The work of Andrey Bolibrukh

on isomonodromic deformations
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At the same time, for specific linear systems related to the Painlevé equations, it is possible to perform a rigorous study of the inverse problem on the basis of analytic considerations only.”
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A. A. Bolibrukh, A. R. Its & A. A. Kapaev
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Main themes
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- Isomonodromy and integrability.
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- Possible general form of an isomonodromic deformation of a Fuchsian system.
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- The Schlesinger system and the Painlevé property.
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  - Equation for the “Theta divisor”.
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- The Schlesinger system and the Painlevé property.
  - Equation for the “Theta divisor”.
  - Bounds for the order of the pole of the solutions along the “Theta divisor”.
Main themes— continuation

- Isomonodromic confluences.
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- Preservation of regularity at the confluence point.
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- Isomonodromy and irregular singularities
What is an isomonodromic deformation?
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\[ X = \mathbb{P}^1 \times T \]
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What is an isomonodromic deformation?

\[ \mathbb{P}^1 \times T \]

\[ Y = \bigcup_i Y_i \]

\[ X = \overline{X} \setminus Y \]
Equations

$$\alpha_i(t^o) \rightarrow Y_i \rightarrow \mathbb{P}^1$$

$$\pi$$

$$t^o \rightarrow t \rightarrow T$$
Fuchsian system

\[
\frac{du}{dx} = \sum_{i=1}^{n} \frac{A_i^0}{x - a_i(t^o)} \cdot u
\]

\(A_i^0: d \times d\) constant matrices
Equations

Matrix of 1-forms

\[ \Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx \]

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Isomonodromic deformation parametrized by \( T \):
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Isomonodromic deformation parametrized by \( T \):

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\Omega = \sum_{i=1}^{n} \frac{A_i(t)}{x - a_i(t)} \cdot dx + \sum_j \Omega_j(x, t) \, dt_j,
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Equations

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\( A_i(t^o) = A^o_i \quad A_i(t) \) holomorphic
Equations

Matrix of 1-forms

\[ \Omega^o = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i(t^o)} \cdot dx \]

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\[ A_i(t^o) = A_i^o \quad A_i(t) \text{ holomorphic} \]

\[ \Omega_j(x, t) \text{ is meromorphic with poles along } Y \]

(regular deformation)

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**Equations**

Matrix of 1-forms

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\[A_i(t^o) = A_i^o \quad A_i(t) \text{ holomorphic}\]

\[\Omega_j(x, t) \text{ is holomorphic (logarithmic deformation)}\]
Vector bundles
Vector bundles

\( E^o \) holomorphic vector bundle on \( \mathbb{P}^1 \)
Vector bundles

$E^o$ holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \rightarrow \Omega^1_{\mathbb{P}^1}(\ast Y^o) \otimes E^o$

integrable meromorphic connection with regular singularities along $Y^o$
Vector bundles

$E^o$ holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \rightarrow \Omega^1_{\mathbb{P}^1}(\log Y^o) \otimes E^o$

integrable meromorphic connection with logarithmic poles along $Y^o$
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Isomonodromic deformation parametrized by $T$:
Vector bundles

$E^o$ holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega_{\mathbb{P}^1}^1(*Y^o) \otimes E^o$

integrable meromorphic connection with regular singularities along $Y^o$

Isomonodromic deformation parametrized by $T$:

$\nabla : E \to \Omega_{\mathcal{X}}^1(*Y) \otimes E, \quad \nabla \circ \nabla = 0, \quad \nabla |_{E^o} = \nabla^o$

integrable meromorphic connection with regular singularities along $Y$
Vector bundles

$E^o$ holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \rightarrow \Omega^1_{\mathbb{P}^1}(\log Y^o) \otimes E^o$

integrable meromorphic connection with logarithmic poles along $Y^o$

Isomonodromic deformation parametrized by $T$:

$\nabla : E \rightarrow \Omega^1_X(\ast Y) \otimes E$, $\nabla \circ \nabla = 0$, $\nabla|_{E^o} = \nabla^o$

integrable meromorphic connection with regular singularities along $Y$ and each $\nabla_t$ on $E_t$ has logarithmic poles along $Y_t$
**Vector bundles**

$E^o$ holomorphic vector bundle on $\mathbb{P}^1$

$\nabla^o : E^o \to \Omega^1_{\mathbb{P}^1}(\log Y^o) \otimes E^o$

integrable meromorphic connection

with logarithmic poles along $Y^o$

Isomonodromic deformation parametrized by $T$:

$\nabla : E \to \Omega^1_X(\log Y) \otimes E$, \quad $\nabla \circ \nabla = 0$, \quad $\nabla|_{E^o} = \nabla^o$

integrable meromorphic connection with logarithmic poles along $Y$
The Schlesinger system

- a finite set of distinct points $a^o = \{a_1^o, \ldots, a_n^o\}$ on $\mathbb{P}^1 \setminus \{\infty\}$,
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- a system $\frac{du}{dx} = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i^o} \cdot u$, $\sum_i A_i^o = 0$. 
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\( \iff \) a logarithmic connection \( \nabla^0 \) on the trivial bundle \( E^0 \) of rank \( d \) on \( \mathbb{P}^1 \), with poles at \( a^0 \).
The Schlesinger system

- a finite set of distinct points $a^o = \{a^o_1, \ldots, a^o_n\}$ on $\mathbb{P}^1 \setminus \{\infty\}$,

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$\iff$ a logarithmic connection $\nabla^o$ on the trivial bundle $E^o$ of rank $d$ on $\mathbb{P}^1$, with poles at $a^o$.

- $T$: universal cover of $(\mathbb{P}^1)^n \setminus$ diagonals, with base point $\tilde{a}^o$. 

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The Schlesinger system

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- $T$: universal cover of $(\mathbb{P}^1)^n \setminus$ diagonals, with base point $\tilde{a}^o$.

**Theorem (Malgrange).** There exists a unique vector bundle $E$ on $\mathbb{P}^1 \times T$ equipped with an integrable logarithmic connection $\nabla$ having poles along the hypersurfaces $Y_i$, and with an identification $(E, \nabla)|_{\mathbb{P}^1 \times \{\tilde{a}^o\}} \sim (E^o, \nabla^o)$. 

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The Schlesinger system— continuation

Divisor $\Theta = \{ \tilde{\alpha} \in T \mid E_{\tilde{\alpha}} \text{ is not trivial} \}$,
The Schlesinger system— continuation

- Divisor \( \Theta = \{ \tilde{a} \in T \mid E_{\tilde{a}} \text{ is not trivial} \} \),

- The matrix of \( \nabla \) in a basis of \( E(\ast \Theta) \) extending that of \( E^o \) is:

\[
\sum_{i=1}^{n} A_i(\tilde{a}) \frac{d(x - \tilde{a}_i)}{(x - \tilde{a}_i)} + \sum_{i=1}^{n} B_i(\tilde{a}_i) d\tilde{a}_i
\]
The Schlesinger system— continuation

- Divisor $\Theta = \{ \tilde{a} \in T \mid E_{\tilde{a}} \text{ is not trivial} \}$,

- The matrix of $\nabla$ in a basis of $E(\ast \Theta)$ extending that of $E^\circ$ is:

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\sum_{i=1}^{n} A_i(\tilde{a}) \frac{d(x - \tilde{a}_i)}{(x - \tilde{a}_i)} + \sum_{i=1}^{n} B_i(\tilde{a}_i) d\tilde{a}_i
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- On can choose the basis such that $B_i \equiv 0$. 

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The Schlesinger system— continuation

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The Schlesinger system— continuation

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- On can choose the basis such that $B_i \equiv 0$.

- The Schlesinger system (integrability condition):

$$dA_i = \sum_{j \neq i} [A_i, A_j] \frac{d(\tilde{a}_i - \tilde{a}_j)}{(\tilde{a}_i - \tilde{a}_j)}, \quad i = 1, \ldots, n.$$
Corollary. The solutions of the Schlesinger system with initial value $A^o_i$ at $\tilde{a}^o$ are meromorphic on $T$ with poles along $\Theta$ at most.
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Behaviour of the solutions to the Schlesinger system near the polar set $\Theta$?
Corollary. The solutions of the Schlesinger system with initial value \( A_i^0 \) at \( \tilde{a}^0 \) are meromorphic on \( T \) with poles along \( \Theta \) at most.

Behaviour of the solutions to the Schlesinger system near the polar set \( \Theta \)?

Andrey has given a method to produce examples and describe in concrete terms this behaviour.
Local equation for $\Theta$
Local equation for $\Theta$

Initial data $\tilde{a}^o$, $A_i^o$
Local equation for \( \Theta \)

Initial data \( \tilde{\alpha}^o, A^o_i \sim (E, \nabla) \) on \( \mathbb{P}^1 \times T \),
Local equation for $\Theta$

Initial data $\tilde{a}^o, A_i^o \leadsto (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$. 
Local equation for $\Theta$

Initial data $\tilde{a}^0, A_i^0 \leadsto (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. 
Local equation for $\Theta$

Initial data $\tilde{a}^0, A_i^o \leadsto (E, \nabla)$ on $\mathbb{P}^1 \times T, \Theta \subset T$.

Take $\alpha^* \in \Theta$. Hence $E_{\alpha^*} \simeq \bigoplus_{j=1}^d \mathcal{O}(-k_j)$,
Local equation for $\Theta$

Initial data $\tilde{a}^0, A^0_i \rightsquigarrow (E, \nabla)$ on $\mathbb{P}^1 \times T$, $\Theta \subset T$.

Take $a^* \in \Theta$. Hence $E_{a^*} \cong \bigoplus_{j=1}^d \mathcal{O}(-k_j)$, with $k_1 \leq \cdots \leq k_d$ and $\deg E_{a^*} = -(k_1 + \cdots + k_d) = 0$. 
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Typical example: $E_{a^*} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$
Local equation for $\Theta$

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There exists a holomorphic subbundle $E_{\alpha^*}^{(0)}$ of the meromorphic bundle $E_{\alpha^*}[\ast \infty]$ which is trivial and on which the connection $\nabla$ has only logarithmic poles.
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Matrix of the connection: $\sum_{i=1}^n \frac{B_i^{(0)}(\alpha^*)}{x - \alpha_i^*} \, dx$
Local equation for $\Theta$

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Matrix of the connection: $\sum_{i=1}^{n} \frac{B_i^{(0)}(a^*)}{x - a_i^*} \ dx$

$\infty = \text{apparent singularity}$ and

$\sum_i B_i^{(0)}(a^*) = \text{diag}(k_1, \ldots, k_d) =: K^{(0)}$
Local equation for $\Theta$— continuation

Malgrange’s theorem applied to $E_{a^*}^{(0)}$ near $a^*$
Local equation for $\Theta$— continuation

Malgrange’s theorem applied to $E_{a^*}^{(0)}$ near $a^*$

$\sim (E^{(0)}, \nabla)$ trivial on $\mathbb{P}^1 \times \text{nb}(a^*)$, 
Local equation for $\Theta$ — continuation

Malgrange’s theorem applied to $E^{(0)}_{a^*}$ near $a^*$

$\leadsto (E^{(0)}, \nabla)$ trivial on $\mathbb{P}^1 \times \text{nb}(a^*)$,

matrix of $\nabla$:

$$\sum_i B^{(0)}_i(a) \frac{d(x - a_i)}{(x - a_i)}$$,

$$\sum_i B^{(0)}_i(a) \equiv K^{(0)}$$

and the $B^{(0)}_i(a)$ satisfy the Schlesinger system.
Local equation for $\Theta$— continuation

Malgrange’s theorem applied to $E_{a^*}^{(0)}$ near $a^*$

$\sim (E^{(0)}, \nabla)$ trivial on $\mathbb{P}^1 \times \text{nb}(a^*)$,

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and the $B_i^{(0)}(a)$ satisfy the Schlesinger system.

**Lemma 1.** There exists $\ell, m \in \{1, \ldots, d\}$ such that $k_m - k_\ell \geq 2$ and $i \in \{1, \ldots, n\}$ such that the $(\ell, m)$-entry $B_{i,\ell m}^{(0)}(a)$ does not vanish identically.
Lemma 2. Fix \( \ell, m \in \{1, \ldots, d\} \) such that \( k_m - k_\ell \geq 2 \) and \( B_{i,\ell m}^{(0)}(a) \neq 0 \).
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Put $\tau^{(0)}(a) = \sum_i B_{i,\ell m}^{(0)}(a) a_i$ and $\Theta^{(0)} = \{ \tau^{(0)} = 0 \}$. 
Lemma 2. Fix $\ell, m \in \{1, \ldots, d\}$ such that $k_m - k_\ell \geq 2$ and $B_{i, \ell m}^{(0)}(a) \not\equiv 0$.

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Then there exists an extension $E^{(1)}[\ast \Theta^{(0)}]$ of $E[\ast(\infty \times T) \cup \Theta^{(0)}]$ such that, out of $\Theta^{(0)}$, 

Local equation for $\Theta$— continuation
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Lemma 2. Fix $\ell, m \in \{1, \ldots, d\}$ such that $k_m - k_\ell \geq 2$ and $B_{i,\ell m}^{(0)}(a) \neq 0$.

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Then there exists an extension $E^{(1)}[\ast \Theta^{(0)}]$ of $E[\ast(\infty \times T) \cup \Theta^{(0)}]$ such that, out of $\Theta^{(0)}$,

- for any $a \in T \setminus \Theta^{(0)}$, the bundle $E^{(1)}[\ast \Theta^{(0)}]_a$ is trivial,
- the connection $\nabla$ is logarithmic on $E^{(1)}[\ast \Theta^{(0)}]$ with poles on $Y_1 \cup \cdots \cup Y_n \cup (\infty \times T)$ and its residue along $\infty \times T$ is $-K^{(1)} = -\text{diag}(k_1^{(1)}, \ldots, k_d^{(1)})$ with
  \[
  \sum_{j=1}^{d} (k_j^{(1)})^2 \leq \sum_{j=1}^{d} (k_j^{(0)})^2 - 2.
  \]
Local equation for $\Theta$— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[\ast \Theta^{(0)}] = E[\ast \Theta^{(0)}]$ and $\Theta \subset \Theta_0$. 
Local equation for $\Theta$— continuation

- If $K^{(1)} = 0$, then $E^{(1)}[\ast \Theta^{(0)}] = E[\ast \Theta^{(0)}]$ and $\Theta \subset \Theta_0$.

- If $K^{(1)} \neq 0$, the matrix of $\nabla$: $\sum_i B_i^{(1)}(a) \frac{d(x - a_i)}{(x - a_i)}$, where $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.
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$\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.

- Apply Lemma 1.
Local equation for $\Theta$— continuation

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Apply Lemma 1.

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- If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.

- If $K^{(1)} \neq 0$, the matrix of $\nabla$: $\sum_i B_i^{(1)}(a) \frac{d(x - a_i)}{(x - a_i)}$,

  $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.

  - Apply Lemma 1.

  - Get $\tau^{(1)}$ and $\Theta^{(1)} \supset \Theta^{(0)}$. 

Local equation for $\Theta$— continuation

If $K^{(1)} = 0$, then $E^{(1)}[*\Theta^{(0)}] = E[*\Theta^{(0)}]$ and $\Theta \subset \Theta_0$.

If $K^{(1)} \neq 0$, the matrix of $\nabla: \sum_i B_i^{(1)}(a) \frac{d(x - a_i)}{x - a_i}$, $\sum_i B_i^{(1)}(a) \equiv K^{(1)}$ and the $B_i^{(1)}(a)$ satisfy the Schlesinger system.

Apply Lemma 1.

Get $\tau^{(1)}$ and $\Theta^{(1)} \supset \Theta^{(0)}$.

Apply Lemma 2 and get $E^{(2)}[*\Theta^{(1)}]$ and $K^{(2)}$. 
Local equation for $\Theta$— continuation

If $K^{(1)} = 0$, then $E^{(1)}[\ast \Theta^{(0)}] = E[\ast \Theta^{(0)}]$ and $\Theta \subset \Theta_0$.

If $K^{(1)} \neq 0$, the matrix of $\nabla: \sum_i B_i^{(1)}(a) \frac{d(x - a_i)}{(x - a_i)}$,

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- Apply Lemma 1.
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etc. Get $\tau^{(\nu)}$, $\Theta^{(\nu)} \supset \Theta^{(\nu - 1)}$, $E^{(\nu+1)}[\ast \Theta^{(\nu)}]$, $K^{(\nu+1)} = 0$. 
Local equation for $\Theta$— continuation

If $K^{(1)} = 0$, then $E^{(1)}[\ast \Theta^{(0)}] = E[\ast \Theta^{(0)}]$ and $\Theta \subset \Theta_0$.

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- Apply Lemma 1.
- Get $\tau^{(1)}$ and $\Theta^{(1)} \supset \Theta^{(0)}$.
- Apply Lemma 2 and get $E^{(2)}[\ast \Theta^{(1)}]$ and $K^{(2)}$.

etc. Get $\tau^{(\nu)}$, $\Theta^{(\nu)} \supset \Theta^{(\nu-1)}$, $E^{(\nu+1)}[\ast \Theta^{(\nu)}]$, $K^{(\nu+1)} = 0$.

Then $E^{(\nu+1)}[\ast \Theta^{(\nu)}] = E[\ast \Theta^{(\nu)}]$ and $\Theta \subset \Theta^{(\nu)}$. 
A picture illustrating the method
A picture illustrating the method
A picture illustrating the method
A picture illustrating the method

\[ E^{(1)}[\ast \Theta^{(0)}] \]
A picture illustrating the method

\[ E^{(2)}[\ast \Theta^{(1)}] \]
A picture illustrating the method

\[ E[\ast \Theta^{(\nu)}] \]

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Theorem. Set \( \tilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \cdots \tau^{(\nu)} \).

Then \( \tilde{\tau} \) is a local equation for \( \Theta \).
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Then $\tilde{\tau}$ is a local equation for $\Theta$.

Sketch of the proof.
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Sketch of the proof.

\[
\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \text{tr} \left( B_i^{(\mu)}(a) B_j^{(\mu)}(a) \right) \frac{d(a_i - a_j)}{(a_i - a_j)}.
\]
Theorem. Set $\tilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \cdots \tau^{(\nu)}$.

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Sketch of the proof.

$$\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \text{tr}(B_i^{(\mu)}(a)B_j^{(\mu)}(a)) \frac{d(a_i - a_j)}{(a_i - a_j)}.$$ 

$$\omega^{(\nu)} = \frac{d\tau}{\tau} \text{ with } \Theta = \{\tau = 0\} \text{ (Theorem of Miwa).}$$
**Theorem.** Set \( \tilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \cdots \tau^{(\nu)} \).

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**Sketch of the proof.**

\[
\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} \text{tr} (B_i^{(\mu)}(a) B_j^{(\mu)}(a)) \frac{d(a_i - a_j)}{(a_i - a_j)}.
\]

\[
\omega^{(\nu)} = \frac{d\tau}{\tau} \quad \text{with} \quad \Theta = \{ \tau = 0 \} \quad \text{(Theorem of Miwa)}.
\]

\[
\omega^{(0)} \quad \text{is holomorphic and closed} \quad \text{(Schlesinger)}.
\]
Theorem. Set $\tilde{\tau} = \tau^{(0)} \cdot \tau^{(1)} \ldots \tau^{(\nu)}$.

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Sketch of the proof.

$$\omega^{(\mu)} = \frac{1}{2} \sum_{i \neq j} tr(B_i^{(\mu)}(a)B_j^{(\mu)}(a)) \frac{d(a_i - a_j)}{(a_i - a_j)}.$$ 

- $\omega^{(\nu)} = \frac{d\tau}{\tau}$ with $\Theta = \{\tau = 0\}$ (Theorem of Miwa).
- $\omega^{(0)}$ is holomorphic and closed (Schlesinger).
- $\omega^{(\mu)} - \omega^{(\mu - 1)} = \frac{d\tau^{(\mu - 1)}}{\tau^{(\mu - 1)}}$. 