
ERRATA TO “POLARIZABLE TWISTOR \mathcal{D} -MODULES”

by

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- (1) On page 21, line 5 and page 22, line 5, replace $L_{z_o}^*$ with $L_{z_o}^*$.
- (2) On page 30, Lemma 1.5.3 and in its proof, replace $\mathcal{C}_{X|\mathbf{S}}^{\infty, \text{an}}$ with $\mathcal{C}_{\mathcal{X}|\mathbf{S}}^{\infty, 0}$.
- (3) On page 32, line 8, the isomorphism $\mathcal{T}^*(-k) \rightarrow \mathcal{T}(k)^*$ is not the morphism obtained by adjunction of (1.6.3), but the inverse morphism obtained from (1.6.3) where we replace \mathcal{T} by \mathcal{T}^* . The choice of (1.6.3) is universal and holds for any \mathcal{T} .
- (4) On page 33, 4th line of 1.6.b, replace $\mathcal{C}_{X|\mathbf{S}}^{\infty, \text{an}}$ with $\mathcal{C}_{\mathcal{X}|\mathbf{S}}^{\infty, 0}$.
- (5) On page 48, the text of Remark 2.2.1 has to be replaced by the following text:

Remark 2.2.1. — We have seen that the sesquilinear pairing C takes values in $\mathcal{C}_{\mathcal{X}|\mathbf{S}}^{\infty, 0}$, according to Lemma 1.5.3. So the restriction to x_o of each component of the smooth twistor structure is well defined. Then, according to (2.1.1), C takes values in $\mathcal{C}_{X|\mathbf{S}}^{\infty, \text{an}}$. It is also nondegenerate and gives a gluing of \mathcal{H}^{l*} with $\overline{\mathcal{H}}^m$, defining thus a $\mathcal{C}_{X \times \mathbb{P}^1}^{\infty, \text{an}}$ -bundle $\widetilde{\mathcal{H}}$ on $X \times \mathbb{P}^1$.

- (6) The statement of Lemma 3.6.33 (which is not used in the text) has to be replaced with

$$\langle \phi_{t,0} C([m'_0], \overline{[m''_0]}), \bullet \rangle = \text{Res}_{s=0} \frac{-1}{s} \langle (|t|^{2s} - s) C(m'_0, \overline{m''_0}), \bullet \wedge \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle.$$

Proof. — We write $m''_0 = \partial_t m''_{-1} + \mu''_{<0}$. By definition,

$$\begin{aligned} \langle \phi_{t,0} C([m'_0], \overline{[m''_0]}), \varphi \rangle &= \text{Res}_{s=0} \langle C(m'_0, \overline{\partial_t m''_{-1}}), \varphi \wedge I_{\widehat{\chi}} \chi \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\ &= \text{Res}_{s=0} \langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge (\partial_t I_{\widehat{\chi}}) \chi \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\ &= -z^{-1} \text{Res}_{s=-1} \langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge t|t|^{2s} \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle, \end{aligned}$$

by (3.6.23). On the other hand,

$$\begin{aligned}
\text{Res}_{s=0} \frac{-1}{s} \langle |t|^{2s} C(m'_0, \overline{\partial_t m''_{-1}}), \varphi \wedge \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\
&= \text{Res}_{s=-1} \frac{-1}{s+1} \langle C(m'_0, \overline{\partial_t m''_{-1}}), \varphi \wedge |t|^{2(s+1)} \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\
&= \text{Res}_{s=-1} \frac{1}{s+1} \langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge \partial_t (|t|^{2(s+1)} \chi(t)) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\
&= -z^{-1} \text{Res}_{s=-1} \langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge t |t|^{2s} \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\
&\quad + \text{Res}_{s=-1} \frac{1}{s+1} \langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge |t|^{2(s+1)} \partial_t \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\
&= -z^{-1} \text{Res}_{s=-1} \langle C(m'_0, \overline{m''_{-1}}), \varphi \wedge t |t|^{2s} \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle \\
&\quad - \langle C(m'_0, \overline{\partial_t m''_{-1}}), \varphi \wedge \chi \frac{i}{2\pi} dt \wedge d\bar{t} \rangle
\end{aligned}$$

and

$$\text{Res}_{s=0} \frac{-1}{s} \langle |t|^{2s} C(m'_0, \overline{\mu''_{<0}}), \varphi \wedge \chi(t) \frac{i}{2\pi} dt \wedge d\bar{t} \rangle = -\langle C(m'_0, \overline{\mu''_{<0}}), \varphi \wedge \chi \frac{i}{2\pi} dt \wedge d\bar{t} \rangle. \quad \square$$

(7) On page 119, in the statement of Corollary 4.2.9, replace $w + 1$ with w .

(8) On page 121, the argument given on lines 10–14 is not correct, as the inverse image by the projection is not known to be a polarizable twistor \mathcal{D} -module. One can argue as follows.

Choose a finite morphism $\pi : Z \rightarrow Z'$ with Z' smooth and projective (a projective line, for instance) and consider the composed morphism $\nu \circ \pi : \tilde{Z} \rightarrow Z'$. On $Z^o \subset \tilde{Z}$, the object $(\mathcal{T}, \mathcal{S})$ defines a harmonic bundle (H, D''_E, θ_E, h) in the sense of C. Simpson [1], according to the correspondence of Lemma 2.2.2 on Z^o . We can restrict Z^o so that $\pi : Z^o \rightarrow Z'^o$ is a finite covering. We wish to show that the eigenvalues of the Higgs field are (multivalued) meromorphic one-forms, with a pole of order at most one at each puncture, and a purely imaginary residue at any such punctures. Indeed, this will imply that the harmonic bundle (H, D''_E, θ_E, h) on Z^o is tame on \tilde{Z} , and that its parabolic filtration at the punctures is the trivial one, so, by [1], the corresponding local system is semisimple.

It is then enough to prove that such a property is satisfied for the direct image $\pi_*(H, D''_E, \theta_E, h)$ on Z'^o , as locally the covering is trivial (in a local coordinate t on \tilde{Z} and t' on Z' for which $\pi(t) = t' = t^q$, we have $dt'/t' = qdt/t$, and, if the eigenvalues of θ'_E are written as $\alpha(t)dt/t$, the eigenvalues of $\pi_*\theta'_E$ are of the form $\frac{1}{q}\alpha(\zeta t)\frac{dt'}{t'}$, with $\zeta^q = 1$; hence the condition on eigenvalues is satisfied for θ'_E if and only if it is satisfied for $\pi_*\theta'_E = \theta'_{\pi_*E}$).

Now, a particular case of Theorem 6.1.1 (the case when π is finite) implies that $\pi_+(\mathcal{T}, \mathcal{S})$ is an object of $\text{MT}^{(r)}(Z, 0)^{(\text{p})}$, and we apply the correspondence of Theorem 5.0.1.

(9) On page 135, the line after (5.3.5), read $\mathcal{O}_{\mathcal{X}}$ instead of $\mathcal{O}_{\mathcal{Y}}$.

(10) On page 156, line -1 and page 157, line 1, replace $n_j + \beta_j = -1$ by $n_j + \beta_j = 0$, and $\ell_z(n_j + \beta_j) = -1$ by $\ell_z(n_j + \beta_j) = 0$. This does not affect the reasoning.

(11) On page 167, line 2: it is implicitly understood that $\omega_{\beta,\ell,k}$ is holomorphic even at $t = 0$, although the previous reasoning only gives the holomorphy away from $t = 0$. The argument that $(\mathcal{D}'_z \eta_{\neq(0,0)})_{\neq(0,0)}$ is L^2 has to be corrected. I thank T. Mochizuki for pointing out the mistake and providing the following proof.

(a) Let us set $\tilde{\omega}_{\beta,\ell,k} = t\omega_{\beta,\ell,k}$, which is holomorphic on $D^* \times \text{nb}(z_o)$. Assume first (see (b) below) we have proved that $\tilde{\omega}_{\beta,\ell,k} e_{\beta,\ell,k}^{\prime(z_o)}$ is L^2 when we fix z in $\text{nb}(z_o)$. Then, if we expand $\tilde{\omega}_{\beta,\ell,k} = \sum_{n \in \mathbb{Z}} \tilde{\omega}_{\beta,\ell,k,n}(z) t^n$, we claim that the coefficients $\tilde{\omega}_{\beta,\ell,k,n}(z)$ identically vanish when $n \leq -1$. In order to prove this, we can argue with z fixed. The L^2 condition we assume is that, for any $n \in \mathbb{Z}$, $|\tilde{\omega}_{\beta,\ell,k,n}(z)| r^{n+\ell_z(q_{\beta,\zeta_o}+\beta)} L(r)^{\ell/2-1} \in L^2_{\text{loc}}(d\theta dr/r)$. But when $n \leq -1$ and $a < 1$ (as is $\ell_z(q_{\beta,\zeta_o} + \beta)$ for z near z_o), $r^{n+a} L(r)^{k/2}$ does not belong to $L^2_{\text{loc}}(d\theta, dr/r)$, hence the coefficients $\tilde{\omega}_{\beta,\ell,k,n}(z)$ have to vanish when $n \leq -1$.

In order to conclude, we want to show that $\tilde{\omega}_{\beta,\ell,k} e_{\beta,\ell,k}^{\prime(z_o)} dt/t$ is L^2_{loc} , while we have only assumed that $\tilde{\omega}_{\beta,\ell,k} e_{\beta,\ell,k}^{\prime(z_o)}$ is so. If $\ell_{z_o}(q_{\beta,\zeta_o} + \beta) \neq 0$, multiplying by $L(r)$ will not cause an escape from the L^2 space, as the L^2 condition is governed by terms like $r^{n+\ell_z(q_{\beta,\zeta_o}+\beta)}$. If $\ell_{z_o}(q_{\beta,\zeta_o} + \beta) = 0$, the previous argument is not valid if $n = 0$. But we precisely considered the $\neq(0,0)$ parts, so the corresponding coefficient $\tilde{\omega}_{\beta,\ell,k,0}(z)$ is identically 0 by definition.

(b) Let us now fix $z \in \text{nb}(z_o)$, that we still denote by z_o for simplicity. The operator $D_E + z_o \theta''_E - \bar{z}_o \theta'_E = \mathcal{D}''_{z_o} + \delta'_{z_o}$ is compatible the harmonic metric h on H by definition, and we have $\mathcal{D}'_{z_o} = z_o \delta'_{z_o} + (1 + |z_o|^2) \theta'_E$. If we know (cf. (c) below) that $(\mathcal{D}''_{z_o} + \delta'_{z_o})(\eta_{\neq(0,0)})$ is a section of $\mathcal{L}^1_{(2)}(H, h)$ then, by the definition of η , the same property holds for $\delta'_{z_o}(\eta_{\neq(0,0)})$. On the other hand, by the expression of Θ'_{z_o} given before (6.2.7), $L(t)^{-1} \theta'_{z_o}(\eta_{\neq(0,0)})$ is also in $\mathcal{L}^1_{(2)}(H, h)$ (the term $L(t)^{-1}$ is here to compensate the norm of dt/t). Therefore, we find that $L(t)^{-1} \mathcal{D}'_{z_o}(\eta_{\neq(0,0)})$ is in $\mathcal{L}^1_{(2)}(H, h)$ and finally, by definition of ω , that $L(t)^{-1} \omega$ is in $\mathcal{L}^1_{(2)}(H, h)$, so the assumption in (a) above is fulfilled.

(c) As $D_{z_o} \stackrel{\text{def}}{=} \mathcal{D}''_{z_o} + \delta'_{z_o}$ is compatible with h , we have, for a C^∞ section e of H on D^* :

$$0 = d^2 h(e, \bar{e}) = 2 \|D_{z_o} e\|_h^2 + h(R_{z_o} e, \bar{e}) + h(e, \overline{R_{z_o} e}),$$

where R_{z_o} denotes the curvature operator of D_{z_o} , and where the (fiberwise) norm of $D_{z_o} e$ is computed with the metric h and the Poincaré metric (for the 1-form components). Arguing as in [1, page 737], we find the the L^2 norm of the operator R_{z_o} with respect to the metric h and the Poincaré metric is bounded by a constant. It follows that $\|D_{z_o} e\|_h \leq C \|e\|_h$ and therefore, if e moreover is a local section of $\mathcal{L}^0_{(2)}(H, h)$, then $D_{z_o} e$ is a local section of $\mathcal{L}^1_{(2)}(H, h)$. By density, we conclude that this holds for any local section of $\mathcal{L}^0_{(2)}(H, h)$. We apply this to $\eta_{\neq(0,0)}$ to get (b).

References

- [1] C. SIMPSON – Harmonic bundles on noncompact curves, *J. Amer. Math. Soc.* **3** (1990), p. 713–770.

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