Erratum to
Hodge Theory of the Middle Convolution

by
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Abstract

We give a correction to the statement of Theorem 3.2.3 of [2].

2010 Mathematics Subject Classification: 14D07, 32G20, 32S40, 34M99.

Keywords: Middle convolution, rigid local system, Katz algorithm, Hodge theory.

Theorem 3.2.3 of [2] is incorrectly stated. The correct statement is as follows. Given $\lambda \in S^1$, we set $\lambda = \exp(-2\pi i' \alpha)$ with $\alpha' \in (0, 1]$ (not $[0, 1)$). With this in mind, we have the following theorem.

Theorem 3.2.3 ([10, Thm. 5.4]).

$$\gr^k_F \phi_{s,\lambda}(M_1 \boxtimes M_2) = \bigoplus_{\lambda_1 \lambda_2 = \lambda} \left\{ \begin{array}{ll} \bigoplus_{j+k=p-1} \gr^j_F \phi_{j,\lambda_1} M_1 \otimes \gr^k_F \phi_{k,\lambda_2} M_2 & \text{if } \alpha'_1 + \alpha'_2 \in (0, 1], \\
\bigoplus_{j+k=p} \gr^j_F \phi_{j,\lambda_1} M_1 \otimes \gr^k_F \phi_{k,\lambda_2} M_2 & \text{if } \alpha'_1 + \alpha'_2 \in (1, 2]. \end{array} \right.$$
We make clear below the side-changing relations to relate our setting to that of \([10]\). Assume \((M, F^*M)\) is a polarizable complex Hodge module on the disc \(\Delta\) as defined in \([2, \S 3.2]\), and that \(M\) is a minimal extension at the origin. Let \(V^*M\) be its \(V\)-filtration (cf. the notation in \([2, \S 2.2]\)).

Since \(\Delta\) has a global coordinate, we can identify the associated right \(\mathcal{D}_{\Delta}\)-module with \(M\) on which \(\mathcal{D}_0\) acts in a transposed way. We denote it by \(M'\). The \(V\)-filtration and the \(F\)-filtration are now denoted increasingly. We have the following relations:

\[
F_p M' = F^{-p-1} M, \quad V_\gamma M' = V^{-\gamma -1} M.
\]

By the definition in \([9]\), we have, for \(\lambda \in S^1\) and \(\lambda = \exp(2\pi i \gamma)\) with \(\gamma \in [-1, 0)\),

\[
(*) \quad F_p \psi_\lambda M' := F_{p-1} \text{gr}_V^\phi M' = F^{-p} \text{gr}_V^\beta M \quad (\beta = -\gamma - 1),
\]

\[
F_p \phi_1 M' := F_p \text{gr}_V^\phi M' = F^{-p-1} \text{gr}_V^{-1} M.
\]

Due to our previous definition of \(F^\alpha \psi_\lambda M\) and \(F^\beta \phi_1 M\) (given before Theorem 2.2.4 and Proposition 2.2.5), we find that

\[
F_p \psi_\lambda M' = F^{-p} \psi_\lambda M, \quad F_p \phi_1 M' = F^{-p} \phi_1 M.
\]

Lastly, the theorem of Saito (for filtered right \(\mathcal{D}_{\Delta}\)-modules) gives, setting \(\lambda = \exp(-2\pi i \beta)\) with \(\beta \in (-1, 0]\) (since we are now interested in vanishing cycles),

\[
gr^F_{p} \phi_{s,\lambda} (M_1' \otimes M_2') = \bigoplus_{(\lambda_1, \lambda_2) \atop \lambda_1 + \lambda_2 = \lambda} \text{gr}^F_{j+k=p+1} \phi_{t_1, \lambda_1} M_1' \otimes \text{gr}^F_{j+k=p} \phi_{t_2, \lambda_2} M_2'
\]

We now replace \(\beta, \beta_1, \beta_2\) by \(\alpha', \alpha'_1, \alpha'_2 \in (0, 1]\) (by adding 1 to each number). The previous formula is immediately translated to the above statement by replacing \(M'\) with \(M\) and increasing \(F\)-filtrations with decreasing ones.

In the setting of Theorem 3.1.2(2), we have \(\alpha'_2 = \alpha_o \in (0, 1]\), and \(\text{gr}^k_F \phi_{t_2, \lambda_0} M_o = 0\) unless \(k = 0\). For \(\alpha', \alpha'_1 \in (0, 1]\), we have

\[
\alpha' = \alpha'_1 + \alpha_o \iff \alpha' \in (0, 1] \cap (\alpha_o, \alpha_o + 1] = (\alpha_o, 1].
\]

If \(\alpha'_1 + \alpha_o \in (1, 2]\), we must set \(\alpha' = \alpha'_1 + \alpha_o - 1\), and similarly \(\alpha' \in (0, \alpha_o]\). We thus find the above expression for \(F^\beta_{s,\lambda,\beta}(MC_\lambda(M))\) depending on the position of \(\alpha'\). Going back to \(\alpha \in [0, 1)\), the condition becomes as stated in \([2]\).

**Remark 1** (Suggested by the referee). The formula of Theorem 3.2.3 is essentially the same as that given in \([11]\). The referee emphasizes that the results of
[10], [11] involve \( \mathbb{Q} \)-mixed Hodge modules, while Theorem 3.2.3 concerns polarizable complex Hodge modules as defined in [2, §3.2]. Fortunately, the last version of [4] proves a Thom–Sebastiani-type theorem for filtered \( \mathcal{D} \)-modules in a sufficiently general case including our case, where the \( V \)-filtration is indexed by \( \mathbb{R} \).

In [2, §2], we have used the (still unpublished) results of Schmid in the context of polarizable variations of real or complex Hodge structures of some weight, according to [13] (cf. also [1, §1.11]) in order to ensure that, by taking their intermediate extensions, we obtain a polarizable complex Hodge module as defined in [2, §3.2]. Recall that another proof is given in [7, §3.a–3.g] relying on the theory of tame harmonic bundles on curves [12].

**Remark 2.** Since we are interested only in proving Theorem 3.1.2 of [2], we will indicate precisely a direct proof of 3.1.2(2) via twistor D-modules, avoiding Thom–Sebastiani in its local form, and using instead the stationary phase formula proved in [8, (A.11) & (A.12)].

To a filtered \( \mathbb{C}[t][\partial_t] \)-module \( (M,F^\bullet M) \) we associate the Rees module \( R_F M := \bigoplus_p F^p M z^{-p} \), where \( z \) is a new variable. It is endowed with the action of \( z^2\partial_z \) such that, for \( m \in F^p M \), we have \( z^2\partial_z (m z^{-p}) = -pz n z^{-(p-1)} \). To a variation of polarized complex Hodge structure \( (V, \nabla, F^\bullet V) \) of weight 0 on \( \mathbb{A}^1 \setminus \mathbf{x} \) is associated a polarized pure twistor \( \mathcal{D}_{\mathbb{P}^1} \)-module \( \mathcal{T} \) of weight 0 whose restriction to \( \mathbb{A}^1 \setminus \mathbf{x} \) is \( (R_F V, R_F V, R_F k) \), where the sesquilinear \( R_F k \) is obtained by the Rees procedure from the flat sesquilinear pairing \( k \) inducing the polarization (cf. [7, §3]). Then \( \mathcal{T} \) is also endowed with a compatible action of \( z^2\partial_z \): one says that it is integrable.

Note that in [7, §3] the construction of \( (\mathcal{T}, z^2\partial_z) \) uses the \( \mathbb{R} \)-variant of Schmid’s results. In order to avoid this, we can use the property that the Hodge metric is a tame harmonic metric and then use the extension property of [12] (cf. also [6, Thm. 5.0.1], [5, Thm. 1.22], both in the simpler case of integrable objects).

The formulas (A.11) and (A.12) of [8] need to be modified in order to take care of the shift by 1 in the definition \( (\ast) \) of \( F^p \phi_1 M \), and of the shift of the filtration by the push-forward by a closed immersion, as explained in [3, (1.2.4)]. Here, the codimension-1 inclusion \( i_0 \) used in Lemma A.10 of [8] produces a shift by 1 in the formulas. With this slight change of convention, compatible with that of [9], (A.11) and (A.12) of [8] read, at \( x_i = 0 \) and with an adaptation of the notation,

\[
(P_t \phi_{1,\lambda} \mathcal{T}, z^2\partial_z) \simeq (P_t \psi_{\tau,\lambda} F\mathcal{T}, z^2\partial_z - \beta z)
\]

if \( \lambda = \exp(-2\pi i \beta) \) and \( \beta \in (-1,0] \).
For $\chi = \lambda_o$, the meromorphic flat bundle $L_\chi$ defines a polarized pure twistor $\mathcal{D}$-module $\mathcal{T}_\chi$ of weight 0. We then have, setting $\beta_o = \alpha_o - 1 \in (-1, 0)$,

$$
(P_t \phi_{t, \lambda}(MC_\chi \mathcal{T}), z^2 \partial_z) 
\simeq (P_t \psi'_{t, \lambda}(F(MC_\chi \mathcal{T})), z^2 \partial_z - \beta z) \quad (\beta \in (-1, 0)) 
\simeq (P_t \psi'_{t, \lambda}(F_{\mathcal{T}} \otimes F_{\mathcal{F}_\chi}), z^2 \partial_z - \beta z) 
\simeq (P_t \phi_{t, \lambda}(\lambda_o, \mathcal{F}), z^2 \partial_z - (\beta - \beta_o)z) \otimes (\psi'_{t, \lambda}(\mathcal{F}_\chi), z^2 \partial_z - \beta_o z) 
\simeq (P_t \phi_{t, \lambda}(\mathcal{F}_\chi), z^2 \partial_z (-z)) \otimes (\phi_{t, \lambda}(\mathcal{F}_\chi), z^2 \partial_z)
\simeq (P_t \phi_{t, \lambda}(\mathcal{F}_\chi), z^2 \partial_z (-z)),
$$

where $(-z)$ means that we add $-z$ if $\beta \in (\beta_o, 0]$, that is, going back to the notation $\alpha'$, if $\alpha' \in (\alpha_o, 1]$. The $\mathbb{C}[z]$-module part of each side is $R_F P_t \phi_{t, \lambda}(MC_\chi M)$ (resp. $R_F P_t \phi_{t, \lambda}(\lambda_o, M)$) and we recover $F^p P_t \phi_{t, \lambda}(MC_\chi M)$ (resp. $F^p P_t \phi_{t, \lambda}(\lambda_o, M)$) by considering $\text{Ker}(z^2 \partial_z + Pz)$. In such a way we obtain $3.1.2(2)'$ at $x_i = 0$.

A similar formula applies at every singularity $x_i$ of $M$ after a twist by $e^{x_i/\tau'z}$ and gives $3.1.2(2)'$ at any $x_i$.

Acknowledgements. We thank Nicolas Martin for pointing out the mistake in the statement of Theorem 3.2.3. We thank the referee for his/her accurate comments.

References

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