A COUNTER-EXAMPLE TO LEVELT-TURRITTIN
IN DIMENSION TWO

by

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Let \( \mathbb{P}^1 \) be the complex projective line covered with two charts \( \mathbb{A}^1_0 \) (coordinate \( x \)) and \( \mathbb{A}^1_\infty \) (coordinate \( y \)), so that \( y = 1/x \) on \( \mathbb{A}^1_0 \cap \mathbb{A}^1_\infty \). Let \( \mathbb{A}^1 \) be the affine line with coordinate \( z \).

We denote by \( S_\pm \) the hyperbolas with equation \( xz = \pm 1 \) in \( \mathbb{A}^1_0 \times \mathbb{A}^1 \), and by \( \overline{S}_\pm \) their closure in \( \mathbb{P}^1 \times \mathbb{A}^1 \). The intersection of \( \overline{S}_\pm \) with \( \mathbb{A}^1_\infty \times \mathbb{A}^1 \) is nothing but the two lines \( y = \pm z \). We set \( S = S_+ \cup S_- \).

**Lemma.** The fundamental group \( \pi_1(\mathbb{P}^1 \times \mathbb{A}^1 \setminus S) \) is generated by \( \mathbb{Z} \oplus \mathbb{Z} \) (loops around \( S_+ \) and \( S_- \)) modulo the relation that the sum of these loops is zero.

**Proof.** Direct application of van Kampen applied to \( \mathbb{A}^1_0 \times \mathbb{A}^1 \setminus S \) glued with a neighbourhood of \( \{0\} \times \mathbb{A}^1 \).

Let us fix a complex number \( \lambda \neq 0, 1 \), and let \( \overline{\mathcal{F}}_\lambda \) be the rank-one local system on \( \mathbb{P}^1 \times \mathbb{A}^1 \setminus S \) with local monodromy \( \lambda \) around \( \mathcal{S}_+ \), and hence monodromy \( 1/\lambda \) around \( \mathcal{S}_- \). Since \( \lambda \neq 1 \), the nearby cycle sheaf \( \psi_{y-z}{\overline{\mathcal{F}}}_\lambda \) and the vanishing cycle sheaf \( \phi_{y+z}{\overline{\mathcal{F}}}_\lambda \) coincide on \( \mathcal{S}_+ \), and both are equal to the rank-one local system with monodromy \( 1/\lambda \) on \( \mathcal{S}_+ \). Similarly, \( \psi_{y+z}{\overline{\mathcal{F}}}_\lambda \) and \( \phi_{y-z}{\overline{\mathcal{F}}}_\lambda \) coincide on \( \mathcal{S}_- \) and both are equal to the rank-one local system with monodromy \( \lambda \) on \( \mathcal{S}_- \).

We denote by \( \mathcal{L}_\lambda \) the restriction of \( \overline{\mathcal{F}}_\lambda \) to \( \mathbb{A}^1_0 \times \mathbb{A}^1 \setminus S \). A similar assertion holds for the nearby and vanishing cycles along \( \mathcal{S}_\pm \). Let \( js : \mathbb{A}^1_0 \times \mathbb{A}^1 \setminus S \hookrightarrow \mathbb{A}^1_0 \times \mathbb{A}^1 \) denote the inclusion. The minimal extension \( \mathcal{F}_\lambda := js_! \mathcal{L}_\lambda \) is also equal to \( js_! \mathcal{L}_\lambda \) and to \( Rjs_* \mathcal{L}_\lambda \) since \( \lambda \neq 1 \), and is an irreducible perverse sheaf on \( \mathbb{A}^1_0 \times \mathbb{A}^1 \). It corresponds to an irreducible regular holonomic \( \mathbb{C}[x,z](\partial_x, \partial_z) \)-module \( M_\lambda \).

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Let $F^\Lambda_M$ be the partial Laplace transform of $M_\Lambda$ with respect to the variable $x$, which is a $\mathbb{C}[[x,z]](\partial_x,\partial_z)$-module. We have $F^\Lambda_M = M_\Lambda$ as a $\mathbb{C}[z]$-module, and the action of $\xi$ (resp. $\partial_z$) is that of $\partial_x$ (resp. $-x$).

**Lemma.** $F^\Lambda_M$ is a holonomic $\mathbb{C}[[x,z]](\partial_x,\partial_z)$-module with singular set equal to $\{z = 0\} \cup \{\xi = 0\} \cup \{\xi = \infty\}$. It is generically regular along $\xi = 0$. The corresponding local system on $\mathbb{A}^2 \setminus \{z = 0\}$ has rank two. Its monodromy around $\xi = 0$ is unipotent with one Jordan block. The restriction of $F^\Lambda_M(1/z)$ to $\xi = \xi_0 \neq 0$ is irregular with Levelt-Turrittin decomposition

$$
\mathbb{C}(z) \otimes_{\mathbb{C}[z]} F^\Lambda_M(1/z)_{|\xi = \xi_0} = (\hat{\partial}^{\xi_0/z} \otimes \hat{R}_+) \oplus (\hat{\partial}^{-\xi_0/z} \otimes \hat{R}_-)
$$

where $R_+$ and $R_-$ are $\mathbb{C}(z)$-vector spaces with regular connection corresponding to $\phi_{x-z} \mathcal{L}_\Lambda$ and $\phi_{x+z} \mathcal{L}_\Lambda$ respectively.

**Proof.** Analytically away from $z = 0$, $F^\Lambda_M$ is a holonomic $\mathcal{O}_{\mathbb{A}^2 \times \mathbb{C}[[z]]}$-module. It is easy to check, by restricting to $z = z_0 \neq 0$, that it has regular singularity along $\xi = 0$, irregular singularity along $\xi = \infty$, and the corresponding local system on $\xi \neq 0$ has rank two. To compute the monodromy around $\xi = 0$, we can also restrict to $z = z_0$.

We denote by $M_\Lambda^\alpha$ the corresponding $\mathbb{C}[z](\partial_x)$-module. Then, since $M_\Lambda^\alpha$ is irreducible, so is $F^\Lambda_M^\alpha$, and in particular it is a minimal extension at $\xi = 0$.

On the one hand, since the monodromy at infinity of $\mathcal{L}_\Lambda$ is equal to the identity by construction, the monodromy $T_\xi$ of the nearby cycle space $\psi_\xi DR F^\Lambda_M^\alpha$ has only 1 as an eigenvalue. It is therefore unipotent, and written as $\exp(-2\pi i N)$ for some nilpotent operator $N$.

On the other hand, since $F^\Lambda_M^\alpha$ is a minimal extension at $\xi = 0$ and has regular singularity, $DR F^\Lambda_M^\alpha = j_{\xi,*}(DR F^\Lambda_M^\alpha)_{|\xi \neq 0}$, and thus $\phi_\xi DR F^\Lambda_M^\alpha$ is identified with the image of $N$. But $\phi_\xi DR F^\Lambda_M^\alpha$ is also identified, by a standard analysis of the local Fourier transform $\mathcal{F}(\infty,0)$ (cf. [1, 2]) with the nearby cycle space of $\mathcal{L}_\Lambda$ at infinity. In particular, it has dimension one. This forces $N$ to have exactly one Jordan block.

The restriction at $\xi = \xi_0$ of $F^\Lambda_M$ is by definition the push-forward by $p : (x,z) \mapsto z$ of $\delta^{z\xi_0} \otimes M_\Lambda$. Up to a change of the variable $x$, we reduce to the case where $\xi_0 = 1$.

We can then apply [4] Th. 1 & 2 to obtain the second statement. \qed

**Proposition.** The rank-two free $\mathbb{C}[[\xi,z]][1/\xi z]$-module with connection $F^\Lambda_M(1/\xi z)$ does not have a Levelt-Turrittin decomposition at $(\xi = 0, z = 0)$, even after a finite ramification along $\xi = 0$ and $z = 0$.

**Proof.** By contradiction. We first prove the non-existence of a Levelt-Turrittin decomposition. We will then give the argument in order to treat ramification.

Assume that $\tilde{N} := \mathbb{C}[[\xi,z]] \otimes_{\mathbb{C}[[\xi,z]]} F^\Lambda_M(1/\xi z)$ has a Levelt-Turrittin decomposition. We can write it as $\tilde{N} = (\hat{\partial}^{\xi_0} \otimes \hat{R}_+) \oplus (\hat{\partial}^{-\xi_0} \otimes \hat{R}_-)$, with at most two terms
since $\tilde{N}$ has rank two. Here, $\varphi_{\pm} \in \mathbb{C}[\xi, z][1/\xi z]/\mathbb{C}[\xi, z]$ and $\hat{R}_{\pm}$ are free (maybe zero) $\mathbb{C}[\xi, z][1/\xi z]$-modules with flat regular connection.

According to [3, Prop. I.1.2.4.1], $\varphi_{\pm}$ are convergent. Moreover, it is easy to find a finite sequence $e$ of point blow up over the origin so that that the pull-back decomposition is good at each point of the exceptional divisor. Then, according to [3, Prop. 2.19] applied to $\varphi_{\pm} \circ e$ along the strict transform of $\{z = 0\}$ by $e$, and according to the previous lemma, the restriction of $\varphi_{\pm}$ to $\xi = \xi_0$, $\xi_0 \neq 0$ and small enough, is equal to $\pm \xi_0/z$. This being true for each such $\xi_0$, this implies that $\varphi_{\pm} \mp \xi/z$ has a pole at most along $\xi = 0$. Arguing similarly with the strict transform of $\{\xi = 0\}$, along which generically the connection has a regular singularity, gives $\varphi_{\pm} = \pm \xi/z$. In particular, the decomposition of $\tilde{N}$ has exactly two terms.

We now consider the moderate nearby cycles of $F_{\lambda}$ and $\tilde{N}$ along $\xi = 0$, given by the $V$-filtration construction. The uniqueness of the $V$-filtration implies that $(\psi_\xi(\tilde{N}), T_\xi)$ is the formalization with respect to $z$ of $(\psi_\xi(F_{\lambda}(1/\xi z)), T_\xi)$, where $T_\xi$ denotes the monodromy operator given by this construction.

On the one hand, when restricted analytically to a punctured disc $\Delta^\ast$ with coordinate $z$ in $\{\xi = 0\}$, in the neighbourhood of which $F_{\lambda}$ has a regular singularity along $\xi = 0$, $(\psi_\xi(F_{\lambda}), T_\xi)$ is a bundle with connection on $\Delta^\ast$, corresponding to a rank-two local system with monodromy operator $T_z$. Both operators $T_\xi$ and $T_z$ commute, and we recall that $T_\xi$ is unipotent with only one Jordan block. It follows that $(\psi_\xi(F_{\lambda}), T_\xi)_{\Delta^\ast}$ does not split as the direct sum of two objects of rank one of the same kind.

On the other hand, we will prove:

**Lemma.** The $\mathbb{C}[z](\partial_z)$-module $\psi_\xi(\hat{\partial}^{\pm \xi/z} \otimes \hat{R}_{\pm})$ has a regular singularity at $z = 0$.

**Proof.** Let us work with the + case for instance. Then $\hat{\partial}^{\xi/z} \otimes \hat{R}_+$ has a generator $e$ which satisfies $\partial_z e = (1/z)e$ and $\partial_z e = [(\alpha z - \xi)/z^2]e$ for some $\alpha \notin \mathbb{Z}$. We consider the $V$-filtration $U_\alpha(\hat{\partial}^{\xi/z} \otimes \hat{R}_+)$ (with respect to $\xi = 0$) generated by $e$, so that $e \in U_0(\hat{\partial}^{\xi/z} \otimes \hat{R}_+)$. Our goal is to show that the class $[e]$ of $e$ in $\text{gr}_0(\hat{\partial}^{\xi/z} \otimes \hat{R}_+)$ satisfies a regular differential equation with respect to $z$.

We have $[(\alpha z - \xi^2)/z^2]e = \xi \partial_z e \in U_{-1}(\hat{\partial}^{\xi/z} \otimes \hat{R}_+)$. Then one checks that $(z\partial_z - \alpha)(z\partial_z + 1)e \in U_{-1}(\hat{\partial}^{\xi/z} \otimes \hat{R}_+)$, that is, $(z\partial_z - \alpha)(z\partial_z + 1)[e] = 0$, as wanted. \(\square\)

It follows that $\psi_\xi(\tilde{N})(1/z)$ has a regular singularity at $z = 0$, hence so has $\psi_\xi(F_{\lambda})(1/z)$ and, since $(\psi_\xi(\tilde{N})(1/z), T_\xi)$ splits as the direct sum of two regular holonomic $\mathbb{C}(z)$-vector spaces of rank one with connection and monodromy operator $T_\xi$, so does $(\psi_\xi(F_{\lambda})(1/z), T_\xi)$. This leads to a contradiction.

For the ramified case, we are lead to a contradiction in a similar way, by replacing $\hat{\partial}^{\pm \xi/z}$ with $\hat{\partial}^{\pm \xi'/z'}$ in the lemma, with $p, q \geq 1$. The proof of the corresponding statement is similar. \(\square\)
References


