$  at  $g(G_+ \cap H)$  with  $\mathfrak{g}_+ / (\mathfrak{g}_+ \cap {}^s\mathfrak{h})$  via left-invariant vector fields. The element  $\xi_g$  of  $(\mathfrak{g}_+ / (\mathfrak{g}_+ \cap {}^s\mathfrak{h}))^{\otimes 2}$  corresponding to the value of  $P'_{G_+/(G_+ \cap H)}$  at  $g(G_+ \cap H)$  is then  $\xi_{g \cap {}^s\mathfrak{h}}$ .

**Proposition 1.3** *The bivectors  $P_{G_+/(G_+ \cap H)}$  and  $P'_{G_+/(G_+ \cap H)}$  are equal.*

*Proof.* We have to show that the injection  $i : G_+/(G_+ \cap H) \rightarrow G/H$  is compatible with the bivectors  $P'_{G_+/(G_+ \cap H)}$  and  $P_{G/H}$ . The differential of  $i$  at  $g(G_+ \cap H)$ , where  $g$  belongs to  $G_+$ , induces the canonical injection  $\iota$  from  $\mathfrak{g}_+ / (\mathfrak{g}_+ \cap {}^g \mathfrak{h})$  to  $\mathfrak{g}/{}^g \mathfrak{h}$ . Let us show that for any  $g$  in  $G_+$ , the injection  $\iota \otimes \iota$  maps  $\xi_g \in (\mathfrak{g}_+ / (\mathfrak{g}_+ \cap {}^g \mathfrak{h}))^{\otimes 2}$  to the class of  $r_{\mathfrak{g}_+, \mathfrak{g}_-}$  in  $(\mathfrak{g}/{}^g \mathfrak{h})^{\otimes 2}$ . We have to verify the commutativity of the diagram

$$\begin{array}{ccc} (\mathfrak{g}_+ \cap {}^g \mathfrak{h})^\perp & \xleftarrow{\iota^*} & ({}^g \mathfrak{h})^\perp \\ \bar{\xi}_g \downarrow & & \bar{r}_g \downarrow \\ \mathfrak{g}_+ / (\mathfrak{g}_+ \cap {}^g \mathfrak{h}) & \xrightarrow{\iota} & \mathfrak{g}/{}^g \mathfrak{h} \end{array}$$

where the horizontal maps are the natural injection and restriction maps, and  $\bar{r}_g$  is defined by  $\bar{r}_g(a) =$  the class of  $\langle r_{\mathfrak{g}_+, \mathfrak{g}_-}, a \otimes id \rangle_{\mathfrak{g} \otimes \mathfrak{g}} \bmod {}^g \mathfrak{h}$ , for  $a \in ({}^g \mathfrak{h})^\perp$ . Let  $a$  belong to  $({}^g \mathfrak{h})^\perp$ . By the maximal isotropy of  ${}^g \mathfrak{h}$ , the element  $a$  can be identified with an element of  ${}^g \mathfrak{h}$ . Let us write  $a = a_+ + a_-$ , with  $a_\pm \in \mathfrak{g}_\pm$ . Then  $\iota^*(a) = a_-$ ,  $\bar{\xi}_g(a_-) = a_+ + (\mathfrak{g}_+ \cap {}^g \mathfrak{h})$  by the definition of  $\bar{\xi}_g$  and because  $a \in {}^g \mathfrak{h}$ , and  $\iota(a_+ + \mathfrak{g}_+ \cap {}^g \mathfrak{h}) = a_+ + {}^g \mathfrak{h}$ . On the other hand,  $\bar{r}_g(a) = a_+ + {}^g \mathfrak{h}$ , so the diagram commutes.  $\blacksquare$

Moreover, it is a result of Drinfeld ([8], Remark 2) that the formal Poisson homogeneous spaces for  $(G_+, P_{G_+})$  are all of the type described in Proposition 1.2.

**Remark 3** The  $(G_+, P_{G_+})$ -Poisson homogeneous space

$$(G_+ / (G_+ \cap H), P_{G_+ / (G_+ \cap H)})$$

is of group type if and only if for some  $g \in G_+$ ,  ${}^g \mathfrak{h}$  is graded for the Manin triple decomposition, see Remark 2.

**Remark 4** *Conjugates of Manin quadruples.* Let us denote the conjugate of an element  $x$  in  $G$  by an element  $h$  in  $G$  by  ${}^s x = gxg^{-1}$ . For  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  a Manin quadruple, and  $g$  in  $G$ ,  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, {}^s \mathfrak{h})$  is also a Manin quadruple. If  $g$  belongs to  $H$ , this is the same quadruple; and if  $g$  belongs to  $G_+$ , the Poisson structure induced by  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, {}^s \mathfrak{h})$  on  $G_+ / (G_+ \cap {}^s H)$  is isomorphic to that induced by  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  on  $G_+ / (G_+ \cap H)$ , via conjugation by  $g$ . It follows that Poisson homogeneous space structures induced by  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  are indexed by elements of the double quotient  $G_+ \backslash G/H$ .

**Remark 5** The first examples of Poisson homogeneous spaces which are not of group type were studied in [3] and [13], under the name of affine Poisson structures. In these cases, the stabilizer of a point is trivial.

## 2. Quantization of $G/H$

In this section, we introduce axioms for the quantization of a quasitriangular Manin pair. We then show that any solution to this quantization problem leads to a quantization of the Poisson homogeneous space  $G/H$  constructed in Section 1.4.

### 2.1. Definition of quantization of Poisson homogeneous spaces

Let  $(\Gamma, P_\Gamma)$  be a Poisson-Lie group and let  $(X, P_X)$  be a formal Poisson homogeneous space over  $(\Gamma, P_\Gamma)$ . Let  $(\mathcal{A}, \Delta_{\mathcal{A}})$  be a quantization of the enveloping algebra of the Lie algebra of  $\Gamma$ , and let  $\mathcal{A}^{opp}$  denote the opposite algebra to  $\mathcal{A}$ .

**Definition 2.1** A quantization of the Poisson homogeneous space  $(X, P_X)$  is a  $\mathbb{C}[[\hbar]]$ -algebra  $\mathcal{X}$ , such that

- 1)  $\mathcal{X}$  is a quantization of the Poisson algebra  $(\mathcal{O}_X, P_X)$  of formal functions on  $X$ , and
- 2)  $\mathcal{X}$  is equipped with an algebra-module structure over  $(\mathcal{A}^{opp}, \Delta)$ , whose reduction mod  $\hbar$  coincides with the algebra-module structure of  $\mathcal{O}_X$  over  $((U\mathfrak{g})^{opp}, \Delta_0)$ , where  $\Delta_0$  is the coproduct of  $U\mathfrak{g}$ .

Condition 1 means that there is an isomorphism of  $\mathbb{C}[[\hbar]]$ -modules from  $\mathcal{X}$  to  $\mathcal{O}_X[[\hbar]]$ , inducing an algebra isomorphism between  $\mathcal{X}/\hbar\mathcal{X}$  and  $\mathcal{O}_X$ , and inducing on  $\mathcal{O}_X$  the Poisson structure defined by  $P_X$ .

The first part of condition 2 means that  $\mathcal{X}$  has a module structure over  $\mathcal{A}^{opp}$ , such that for  $x, y \in \mathcal{X}$ ,  $a \in \mathcal{A}$ , and  $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$ ,  $a(xy) = \sum a^{(1)}(x)a^{(2)}(y)$ .

#### Conventions

We will say that a  $\mathbb{C}[[\hbar]]$ -module  $V$  is *topologically free* if it is isomorphic to  $W[[\hbar]]$ , where  $W$  is a complex vector space. We denote the canonical projection of  $V$  onto  $V/\hbar V$  by  $v \mapsto v \bmod \hbar$ . In what follows, all tensor products of  $\mathbb{C}[[\hbar]]$ -modules are  $\hbar$ -adically completed. When  $E$  is a  $\mathbb{C}[[\hbar]]$ -module, we denote by  $E^*$  its dual  $\text{Hom}_{\mathbb{C}[[\hbar]]}(E, \mathbb{C}[[\hbar]])$ . For a subset  $\mathcal{S}$  of an algebra  $\mathcal{A}$ , we denote by  $\mathcal{S}^\times$  the group of invertible elements of  $\mathcal{S}$ . When  $\mathcal{A}, \mathcal{B}$  are two Hopf or quasi-Hopf algebras with unit elements  $1_{\mathcal{A}}, 1_{\mathcal{B}}$  and counit maps  $\varepsilon_{\mathcal{A}}, \varepsilon_{\mathcal{B}}$ , and  $\mathcal{S}$  is a subset of  $\mathcal{A} \otimes \mathcal{B}$ , we denote by  $\mathcal{S}_0^\times$  the subgroup of  $\mathcal{S}^\times$  with elements  $x$  such that  $(\varepsilon_{\mathcal{A}} \otimes id)(x) = 1_{\mathcal{B}}$  and  $(id \otimes \varepsilon_{\mathcal{B}})(x) = 1_{\mathcal{A}}$ . We also denote  $\text{Ker } \varepsilon_{\mathcal{A}}$  by  $\mathcal{A}_0$ .

### 2.2. Quantization of quasitriangular Manin pairs

**Definition 2.2** A quantization of a quasitriangular Manin pair  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$  is the data of

- 1) a quasitriangular Hopf algebra  $(A, \Delta, \mathcal{R})$  quantizing  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ ,
- 2) a subalgebra  $B \subset A$  and an element  $F$  in  $(A \otimes A)_0^\times$ , such that
  - a)  $B \subset A$  is a flat deformation of the inclusion  $U\mathfrak{h} \subset U\mathfrak{g}$ ,
  - b)  $F\Delta(B)F^{-1} \subset B \otimes B$ , and  $F^{(12)}(\Delta \otimes id)(F) (F^{(23)}(id \otimes \Delta)(F))^{-1} \in B^{\otimes 3}$ ,
  - c) there exists a Lagrangian complement  $L$  of  $\mathfrak{h}$  in  $\mathfrak{g}$ , such that

$$\left( \frac{1}{\hbar}(F - F^{(21)}) \bmod \hbar \right) = r - r_{\mathfrak{h}, L},$$

where  $r_{\mathfrak{h},L} = \sum_i \varepsilon^i \otimes \varepsilon_i$ , and  $(\varepsilon^i), (\varepsilon_i)$  are dual bases of  $\mathfrak{h}$  and  $L$ .

Observe that condition 1 implies that  $(\frac{1}{\hbar}(\mathcal{R} - 1) \bmod \hbar) = r$ . In condition 2,  $(A \otimes A)_0^\times = 1 + \hbar(A_0 \otimes A_0)$ . Condition 2a means that the  $\mathbb{C}[[\hbar]]$ -module isomorphism between  $A$  and  $U\mathfrak{g}[[\hbar]]$  arising from condition 1 can be chosen in such a way that it induces an isomorphism between  $B$  and  $U\mathfrak{h}[[\hbar]]$ .

Let  $(A, \Delta_B, \Phi_B)$  be the quasi-Hopf algebra obtained by twisting the Hopf algebra  $(A, \Delta)$  by  $F$ . Then, by definition,  $\Delta_B(x) = F\Delta(x)F^{-1}$ , for any  $x \in A$ , and  $\Phi_B = F^{(12)}(\Delta \otimes id)(F) (F^{(23)}(id \otimes \Delta)(F))^{-1}$ . Condition 2b expresses the fact that  $B \subset A$  is a sub-quasi-Hopf algebra of  $A$ .

Condition 2c expresses the fact that the classical limit of  $(B, \Delta_B, \Phi_B)$  is  $\mathfrak{h}$  equipped with the Lie quasi-bialgebra structure associated with  $L$ .

### 2.3. Quantization of $(G/H, P_{G/H})$

Let us denote the counit map of  $A$  by  $\varepsilon$ .

**Theorem 2.1** *Assume that  $((A, \Delta, \mathcal{R}), B, F)$  is a quantization of the quasitriangular Manin pair  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$ . Let  $(A^*)^B$  be the subspace of  $A^*$  consisting of the forms  $\ell$  on  $A$  such that  $\ell(ab) = \ell(a)\varepsilon(b)$  for any  $a \in A$  and  $b \in B$ .*

a) *For  $\ell, \ell'$  in  $(A^*)^B$ , define  $\ell * \ell'$  to be the element of  $A^*$  such that*

$$(\ell * \ell')(a) = (\ell \otimes \ell')(\Delta(a)F^{-1}),$$

*for any  $a$  in  $A$ . Then  $*$  defines an associative algebra structure on  $(A^*)^B$ .*

b) *For  $a \in A$  and  $\ell \in (A^*)^B$ , define  $a\ell$  to be the form on  $A$  such that  $(a\ell)(a') = \ell(aa')$ , for any  $a' \in A$ . This map defines on  $((A^*)^B, *)$  a structure of an algebra-module over the Hopf algebra  $(A^{opp}, \Delta)$ .*

c) *The algebra  $((A^*)^B, *)$  is a quantization of the Poisson algebra  $(\mathcal{O}_{G/H}, P_{G/H})$ .*

*With its algebra-module structure over  $(A^{opp}, \Delta)$ ,  $((A^*)^B, *)$  is a quantization of the Poisson homogeneous space  $(G/H, P_{G/H})$ .*

*Proof.* Let  $\ell$  and  $\ell'$  belong to  $(A^*)^B$ . Then for any  $a \in A, b \in B$ ,

$$\begin{aligned} (\ell * \ell')(ab) &= (\ell \otimes \ell')(\Delta(a)\Delta(b)F^{-1}) = (\ell \otimes \ell')(\Delta(a)F^{-1}\Delta_B(b)) \\ &= \varepsilon(b)(\ell \otimes \ell')(\Delta(a)F^{-1}) = \varepsilon(b)(\ell * \ell')(a), \end{aligned}$$

where the third equality follows from the fact that  $\Delta_B(B) \subset B \otimes B$  and  $(\varepsilon \otimes \varepsilon) \circ \Delta_B = \varepsilon$ . It follows that  $\ell * \ell'$  belongs to  $(A^*)^B$ .

Let  $\ell, \ell'$  and  $\ell''$  belong to  $(A^*)^B$ . Then for any  $a$  in  $A$ ,

$$((\ell * \ell') * \ell'')(a) = (\ell \otimes \ell' \otimes \ell'') \left( (\Delta \otimes id) \circ \Delta(a) (\Delta \otimes id) (F^{-1}) (F^{(12)})^{-1} \right), \quad (2)$$

and

$$(\ell * (\ell' * \ell''))(a) = (\ell \otimes \ell' \otimes \ell'') \left( (id \otimes \Delta) \circ \Delta(a) (id \otimes \Delta) (F^{-1}) (F^{(23)})^{-1} \right).$$

By the coassociativity of  $\Delta$  and the definition of  $\Phi_B$ , this expression is equal to  $(\ell \otimes \ell' \otimes \ell'') ((\Delta \otimes id) \circ \Delta(a)(\Delta \otimes id)(F^{-1})(F^{(12)})^{-1}\Phi_B^{-1})$ , and since  $\Phi_B$  belongs to  $B^{\otimes 3}$  and  $\varepsilon^{\otimes 3}(\Phi_B) = 1$ , this is equal to the right-hand side of (2). It follows that  $(\ell * \ell') * \ell'' = \ell * (\ell' * \ell'')$ , so  $*$  is associative. Moreover,  $\varepsilon$  belongs to  $(A^*)^B$  and is the unit element of  $((A^*)^B, *)$ . This proves part a of the theorem.

Part b follows from the definitions.

Let us prove that  $(A^*)^B$  is a flat deformation of  $(U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h})^*$ . Let us consider a Lagrangian complement  $L$  of  $\mathfrak{h}$  in  $\mathfrak{g}$ . Let  $\text{Sym}$  denote the symmetrisation map from the symmetric algebra  $S(\mathfrak{g})$  of  $\mathfrak{g}$  to  $U\mathfrak{g}$ , i.e., the unique linear map such that  $\text{Sym}(x^l) = x^l$  for any  $x \in \mathfrak{g}$  and  $l \geq 0$ , and let us define  $\tilde{L}$  to be  $\text{Sym}(S(L))$ . Thus  $\tilde{L}$  is a linear subspace of  $U\mathfrak{g}$ , and inclusion of  $\tilde{L} \otimes U\mathfrak{h}$  in  $U\mathfrak{g} \otimes U\mathfrak{g}$  followed by multiplication induces a linear isomorphism from  $\tilde{L} \otimes U\mathfrak{h}$  to  $U\mathfrak{g}$ . It follows that the restriction to  $\tilde{L}$  of the projection  $U\mathfrak{g} \rightarrow U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h}$  is an isomorphism, which defines a linear isomorphism between  $\mathcal{O}_{G/H} = (U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h})^*$  and  $\tilde{L}^*$ .

Let us fix an isomorphism of  $\mathbb{C}[[\hbar]]$ -modules from  $A$  to  $U\mathfrak{g}[[\hbar]]$ , inducing an isomorphism between  $B$  and  $U\mathfrak{h}[[\hbar]]$  and let us define  $C$  to be the preimage of  $\tilde{L}[[\hbar]]$ . Thus  $C$  is isomorphic to  $\tilde{L}[[\hbar]]$ , and inclusion followed by multiplication induces a linear isomorphism between  $C \otimes B$  and  $A$ , as does any morphism between two topologically free  $\mathbb{C}[[\hbar]]$ -modules  $E$  and  $F$ , which induces an isomorphism between  $E/\hbar E$  and  $F/\hbar F$ . Therefore, restriction of linear forms to  $C$  induces an isomorphism between  $(A^*)^B$  and  $C^*$ . It follows that  $(A^*)^B$  is isomorphic to  $(\tilde{L}[[\hbar]])^*$ , which is in turn isomorphic to  $\tilde{L}^*[[\hbar]]$ . Since  $\tilde{L}^*$  is isomorphic to  $\mathcal{O}_{G/H}$ ,  $(A^*)^B$  is isomorphic to  $\mathcal{O}_{G/H}[[\hbar]]$ .

Let us fix  $\ell, \ell'$  in  $(A^*)^B$ , and let us compute  $(\frac{1}{\hbar}(\ell * \ell' - \ell' * \ell) \bmod \hbar)$ . Let us set  $f = (\frac{1}{\hbar}(F - 1) \bmod \hbar)$ . For  $a$  in  $A$ ,

$$\begin{aligned} \frac{1}{\hbar}(\ell * \ell' - \ell' * \ell)(a) &= (\ell \otimes \ell') \left( \frac{1}{\hbar}(\Delta(a)F^{-1} - \Delta'(a)(F^{(21)})^{-1}) \right) \\ &= (\ell \otimes \ell') \left( \frac{1}{\hbar}(\Delta(a)F^{-1} - \mathcal{R}\Delta(a)\mathcal{R}^{-1}(F^{(21)})^{-1}) \right) \\ &= (\ell \otimes \ell') \left( -r\Delta_0(a_0) + \Delta_0(a_0)(r + f^{(21)} - f) \right) + o(\hbar), \end{aligned}$$

where  $\Delta_0$  is the coproduct of  $U\mathfrak{g}$  and  $a_0$  is the image of  $a$  in  $A/\hbar A = U\mathfrak{g}$ . Since, by condition 2c of Definition 2.2,  $r - f + f^{(21)}$  is equal to  $r_{\mathfrak{h},L}$ , it belongs to  $\mathfrak{h} \otimes \mathfrak{g}$  and since  $\ell$  and  $\ell'$  are right  $B$ -invariant,  $(\frac{1}{\hbar}(\ell * \ell' - \ell' * \ell)(a) \bmod \hbar) = (\ell \otimes \ell')(-r\Delta_0(a_0))$ , which is the Poisson bracket defined by  $r^L$  on  $G/H$ . This ends the proof of part c of the theorem.

Since  $(F \bmod \hbar) = 1$ , the reduction modulo  $\hbar$  of the algebra-module structure of  $((A^*)^B, *)$  over  $(A^{opp}, \Delta)$  is that of  $\mathcal{O}_X$  over  $((U\mathfrak{g})^{opp}, \Delta_0)$ . This ends the proof of the theorem.  $\blacksquare$

**Remark 6** If  $((A, \Delta, \mathcal{R}), B, F)$  is a quantization of a quasitriangular Manin pair  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$ , and if  $F_0$  is an element of  $(B \otimes B)_0^\times$ , then  $((A, \Delta, \mathcal{R}), F_0 F)$  is a quantization of the same Manin pair. We observe that the product  $*$  on  $(A^*)^B$  is independent of such a modification of  $F$ .

**Remark 7** In the case of a Manin quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  where  $\mathfrak{h}$  is graded for the Manin triple decomposition (see Remark 2),  $r_{\mathfrak{g}_+, \mathfrak{g}_-} - r_{\mathfrak{h}, L} = f - f^{(21)}$  belongs to  $\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$ . The corresponding quantum condition is that

$$F \in 1 + \hbar(B_0 \otimes A_0 + A_0 \otimes B_0). \quad (3)$$

This is the case is [9], where  $F$  belongs to  $(B \otimes A)_0^\times = 1 + \hbar(B_0 \otimes A_0)$ .

When condition (3) is fulfilled, the product  $*$  is the restriction to  $(A^*)^B$  of the usual product on  $A^*$ , defined as the dual map to  $\Delta$ .

#### 2.4. Relations in $(A^*)^B$

It is well known that the matrix coefficients of the representations of a quasitriangular Hopf algebra can be organized in  $L$ -operators, satisfying the so-called  $RLL$  relations. We recall this construction and introduce analogues of these matrix coefficients and of the  $RLL$  relations for the algebra  $(A^*)^B$ .

Recall that there is an algebra structure on  $A^*$ , where the product is  $(\ell, \ell') \mapsto \ell \ell'$ , such that for any  $a \in A$ ,  $(\ell \ell')(a) = (\ell \otimes \ell')(\Delta(a))$ .

Let  $\text{Rep}(A)$  be the category of modules over  $A$ , which are free and finite dimensional over  $\mathbb{C}[[\hbar]]$ . There is a unique map

$$\bigoplus_{V \in \text{Rep}(A)} (V^* \otimes V) \rightarrow A^*, \quad \kappa \mapsto \ell_\kappa,$$

such that for any object  $(V, \pi_V)$  in  $\text{Rep}(A)$ , and any  $\xi \in V^*$  and  $v \in V$ ,  $\ell_{\xi \otimes v}(a) = \xi(\pi_V(a)v)$ , for any  $a \in A$ . Define  $\text{Coeff}(A)$  to be the image of this map. Then  $\text{Coeff}(A)$  is a subalgebra of  $A^*$ . Moreover, there is a Hopf algebra structure on  $\text{Coeff}(A)$ , with coproduct  $\Delta_{\text{Coeff}(A)}$  and counit  $\varepsilon_{\text{Coeff}(A)}$ , uniquely determined by the rules

$$\Delta_{\text{Coeff}(A)}(\ell_{\xi \otimes v}) = \sum_i \ell_{\xi \otimes v_i} \otimes \ell_{\xi^i \otimes v}, \quad \varepsilon_{\text{Coeff}(A)}(\ell_{\xi \otimes v}) = \xi(v),$$

where  $(v_i)$  and  $(\xi^i)$  are dual bases of  $V$  and  $V^*$ . The duality pairing between  $A$  and  $A^*$  then induces a Hopf algebra pairing between  $(A, \Delta)$  and  $(\text{Coeff}(A), \Delta_{\text{Coeff}(A)})$  (see [1]).

For  $V$  an object of  $\text{Rep}(A)$ , define  $L_V$  to be the element of  $\text{End}(V) \otimes \text{Coeff}(A)$  equal to  $\sum_i \kappa^i \otimes \ell_{\kappa_i}$ , where  $(\kappa^i)$  and  $(\kappa_i)$  are dual bases of  $\text{End}(V)$  and  $V^* \otimes V$ . It follows from  $\mathcal{R}\Delta = \Delta'\mathcal{R}$  that the relation

$$R_{V,W}^{(12)} L_V^{(1a)} L_W^{(2a)} = L_W^{(2a)} L_V^{(1a)} R_{V,W}^{(12)}$$

is satisfied in  $\text{End}(V) \otimes \text{End}(W) \otimes \text{Coeff}(A)$ , where the superscripts 1, 2, and  $a$  refer to the successive factors of the tensor product. Moreover,

$$(id_V \otimes \Delta_{\text{Coeff}(A)})(L_V) = L_V^{(1a)}L_V^{(1a')}$$

holds in  $\text{End}(V) \otimes \text{Coeff}(A)^{\otimes 2}$ , where the superscripts 1,  $a$  and  $a'$  refer to the successive factors of this tensor product.

For  $(V, \pi_V)$  an  $A$ -module, we set  $V^B = \{v \in V \mid \forall b \in B, \pi_V(b)(v) = \varepsilon(b)v\}$ . There is a unique map

$$\bigoplus_{V \in \text{Rep}(A)} (V^* \otimes V^B) \rightarrow (A^*)^B, \quad \kappa \mapsto \tilde{\ell}_\kappa$$

such that for  $\xi \in V^*$  and  $v \in V^B$ ,  $\tilde{\ell}_{\xi \otimes v}(a) = \xi(\pi_V(a)v)$ , for any  $a \in A$ . Define  $\text{Coeff}(A, B)$  to be the image of this map.

For  $V$  an object of  $\text{Rep}(A)$ ,  $V^B$  is a free, finite-dimensional  $\mathbb{C}[[\hbar]]$ -module. It follows that the dual of  $V^* \otimes V^B$  is  $(V^B)^* \otimes V$ , which may be identified with  $\text{Hom}_{\mathbb{C}[[\hbar]]}(V^B, V)$ . Define  $\tilde{L}_V$  to be the element of  $\text{Hom}_{\mathbb{C}[[\hbar]]}(V^B, V) \otimes \text{Coeff}(A, B)$  equal to  $\sum_i \kappa^i \otimes \tilde{\ell}_{\kappa_i}$ , where  $(\kappa^i)$  and  $(\kappa_i)$  are dual bases of  $\text{Hom}_{\mathbb{C}[[\hbar]]}(V^B, V)$  and  $V^* \otimes V^B$ .

When  $(V, \pi_V)$  and  $(W, \pi_W)$  are objects of  $\text{Rep}(A)$ , let  $R_{V,W}$  be the element of  $\text{End}_{\mathbb{C}[[\hbar]]}(V \otimes W)$  equal to  $(\pi_V \otimes \pi_W)(\mathcal{R})$ , where  $\mathcal{R}$  is the  $R$ -matrix of  $A$ . Recall that the twist of  $\mathcal{R}$  by  $F$  is  $\mathcal{R}_B = F^{(21)}\mathcal{R}F^{-1}$ , and set  $R_{B;V,W} = (\pi_V \otimes \pi_W)(\mathcal{R}_B)$ .

**Proposition 2.1** *Coeff(A, B) is a subalgebra of  $(A^*)^B$ . For any objects  $V$  and  $W$  in  $\text{Rep}(A)$ , the relation*

$$R_{V,W}^{(12)}\tilde{L}_V^{(1a)}\tilde{L}_W^{(2a)} = \tilde{L}_W^{(2a)}\tilde{L}_V^{(1a)}(R_{B;V,W}^{(12)})|_Z$$

is satisfied in  $\text{Hom}_{\mathbb{C}[[\hbar]]}(Z, V \otimes W) \otimes \text{Coeff}(A, B)$ , where  $Z$  is the intersection  $(V^B \otimes W^B) \cap R_{B;V,W}^{-1}(V^B \otimes W^B)$ . In this equality, the left-hand side is an element of  $\text{Hom}_{\mathbb{C}[[\hbar]]}(V^B \otimes W^B, V \otimes W) \otimes \text{Coeff}(A, B)$ , viewed as an element of  $\text{Hom}_{\mathbb{C}[[\hbar]]}(Z, V \otimes W) \otimes \text{Coeff}(A, B)$  by restriction.

Recall that an algebra-comodule  $\mathcal{X}$  over a Hopf algebra  $(\mathcal{A}, \Delta_{\mathcal{A}})$  is the data of an algebra structure over  $\mathcal{X}$  and a left comodule structure of  $\mathcal{X}$  over  $(\mathcal{A}, \Delta_{\mathcal{A}})$ ,  $\Delta_{\mathcal{X}, \mathcal{A}} : \mathcal{X} \rightarrow \mathcal{A} \otimes \mathcal{X}$ , which is also a morphism of algebras.

**Proposition 2.2** *There is a unique algebra-comodule structure on  $\text{Coeff}(A, B)$  over  $(\text{Coeff}(A), \Delta_{\text{Coeff}(A)})$ , compatible with the algebra-module structure of  $(A^*)^B$  over  $(A^{opp}, \Delta)$ . The relation*

$$(id_V \otimes \Delta_{\text{Coeff}(A, B), \text{Coeff}(A)})(\tilde{L}_V) = L_V^{(1a)}\tilde{L}_V^{(1a')}$$

is satisfied in  $\text{End}(V) \otimes \text{Coeff}(A) \otimes \text{Coeff}(A, B)$ , where the superscripts 1,  $a$ , and  $a'$  refer to the successive factors of this tensor product.

### 3. Quantization of $G_+/(G_+ \cap H)$

In this section, we state axioms for the quantization of a Manin quadruple, and show that any such quantization gives rise to a quantization of the Poisson homogeneous space  $G_+/(G_+ \cap H)$  constructed in [8] (see Propositions 1.2 and 1.3).

#### 3.1. Quantization of Manin quadruples

Let us fix a Manin quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ . Recall that  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  gives rise to a quasitriangular Manin pair  $(\mathfrak{g}, r, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$ , if we set  $r = r_{\mathfrak{g}_+, \mathfrak{g}_-} = \sum_i e^i \otimes e_i$ , where  $(e^i)$  and  $(e_i)$  are dual bases of  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$ .

**Definition 3.1** A quantization of a Manin quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$  is the data of

- 1) a quantization  $((A, \Delta, \mathcal{R}), B, F)$  of the quasitriangular Manin pair  $(\mathfrak{g}, r_{\mathfrak{g}_+, \mathfrak{g}_-}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, \mathfrak{h})$ ,
- 2) a Hopf subalgebra  $A_+$  of  $(A, \Delta)$  such that
  - a)  $A_+ \subset A$  is a flat deformation of  $U\mathfrak{g}_+ \subset U\mathfrak{g}$ ,
  - b)  $B \cap A_+ \subset A_+$  is a flat deformation of the inclusion  $U(\mathfrak{h} \cap \mathfrak{g}_+) \subset U\mathfrak{g}_+$ ,
  - c)  $F$  satisfies

$$F^{-1} \in ((AB_0 + A_+) \otimes A + A \otimes AB_0) \cap (AB_0 \otimes A + A \otimes (AB_0 + A_+)). \quad (4)$$

It follows from condition 2c of Definition 2.2 and the beginning of the proof of Proposition 1.2 that  $(\frac{1}{\hbar}(F - F^{(21)}) \bmod \hbar)$  belongs to  $\mathfrak{g}_+ \otimes \mathfrak{g}_+ + \mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h} = ((\mathfrak{h} + \mathfrak{g}_+) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}) \cap (\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes (\mathfrak{h} + \mathfrak{g}_+))$ . Therefore condition (4) is natural. It is equivalent to the condition that  $F$  belong to the product of subgroups of  $(A \otimes A)_0^\times$

$$(1 + \hbar(AB_0 \otimes A_0 + A_0 \otimes AB_0)) (1 + \hbar(A_+)_0 \otimes (A_+)_0).$$

*Example*

Recall that (1) is a graded Manin quadruple. In [9], a quantization of this quadruple was constructed for the case where  $\bar{\mathfrak{g}} = \mathfrak{sl}_2$ .

**Remark 8** In the case where  $\mathfrak{g}_+ \cap \mathfrak{h} = 0$ , which corresponds to a homogeneous space over  $G_+$  with trivial stabilizer,  $F$  automatically satisfies condition (4). Indeed, in that case, multiplication induces an isomorphism  $A_+ \otimes B \rightarrow A$ , therefore  $A = AB_0 + A_+$ .

#### 3.2. Quantization of $(G_+/(G_+ \cap H), \mathcal{P}_{G_+/(G_+ \cap H)})$

Let  $I_0$  be the subspace of  $\mathcal{O}_{G/H}$  equal to  $\mathcal{O}_{G/H} \cap (U\mathfrak{g}_+)^\perp$ . It follows from Lemma 1.1 that  $I_0$  is an ideal of  $\mathcal{O}_{G/H}$ , and that the algebra  $\mathcal{O}_{G_+/(G_+ \cap H)}$  can be identified as a Poisson algebra with the quotient  $\mathcal{O}_{G/H}/I_0$ .

Let  $I$  be the subspace of  $(A^*)^B$  defined as the set of all linear forms  $\ell$  on  $A$  such that  $\ell(a_+) = 0$  for any  $a_+ \in A_+$ . Therefore

$$I = (A^*)^B \cap A_+^\perp.$$

**Theorem 3.1** *Assume that  $((A, \Delta, \mathcal{R}), A_+, B, F)$  is a quantization of the Manin quadruple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ . Then  $I$  is a two-sided ideal in  $(A^*)^B$ , the algebra  $(A^*)^B/I$  is a flat deformation of  $\mathcal{O}_{G_+/(G_+ \cap H)}$ , and is a quantization of the formal Poisson space  $(G_+/(G_+ \cap H), P_{G_+/(G_+ \cap H)})$ .*

*Moreover,  $I$  is preserved by the action of  $A_+^{opp}$ , and  $(A^*)^B/I$  is an algebra-module over  $(A_+^{opp}, \Delta)$ . With this algebra-module structure,  $(A^*)^B/I$  is a quantization of the  $(G_+, P_{G_+})$ -Poisson homogeneous space  $(G_+/(G_+ \cap H), P_{G_+/(G_+ \cap H)})$ .*

*Proof.* Let us fix  $\ell$  in  $I$  and  $\ell^l$  in  $(A^*)^B$ . For any  $a_+$  in  $A_+$ , we have

$$(\ell * \ell^l)(a_+) = (\ell \otimes \ell^l)(\Delta(a_+)F^{-1}) = 0$$

because  $\Delta(A_+) \subset A_+ \otimes A_+$  and by assumption (4) on  $F$ . In the same way,  $(\ell^l * \ell)(a_+) = 0$ . Therefore  $I$  is a two-sided ideal in  $(A^*)^B$ .

Let us prove that  $(A^*)^B/I$  is a flat deformation of  $\mathcal{O}_{G_+/(G_+ \cap H)}$ . To this end, we will identify the  $\mathbb{C}[[\hbar]]$ -modules  $(A^*)^B/I$  with  $(A_+^*)^{A_+ \cap B}$ , to which we apply the result of Theorem 2.1.

Recall that  $B_0$  and  $(A_+ \cap B)_0$  denote the augmentation ideals of  $B$  and  $A_+ \cap B$ . Thus  $(A^*)^B$  is equal to  $(A/AB_0)^*$ , where  $AB_0$  is the image of the product map  $A \otimes B_0 \rightarrow A$ . In the same way,  $(A_+^*)^{A_+ \cap B}$  is equal to  $(A_+/A_+(A_+ \cap B)_0)^*$ . Let us show that  $(A^*)^B/I$  is equal to  $(A_+/A_+(A_+ \cap B)_0)^*$ . Restriction of a linear form to  $A_+$  induces a linear map  $\rho : (A/AB_0)^* \rightarrow (A_+/A_+(A_+ \cap B)_0)^*$ . Moreover, the kernel of  $\rho$  is  $I$ , therefore  $\rho$  induces an injective map

$$\tilde{\rho} : (A/AB_0)^*/I \rightarrow (A_+/A_+(A_+ \cap B)_0)^*.$$

Let us now show that  $\tilde{\rho}$  is surjective. For this, it is enough to show that the restriction map  $\rho : (A/AB_0)^* \rightarrow (A_+/A_+(A_+ \cap B)_0)^*$  is surjective.  $(A/AB_0)^*$  and  $(A_+/A_+(A_+ \cap B)_0)^*$  are topologically free  $\mathbb{C}[[\hbar]]$ -modules, and the map from  $(A/AB_0)^*/\hbar(A/AB_0)^*$  to  $(A_+/A_+(A_+ \cap B)_0)^*/\hbar(A_+/A_+(A_+ \cap B)_0)^*$  coincides with the canonical map from  $\mathcal{O}_{G/H}$  to  $\mathcal{O}_{G_+/(G_+ \cap H)}$  which, by Lemma 1.1, is surjective. Therefore  $\rho$  is surjective, and so is  $\tilde{\rho}$ . It follows that  $(A_+^*)^{A_+ \cap B}$  is a flat deformation of  $\mathcal{O}_{G_+/(G_+ \cap H)}$ .

There is a commutative diagram of algebras

$$\begin{array}{ccc} (A^*)^B & \rightarrow & (A^*)^B/I \\ \downarrow & & \downarrow \\ \mathcal{O}_{G/H} & \rightarrow & \mathcal{O}_{G_+/(G_+ \cap H)} \end{array}$$

where the vertical maps are projections  $X \rightarrow X/\hbar X$ . Since the projection  $\mathcal{O}_{G/H} \rightarrow \mathcal{O}_{G_+/(G_+ \cap H)}$  is a morphism of Poisson algebras, where  $\mathcal{O}_{G/H}$  and  $\mathcal{O}_{G_+/(G_+ \cap H)}$  are equipped with  $P_{G/H}$  and  $P_{G_+/(G_+ \cap H)}$ , the classical limit of  $(A^*)^B/I$  is  $(\mathcal{O}_{G_+/(G_+ \cap H)}, P_{G_+/(G_+ \cap H)})$ .

Finally, the algebra-module structure of  $(A^*)^B$  over  $(A^{opp}, \Delta)$  induces by restriction an algebra-module structure on  $(A^*)^B$  over  $(A_+^{opp}, \Delta)$ , and since  $I$  is preserved by the action of  $A_+^{opp}$ ,  $(A^*)^B/I$  is also an algebra-module over  $(A_+^{opp}, \Delta)$ . ■

**Remark 9** For  $u \in A^\times$ , set  ${}^u B = uBu^{-1}$  and  ${}^u F = (u \otimes u)\Delta_B(u)^{-1}F$ . Let  $((A^*)^B, *_{u})$  be the algebra-module over  $(A^{opp}, \Delta)$  corresponding to  $(A, {}^u B, {}^u F)$ . There is an algebra-module isomorphism  $i_u : ((A^*)^B, *) \rightarrow ((A^*)^B, *_{u})$ , given by  $(i_u \ell)(a) = \ell(au)$ , for any  $a \in A$ .

If  $u$  does not lie in  $A_+$ , there is no reason for  $A_+ \cap {}^u B$  to be a flat deformation of its classical limit, nor for  ${}^u F$  to satisfy (4). But, if the conditions of Theorem 3.1 are still valid for  $(A, {}^u B, {}^u F)$ , the resulting algebra-module over  $(A_+^{opp}, \Delta_+)$  can be different from the one arising from  $(A, B, F)$ . We will see an example of this situation in Section 4.1.

**Remark 10** In their study of preferred deformations, Bonneau *et al.* studied the case of quotients of compact, connected Lie groups ([2]). See also the article [15] and references that it cites.

In [5], Donin, Gurevich, and Shnider used quasi-Hopf algebra techniques to construct quantizations of some homogeneous spaces. More precisely, they classified the Poisson homogeneous structures on the semisimple orbits of a simple Lie group with Lie algebra  $\mathfrak{g}_0$ , and constructed their quantizations using Drinfeld's series  $F_\hbar$  relating the Hopf algebra  $U_\hbar \mathfrak{g}_0$  to a quasi-Hopf algebra structure on  $U_{\mathfrak{g}_0}[[\hbar]]$  involving the Knizhnik-Zamolodchikov associator.

In [16], Parmentier also used twists to propose a quantization scheme of Poisson structures on Lie groups, generalizing the affine Poisson structures.

## 4. Examples

In view of Remark 7, we can only find nontrivial applications of the above results in the case of a nongraded  $\mathfrak{h}$ . In this section, we shall construct quantizations of some nongraded Manin quadruples.

### 4.1. Finite-dimensional examples

Let us set  $\bar{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$ ; let  $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$  be the Cartan decomposition of  $\bar{\mathfrak{g}}$ , and let  $(\bar{e}_+, \bar{h}, \bar{e}_-)$  be the Chevalley basis of  $\bar{\mathfrak{g}}$ , so  $\bar{\mathfrak{n}}_\pm = \mathbb{C}e_\pm$  and  $\bar{\mathfrak{h}} = \mathbb{C}\bar{h}$ . Let  $\langle \cdot, \cdot \rangle_{\bar{\mathfrak{g}}}$  be the invariant symmetric bilinear form on  $\bar{\mathfrak{g}}$  such that  $\langle \bar{h}, \bar{h} \rangle_{\bar{\mathfrak{g}}} = 1$ . Set  $\mathfrak{g} = \bar{\mathfrak{g}} \times \bar{\mathfrak{g}}$ ,  $\langle (x, y), (x', y') \rangle_{\mathfrak{g}} = \langle x, x' \rangle_{\bar{\mathfrak{g}}} - \langle y, y' \rangle_{\bar{\mathfrak{g}}}$ . Set  $\mathfrak{g}_+ = \{(x, x), x \in \bar{\mathfrak{g}}\}$  and  $\mathfrak{g}_- = \{(\eta + \xi_+, -\eta + \xi_-), \xi_\pm \in \bar{\mathfrak{n}}_\pm, \eta \in \bar{\mathfrak{h}}\}$ . Then  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  is a Manin triple. Let  $(e_+, h, e_-, e_+^*, h^*, e_-^*)$  be the basis of  $\mathfrak{g}$ , such that  $x = (\bar{x}, \bar{x})$  for  $x \in \{e_+, h, e_-\}$ ,  $e_+^* = (\bar{e}_+, 0)$ ,  $h^* = (\bar{h}, -\bar{h})$  and  $e_-^* = (0, \bar{e}_-)$ .

A quantization of the Manin triple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  is the algebra  $A$  with generators again denoted  $(e_+, h, e_-, e_+^*, h^*, e_-^*)$ , and relations

$$\begin{aligned} [h, e_\pm] &= \pm 2e_\pm, [h^*, e_\pm^*] = 2e_\pm^*, [h, e_\pm^*] = \pm 2e_\pm^*, [h^*, e_\pm] = -2(e_\pm - 2e_\pm^*), \\ [e_+, e_-] &= \frac{q^h - q^{-h}}{q - q^{-1}}, [e_+, e_-^*] = \frac{q^h - q^{h^*}}{q - q^{-1}}, [e_+^*, e_-] = \frac{q^{h^*} - q^{-h}}{q - q^{-1}}, [e_+^*, e_-^*] = 0, \\ [h, h^*] &= 0, \quad e_\pm^* e_\pm - q^{-2} e_\pm e_\pm^* = (1 - q^{-2})(e_\pm^*)^2, \end{aligned}$$

where we set  $q = \exp(\hbar)$ . We define  $A_+$  (resp.,  $A_-$ ) to be the subalgebra of  $A$  generated by  $e_+, h, e_-$  (resp.,  $e_+^*, h^*, e_-^*$ ). There is a unique algebra map  $\Delta : A \rightarrow A \otimes A$ , such that

$$\begin{aligned} \Delta(e_+) &= e_+ \otimes q^h + 1 \otimes e_+, \Delta(e_-) = e_- \otimes 1 + q^{-h} \otimes e_-, \Delta(h) = h \otimes 1 + 1 \otimes h, \\ \Delta(e_+^*) &= (e_+^* \otimes q^{h^*}) (1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*)^{-1} + 1 \otimes e_+^*, \\ \Delta(e_-^*) &= e_-^* \otimes 1 + (q^{h^*} \otimes e_-^*) (1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*)^{-1}, \\ \Delta(q^{h^*}) &= (q^{h^*} \otimes q^{h^*}) (1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*)^{-1} \times \\ &\quad \times (1 - q^{-3}(q - q^{-1})^2 e_+^* \otimes e_-^*)^{-1}. \end{aligned}$$

Then  $A_+$  and  $A_-$  are Hopf subalgebras of  $A$ , and  $(A, A_+, A_-, \Delta)$  is a quantization of the Manin triple  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ . The algebra  $A$  is the Drinfeld double of  $(A_+, \Delta)$  and its  $R$ -matrix is

$$\mathcal{R} = \exp_{q^2} \left( -(q - q^{-1}) e_-^* \otimes e_+ \right) q^{\frac{1}{2} h^* \otimes h} \exp_{q^2} \left( -(q - q^{-1}) e_+^* \otimes e_- \right)^{-1},$$

where  $\exp_{q^2}(z) = \sum_{n \geq 0} \frac{z^n}{[n]!}$  and  $[n]! = \prod_{k=1}^n (1 + q^2 + \dots + q^{2k-2})$ .

Let us fix  $\alpha \in \mathbb{C}$  and define  $\mathfrak{h}_\alpha$  to be the subalgebra  $\text{Ad}(e^{\alpha h^*})(\mathfrak{g}_+)$  of  $\mathfrak{g}$ . The linear space  $\mathfrak{h}_\alpha$  is spanned by  $h$  and  $e_\pm + \beta e_\pm^*$ , where  $\beta = e^{4\alpha} - 1$ . When  $\beta \neq 0$ ,  $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h}_\alpha)$  are nongraded Manin quadruples. We now assume  $\beta \neq 0$ .

Here are quantizations of these Manin quadruples. Let us define  $B_\alpha$  to be the subalgebra of  $A$  generated by  $h$  and  $e_\pm + \beta e_\pm^*$ . Let us set

$$F_\alpha = \Psi_\alpha(e_+^* \otimes e_-^*), \quad \text{with } \Psi_\alpha(z) = \frac{\exp_{q^2} \left( -(q - q^{-1}) e^{-4\alpha z} \right)}{\exp_{q^2} \left( -(q - q^{-1}) z \right)}.$$

**Proposition 4.1**  $((A, \Delta, \mathcal{R}), A_\pm, B_\alpha, F_\alpha)$  are quantizations of the Manin quadruples  $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h}_\alpha)$ .

*Proof.* We have  $B_\alpha \cap A_+ = \mathbb{C}[h][[\hbar]]$ , and  $B_\alpha \cap A_- = \mathbb{C}[[\hbar]]$ . Let us set  $u_\alpha = e^{\alpha h^*}$ . Then  $\Delta(u_\alpha) = F^*(u_\alpha \otimes u_\alpha)(F^*)^{-1}$ , where  $F^* = \exp_{q^2} \left( -(q - q^{-1}) e_+^* \otimes e_-^* \right)$ . It follows that  $F_\alpha = (u_\alpha \otimes u_\alpha) \Delta(u_\alpha)^{-1}$ , which implies that  $F_\alpha$  satisfies the cocycle identity.

Moreover,  $B_\alpha = u_\alpha A_+ u_\alpha^{-1}$ , therefore  $F_\alpha \Delta(B_\alpha) F_\alpha^{-1} \subset B_\alpha^{\otimes 2}$ . In fact,  $A$  equipped with the twisted coproduct  $F_\alpha \Delta F_\alpha^{-1}$  is a Hopf algebra, and thus  $B_\alpha$  is a Hopf subalgebra of  $(A, F_\alpha \Delta F_\alpha^{-1})$ .

Let us now show that  $F_\alpha$  satisfies both condition (4), and the similar condition where  $A_+$  is replaced by  $A_-$ . This follows from the conjunction of

$$\Delta(u_\alpha) \in (1 + \hbar(A_-)_0 \otimes (A_+)_0) (u_\alpha \otimes u_\alpha) (1 + \hbar A_0 \otimes (A_+)_0) \quad (5)$$

and

$$\Delta(u_\alpha) \in (1 + \hbar(A_+)_0 \otimes (A_-)_0) (u_\alpha \otimes u_\alpha) (1 + \hbar(A_+)_0 \otimes A_0). \quad (6)$$

Then

$$\Delta(u_\alpha) = F'(u_\alpha \otimes u_\alpha) (F')^{-1} = F''(u_\alpha \otimes u_\alpha) (F'')^{-1}, \quad (7)$$

where

$$F' = \exp_{q^2}(-(q - q^{-1})e_+^* \otimes e_-), \quad F'' = \exp_{q^2}(-(q - q^{-1})e_+ \otimes e_-^*).$$

The first equality of (7) proves (5), and the second one proves (6).  $\blacksquare$

One may expect that the algebra-module over  $(A_+^{opp}, \Delta)$  constructed by means of  $((A, \Delta, \mathcal{R}), A_+, B_\alpha, F_\alpha)$  in Theorem 3.1 is a formal completion of the function algebra over a Podleś sphere ([17]).

**Remark 11** One shows that for  $x \in A_-$ ,  $\Delta(x) = F^* \tilde{\Delta}(x) (F^*)^{-1}$ , where  $\tilde{\Delta}$  is the coproduct on  $A_-$  defined by  $\tilde{\Delta}(e_+^*) = e_+^* \otimes q^{h^*} + 1 \otimes e_+^*$ ,  $\tilde{\Delta}(q^{h^*}) = q^{h^*} \otimes q^{h^*}$ , and  $\tilde{\Delta}(e_-^*) = e_-^* \otimes 1 + q^{h^*} \otimes e_-^*$ . The completion of the Hopf algebra  $(A_-, \Delta)$  with respect to the topology defined by its augmentation ideal should be isomorphic to the formal completion at the identity of the quantum coordinate ring of  $SL_2$ . The Hopf algebra  $A$  was obtained by a method similar to that of Drinfeld's "new realizations".

#### 4.2. The case of loop algebras

Let us return to the situation of the Manin quadruple (1), in the case where  $\mathcal{K}$  is a field of Laurent series and  $R$  is a ring of functions on an affine curve. In this situation, one can consider the following problems.

1) If the double quotient  $G_+ \backslash G/H$  is equipped with the zero Poisson structure, the projection  $G/H \rightarrow G_+ \backslash G/H$  is Poisson. The ring of formal functions on this double quotient is  $\mathcal{O}_{G_+ \backslash G/H} = (U\mathfrak{g}/((U\mathfrak{g})\mathfrak{h} + \mathfrak{g}_+ U\mathfrak{g}))^*$ . On the other hand,  $(A^*)^{A_+, B} = \{\ell \in A^* | \forall a_+ \in A_+, b \in B, \ell(a_+ ab) = \ell(a) \varepsilon(a_+) \varepsilon(b)\}$  is a subalgebra of  $(A^*)^B$ . It is commutative because the  $R$ -matrix  $\mathcal{R}$  of  $(A, \Delta)$  belongs to  $(A_+ \otimes A_-)_0^\times$  and the twisted  $R$ -matrix  $F^{(21)} \mathcal{R} F^{-1}$  belongs to  $(B \otimes A)_0^\times$  (see [9]).

It would be interesting to describe the algebra inclusion  $(A^*)^{A_+, B} \subset (A^*)^B$ , to see whether it is a flat deformation of  $\mathcal{O}_{G_+ \backslash G/H}$ , and when the level is critical,

to describe the action of the quantum Sugawara field by commuting operators on  $(A^*)^{A+,B}$  and  $(A^*)^B$ . These operators could be related to the operators constructed in [10].

2) For  $g$  fixed in  $G$ , let  ${}^g x$  denote the conjugate  $gxg^{-1}$  of an element  $x$  in  $G$  by  $g$ . One would like to describe the Poisson homogeneous spaces  $G_{\pm}/(G_{\pm} \cap {}^g H)$ , and to obtain quantizations of the Manin quadruples  $(\mathfrak{g}, \mathfrak{g}_{\pm}, \mathfrak{g}_{\mp}, {}^g \mathfrak{h})$ . A natural idea would be to start from the quantization of the Manin quadruple  $(\mathfrak{g}, \mathfrak{g}_{\pm}, \mathfrak{g}_{\mp}, \mathfrak{h})$  obtained in [9], and to apply to  $B$  a suitable automorphism of  $A$  which lifts the automorphism  $\text{Ad}(g)$  of  $U\mathfrak{g}$ .

We hope to return to these questions elsewhere.

## References

1. Andruskiewitsch, N.: Some exceptional compact matrix pseudogroups, *Bull. Soc. Math. France* **120** (1992), 297–325.
2. Bonneau, P., Flato, M., Gerstenhaber, and M., Pinczon, G.: The hidden group structure of quantum groups: strong duality, rigidity, and preferred deformations, *Comm. Math. Phys.* **161** (1994), 125–156.
3. Dazord, P. and Sondaz, D.: Groupes de Poisson affines, in: P. Dazord and A. Weinstein, (eds.), *Symplectic geometry, groupoids, and integrable systems (Berkeley 1989)*, Math. Sci. Res. Inst. Publ., 20, Springer-Verlag, New York, 1991.
4. Delorme, P.: Sur les triples de Manin pour les algèbres réductives complexes, preprint math/9912055.
5. Donin, J., Gurevich, D., and Shnider, S.: Double quantization on some orbits in the coadjoint representations of simple Lie groups, *Comm. Math. Phys.* **204** (1999), 39–60.
6. Drinfeld, V.: Quantum groups, in: *Proceedings of the International Congress of Mathematicians (Berkeley 1986)*, Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.
7. Drinfeld, V.: Quasi-Hopf algebras, *Leningrad Math. J.* **1** (1990), 1419–1457.
8. Drinfeld, V.: On Poisson homogeneous spaces of Poisson-Lie groups, *Theoret. and Math. Phys.* **95** (1993), 524–525.
9. Enriquez, B. and Rubtsov, V.: Quasi-Hopf algebras associated with  $\mathfrak{sl}_2$  and complex curves, *Israel J. Math.* **112** (1999), 61–108.
10. Enriquez, B. and Felder, G.: Commuting differential and difference operators associated with complex curves II, preprint math/9812152.
11. Etingof, P. and Kazhdan, D.: Quantization of Poisson algebraic groups and Poisson homogeneous spaces, in: A. Connes, K. Gawedzki, and J. Zinn-Justin, (eds.), *Symétries quantiques (Les Houches 1995)*, North-Holland, Amsterdam, 1998, pp. 935–946.
12. Karolinsky, E.: A classification of Poisson homogeneous spaces of compact Poisson-Lie groups, *Dokl. Math.*, **57** (1998), 179–181. See also math/9901073.
13. Lu, J.-H.: *Multiplicative and affine Poisson structures on Lie groups*, Ph.D. thesis, Univ. of Berkeley, 1990.
14. Lu, J.-H.: Classical dynamical  $r$ -matrices and homogeneous Poisson structures on  $G/H$  and  $K/T$ , preprint math/9909004.
15. Noumi, M.: Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, *Adv. Math.* **123** (1996), 16–77.
16. Parmentier, S.: On coproducts of quasi-triangular Hopf algebras, *St. Petersburg Math. J.* **6** (1995), 879–894.
17. Podleś, P.: Quantum spheres, *Lett. Math. Phys.* **14** (1987), 193–202.

