SOME PROPERTIES AND APPLICATIONS OF
BRIESKORN LATTICES

by

Claude Sabbah

Abstract. After reviewing the main properties of the Brieskorn lattice in the framework of tame regular functions on smooth affine complex varieties, we prove a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

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1. Introduction

The Brieskorn lattice, introduced by Brieskorn in [Bri70] in order to provide an algebraic computation of the Milnor monodromy of a germ of complex hypersurface with an isolated singularity, has also proved central in the Hodge theory for vanishing cycles of such a singularity, as emphasized by Pham [Pha80, Pha83]. Hodge theory for vanishing cycles, as developed by Steenbrink [Ste76, Ste77, SS85] and Varchenko [Var82], makes it an analogue of the Hodge filtration in this context, and fundamental results have been obtained by M. Saito [Sai89] in order to characterize it among other lattices in the Gauss-Manin system of an isolated singularity of complex hypersurface. As such, it leads to the definition of a period mapping, as introduced and studied with much detail by K. Saito for some singularities [Sai83]. It is also a basic constituent of the period mapping restricted to the $\mu$-constant stratum [Sai91], where a natural Torelli problem occurs (see [Sai91], [Her99]).

2010 Mathematics Subject Classification. 14F40, 32S35, 32S40.
Key words and phrases. Brieskorn lattice, irregular Hodge filtration, irregular Hodge numbers, tame function.
For a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity, denoting by $t$ the coordinate on the target space $\mathbb{C}$, the space

\[(1.1) \quad \Omega^{n+1}_{\mathbb{C}^{n+1}, 0}/df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1}, 0}\]

is naturally endowed with a $\mathbb{C}(t)$-module structure (where $t$ acts as the multiplication by $f$), and the Brieskorn lattice is the $\mathbb{C}(t)$-module (see [Bri70] p. 125)

\[(1.2) \quad ''H^p_{f,0} = \left( \Omega^{n+1}_{\mathbb{C}^{n+1}, 0}/df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1}, 0} \right)/\mathbb{C}(t)\text{-torsion}.\]

Brieskorn shows that (1.2) is free of finite rank equal to the Milnor number $\mu(f, 0)$, and Sebastiani [Seb70] shows the torsion freeness of (1.1), which can thus also serve as an expression for ''$H^p_{f,0}$. It is also endowed with a meromorphic connection $\nabla$ having a pole of order at most two at $t = 0$, and the $\mathbb{C}(t)$-vector space with connection generated by ''$H^p_{f,0}$ is isomorphic to the Gauss-Manin connection, which has a regular singularity there. ''$H^p_{f,0}$ is thus a $\mathbb{C}(t)$-lattice of this $\mathbb{C}(t)$-vector space. While the action of $\nabla_{\partial_t}$, simply written as $\partial_t$, introduces a pole, there is a well-defined action of its inverse $\partial_t^{-1}$ that makes ''$H^p_{f,0}$ a module over the ring of $\mathbb{C}(\{\partial_t^{-1}\})$ of 1-Gevrey series (i.e., formal power series $\sum_{n \geq 0} a_n \partial_t^{-n}$ such that the series $\sum_n a_n u^n/n!$ converges). It happens to be also free of rank $\mu$ over this ring ([Mal74, Mal75]). The relation between the rings $\mathbb{C}(t)$ and $\mathbb{C}(\{\partial_t^{-1}\})$ is called microlocalization. In the global case below, we will use instead the Laplace transformation. The mathematical richness of this object leads to various generalizations.

For non-isolated hypersurface singularities, the objects with definition as in (1.2) (but in various degrees) have been introduced by Hamm in his Habilitationsschrift (see [Ham75, §II.5]), who proved that they are $\mathbb{C}(t)$-free of finite rank, but do not coincide with (1.1) in general. A natural $\mathbb{C}(\{\partial_t^{-1}\})$-structure still exists on (1.1), and Barlet and Saito [BS07] have shown that the $\mathbb{C}(t)$-torsion and the $\mathbb{C}(\{\partial_t^{-1}\})$-torsion coincide, so that ''$H^p_{f,0}$ remains $\mathbb{C}(\{\partial_t^{-1}\})$-free of finite rank.

The Brieskorn lattice has also a global variant. On the one hand, the Brieskorn lattice for tame regular functions on smooth affine complex varieties (see Section 2) is a direct analogue of the case of an isolated singularity, but the double pole of the action of $t$ with respect to the variable $\partial_t$ cannot in general be reduced to a simple one by a meromorphic (even formal) gauge transformation i.e., the Gauss-Manin system with respect to the variable $\partial_t^{-1}$ has in general an irregular singularity. The properties of the Brieskorn module for regular functions on affine manifolds which are not tame have been considered by Dimca and M. Saito [DS01].

On the other hand, given a projective morphism $f : X \to \mathbb{A}^1$ on a smooth quasi-projective variety $X$, the Brieskorn modules, defined as the hypercohomology $\mathbb{C}[\partial_t^{-1}]$-modules of the twisted de Rham complex $(\Omega^n_X[\partial_t^{-1}], d - \partial_t^{-1} df)$, have been shown to be $\mathbb{C}[\partial_t^{-1}]$-free ([Barannikov-Kontsevich, see Sab99b]), and a similar result holds when one replaces $\Omega^n_X$ with $\Omega^n_X(\log D)$ for some divisor with normal crossings. More generally, one can adapt the definition of the Brieskorn modules for the twisted de Rham complex attached to a mixed Hodge module, and the $\mathbb{C}[\partial_t^{-1}]$-freeness still holds, so that they can be called Brieskorn lattices (see loc. cit.). This enables one to use the push-forward operation by the map $f$ and reduce the study to that of Brieskorn lattices attached to mixed Hodge modules on the affine line, as for example
the mixed Hodge modules that the Gauss-Manin systems of $f$ underlie. In such a way, the Brieskorn lattice has a purely Hodge-theoretic definition, which does not refer to the underlying geometry, and can thus be attached, for example, to any polarizable variation of Hodge structure on a punctured affine line (see [Sab08, §1.d]).

The Brieskorn lattice of tame functions is of particular interest and has been considered in [Sab06] for example. The Brieskorn lattice for families of such functions, considered in [DS03], has been investigated with much care for families of Laurent polynomials in relation with mirror symmetry by Reichelt and Reichelt-Severinde [RS15, Rei14, Rei15, RS17].

Lastly, in the global setting as above, the pole of order two of the action of $t$ with respect to the variable $\partial t^{-1}$ produces in general a truly irregular singularity, and the Brieskorn lattice is an essential tool to produce the irregular Hodge filtration attached to such a singularity (see [SY15, Sab17]).

The contents of this article is as follows. In Section 2, we review known results on the Brieskorn lattice for a tame function. We show in Section 3 how these results enables one to obtain a simple proof of a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

Acknowledgements. I thank the referee for his/her careful reading of the manuscript and interesting suggestions and Claus Hertling for pointing out Lemma 2.4.

2. The Brieskorn lattice of a tame function

In this section, we review the main properties of the Brieskorn lattice attached to a tame function on an affine manifold, following [Sab99a, Sab06, DS03].

Let $U$ be a smooth complex affine variety of dimension $n$ and let $f \in \mathcal{O}(U)$ be a regular function on $U$. There are various notions of tameness for such a function, which are not known to be equivalent, but for what follows they have the same consequences. One of the definitions, given by Katz in [Kat90, Th.14.13.3], is that the cone of $f! \mathbb{C}U \rightarrow \mathbb{R}f_* \mathbb{C}U$ should have constant cohomology on $\mathbb{A}^1$. We will use the notion of a weakly tame function, as defined in [NS99], that is, either cohomologically tame or M-tame.

We assume that $f$ is weakly tame. Let $\theta$ be a new variable. The Brieskorn lattice attached to $f$ is the $\mathbb{C}[\theta]$-module

$$G_0 := \Omega^n(U)[\theta]/(\theta d - df)\Omega^{n-1}(U)[\theta].$$

An expression like (1.1) also exists if $U$ is the affine space $\mathbb{A}^{n+1}$, but the above one is valid for any smooth affine variety $U$. The variable $\theta$ is for $\partial t^{-1}$. We already notice that

$$(2.1)\quad G_0/\theta G_0 \simeq \Omega^n(U)/df \wedge \Omega^{n-1}(U)$$

has dimension equal to the sum $\mu = \mu(f)$ of the Milnor numbers of $f$ at all its critical points in $U$. The following properties are known in this setting.

1. The algebraic Gauss-Manin systems $\mathcal{H}^k f_* \mathcal{O}_U$ are isomorphic to powers of the $\mathbb{C}[t](\partial t)$-module $(\mathbb{C}[t], \partial t)$, except for $k = 0$, so their localized Laplace transforms vanish except that for $k = 0$. If we regard the Laplace transform of $\mathcal{H}^0 f_* \mathcal{O}_U$ as a
\[ G = \Omega^n(U)[\tau, \tau^{-1}]/(d - \tau df)\Omega^{n-1}(U)[\tau, \tau^{-1}] \]

(2) Setting \( \theta = \tau^{-1} \), we write

\[ G = \Omega^n(U)[\theta, \theta^{-1}]/(\theta d - df)\Omega^{n-1}(U)[\theta, \theta^{-1}] \]

and there is therefore a natural morphism \( G_0 \to G \). This morphism is \textit{injective}, so that \( G_0 \) is a \textit{free} \( \mathbb{C}[\theta] \)-module of rank \( \mu \) such that \( \mathbb{C}[\theta, \theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0 = G \), i.e., \( G_0 \) is a \( \mathbb{C}[\theta] \)-lattice of \( G \), on which the restriction of the Gauss-Manin connection has a pole of order at most two. Moreover, the action of \( \partial_\theta \) on the class \( [\omega] \) of \( \omega \in \Omega^n(U) \) in \( G_0 \) is given by

\[ \theta^2 \partial_\theta [\omega] = [f \omega] \]

and the action of \( \theta^2 \partial_\theta \) on a polynomial \( \sum_{k \geq 0} [\omega_k \theta^k] \) is obtained by the usual formulas.

(3) Let \( V_\bullet G \) be the (increasing) \( V \)-filtration of \( G \) with respect to the function \( \tau \) (recall that \( G \) has a regular singularity at \( \tau = 0 \), while that at infinity is usually irregular). It is a filtration by free \( \mathbb{C}[\tau] \)-modules of rank \( \mu \) indexed by \( \mathbb{Q} \). The jumping indices of the induced filtration \( V_\bullet(G_0/\theta G_0) \), together with their multiplicities (the dimension of \( \text{gr}_V^p(G_0/\theta G_0) \)) form the \textit{spectrum} of \( f \) at \( \infty \). The jumping indices are contained in the interval \([0, n] \cap \mathbb{Q}\) and the spectrum is symmetric with respect to \( n/2 \).

(4) On the other hand, for \( \alpha \in [0, 1) \cap \mathbb{Q} \), the vector space \( \text{gr}_V^\alpha G \) is endowed with the nilpotent endomorphism \( N \) induced by the action of \((\tau \partial_\tau + \alpha)\) and with the increasing filtration \( \text{gr}_V^\alpha G \) naturally induced by the filtration \( G_p = \theta^{-p}G_0 \), i.e.,

\[ G_p \text{gr}_V^\alpha G = (G_p \cap V_\alpha G)/(G_p \cap V_{< \alpha} G) \]

where the intersections are taken in \( G \). As a consequence, we have isomorphisms \((p \in \mathbb{Z}, \alpha \in [0, 1))\)

\[ \text{gr}_p \text{gr}_V^\alpha G \xrightarrow{\theta^p} \text{gr}_p \text{gr}_V^{\alpha+p} (G_0/\theta G_0) \]

(5) The \( \mathbb{C} \)-vector space \( H_{\neq 1} := \bigoplus_{\alpha \in (0, 1) \cap \mathbb{Q}} \text{gr}_V^\alpha G \), resp. \( H_1 := \text{gr}_V^0 G \), endowed with

- the filtration

\[ F^p H_{\neq 1} := \bigoplus_{\alpha \in (0, 1) \cap \mathbb{Q}} G_{n-1-p} \text{gr}_V^\alpha G \quad \text{resp.} \quad F^p H_1 = G_{n-p} \text{gr}_V^0 G \]

- and the weight filtration \( W_\bullet = M(N)[n-1] \) (resp. \( M(N)[n] \)), i.e., the monodromy filtration of \( N \) centered at \( n-1 \) (resp. \( n \)),

is part of a mixed Hodge structure. In particular, \( N \) strictly shifts by one the filtration \( \text{gr}_V^\alpha G \) and acts on the graded space \( \text{gr}_V^\alpha G \) as the degree-one morphism induced by \( -\tau \partial_\tau \). We therefore have a commutative diagram, for any \( \alpha \in [0, 1) \) and \( p \in \mathbb{Z} \), (see \textbf{Var81} and \textbf{SS85} §7 in the singularity case):

\[ \begin{array}{ccc}
\text{gr}_V^\alpha G & \xrightarrow{\theta^p} & \text{gr}_V^{\alpha+p} (\Omega^n(U)/df \wedge \Omega^{n-1}(U)) \\
\text{gr}_V^{\alpha+1} G & \xrightarrow{\theta^p+1} & \text{gr}_V^{\alpha+p+1} (\Omega^n(U)/df \wedge \Omega^{n-1}(U)) \\
\end{array} \]

(2.2)

by using the relation \( -\tau \partial_\tau = \theta \partial_\theta \).
To see this, write the commutative diagram

\[
\begin{array}{c}
\text{gr}_p G \
\downarrow_{\theta \partial_b - \alpha} \quad \downarrow_{\theta \partial_b - (\alpha + p)} \quad \downarrow_{\theta} \\
\text{gr}_{p+1} G \end{array}
\]

\[
\begin{array}{c}
\text{gr}_p V G \
\downarrow_{\theta} \quad \downarrow_{\theta} \\
\text{gr}_{p+1} V G
\end{array}
\]

and use that in the vertical morphisms, the constant part \(\alpha\) or \(\alpha + p\) induces the morphism 0.

(6) Recall that a mixed Hodge structure \((H_\mathbb{Q}, F^* H_\mathbb{C}, W_\bullet H_\mathbb{Q})\) is said to be of Hodge-Tate type if

\[
\begin{align*}
&\text{(a) the filtration } W_\bullet \text{ has only even jumping indices} \\
&\text{(b) and } W_{2\mathbb{Q}} H_\mathbb{C} \text{ is opposite to } F^* H_\mathbb{C}.
\end{align*}
\]

The description of the mixed Hodge structure given in [5] implies the following criterion. We will set \(\nu = n - 1\) when considering \(H_{\neq 1}\) and \(\nu = n\) when considering \(H_1\). We will then denote by \(H\) either \(H_{\neq 1}\) or \(H_1\).

**Corollary 2.3.** The mixed Hodge structure that the triple \((H, F^* H, W_\bullet H)\) underlies is of Hodge-Tate type if and only if, for any integer \(k\) such that \(0 \leq k \leq [\nu/2]\), the \((\nu - 2k)\)th power of \(N\) induces an isomorphism

\[
[N]^{\nu-2k} : \text{gr}^G_{\nu-k} H \sim \text{gr}_k G H.
\]

**Proof.** We define the filtration \(W'_\bullet H\) indexed by \(2\mathbb{Z}\) by the formula \(W_{2k}' H = G_{\nu-k} H\), so that \(G_k H = W_{2(\nu-k)} H\). If we set \(\ell = \nu - 2k\) for \(0 \leq k \leq [\nu/2]\), we have \(0 \leq \ell \leq \nu\) and the isomorphism in the corollary is written

\[
[N]^\ell : \text{gr}_{\nu+\ell} W' H \sim \text{gr}_{\nu-\ell} W' H.
\]

We can conclude that \(W'_\bullet H = W_\bullet H\) if we know that \(N^{\nu+1} = 0\), that is, \(\text{gr}^{G}_{\nu+1} H = 0\). This is a consequence of the positivity of the spectrum [Sab06, Cor. 13.2], which says that, if \(\alpha \in [0, 1]\), we have \(\text{gr}^G_{\nu} \text{gr}^{V}_{\nu} G = 0\) for \(k \notin [0, \nu] \cap \mathbb{N}\).

The following lemma was pointed out to me by Claus Hertling.

**Lemma 2.4.** A mixed Hodge structure \((H_\mathbb{Q}, F^* H_\mathbb{C}, W_\bullet H_\mathbb{Q})\) is Hodge-Tate if and only if we have, for all \(p \in \frac{1}{2} \mathbb{Z}\),

\[
\dim \text{gr}_p^F H_\mathbb{C} = \dim \text{gr}_{2p}^W H_\mathbb{Q}.
\]

**Proof.** Indeed, one direction is clear. Conversely, if the equality of dimensions holds, then [6a] holds since \(F^* H\) has only integral jumps; moreover, up to a Tate twist, one can assume that \(W_{< \alpha} H = 0\), so \(\text{gr}^{F}_{\ell} H = 0\) for \(k < 0\). It is enough to prove that \(\text{gr}^{F}_{\ell} \text{gr}^{W}_{2\ell} H = 0\) for all \(p \neq \ell\). We prove this by induction on \(\ell\). If \(\ell = 0\), the result follows from the property that \(F^p H = 0\) for \(p < 0\) and Hodge symmetry. Assume the result up to \(\ell\). For \(j \leq \ell\) we thus have \(\dim \text{gr}^{F}_{j} \text{gr}^{W}_{2j} H = \dim \text{gr}^{W}_{2j} H = \dim \text{gr}^{F}_{j} H\) (the latter equality by the assumption), and therefore \(\text{gr}^{W}_{2i} \text{gr}^{F}_{i} H = 0\) for \(i \neq j\). In particular, taking \(i = \ell + 1\), we have \(\text{gr}^{F}_{\ell} \text{gr}^{W}_{2(\ell + 1)} H = 0\) for all \(p \leq \ell\). By Hodge symmetry, we obtain \(\text{gr}^{F}_{p} \text{gr}^{W}_{2(\ell + 1)} H = 0\) for all \(p \neq \ell + 1\), as wanted.
(7) We now consider the case where $U = (\mathbb{C}^*)^n$, endowed with coordinates $x = (x_1, \ldots, x_n)$. Let $f \in \mathbb{C}[x, x^{-1}]$ be a Laurent polynomial in $n$ variables, with Newton polyhedron $\Delta(f)$. We assume that $f$ is nondegenerate with respect to its Newton polyhedron and convenient (see [Kou76]). In particular, 0 belongs to the interior of its Newton polyhedron. It is known that such a function is $M$-tame.

For any face $\sigma$ of dimension $n-1$ of the boundary $\partial \Delta(f)$, we denote by $L_\sigma$ the linear form with coefficients in $\mathbb{Q}$ such that $L_\sigma \equiv 1$ on $\sigma$. For $g \in \mathbb{C}[x, x^{-1}]$, we set $\deg_\sigma(g) = \max_{m \in L_\sigma(m)}$, where the max is taken on the exponents of monomials $x^m$ appearing in $g$, and $\deg_\sigma(g) = \max_{m \in L_\sigma(m)}$. We denote the volume form $dx_1/x_1 \wedge \cdots \wedge dx_n/x_n$ by $\omega$, giving rise to an identification $\mathbb{C}[x, x^{-1}] \overset{\sim}{\longrightarrow} \Omega^n(U)$ and $\mathbb{C}[x, x^{-1}]/(\partial f) \overset{\sim}{\longrightarrow} G_0/\theta G_0$ (see [2.1]).

The Newton increasing filtration $N_\beta \Omega^n(U)$ indexed by $\beta$ is defined by

\[ N_\beta \Omega^n(U) := \{ g \omega \in \Omega^n(U) \mid \deg_\beta(g) \leq \beta \}. \]

We have $N_\beta \Omega^n(U) = 0$ for $\beta < 0$ and $N_0 \Omega^n(U) = \mathbb{C} \cdot \omega$. We can extend this filtration to $\Omega^n(U)[\theta]$ by setting

\[ N_\beta \Omega^n(U)[\theta] := N_\beta \Omega^n(U) + \theta N_{\beta-1} \Omega^n(U) + \cdots + \theta^k N_{\beta-k} \Omega^n(U) + \cdots \]

and then naturally induce this filtration on $G_0$, to obtain a filtration $N_\beta G_0$ and then on $G_0/\theta G_0$. We have

\[ N_\beta G_0 = N_\beta \Omega^n(U)[\theta] \] and

\[ N_\beta(G_0/\theta G_0) = N_\beta (G_0/\theta G_0). \]

Corollary 2.3 now reads, according to (2.2) and by using the above identification through multiplication by $\omega$:

**Corollary 2.6.** The mixed Hodge structure that the triple $(H, F^* H, W_* H)$ underlies is of Hodge-Tate type if and only if, for any integer $k$ such that $0 \leq k \leq [\nu/2]$ ($\nu = n-1$, resp. $n$), we have isomorphisms

\[ \text{gr}_{\alpha+k}^N (\mathbb{C}[x, x^{-1}]/(\partial f)) \overset{\sim}{\longrightarrow} \text{gr}_{\alpha+n-1-k}^N (\mathbb{C}[x, x^{-1}]/(\partial f)) \quad \forall \alpha \in (0, 1), \]

resp.

\[ \text{gr}_k^N (\mathbb{C}[x, x^{-1}]/(\partial f)) \overset{\sim}{\longrightarrow} \text{gr}_{n-k}^N (\mathbb{C}[x, x^{-1}]/(\partial f)). \]

3. On a conjecture of Katzarkov-Kontsevich-Pantev

In this section we use the algebraic Brieskorn lattice of a convenient nondegenerate Laurent polynomial to solve the toric case of the part “$f^{p,q} = h^{p,q}$” of Conjecture 3.6 in [KKP17] (the other equality “$h^{p,q} = i^{p,q}$” is obviously not true by simply considering the case of the standard Laurent polynomial mirror to the projective space $\mathbb{P}^n$, see also another counter-example in [LP18]). We refer to [LP18, Har17, Sha17] for further discussion and positive results on this conjecture.

3.a. The Brieskorn lattice and the conjecture of Katzarkov-Kontsevich-Pantev

Given a smooth quasi-projective variety $U$ and a morphism $f : U \rightarrow \mathbb{A}^1$, every twisted de Rham cohomology $H^{2k}_{\text{DR}}(U, d + df)$, i.e., the $k$th hypercohomology of
the twisted de Rham complex \((Ω^* U, d + df)\), is endowed with a decreasing filtration \(F^n_{\text{DR}}(U, d + df)\) indexed by \(\mathbb{Q}\) (see [Yu14]). For \(\alpha \in [0, 1]\), the filtration indexed by \(\mathbb{Z}\) defined by \(F^n_{\text{DR}}\) can also be computed in terms of the Kontsevich complex \(\Omega^* (\alpha)\) together with its stupid filtration (see [ESY17, Cor. 1.4.5]). The irregular Hodge numbers \(h_{\alpha}^{p,q}(f)\) are defined as

\[
h_{\alpha}^{p,q}(f) := \dim \text{gr} F_{\alpha}^{-p} H^{p+q}_{\text{DR}}(U,d + df).
\]

It is well-known that \(\dim H^{p,q}_{\text{DR}}(U,d + df) = \dim H^k(U, f^{-1}(t))\) for \(|t| > 0\). This space is endowed with a monodromy operator (around \(t = \infty\)), and we will consider the case where this monodromy operator is unipotent. In such a case, the filtration \(F^n_{\text{DR}}(U,d + df)\) is known to jump at integers only, and in (3.1) only \(\alpha = 0\) occurs. We then simply denote this number by \(h^{p,q}(f)\), so that, in such a case,

\[
h^{p,q}(f) := \dim \text{gr} F^n_{\text{DR}} H^{p+q}_{\text{DR}}(U,d + df).
\]

Let \(W_\alpha\) be the monodromy filtration on \(H^k(U, f^{-1}(t))\) centered at \(k\). The conjecture of [KKP17] that we consider is the possible equality (see [LP18, Har17, Sha17])

\[
h^{p,q}(f) = \dim \text{gr}^W_{\alpha} H^{p+q}(U, f^{-1}(t)).
\]

If moreover \(U\) is affine and \(f\) is weakly tame, so that \(H^{p,q}_{\text{DR}}(U,d + df) = 0\) unless \(p + q = n\), [SY15, Cor. 8.19] gives, using the notation of Section 2(3)

\[
h^{p,q}(f) = \begin{cases} \dim \text{gr}^V_{n-p}(G_0(f)/\theta G_0(f)) = \dim \text{gr}^V_{p} \text{gr}^V_0 G & \text{if } p + q = n, \\ 0 & \text{if } p + q \neq n, \end{cases}
\]

and this is the number denoted by \(f^{p,q}\) in [KKP17]. In such a case, we have \(H = H_1\) in the notation of Section 2(3).

The following criterion has been obtained, with a different approach of the irregular Hodge filtration, by Y. Shamoto.

**Proposition 3.3 (Sha17).** Assume \(U\) affine and \(f\) weakly tame with unipotent monodromy operator at infinity. Then (3.2) holds true if and only if the mixed Hodge structure of Section 2(3) on \(H = H_1\) is of Hodge-Tate type.

**Proof.** According to Lemma 2(4) proving the result amounts to identifying the space \(\text{gr}^V_0 G\) endowed with its nilpotent operator \(N\) with the space \(H^n(U, f^{-1}(t))\) endowed with the nilpotent part of the (unipotent) monodromy (up to a nonzero constant). Choosing an extension \(F : X \to \mathbb{P}^1\) of \(f\) as a projective morphism on a smooth variety \(X\) such that \(X \smallsetminus U\) is a divisor, and setting \(F = \mathcal{R} j_* C_{U} (j : U \to X)\), we identify the dimension of \(H^k(U, f^{-1}(t))\) with that of the \(k\)th-hypercohomology on \(X\) of the Beilinson extension \(\Xi_F C\). Then the desired identification is given by [Sab97, Cor. 1.13]. \(\square\)

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\(^{[1]}\) The definition of \(\delta_{i}\) in [SY15] should read \(\dim \text{gr}^V_0 (G_0(f)/\theta G_0(f))\).
3.b. The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, first part

As usual in toric geometry, we denote by $M$ the lattice $\mathbb{Z}^n$ in $\mathbb{C}^n$ and by $N$ its dual lattice. We fix a reflexive simplicial polyhedron $\Delta \subset \mathbb{R} \otimes M$ with vertices in $M$ and having 0 in its interior (it is then the unique integral point in its interior), see \cite{Bat94} §4.1. We denote by $\Delta^*$ the dual polyhedron with vertices in $N$, which is also simplicial reflexive and has 0 in its only interior point, and by $\Sigma \subset N$ the fan dual to $\Delta$, which is also the cone on $\Delta^*$ with apex 0. We assume that $\Sigma$ is the fan of nonsingular toric variety $X$ of dimension $n$, that is, each set of vertices of the same $(n-1)$-dimensional face of $\partial \Delta^*$ is a $\mathbb{Z}$-basis of $N$. We know that

- $X$ is Fano (\cite{Bat94} Th.4.1.9),
- the Chow ring $A^*(X) \simeq H^{2*}(X, \mathbb{Z})$ is generated by the divisor classes $D_v$ corresponding to vertices $v \in V(\Delta^*)$ of $\Delta^*$, i.e., primitive elements on the rays of $\Sigma$ (see \cite{Ful93} p.101),
- we have $c_1(TX) = c_1(K_X^\vee) = \sum_{v \in V(\Delta^*)} D_v$ in $H^{2*}(X, \mathbb{Z})$ (see \cite{Ful93} p.109), which satisfies Hard Lefschetz on $H^{2*}(X, \mathbb{Q})$, by ampleness of $K_X^\vee$.

Let us fix coordinates $x = (x_1, \ldots, x_n)$ such that $\mathbb{Q}[N] = \mathbb{Q}[x, x^{-1}]$. We use the notation of Section 2(7). Due to the reflexivity of $\Delta^*$, $L_y$ has coefficients in $\mathbb{Z}$ (it corresponds to a vertex of $\Delta$). For $g \in \mathbb{C}[x, x^{-1}]$, the $\sigma$-degree $\deg_\sigma(g) = \max_m L_\sigma(m)$ and the $\Delta^*$-degree $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$ are thus nonnegative integers.

**Proposition 3.4.** The case “$p^\sigma = h^{p,q}$” of \cite{KKP17} Conj.3.6 holds true if $f$ is the Laurent polynomial

$$f(x) = \sum_{v \in V(\Delta^*)} x^v \in \mathbb{Q}[x, x^{-1}].$$

The idea of the proof is to notice that the property for the second morphism in Corollary 2.6 to be an isomorphism is exactly the property that $c_1(TX)$ satisfies the Hard Lefschetz property, and thus to identify its source and target as the cohomology of $X$ in suitable degree.

**Lemma 3.5.** For $\Delta$ as above, any Laurent polynomial

$$f_\alpha(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \alpha = (a_v \in V(\Delta^*)$$

is convenient and non-degenerate in the sense of Kouchnirenko.

**Proof.** The Newton polyhedron of $f_\alpha$ is equal to $\Delta^*$, and 0 belongs to its interior. In order to prove the non-degeneracy, we note that the vertices of any $(n-1)$-dimensional face $\sigma$ of $\partial \Delta^*$ form a $\mathbb{Z}$-basis. It follows that, in suitable toric coordinates $y_1, \ldots, y_n$, the restriction $f_{\alpha|\sigma}$ can be written as $y_1 + \cdots + y_n$, and the non-degeneracy is then obvious.

**Proof of Proposition 3.4.** Note that $\deg_{\Delta^*}(f) = 1$, as well as $\deg_{\Delta^*}(x_i \partial f/\partial x_i) = 1$. The Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$ is endowed with the Newton filtration $N_\omega$ induced by the $\Delta^*$-degree $\deg_{\Delta^*}$, and corresponds to $N_\omega(G_0/\theta G_0)$ by multiplication by $\omega$. In
the present setting, [BCS05, Th.1.1] identifies the graded ring \( A^*(X)_\mathbb{Q} \) with the graded ring

\[ \text{gr}^N_*(\mathbb{Q}[x,x^{-1}]/(\partial f)). \]

By applying Hard Lefschetz to \( c_1(TX) \), we deduce that, for every \( k \in \mathbb{N} \) such that \( 0 \leq k \leq \lfloor n/2 \rfloor \), multiplication by the \((n-2k)\)th power of the \( N \)-class \([f]\) of \( f \) induces an isomorphism

\[ [f]^{n-2k} : \text{gr}^N_N(\mathbb{Q}[x,x^{-1}]/(\partial f)) \xrightarrow{\sim} \text{gr}^{N-N}_{n-k}(\mathbb{Q}[x,x^{-1}]/(\partial f)). \]

By Corollary 2.6 for \( H = H_1 \), we deduce the assertion of the proposition from Proposition 3.3.

\[ \square \]

3.c. The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, second part

We now prove the main result of this note.

**Theorem 3.6.** The case “\( f^{p,q} = h^{p,q} \)” of [KKP17, Conj. 3.6] holds true for any Laurent polynomial

\[ f_a(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x,x^{-1}], \quad a = (a_v \in V) \in (\mathbb{C}^*)^{V(\Delta^*)}. \]

**Remark 3.7.** The case where \( n = 3 \) was already proved differently by Y. Shamoto [Sha17, §4.2].

**Proof.** Let us set \( H(f_a) = H_1(f_a) = \text{gr}^Y_0 G(f_a) \), where \( G(f_a) \) is the localized Laplace transform of the Gauss-Manin system for \( f_a \) as in Section 2.2. By Lemma 3.5 we can apply the results of Section 2 to \( f_a \) for any \( a \in (\mathbb{C}^*)^{V(\Delta^*)} \). We will prove that, for fixed \( p \), both terms \( \dim \text{gr}^p_{1-p} H(f_a) \) and \( \dim \text{gr}^p_{2p} H(f_a) \) in Lemma 2.4 are independent of \( a \). Since they are equal if \( a = (1, \ldots, 1) \), after Proposition 3.4 they are equal for any \( a \in (\mathbb{C}^*)^{V(\Delta^*)} \), as wanted.

1. For the first term, we will use [NS99]. We have denoted there \( \dim \text{gr}^p_{\mu N} H(f_a) \) by \( \nu \mu N(f_a) \) and, since \( \text{gr}^Y_0 G = 0 \) for \( \alpha \notin \mathbb{Z} \), it is also equal to the number denoted there by \( \Sigma_{p-1}(f_a) \). By the theorem in [NS99] and Lemma 3.5, \( \Sigma_{p-1}(f_a) \) depends semi-continuously on \( a \). On the other hand, according to [Kou76], \( \dim H(f_a) \) is independent of \( a \) and is computed only in terms of \( \Delta \). Since \( \dim H(f_a) = \sum_{p} \Sigma_{p-1}(f_a) \), each term in this sum is also constant with respect to \( a \).

2. We will prove the local constancy of \( \dim \text{gr}^p_{\mu N} H(f_a) \) near any \( a_0 \in (\mathbb{C}^*)^{V(\Delta^*)} \). As noticed in [DS03, §4], we can apply the results of Section 2 of loc. cit. to \( f_a \). We fix a Stein open set \( \mathcal{B}^0 \) adapted to \( f_a \) as in [DS03, §2a], and fix a neighbourhood \( X \) of \( a_0 \) so that it is also adapted to any \( f_a \) for \( a \) in this neighbourhood. By construction, all the critical points of \( f_{a_0} \) are contained in the interior of \( \mathcal{B}^0 \) if \( X \) is chosen small enough, and since \( \mu(f_{a_0}) \) is constant, the same property holds for \( a \in X \). By using successively Theorem 2.9, Remark 2.11 and Proposition 1.20(1) in [DS03], we deduce that, when \( a \) varies in \( X \), the localized partial Laplace transformed Gauss-Manin systems \( G(f_a) \) form an \( \mathcal{O}_X[^{\tau}[\tau^{-}] \)-free module with integrable connection and regular singularity along \( \tau = 0 \), which is compatible with base change with respect to \( X \).
As a consequence, the monodromy of each $G(f_a)$ around $\tau = 0$ is constant, and the assertion follows.

**Remark 3.8 (suggested by the referee).** If we relax the condition in Section 3.b that the toric Fano variety $X$ is **nonsingular**, then we have to consider the orbifold Chow ring of $X$ as in [BCS05], or the Chen-Ruan orbifold cohomology of $X$. For the cohomology of the untwisted sector (i.e., the usual cohomology), the Hard Lefschetz theorem is still valid (see [Ste77]) and Proposition 3.4 still holds, i.e., (3.2) holds for $f$. Moreover, Part (2) of the proof of Theorem 3.6 also extends to this setting. However, the semicontinuity result of [NS99] used in Part (1) of the proof is not enough to imply the constancy (with respect to $a$) of $\nu_p(f_a)$.

On the other hand, one can also consider the various $h^{p,q}_\alpha(f)$ for $\alpha \in (0,1)\cap \mathbb{Q}$ and, correspondingly, the twisted sectors of the orbifold $X$. In such a case, Hard Lefschetz for $f$ may already give trouble (see [Fer06]).

**References**


SOME PROPERTIES AND APPLICATIONS OF BRIESKORN LATTICES


