A REMARK ON THE MEROMORPHIC EXTENSION OF HORIZONTAL SECTIONS

by

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Abstract. We give a criterion for horizontal sections of a meromorphic connection on \((\mathbb{C}^2, 0)\) with poles along the coordinate axes to extend as meromorphic sections. An application is given to morphisms between wild twistor \(\mathcal{D}\)-modules on the disc.

Let \(M\) be the germ of a meromorphic bundle with a flat connection on the germ \(X = (\mathbb{C}^2, 0)\) equipped with coordinates \(x_1, x_2\), with poles contained in the (germ of) divisor \(D = \{x_1x_2 = 0\}\). In other words, \(M\) is a free \(\mathbb{C}\{x_1, x_2\}[[x_1x_2]^{-1}]\)-module of finite rank equipped with a flat connection \(\nabla : M \to \Omega^1_X \otimes M\). If \(\nabla\) has regular singularities along \(D\), it is well-known that any \(\nabla\)-horizontal section of \(M\) on \(X^* := X \setminus D\) is meromorphic along \(D\), because it has moderate growth (cf. [Del70]). In particular, given two such meromorphic bundles \(M'\) and \(M''\), any morphism of bundles with connection \((M', \nabla)|_{X^*} \to (M'', \nabla)|_{X^*}\) can be extended as a morphism \((M', \nabla) \to (M'', \nabla)\).

Without the assumption of regular singularity, the previous statement is evidently not true in general: it suffices to consider the free module \(M = \mathbb{C}\{x_1, x_2\}[[x_1x_2]^{-1}]\) of rank one and the connection \(\nabla\) such that \(\nabla e^{1/x_1x_2} \cdot 1\) is a horizontal section on \(X^*\), but is not meromorphic.

We wish to give a sufficient condition so that the following extension property is satisfied:

\[\text{(P)} \quad \text{Any } \nabla\text{-horizontal section of } M \text{ on } X^*, \text{ which is meromorphic along } D_2 := \{x_2 = 0\}, \text{ is meromorphic along } D.\]

By Hartogs’ theorem, if \((M, \nabla)\) has regular singularity generically along \(D_1 := \{x_1 = 0\}\), Property (P) holds. We will introduce a less restrictive condition.

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We will say that $M$ has a \textit{good formal decomposition} at the origin if, setting $\widehat{M} = \mathbb{C} [x_1, x_2] \otimes_{\mathbb{C} [x_1, x_2]} M$, there is an isomorphism
\begin{equation}
\widehat{M} \simeq \bigoplus_{\varphi \in \Phi} (\mathcal{E}^\varphi \otimes \hat{\mathcal{R}}^\varphi),
\end{equation}
where (cf. \cite{Sab00} §I.2.1.4, p. 10)
\begin{enumerate}
\item the $\varphi$'s vary in a finite subset $\Phi$ of $\mathbb{C} \{x_1, x_2\}[1/x_1x_2]/\mathbb{C} \{x_1, x_2\}$, are pairwise
distinct and, for any $\varphi, \psi \in \Phi$, the divisor of $\varphi$ and of $\varphi - \psi$ is $\leq 0$;
\item $\mathcal{E}^\varphi$ is the meromorphic bundle with flat connection of rank one having a basis
in which the matrix of $\nabla$ is $d\varphi$; $\hat{\mathcal{R}}^\varphi$ has regular singularities along $D$.
\end{enumerate}

\textbf{Proposition.} If $M$ has a good formal decomposition at the origin and all the $\varphi \in \Phi \setminus \{0\}$ have a pole along $D_2 = \{x_2 = 0\}$, then $M$ satisfies Property (P).

The condition on the polar locus of the $\varphi$'s will prevent us from the example $\varphi = e^{1/x_1}$, for which Property (P) is clearly not satisfied.

\textbf{Proof.} Let us denote by $e : \tilde{X} \to X$ the real blow-up of $X$ along both components
of $D$. Then $\tilde{X}$ is a real analytic space isomorphic to the product $([0, \varepsilon) \times S^1)^2$, equipped with polar coordinates $(\rho_1, \theta_1; \rho_2, \theta_2)$. Let $\mathcal{R}^\varepsilon_{\tilde{X}}$ be the sheaf of functions
which are $C^\infty$ on $\tilde{X}$ and holomorphic on $X^*$. After \cite{Sab00} Th. II.2.1.1, the formal
decomposition can be locally lifted to $\tilde{X}$ with coefficients in $\mathcal{R}^\varepsilon_{\tilde{X}}$, and gives rise to an
analogous decomposition of $M^\varepsilon := \mathcal{R}^\varepsilon_{\tilde{X}} \otimes e^{-1} \mathcal{S}_X e^{-1} M$.

Let us now work in the neighbourhood of some point $\theta^0 = (\theta^0_1, \theta^0_2)$ of the torus $(S^1)^2 = e^{-1}(0)$.

\textbf{Lemma.} Under the assumption of the proposition, let $m$ be a horizontal section of $M$
on the intersection with $X^*$ of a neighbourhood of $\theta^0$ in $\tilde{X}$ (in other words, an open
bi-sector of bi-direction $\theta^0$). If, in some (or any) $\mathcal{R}^\varepsilon_{\tilde{X}^{\theta^0}}$-basis of $M^\varepsilon_{\theta^0}$, the entries
of the section $m$ have moderate growth along $D_2 = \{x_2 = 0\}$, then they also have moderate
growth along $D_1 = \{x_1 = 0\}$.

\textbf{Proof.} As the choice of the local $\mathcal{R}$-basis is irrelevant, we can assume that the basis is
adapted to the $\mathcal{R}$-decomposition into elementary connections, hence we can assume
that $M^\varepsilon_{\theta^0} = (\mathcal{E}^\varphi \otimes \hat{\mathcal{R}}^\varphi)_{\theta^0}$. In a suitable basis of $\mathcal{R}_\varphi$, the entries of $m^\varepsilon := 1 \otimes m$
take the form $e^{-\varphi} x_1^{a_1} x_2^{a_2} (\log x_1)^{k_1} (\log x_2)^{k_2}$ and, by assumption, if $\varphi \neq 0$, then $\varphi$
has a pole along $D_2$. Then, such an entry has moderate growth along $D_2$ in the
neighbourhood of $\theta^0$ if and only if one of the following conditions is satisfied:
\begin{itemize}
\item $\varphi = 0$,
\item $\text{Re } \varphi > 0$ in some neighbourhood of $\theta^0$.
\end{itemize}
If $\varphi = 0$ or if $\varphi$ has no pole along $D_1$, then the corresponding entry has moderate
growth along $D_1$. If $\varphi$ has a pole along $D_1$, it also has a pole along $D_2$ and the
corresponding entry has rapid decay along \( \{ x_1 x_2 = 0 \} \) (all this understood in some neighbourhood of \( \theta^0 \)).

We can now end the proof of the proposition. If \( m \) is a horizontal section of \( M \) on \( X^* \), the entries of which in some \( \mathcal{O}_X[1/x_1 x_2] \)-basis of \( M \) are meromorphic along \( D_2 \), then the entries of \( m^\omega \) have moderate growth along \( D_2 \) in the neighbourhood of any \( \theta^0 \in (S^1)^2 \) (in any \( \mathcal{O}_{X^*} \)-basis of \( M^\omega \)). After the lemma, its entries (in some local \( \mathcal{O}_X \)-basis of \( M \)) have moderate growth along \( D_1 \) in a small sector of bi-direction \( \theta^0 \) for any \( \theta^0 \in (S^1)^2 \). In other words, \( m \) is a (meromorphic) section of \( M \) in the neighbourhood of \( 0 \). It is then meromorphic, according to Hartogs, on its domain of definition.

Let us now consider the situation where the decomposition \( \text{(DEC)} \) exists but is maybe not good, i.e., does not satisfy (1). We associate a Newton polygon \( N(\varphi) \subset \mathbb{R}^2 \) to any exponent \( \varphi \in \mathbb{C}\{x_1, x_2\}[1/x_1 x_2]/\mathbb{C}\{x_1, x_2\} \): this is the convex hull of the union of subsets \((k_1, k_2) + N^2\), where \((k_1, k_2) \) is the exponent of some monomial in \( \varphi \).

**Corollary 1.** If \( M \) has a formal decomposition \( \text{(DEC)} \) at the origin (but maybe not good) and if, for any \( \varphi \in \Phi \setminus \{0\} \), the polygon \( N(\varphi) \) has no vertex \((k_1, k_2) \) with \( k_1 < 0 \) and \( k_2 \geq 0 \), then \( M \) satisfies Property \( (P) \).

**Proof.** We perform a sequence \( \pi \) of toric blowing-up above the origin of \( \mathbb{C}^2 \) in order to reduce to the case where, in any crossing point of the pull-back divisor of \( D \), the pulled-back connection has a good formal decomposition (this is easy, see for example [Sab95, lemme III.1.2.4, p. 83]). The source space of \( \pi \) is covered by a finite number of charts. Typically, each chart has coordinates \( y_1, y_2 \) and the map \( \pi \) is given by formulas like

\[
\begin{align*}
x_1 \circ \pi &= y_1^a y_2^b \\
x_2 \circ \pi &= y_1^c y_2^d,
\end{align*}
\]

with \( ad - bc = 1 \). By assumption, for any \( \varphi, \psi \in \Phi \) the functions \( \varphi \circ \pi \) and \( \varphi \circ \pi - \psi \circ \pi \) are holomorphic or have a non positive divisor.

In the source space of \( \pi \), the dual graph of the pulled-back divisor of \( D \) is a tree of the form \( \cdot - \cdot - \cdot - \cdot - \cdot - \cdot - \cdot - \cdot \) where the extremal vertices correspond to the strict transforms by \( \pi \) of \( D_1 \) and \( D_2 \). A chart of this space corresponds to a subgraph \( \cdot - \cdot - \cdot \) and, in this chart, one can distinguish a divisor “on the side of \( D_1 \)” and a divisor “on the side of \( D_2 \)”. In the coordinates given above, where we assume \( ad - bc = 1 \), the divisor \( \{ y_1 = 0 \} \) is on the side of \( D_1 \) and \( \{ y_2 = 0 \} \) on the side of \( D_2 \).

The proof is done by induction on the length of the tree, starting from the vertex corresponding to the strict transform of \( D_2 \). It is a straightforward application of the proposition, once we have proved the following property:
At any crossing point of the divisor $\pi^{-1}(D)$, and for any $\varphi \in \Phi$, if $\varphi \circ \pi$ has a pole along the divisor on the side of $D_1$, then $\varphi$ also has a pole along the divisor on the side of $D_2$.

By assumption, if $\varphi \neq 0$, it is a minimal finite sum of terms of the form $x_1^{k_1} x_2^{k_2} u(x_1, x_2)$, where $u$ is a holomorphic unit. Moreover, by assumption, if $k_1 < 0$, then $k_2 < 0$. In any chart as above, $ak_1 + ck_2$ and $bk_1 + dk_2$ have the same sign, because of the assumption of good formal decomposition. It is then a matter of checking that, if $bk_1 + dk_2 = 0$, we cannot have $ak_1 + ck_2 < 0$. Let us recall that $a, b, c, d$ are non negative integers such that $ad - bc = 1$. This implies $d > 0$. We then have $k_2 = -bk_1/d$, hence $ak_1 + ck_2 = (ad - bc)k_1/d = k_1/d$; but in such a situation we cannot have $k_1 < 0$, otherwise we would also have $k_2 < 0$ and $bk_1 + dk_2 < 0$, a contradiction.

**Corollary 2.** If $M$ satisfies the property of Corollary [1] after a cyclic ramification around $D_1$ and/or $D_2$, then $M$ satisfies Property (P).

**Proof.** Easy. □

**Corollary 3.** If $M', M''$ both satisfy the assumptions of Corollary [2], any morphism of meromorphic bundles $(M', \nabla)_{|X \smallsetminus D_1} \to (M'', \nabla)_{|X \smallsetminus D_1}$ compatible with the connections can be extended as a morphism $(M', \nabla) \to (M'', \nabla)$.

**Proof.** Such a morphism is a horizontal section of $M' \otimes M''$ on $X \smallsetminus D_1$, hence a horizontal section of $M' \otimes M''$ on $X^*$ with moderate growth along $D_2$. One can immediately check that the hypotheses of Corollary [2] are satisfied by $M' \otimes M''$ if they are satisfied by $M'$ and $M''$. This is enough to conclude. □

**Example.** Here is an example related to wild twistor $\mathcal{D}$-modules (cf. [Sab09]). Let $M$ be a meromorphic bundle with flat connection on $X$. Let us assume that, after a ramification along $D_1$ (that we forget in the following), there exists a finite family $\Phi$ consisting of pairwise distinct $\varphi \in \mathcal{K}^{-1}[x_1]^{-1}$ such that, denoting by $M_{X[D_1]}$ the formalized bundle of $M$ along $D_1$, we have a decomposition

$$M_{X[D_1]} \simeq \oplus_{\varphi \in \Phi} (\mathcal{O}^{x_2} \otimes N_{\varphi}),$$

where $N_{\varphi}$ is a free $\mathcal{O}_{X[D_1]}[1/x_1 x_2]$-module having a basis in which the matrix of $x_1 \partial_{x_1}$ has no pole. Then, formalizing once more with respect to $x_2$, any $N_{\varphi}$ can be decomposed as the direct sum of terms $\mathcal{O}^{x_2} \otimes \hat{R}_{\varphi, \psi}$, with pairwise distinct $\psi \in \mathcal{K}^{-1}[x_2^{-1}]$ and $\hat{R}_{\varphi, \psi}$ with regular singularity (cf. [Sab09] prop.III.2.1.1(2), p.89). The Newton polygon of any nonzero exponent $\psi(x_2) + \varphi(x_1)/x_2$ has no vertex $(k_1, k_2)$ with $k_1 < 0$ and $k_2 \geq 0$. We are thus in the situation of Corollary [1]. We deduce that $M$ satisfies Property (P).

We now use notation and definitions of [Sab09] §4.5, where $X$ is $(\mathbb{C}, 0)$ with coordinate $t$ and $\mathcal{X} = X \times \mathbb{C}$ with coordinates $(t, z)$. We set $\mathcal{X}^* = \mathcal{X} \smallsetminus \{t = 0\}$. 
Corollary 4. Let $\tilde{\mathcal{M}}', \tilde{\mathcal{M}}''$ be two free $\mathcal{O}_X[t^{-1}]$-modules of finite rank, equipped with a compatible action of $\mathcal{R}_X$. Let us also assume that they are integrable $\mathcal{R}_X$-modules, that is, are equipped with a flat meromorphic connection with a pole of Poincaré rank one along $\{z = 0\}$, extending the $z$-connection coming from the $\mathcal{R}_X$-structure. Let us assume that $\tilde{\mathcal{M}}', \tilde{\mathcal{M}}''$ are strictly specializable with ramification and exponential twist along $\{t = 0\}$.

Then, any morphism $\tilde{\mathcal{M}}'_{|\mathcal{Y}^*} \to \tilde{\mathcal{M}}''_{|\mathcal{Y}^*}$, which is compatible with the connections can be extended to a morphism $\tilde{\mathcal{M}}' \to \tilde{\mathcal{M}}''$.

Proof. In the previous notation, we set $x_1 = t$, $x_2 = z$. According to [Sab09] Prop. 4.5.4, we can apply the argument above.

References


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