Fourier transformation of $\mathscr{D}$-modules and applications

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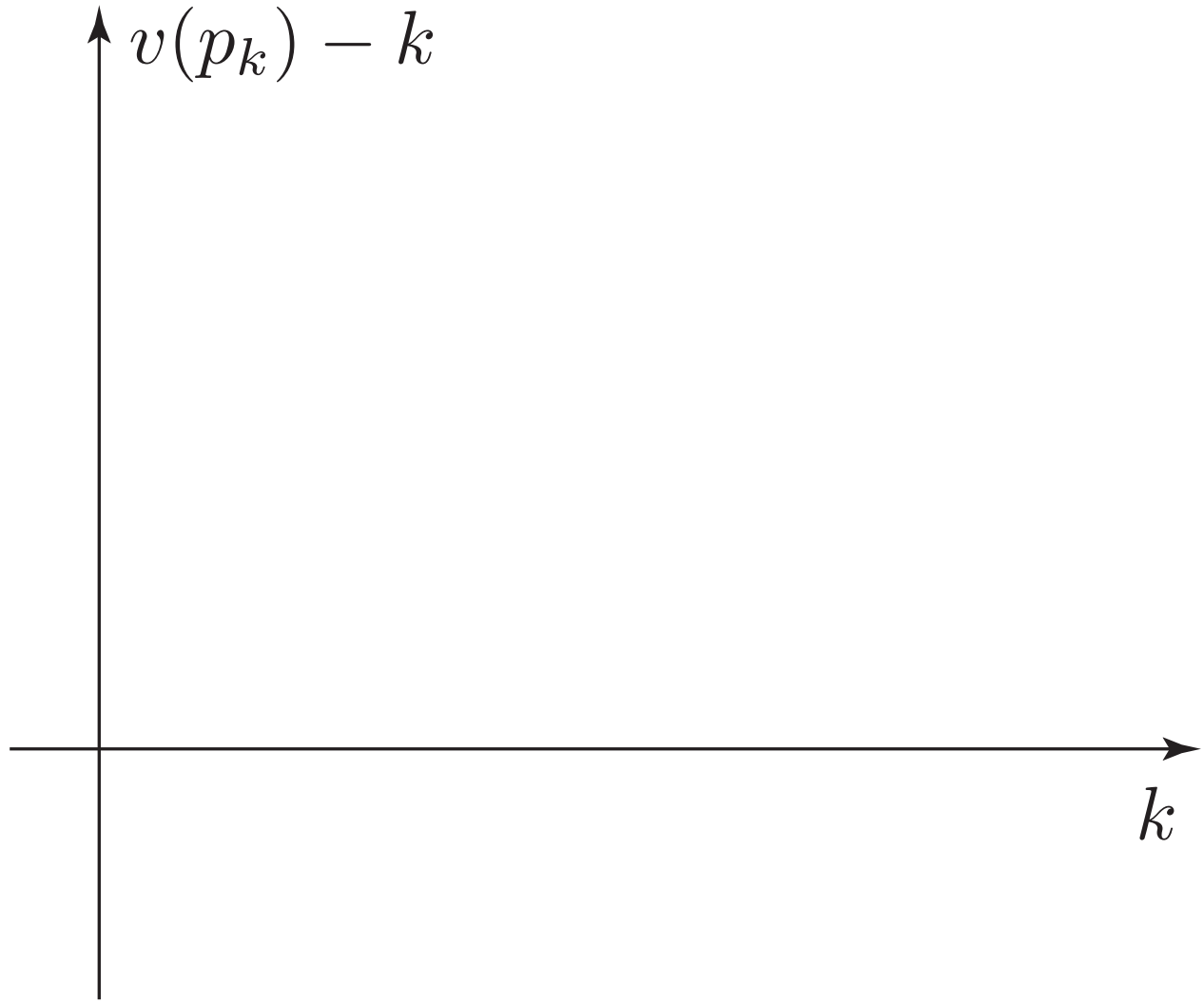

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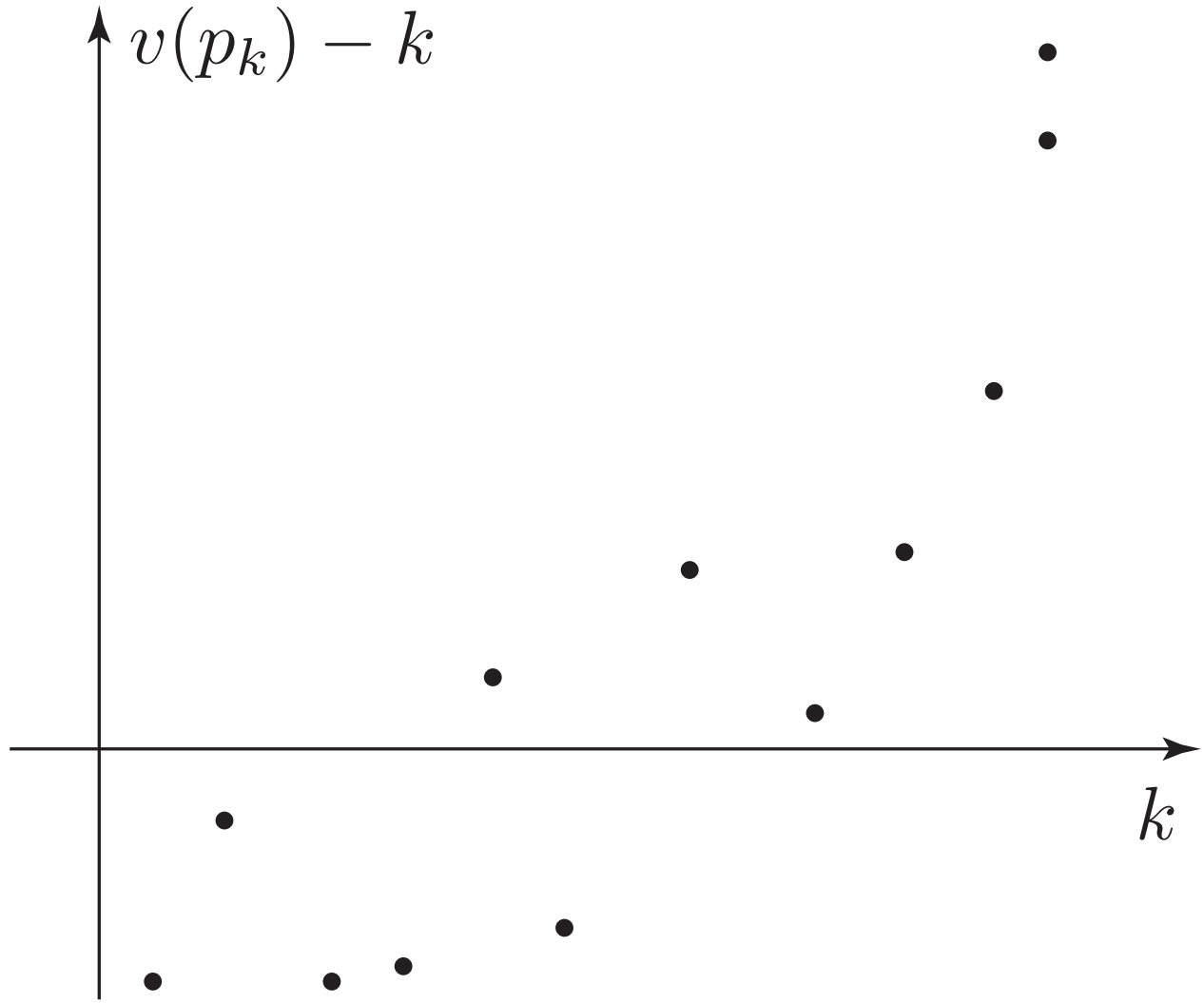

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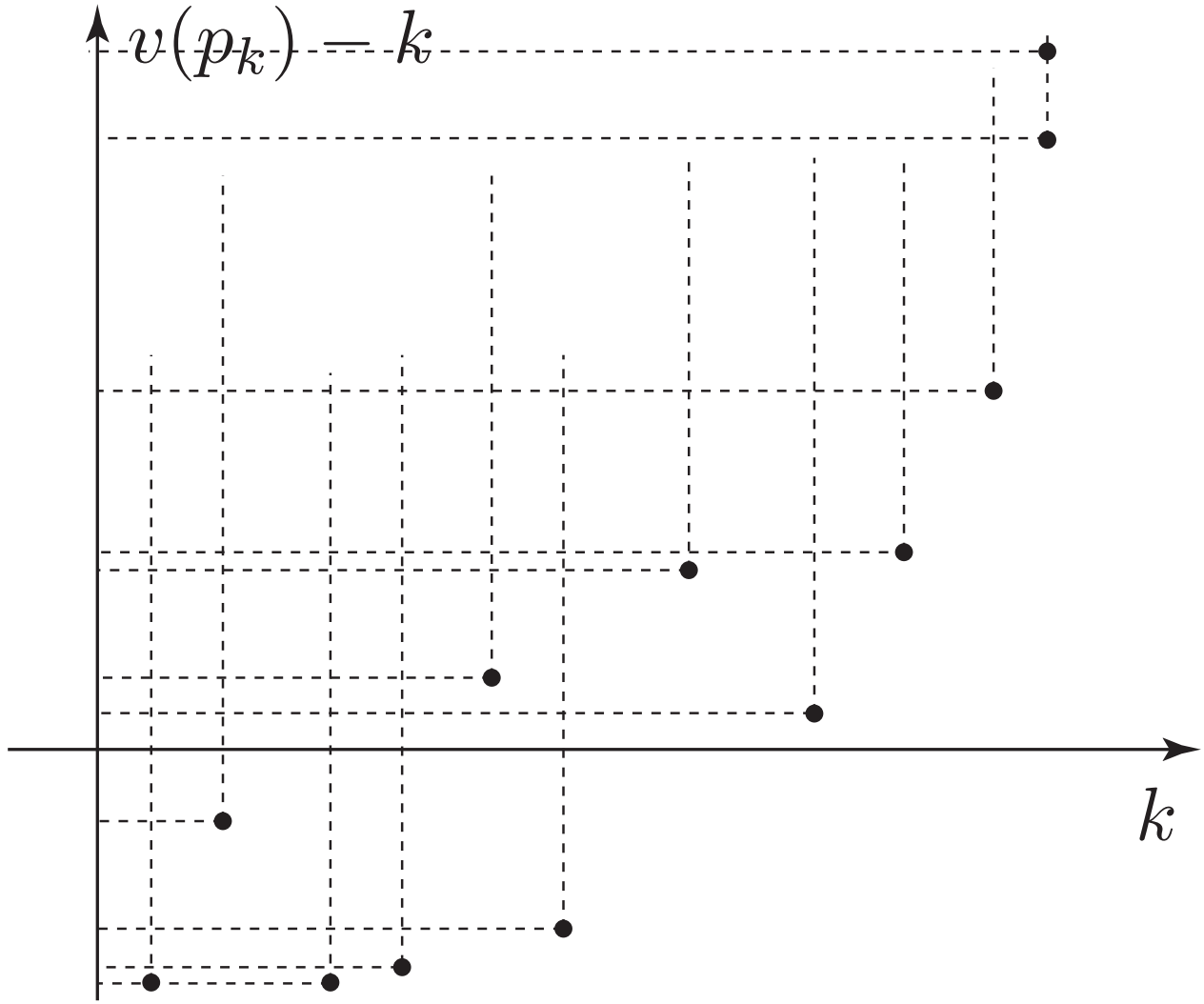

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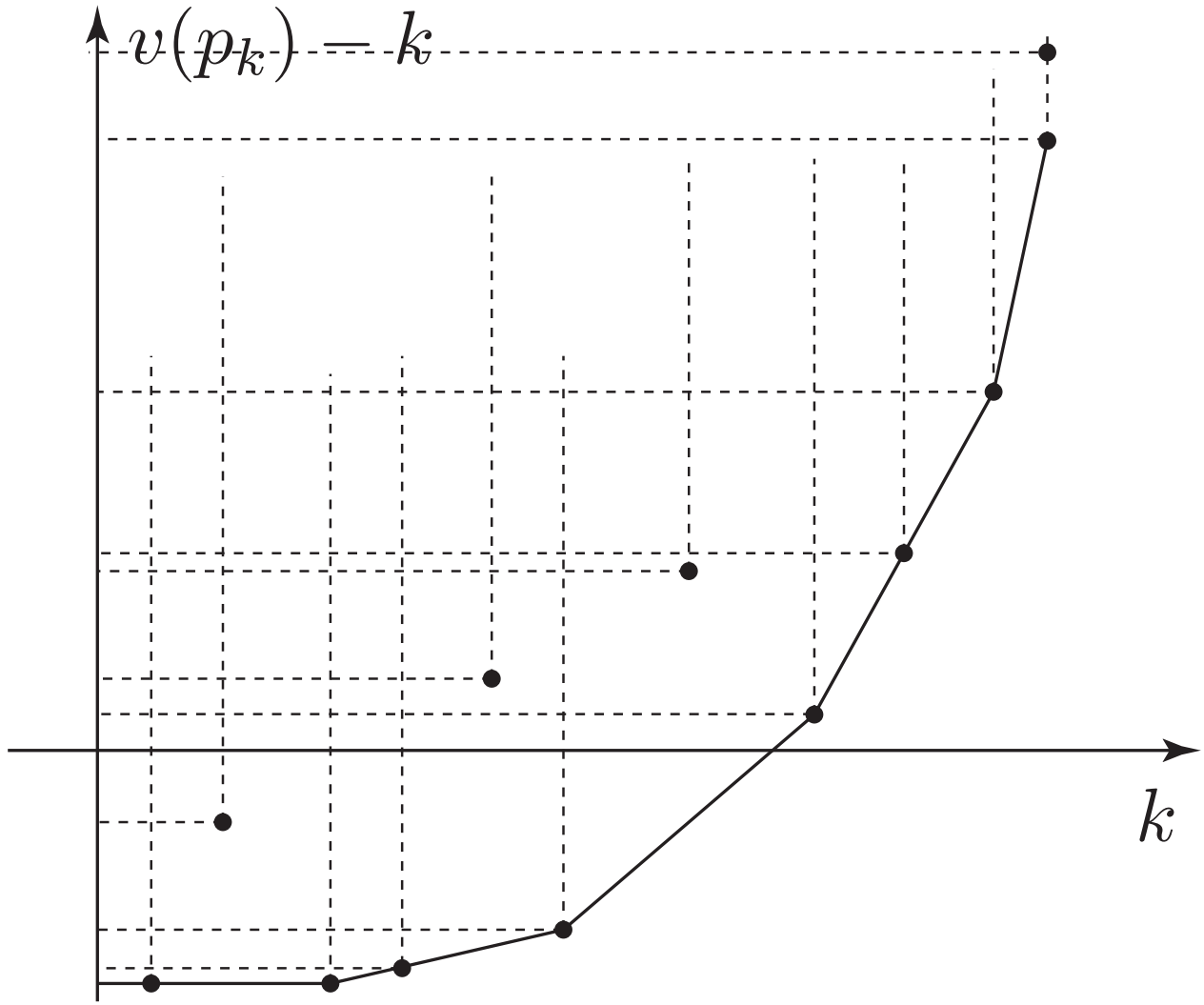

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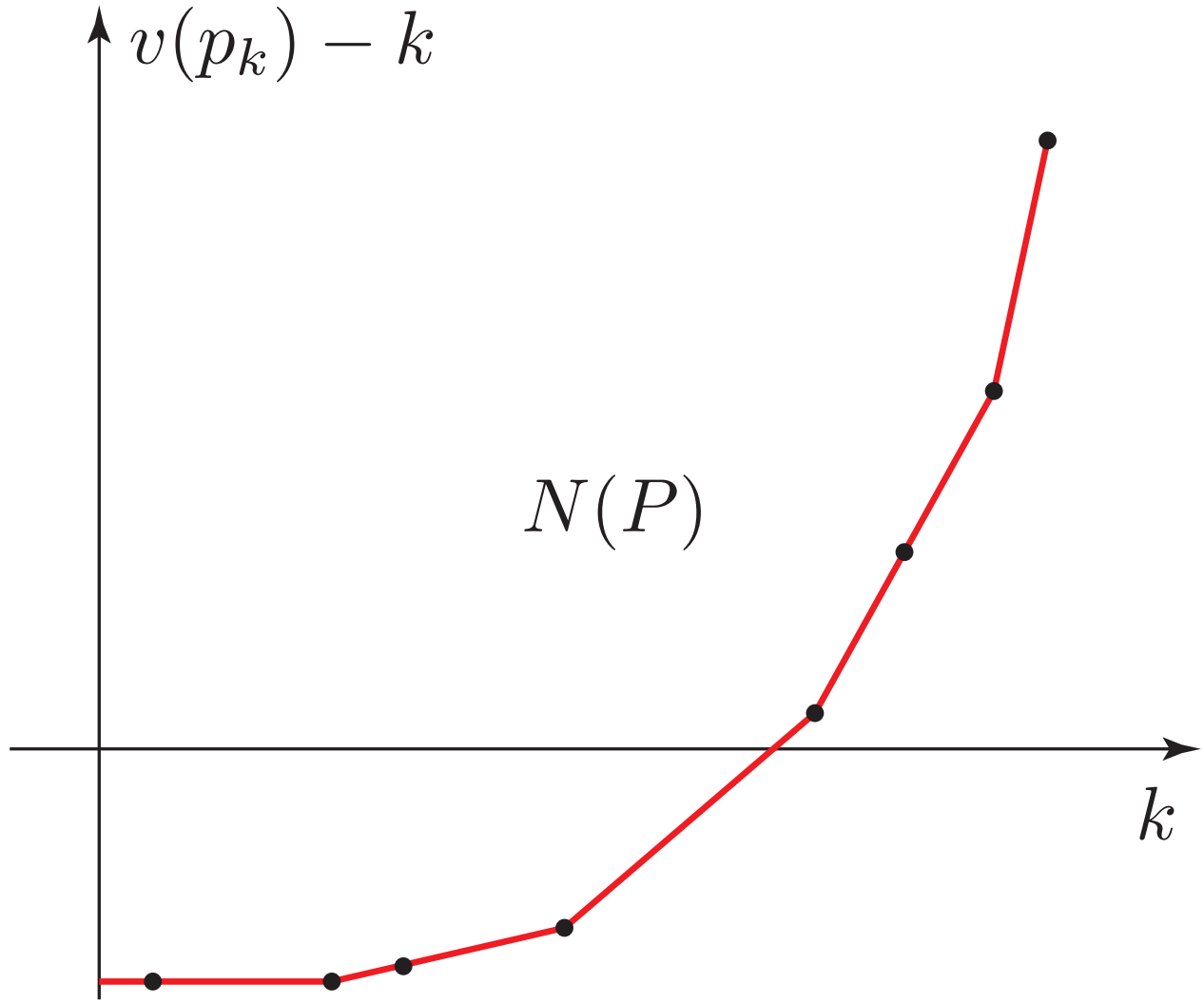

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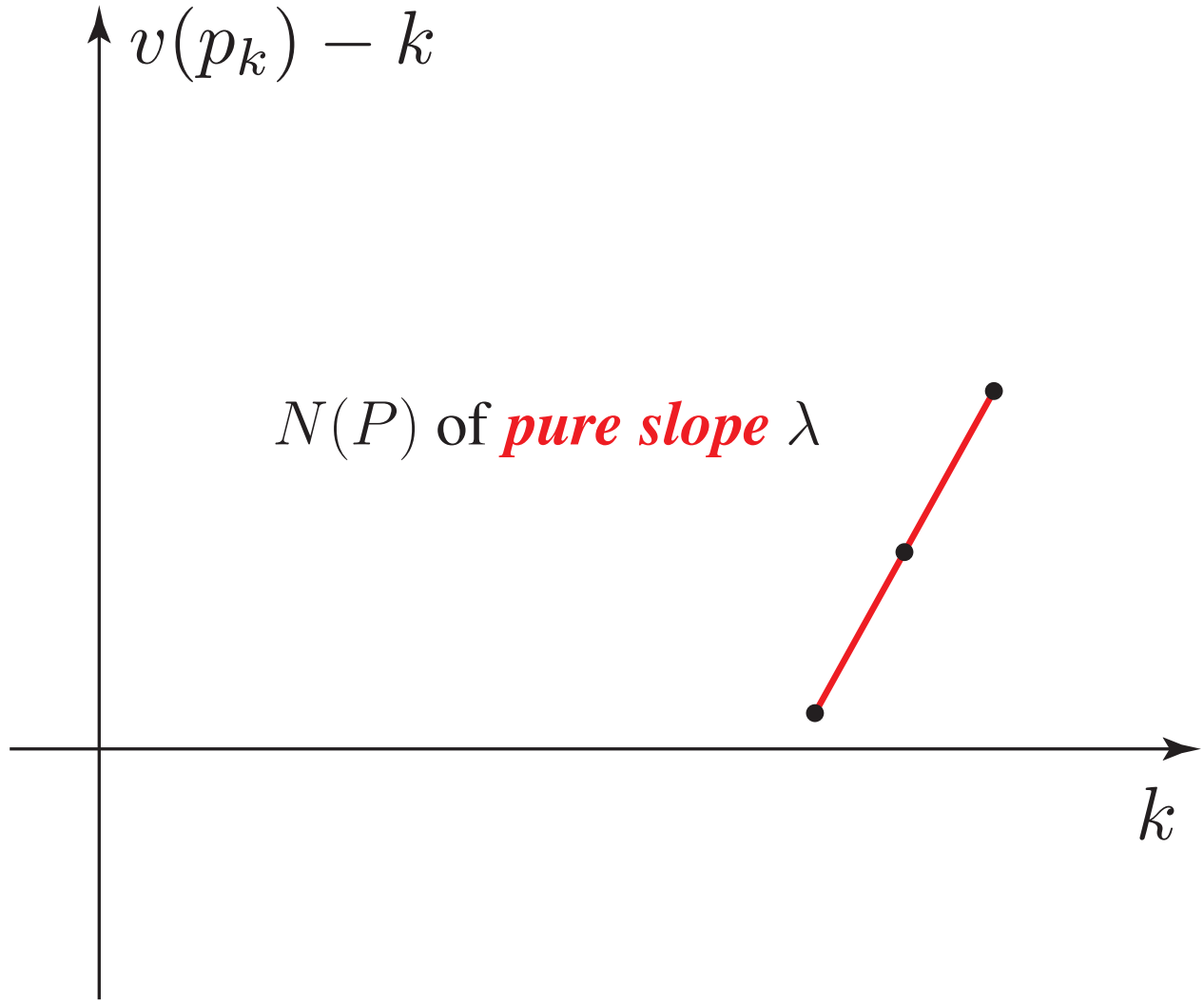

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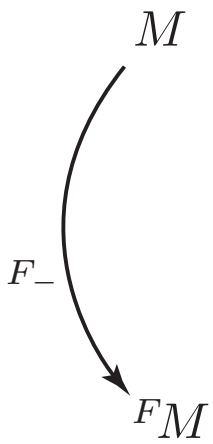

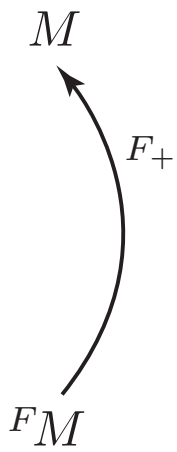

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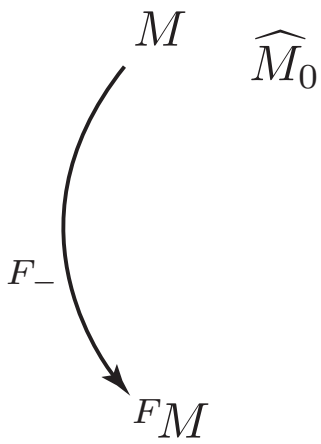

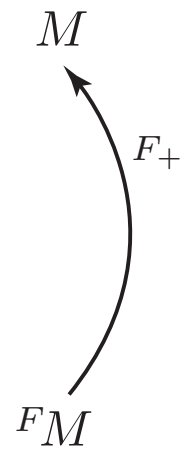

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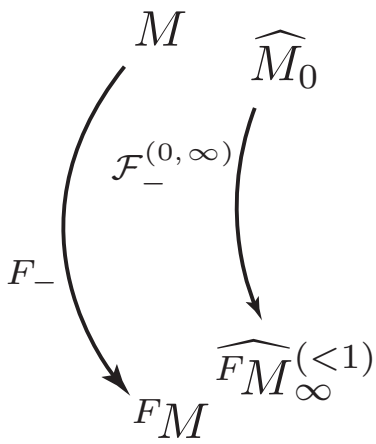

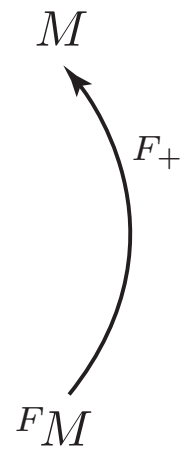

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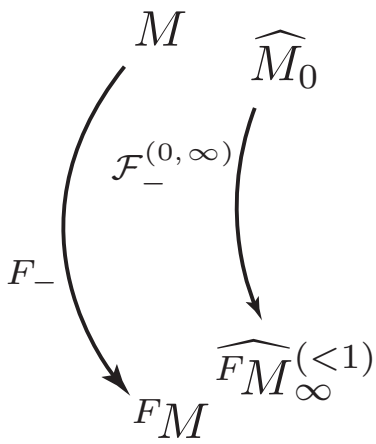

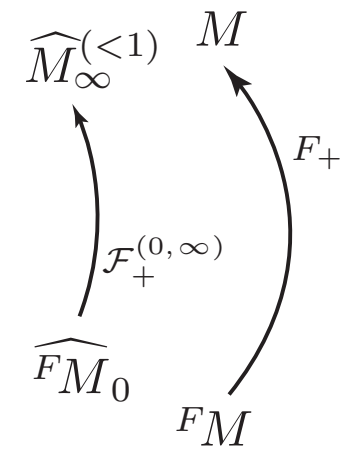

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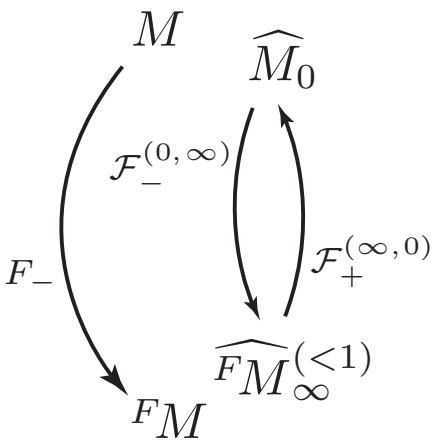

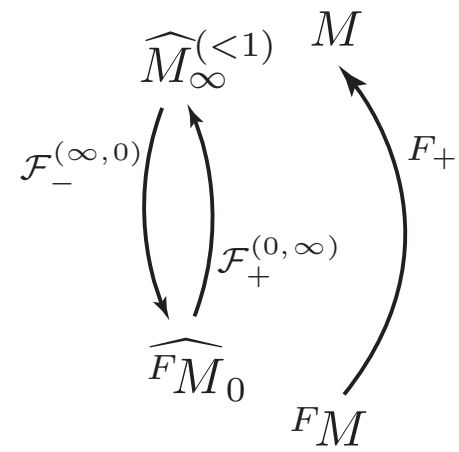

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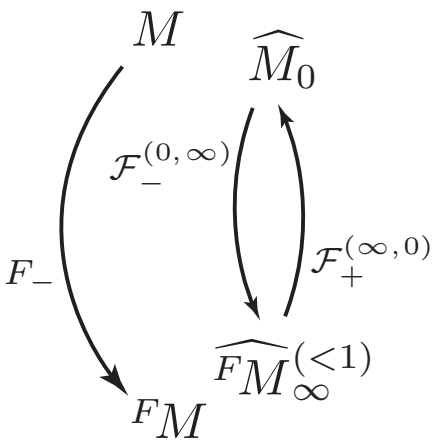

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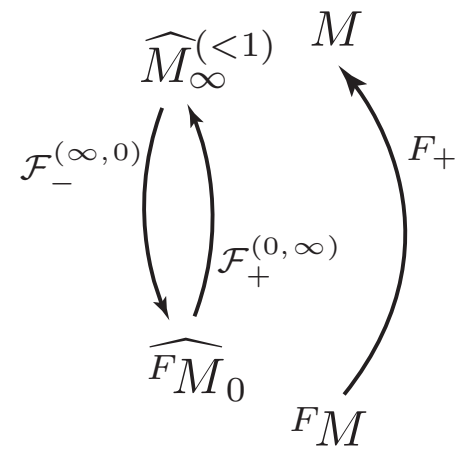

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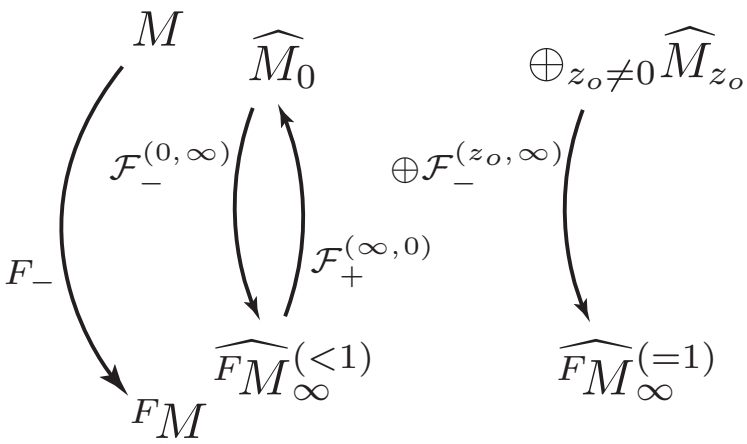

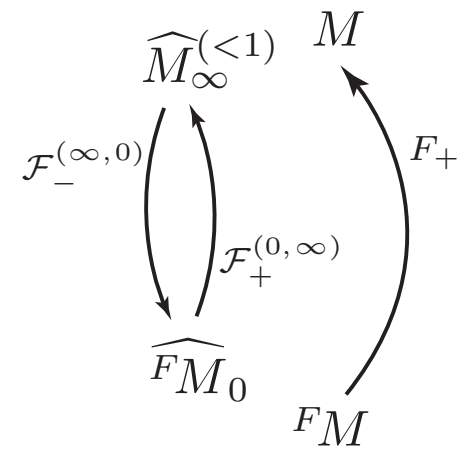

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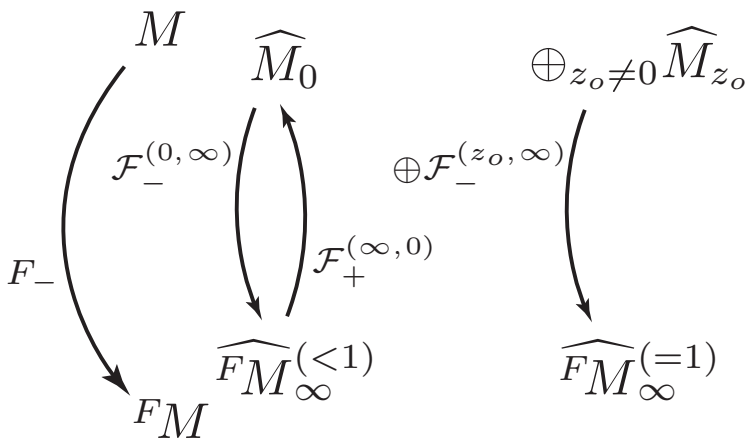

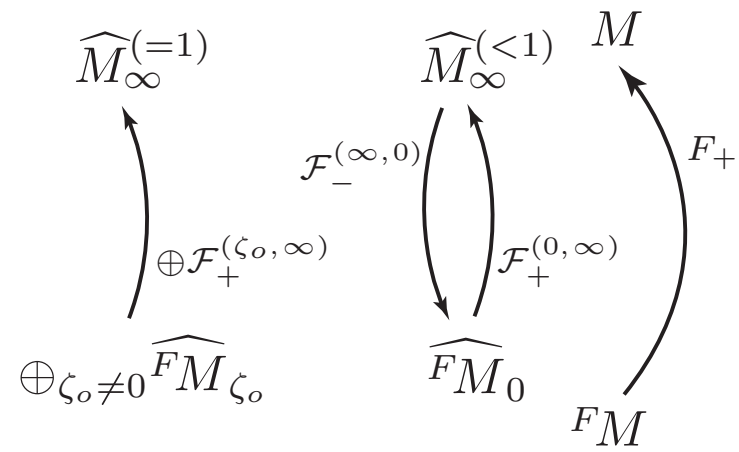

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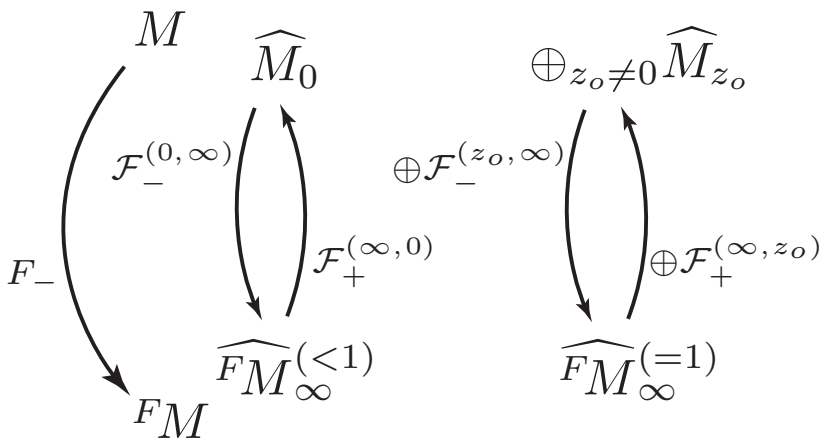

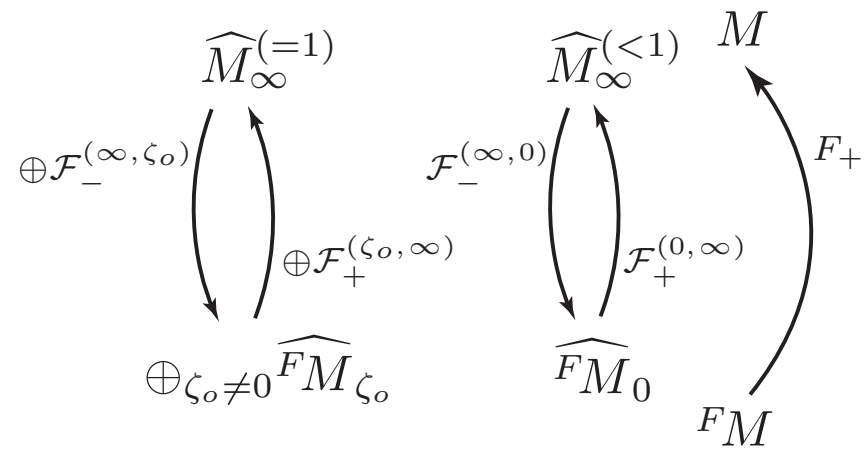

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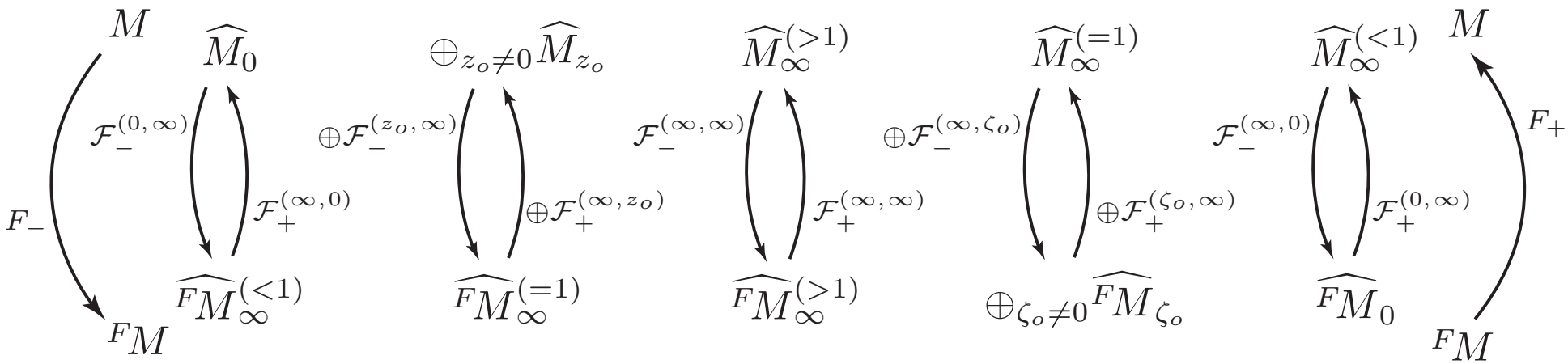

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  $\theta\partial_\theta(1 \otimes f) = 1 \otimes \frac{1}{2\pi i}\log(cv)(f).$

- $\mathscr{F}_\pm^{(0,\infty)}\mathrm{El}(\rho,\varphi,R) = \mathrm{El}({}^F\rho_\pm, {}^F\varphi, {}^FR),$

  - ${}^F\rho_\pm(u) = \mp\frac{\rho'(u)}{\varphi'(u)},$
  - ${}^F\varphi(u) = \varphi(u) - \frac{\rho(u)}{\rho'(u)}\varphi'(u),$
  - ${}^FR \simeq R \otimes L_q, \quad L_q = (\mathbb{C}((u)), d - \frac{q}{2}\frac{du}{u})$

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  - $(j_{!*}M_U)_U = M_U$,
  - $N \subset M$ and $N_U = \begin{cases} M_U \\ 0 \end{cases} \implies N = \begin{cases} M \\ 0 \end{cases}$.

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Middle Fourier transform

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Answer: Invertible when restricted to ***irred***. M_U.

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REMARK: $\otimes$ and μ^* keep the rank cst.

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Middle Fourier, and hence $\star$, ***do not***.

Katz algorithm

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- $\exists$? a compos. λ of Katz transf.s, s.t. $\mathbf{rk}\,\lambda(M_U) < \mathbf{rk}\,M_U$

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Deligne-Simpson & rigidity

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Deligne-Simpson & rigidity

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DEFINITION: M_U is ***rigid*** if it is uniquely determined (up to isom.) by the isom. classes of $\widehat{M_U}_{z_o}$, $z_o \in Z \cup \{\infty\}$.

Deligne-Simpson & rigidity

EXAMPLE (***Rank one***):

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$\Rightarrow$ Any rank one module with ∂_z is rigid.

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DEFINITION 2.7(***Index of rigidity***):

$$\mathrm{rig}\, M_U := \chi(\mathrm{End}(M_U)) - \sum_{z_o \in Z \cup \{\infty\}} h_{z_o}(\mathrm{End}(M_U)).$$

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An example

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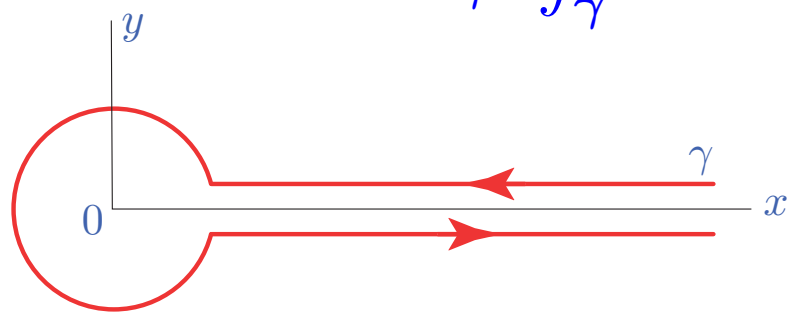

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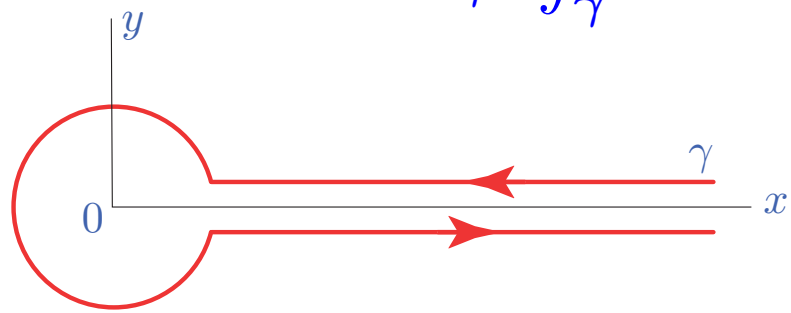

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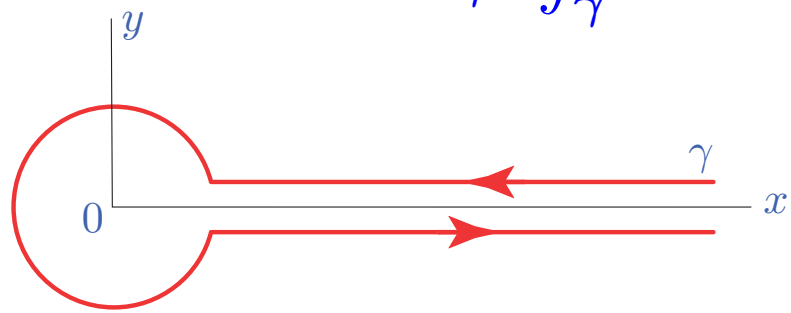

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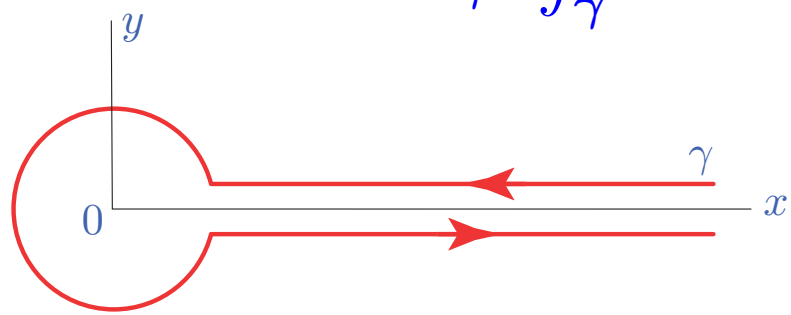

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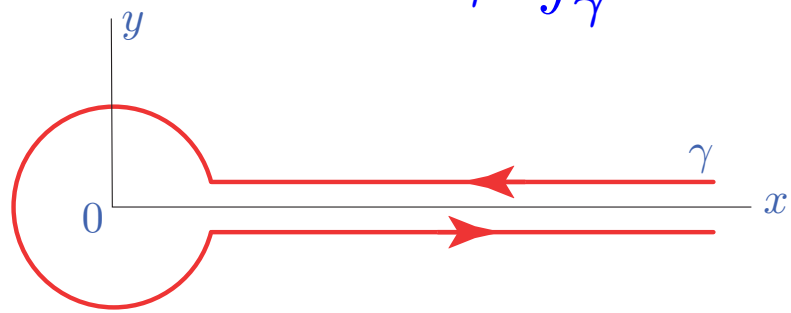

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Express $\det H^0_{\mathrm{DR}}(M) \otimes \det H^{-1}_{\mathrm{DR}}(M)$ in terms of

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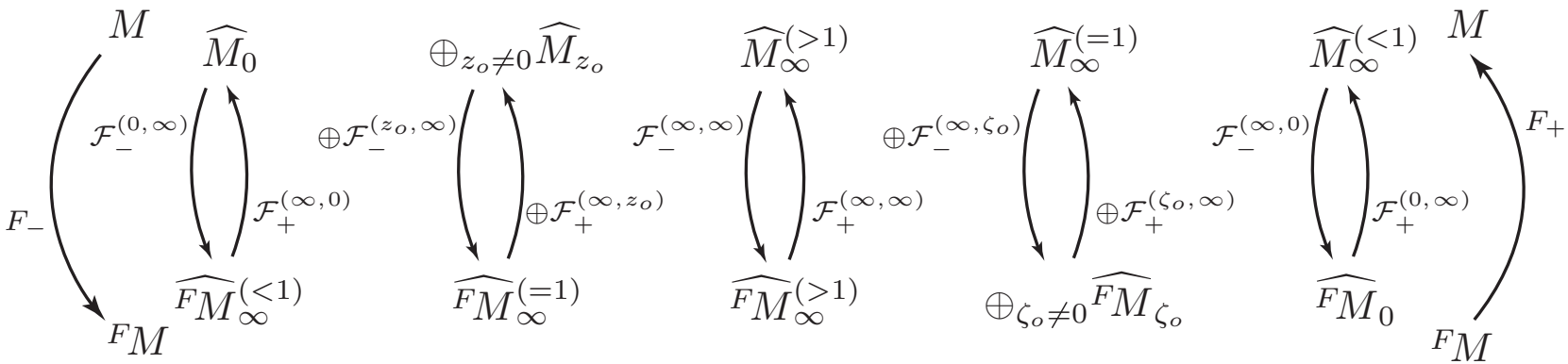

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- $\Rightarrow$ canonical identification $(\det {}^F\mathcal{E})_0 \simeq (\det {}^F\mathcal{E})_\infty$,
- Compute $(\det {}^F\mathcal{E})_\infty$ in terms of $\det F_i$, $\det E_{i,\rho,\varphi}$.

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I recall:

$$\widehat{M}^{(0)}_{z_i} \iff E_i \underset{v_i}{\overset{c_i}{\rightleftarrows}} F_i$$

$$\widehat{M}^{(\lambda)}_{z_i} = \bigoplus_{q/p=\lambda} \mathrm{El}(\rho, \varphi, E_{i,\rho,\varphi}, T_{i,\rho,\varphi})$$

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${}^F E'_U \subset {}^F M_U$ and ${}^F E'_\infty = \bigoplus_i \left[F_i \oplus \bigoplus_{\rho,\varphi} (E_{i,\rho,\varphi})^{p+q} \right]$

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Conclusion:

$$\det H^0_{\mathrm{DR}}(M) \otimes \det H^{-1}_{\mathrm{DR}}(M)$$
$$=$$
$$\bigotimes_i \left[\det F_i \otimes \bigotimes_{\rho,\varphi} (\det E_{i,\rho,\varphi})^{\otimes p+q} \right] \otimes (\det E_\infty)^{-1}$$