# Deformations of Nearly Kähler Manifolds

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# Part I

# Basics on nearly Kähler manifolds

**Motivation :**

- spin geometry : a NK structure in dimension 6 $\iff$ a real Killing spinor (Friedrich, Grunewald) ;
- the theory of connections with skew-symmetric parallel torsion (Cleyton, Swann).

# Nearly Kähler manifolds

Definition. An almost Hermitian manifold $(M, g, J)$ with fundamental form $\omega := g(J.,.)$ is nearly Kähler (or NK) if $(\nabla_X J)X = 0$, $\forall X \in TM$. $M$ is strictly nearly Kähler (SNK) if $\nabla_X J \neq 0$, $\forall X \in TM$.

The projection of the Levi-Civita connection on the space of Hermitian connections $\rightsquigarrow$ canonical Hermitian connection :

$$\bar{\nabla}_X := \nabla_X - \frac{1}{2} J \circ \nabla_X J.$$

Its torsion $T^{\bar{\nabla}} = -J(\nabla J)$ is totally skew-symmetric.

A SNK manifold is neither symplectic nor Hermitian. The obstruction is given by $\nabla J$ : $d\omega = 3\nabla J$, $N^J = J(\nabla J)$.

# Nearly Kähler manifolds

Properties :

- $\nabla J$ (and hence $T^{\bar\nabla}$) is parallel wrt $\bar\nabla$.
- in dimension 4 : NK = Kähler
- in dimension 6 : a NK manifold is either Kähler or SNK and :
  - carries a $\bar\nabla$-parallel $\mathrm{SU}_3$-structure : $(g, J, d\omega)$.
  - carries Killing spinors $\rightsquigarrow$ Einstein with positive scalar curvature.

# Nearly Kähler manifolds

Theorem. An almost Hermitian manifold $(M^6, g, J)$ is SNK iff the Riemannian cone $\bar{M} := (M \times \mathbb{R}_+, t^2 g + dt^2)$ satisfies $\mathrm{Hol}(\bar{M}) \subset G_2$.

Idea of the proof : define a generic 3-form (in the sense of Hitchin) :

$$\varphi := t^2 dt \wedge \omega + \frac{1}{3} t^3 d\omega$$

and check that $\omega$ is parallel $\iff$ $(M^6, g, J)$ is NK.

Corollary. The sphere $(S^6, can)$ is SNK, wrt the almost complex structure induced by

$$\omega = e^{123} + e^{145} + e^{167} + e^{246} + e^{257} + e^{347} + e^{356}.$$

# Examples of NK manifolds

- Kähler manifolds.
- Twistor spaces over QK manifolds with $Scal > 0$, endowed with the non-integrable almost complex structure $\mathcal{J}^-$.
- Naturally reductive 3-symmetric spaces : Homogeneous spaces $G/H$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $H$ is the fix point set of an automorphism $\sigma$ of $G$ of order 3 which defines an almost complex structure on $\mathfrak{m}$ by the relation $\sigma_* = -\frac{1}{2} Id + \frac{\sqrt{3}}{2} J$;
Example : $G = K \times K \times K$, $H = \Delta \simeq K$ (diagonal).
- In dimension 6 :
  - $S^6 = G2/\mathrm{SU}_3$
  - $\mathbb{C}\mathrm{P}^3 = \mathrm{Sp}_2/S^1 \times \mathrm{Sp}_1$, $F(1,2) = \mathrm{SU}_3/S^1 \times S^1$
  - $S^3 \times S^3 = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2/\Delta$

# Decomposition of NK manifolds

Theorem. (Nagy) Every NK manifold is locally a Riemannian product of

- Kähler manifolds.
- Twistor spaces over positive QK manifolds.
- Naturally reductive 3-symmetric spaces.
- 6-dimensional NK manifolds.

Corollary. The classification of homogeneous NK manifolds reduces to the dimension 6.

Theorem. (Butruille) Every homogeneous SNK manifold is a naturally reductive 3-symmetric space.

# Further results

Theorem. (Belgun-M., Nagy) A compact SNK manifold whose canonical Hermitian connection has complex reducible holonomy is isomorphic to $\mathbb{C}\mathrm{P}^3$, $F(1,2)$ (or to a twistor space over a positive QK manifold in dimension greater than 6).

Theorem. (M.-Nagy-Semmelmann) A compact 6-dimensional SNK manifold carrying unit Killing vector fields is isomorphic to $S^3 \times S^3$.

# Part II

# Deformations of nearly Kähler structures

Definition. Let $M$ be a smooth 6-dimensional manifold. A $\mathrm{SU}_3$ structure on $M$ is a reduction of the frame bundle of $M$ to $\mathrm{SU}_3$. It consists of a 5-tuple $s := (g, J, \omega, \psi^+, \psi^-)$ where

- $g$ is a Riemannian metric,
- $J$ is a compatible almost complex structure,
- $\omega$ is the corresponding fundamental 2-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$
- $\psi^+ + i\psi^-$ is a complex volume form of type $(3, 0)$.

# $\mathrm{SU}_3$ structures

Decomposition of the exterior bundles under the $\mathrm{SU}_3$ action :

- $\Lambda^2 M = \Lambda_0^{(1,1)} M \oplus \mathbb{R}\omega \oplus \Lambda^{(2,0)} M.$
- $\Lambda^3 M = \Lambda_0^{(2,1)} M \oplus \mathbb{R}\psi^+ \oplus \mathbb{R}\psi^- \oplus \Lambda^1 M \wedge \omega.$

Lemma. Isomorphisms of $\mathrm{SU}_3$ bundles :

- $TM \to \Lambda^{(2,0)} M, \qquad X \mapsto X \lrcorner \psi^+,$
- $\mathrm{Sym}^+ M \to \Lambda^{(1,1)} M, \qquad h \mapsto g(hJ\cdot, \cdot),$
- $\mathrm{Sym}^- M \to \Lambda_0^{(2,1)} M, \qquad S \mapsto S_\star \psi^+,$

where $_\star$ denotes the Lie algebra extension of an endomorphism.

Definition. Infinitesimal $\mathrm{SU}_3$ deformation $(\dot g, \dot J, \dot\omega, \dot\psi^+, \dot\psi^-)$ of $s$ : tangent vector at 0 to a path of $\mathrm{SU}_3$ structures $s_t$ with $s_0 = s$.

Proposition. 1-1 correspondence between infinitesimal $\mathrm{SU}_3$ deformations and sections of $TM \oplus \mathrm{Sym}^- M \oplus \Lambda^{(1,1)} M \oplus \mathbb{R}$, given by $(\xi, S, \varphi, \mu) \mapsto (\dot g, \dot J, \dot\omega, \dot\psi^+, \dot\psi^-)$ :

$$
(1) \quad \begin{cases}
\dot g = g((h+S)\cdot, \cdot) \\
\dot J = JS + \psi^+_\xi \\
\dot\omega = \varphi + \xi \lrcorner \psi^+ \\
\dot\psi^+ = -\xi \wedge \omega + \lambda\psi^+ + \mu\psi^- - \frac{1}{2} S_\star \psi^+ \\
\dot\psi^- = -J\xi \wedge \omega - \mu\psi^+ + \lambda\psi^- - \frac{1}{2} S_\star \psi^-
\end{cases}
$$

with $g(h\cdot, \cdot) = \varphi(\cdot, J\cdot)$, $\lambda = \frac{1}{4}\mathrm{tr}(h)$ and $g(\psi^+_\xi \cdot, \cdot) ;= \psi^+(\xi, \cdot, \cdot)$.

# Gray structures

A Gray structure on a 6-dimensional manifold $M$ is a SNK structure with $\mathrm{scal} = 30$.

Proposition. (Hitchin, Reyes-Carrión) A Gray structure is a $\mathrm{SU}_3$ structure $(g, J, \omega, \psi^+, \psi^-)$ which satisfies the exterior differential system

$$(2)\quad \begin{cases} d\omega = 3\psi^+ \\ d\psi^- = -2\omega \wedge \omega \end{cases}$$

At the infinitesimal level, the tangent vector at $t = 0$ to a curve of Gray structures $(g_t, J_t, \omega_t, \psi_t^+, \psi_t^-)$ is an infinitesimal $\mathrm{SU}_3$ deformation $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ which satisfies the linearized system :

$$(3)\quad \begin{cases} d\dot{\omega} = 3\dot{\psi}^+, \\ d\dot{\psi}^- = -4\dot{\omega} \wedge \omega. \end{cases}$$

# Infinitesimal Gray deformations

Definition. The set of $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ satisfying (1) and (3) is called virtual tangent space of Gray structures.

Infinitesimal Gray deformations : the quotient of the virtual tangent space by the action (by Lie derivative) of vector fields.

Theorem. Assume that $(M, g)$ is a Gray manifold not isometric to the round sphere $(S^6, can)$. Then the space of infinitesimal Gray deformations is isomorphic to the eigenspace for the eigenvalue 12 of the Laplace operator $\Delta$ acting on *co-closed* primitive $(1, 1)$-forms.

# Idea of the proof

- Since $M$ is not the sphere, the full Gray structure is determined by the metric.
- By the Ebin slice theorem, infinitesimal Gray deformations correspond to solutions of (1) and (3) with $\delta\dot{g} = 0$ and $\mathrm{tr}(\dot{g}) = 0$.
- Show that $\mu = 0$ and $\xi = 0$. The system (1), (3) becomes

$$\begin{cases} d\varphi = -\frac{3}{2} S_\star \psi^+, \\ \delta(S_\star \psi^+) = *d(S_\star \psi^-) = -2 * d\dot{\psi}^- = 8 * (\dot{\omega} \wedge \omega) = -8\varphi \end{cases}$$

- Thus $\Delta\varphi = 12\varphi$, $\delta\varphi = 0$.
- The converse follows from the fact that if $\varphi$ is a co-closed form in $\Omega_0^{(1,1)}M$ then $d\varphi \in \Omega_0^{(2,1)}M$.

# Part III

# The Hermitian Laplace operator on NK manifolds

# The curvature operator

Let $\nabla$ be a metric $G$-connection (not necessarily torsion-free) on some Riemannian $G$-manifold $M$. For every 2-form $\alpha$, the curvature tensor of $\nabla$ maps $\alpha$ to an endomorphism $R(\alpha)$ of $TM$, which induces an endomorphism $R(\alpha)_\star$ of the tensor bundles of $M$ as before. On every tensor bundle one defines the curvature endomorphism

$$q(R) := \sum_k \alpha_{k\star} R(\alpha_k)_\star,$$

where $\alpha_k$ is an ON basis of $\Lambda^2 M$. If the curvature tensor of $\nabla$ satisfies the symmetry by pairs, the curvature endomorphism commutes with all $G$-equivariant endomorphisms between tensor bundles, so in particular it leaves invariant all $\nabla$-parallel sub-bundles.

# Weitzenböck formulas

Let $\nabla$ and $\bar{\nabla}$ be the Levi-Civita resp. the Hermitian connection on a Gray manifold.

Weitzenböck formula : $\Delta = \nabla^*\nabla + q(R)$.

Definition. $\bar{\Delta} = \bar{\nabla}^*\bar{\nabla} + q(\bar{R})$.

$\bar{\Delta}$ is an elliptic self-adjoint operator on $\Lambda_0^{(p,q)}M$.

Definition. If $E$ is a $\mathrm{SU}_3$ representation $\rightsquigarrow$ $EM$ associated bundle. Denote by $E(\lambda)$ the $\lambda$-eigenspace of $\bar{\Delta}$ on $EM$.

# Relationship between Laplacians

**Lemma.**

$$q(R) - q(\bar{R}) = \begin{cases} \mathrm{id} & \text{on} \quad TM \\ -\mathrm{id} & \text{on} \quad \Lambda_0^{(1,1)}M \\ 4\mathrm{id} & \text{on} \quad \Lambda^{(2,0)}M \end{cases}$$

**Lemma.**

$$\begin{aligned} (\nabla^*\nabla - \bar{\nabla}^*\bar{\nabla})\varphi &= \varphi + (J\delta\varphi) \lrcorner \psi^+, \qquad \forall\, \varphi \in \Omega_0^{(1,1)}M, \\ (\nabla^*\nabla - \bar{\nabla}^*\bar{\nabla})X &= -X + P(X), \qquad \forall\, X \in TM, \end{aligned}$$

where $P(X)$ is the vector field corresponding to the projection of $\nabla X$ onto $\Lambda^{(2,0)}M \simeq TM$. In particular, $\Delta = \bar{\Delta}$ on closed 1-forms and on co-closed primitive $(1,1)$-forms.

**Corollary.** The space of infinitesimal Gray deformations is isomorphic to the subspace of $\Lambda_0^{(1,1)}(12)$ consisting of co-closed forms.

# Eigenspaces of $\bar{\Delta}$

Proposition. If $\xi$ is a Killing vector field on $M$ then the projection $\varphi_\xi := (d\xi)_0^{(1,1)}$ of $d\xi$ onto $\Lambda_0^{(1,1)}M$ belongs to $\Lambda_0^{(1,1)}(12)$ and has codifferential $\delta(\varphi_\xi) = 8\xi$.

Proposition. $f \mapsto \varphi_f := (dJdf)_0^{(1,1)}$ maps $\Lambda^0(\lambda)$ into $\Lambda_0^{(1,1)}(\lambda)$, and $\delta(\varphi_f) = (\frac{2\lambda}{3} - 4)Jdf$.

We thus have a mapping

$$\{\text{Killing vector fields}\} \oplus \Lambda^0(12) \to \Lambda_0^{(1,1)}(12)$$

whose image is transverse to the space of infinitesimal Gray deformations.

# Part IV

# Explicit computations on 3-symmetric spaces

# Homogeneous spaces

Let $M = G/K$ be a homogeneous space with compact Lie groups $G$ and $K$, and let $E := G \times_\pi V$ be the vector bundle over $M$ associated to a complex representation $\pi : K \to \mathrm{Aut}(V)$ of $K$. Then by the Peter-Weyl theorem the following isomorphism holds :

$$L^2(E) \simeq \bigoplus_{\gamma \in \hat{G}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, V),$$

where $\hat{G}$ is the set of irreducible $G$-representations.

Let $\mathfrak{g}$ be the Lie algebra of $G$. We denote by $B$ the *Killing form* of $\mathfrak{g}$, $B(X, Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. The Killing form is negative definite if $G$ is semi-simple, which is the case in all examples below.

# The Casimir operator

If $\tau : G \to \mathrm{Aut}(V)$ is a $G$-representation, the *Casimir operator* of $G$ acts on $V$ by the formula

$$\mathrm{Cas}_V = \sum (\tau_* X_i)^2,$$

where $\{X_i\}$ is a $(-B)$-orthonormal basis of $\mathfrak{g}$.

Theorem. Assume that $G$ is semi-simple and let $\bar{\nabla}$ denote the canonical connection on $M = G/K$ with curvature $\bar{R}$. If one consider the Riemannian metric on $M$ induced by $-B$, then

$$q(\bar{R}) = -\mathrm{Cas}_K, \qquad \bar{\nabla}^* \bar{\nabla} = -(\mathrm{Cas}_G - \mathrm{Cas}_K),$$

on $L^2(E)$. In particular, $\bar{\Delta} = -\mathrm{Cas}_G$.

If $M$ is a 3-symmetric Gray manifold, the canonical connection coincides with the Hermitian connection.

# The case $M = S^3 \times S^3$

$M = G/K$ where $G = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2$ and $K = \mathrm{SU}_2$ diagonally embedded into $G$. We denote by $E$ the standard representation of $\mathrm{SU}_2$ on $\mathbb{C}^2$. The irreducible representations of $G$ are

$$V_{a,b,c} = \mathrm{Sym}^a A \otimes \mathrm{Sym}^b B \otimes \mathrm{Sym}^c C,$$

where $A$, $B$, $C$ denote the representations of $G$ on $\mathbb{C}^2$ induced by the standard representation of each factor.

$$\mathrm{Cas}(V_{a,b,c}) = -\frac{3}{2}[a(a+2) + b(b+2) + c(c+2)].$$

Branching rules :

$$V_{a,b,c}|_{\mathrm{SU}_2} = \mathrm{Sym}^a E \otimes \mathrm{Sym}^b E \otimes \mathrm{Sym}^c E = \bigoplus_{\substack{k=a+b+c \ (mod 2) \\ k \leq a+b+c}} p_k \mathrm{Sym}^k E,$$

for some positive integers $p_k$.

# The case $M = S^3 \times S^3$

The bundle $\Lambda^{(1,1)}M \otimes \mathbb{C}$ is associated to the representation of $K$ on $\Lambda^{(1,1)}(\mathfrak{m}) \otimes \mathbb{C} = \mathfrak{m}^{(1,0)} \otimes \mathfrak{m}^{(0,1)} \simeq \mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E = \mathrm{Sym}^4 E \oplus \mathrm{Sym}^2 E \oplus \mathbb{C}$, so the defining representation of $\Lambda_0^{(1,1)} M \otimes \mathbb{C}$ is $V = \mathrm{Sym}^4 E \oplus \mathrm{Sym}^2 E$.
$-\mathrm{Cas}(V_{a,b,c}) = 12 \iff (a, b, c) = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$.
In each case,
$\mathrm{Hom}_K(V_\gamma, V) = \mathrm{Hom}_K(\mathrm{Sym}^2 E, \mathrm{Sym}^4 E \oplus \mathrm{Sym}^2 E) = \mathbb{C}$. The eigenspace $\Lambda_0^{(1,1)}(12)$ has thus dimension 9. Since $\dim(\mathfrak{g}) = \dim \Lambda_0^{(1,1)}(12)$, this shows that there are no infinitesimal Gray deformations in this case.

## The case $M = \mathbb{C}\mathrm{P}^3$

$M = G/K$ where $G = \mathrm{SO}_5$ and $K = \mathrm{U}_2$, embedded into $G$ by $\mathrm{U}_2 \subset \mathrm{SO}_4 \subset \mathrm{SO}_5$. The irreducible representations $V_{a,b}$ of $G$ are classified by the highest weight vector $(a, b)$ with $a, b \in \mathbb{Z}$, $0 \leq b \leq a$.

$$\mathrm{Cas}(V_{a,b}) = -2[a(a+3) + b(b+1)].$$

We see that $\mathrm{Cas}(V_{a,b}) = -12 \iff a = 1$ and $b = 1$.

Let $\mathbb{C}_k$ be the $\mathrm{U}_1$ representation on $\mathbb{C}$ given by $(z, v) \mapsto z^k v$. Since $\mathrm{U}_2 = \mathrm{SU}_2 \times \mathrm{U}_1/\mathbb{Z}_2$, the $\mathrm{U}_2$ representations are

$$E_{a,b} := \mathrm{Sym}^a E \otimes \mathbb{C}_b, \qquad a \in \mathbb{N},\ b \in \mathbb{Z},\ a = b\ (mod 2).$$

# The case $M = \mathbb{CP}^3$

$$V_{1,1}|_{\mathrm{U}_2} = E_{0,0} \oplus E_{0,2} \oplus E_{0,-2} \oplus E_{1,1} \oplus E_{1,-1} \oplus E_{2,0}.$$

The defining representation of $\Lambda_0^{(1,1)} M \otimes \mathbb{C}$ is

$$V = E_{0,0} \oplus E_{0,3} \oplus E_{0,-3} \oplus E_{2,0}.$$

We see that $\mathrm{Hom}_{\mathrm{U}_2}(V_{1,1}, V) = \mathbb{C} \oplus \mathbb{C}$, whence

$$\dim \Lambda_0^{(1,1)}(12) = \dim(V_{1,1}) \times \dim(\mathrm{Hom}_{\mathrm{U}_2}(V_{1,1}, V)) = 20.$$

Similarly,

$$\dim \Lambda^0(12) = \dim(V_{1,1}) \times \dim(\mathrm{Hom}_{\mathrm{U}_2}(V_{1,1}, E_{0,0})) = 10.$$

Since $\dim(\mathfrak{g}) + \dim \Lambda^0(12) = \dim \Lambda_0^{(1,1)}(12)$, there are no infinitesimal Gray deformations on $\mathbb{CP}^3$.

# The case $M = F(1,2)$

$M = G/K$ where $G = \mathrm{SU}_3$ and $K = \mathrm{U}_1 \times \mathrm{U}_1$ is its maximal torus. Denote by $E$ the standard representation of $\mathrm{SU}_3$ on $\mathbb{C}^3$ (notice that $E$ is not self-dual). Then the irreducible representations of $G$ are

$$V_{a,b} = (\mathrm{Sym}^a E \otimes \mathrm{Sym}^b \bar{E})_0.$$

$$\mathrm{Cas}(V_{a,b}) = -2[a(a+2) + b(b+2)].$$

Again, $\mathrm{Cas}(V_{a,b}) = -12 \iff a = 1$ and $b = 1$. Let $\mathbb{C}_{k,l}$ be the $\mathrm{U}_1 \times \mathrm{U}_1$ representation on $\mathbb{C}$ given by $((z_1, z_2), v) \mapsto z_1^k z_2^l v$. Then

$$V_{1,1}|_{\mathrm{U}_1\times\mathrm{U}_1} = 2\mathbb{C}_{0,0} \oplus \mathbb{C}_{1,-1} \oplus \mathbb{C}_{-1,1} \oplus \mathbb{C}_{\pm 2,\pm 1} \oplus \mathbb{C}_{\pm 1,\pm 2}.$$

Similarly, the defining representation of $\Lambda_0^{(1,1)} M \otimes \mathbb{C}$ is

$$V|_{\mathrm{U}_1\times\mathrm{U}_1} = 2\mathbb{C}_{0,0} \oplus \mathbb{C}_{\pm 3,0} \oplus \mathbb{C}_{0,\pm 3} \oplus \mathbb{C}_{\pm 3,\pm 3}.$$

We get :

$$\begin{aligned} \dim \Lambda_0^{(1,1)}(12) = \dim(V_{1,1}) \times \dim(\mathrm{Hom}_K(V_{1,1}, V)) &= 32. \\ \dim \Lambda^0(12) = \dim(V_{1,1}) \times \dim(\mathrm{Hom}_K(V_{1,1}, \mathbb{C}_{0,0})) &= 16. \end{aligned}$$

Thus $\dim(\mathfrak{g}) + \dim \Lambda^0(12) = 24 < 32 = \dim \Lambda_0^{(1,1)}(12)$.

# The case $M = F(1,2)$

It turns out that there exists an 8-dimensional subspace of co-closed forms in $\Lambda_0^{(1,1)}(12)$.

Corollary. The space of infinitesimal Gray deformations of $F(1,2)$ has dimension 8.

What next ? $\rightsquigarrow$ Compute the second jet of a curve of Gray structures on $F(1,2)$ (work in progress).