An equivariant Riemann-Roch theorem for complete, simplicial toric varieties

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Introduction

The theory of toric varieties establishes a now classical connection between algebraic geometry and convex polytopes. In particular, as observed by Danilov in the seventies, finding a closed formula for the Todd class of complete toric varieties would have important consequences for enumeration of lattice points in convex lattice polytopes. Since then, a number of such formulas have been proposed; see [M], [P1], [P2]... The Todd class of complete simplicial toric varieties is computed in [G-G-K], using the Riemann-Roch formula of T. Kawasaki [K].

On the other hand, it has been realized that the sum of values of a function f over all lattice points of a convex lattice polytope P can be obtained from the integral of f over the deformed polytope (where all facets of P are translated independently) by applying to the translation variables, a differential operator of infinite order: the Todd operator. For this, we refer to [K-P] and its subsequent generalizations [K-K], [C-S1], [B-V], [C-S2]... These results are higher-dimensional analogues of the classical Euler-MacLaurin summation formula (the case where P is an interval).

The Todd operator of a convex lattice polytope P is closely related to the Todd class of the projective toric variety associated to the normal fan of P. In the present paper, we explain this connection as follows. We obtain an equivariant Riemann-Roch theorem for any complete, simplicial toric variety X (theorem 4.1). It involves the equivariant Todd class of X, a lift of the

Todd class to the completion of the equivariant cohomology ring. We obtain a closed formula for this equivariant Todd class (theorem 4.2). Generalizing work of Pommersheim [P1], we relate this class to higher Dedekind sums (proposition 4.4). Finally, we show that the generalized Euler-MacLaurin summation formula for convex lattice polytopes, is a consequence of our equivariant Riemann-Roch theorem (theorem 4.5). We refer to [B-V] for a direct, elementary proof of this summation formula in the case of simple polytopes.

Observe that a closed formula for the Todd class of a complete toric variety X must involve some choices, because the rational Chow group of X has no distinguished basis. In contrast, the equivariant cohomology ring of X has a very convenient description when X is simplicial, either as the Stanley-Reisner ring of the corresponding fan Σ (see [B-D-P]), or as the ring of continuous, piecewise polynomial functions on Σ (see 3.2 below). This makes the equivariant Todd class easier to handle than the "usual" Todd class.

Although our results may look fancy, our proofs use little theory. In particular, instead of relying on the equivariant Riemann-Roch theorem for orbifolds (see [V]), we construct explicitely all objects involved in it, e.g. the Grothendieck group of linearized coherent sheaves, and the equivariant Chern character with values in the completion of the equivariant cohomology ring. Then the equivariant Riemann-Roch formula is checked in a straightforward way.

Our results are stated over the field of complex numbers; they should hold for any algebraically closed field, with equivariant cohomology replaced by equivariant Chow group, see [E-G]. However, a full treatment based on equivariant Chow theory would require further developments of this theory.

Notation

We begin with some notation and results concerning the theory of toric varieties; we refer to [O] and [F2] for expositions of this theory. Denote by T a d-dimensional torus, by $M = \mathrm{Hom}(T, \mathbf{C}^*)$ its character group, and by $N = \mathrm{Hom}(\mathbf{C}^*, T)$ the group of one-parameter subgroups of T. There is a natural pairing $M \times N \to \mathbf{Z} : (m, n) \mapsto \langle m, n \rangle$ where $\langle m, n \rangle$ is the integer such that $m(n(t)) = t^{\langle m,n \rangle}$ for all $t \in \mathbf{C}^*$.

We denote by X a toric variety, i.e. a normal variety where T acts with a dense orbit isomorphic to T. Such a variety is described by its fan Σ in $N_{\mathbf{Q}}$. Moreover, $X = X_\Sigma$ has only quotient singularities by finite groups (resp. X is smooth) if and only if each cone in Σ is simplicial (resp. is generated by part of a basis of N).

There is a bijection $\sigma \mapsto X_\sigma$ between cones in Σ and T-stable open affine subsets of X. We denote by Ω_σ the unique closed T-orbit in X_σ; then $\sigma \to \Omega_\sigma$ sets up a bijection from Σ to the set of T-orbits in X. Moreover, we have $\dim(\sigma) = \operatorname{codim}(\Omega_\sigma)$.

For a cone σ, we denote by N_σ the subgroup of N generated by $\sigma \cap N$, and by $\sigma^\perp \subset M_{\mathbf{Q}}$ the set of linear forms on $N_{\mathbf{Q}}$ which vanish identically on σ. We denote by T_σ the subgroup of T with character group $M/M \cap \sigma^\perp$; then T_σ is connected, with group of one-parameter subgroups N_σ. Observe that $\Omega_\sigma = T/T_\sigma$ and that there is a T-equivariant retraction $r_\sigma : X_\sigma \to \Omega_\sigma$. It follows that X_σ is isomorphic to $T \times^{T_\sigma} S_\sigma$ where S_σ is an affine, T_σ-toric variety with a fixed point.

For $0 \leq j \leq d$, we denote by $\Sigma(j)$ the set of j-dimensional cones in Σ. In particular, $\Sigma(1)$ is the set of edges of Σ. For $\tau \in \Sigma(1)$ we denote by n_τ the generator of the semigroup $\tau \cap N$ and by $D_\tau = \overline{\Omega_\tau}$ the T-stable prime divisor associated to τ.

1. Linearized sheaves on toric varieties

1.1. *Existence of resolutions*

Let $\mathcal{F}$ be a coherent sheaf on a toric variety X. Recall that a *T-linearization* of $\mathcal{F}$ is an action of T on $\mathcal{F}$ which is compatible with its structure of an $\mathcal{O}_X$-module. For example, if D is a T-stable (Weil) divisor on X, then the coherent sheaf $\mathcal{O}_X(D)$ has a canonical linearization.

Given a T-linearized sheaf $\mathcal{F}$ and $m \in M$, we denote by $\mathcal{F} \otimes m$ the sheaf $\mathcal{F}$ with its T-linearization twisted by the character m: the T-module $H^0(X_\sigma, \mathcal{F} \otimes m)$ is the tensor product of $H^0(X_\sigma, \mathcal{F})$ with the T-module $\mathbf{C}m$.

Any linearized *locally free* sheaf $\mathcal{E}$ on an affine toric variety is trivial, i.e. $\mathcal{E}$ can be written as a direct sum of sheaves $\mathcal{O}_X \otimes m$ (this follows e.g. from [B-H] 10.1). As a global analogue of this result, we have the following

Theorem. *Let X be a toric variety. Then any coherent, T-linearized sheaf on X has a finite resolution by finite direct sums of T-linearized sheaves $\mathcal{O}_X(D) \otimes m$ where D is a T-stable divisor in X, and $m \in M$. Moreover, any* *coherent sheaf on X has a finite resolution by finite direct sums of sheaves $\mathcal{O}_X(D)$ where D is as before.*

Proof. First we recall how to obtain X as a quotient of a smooth toric variety by a torus; see [A] and [C] for other versions of the following construction.

Let $\mathbf{Z}^{\Sigma(1)} = \oplus_{\tau\in\Sigma(1)} \mathbf{Z}e_\tau$ be the free abelian group on the set $\Sigma(1)$. Set $\tilde{N} := N \times \mathbf{Z}^{\Sigma(1)}$ and denote by $\tilde{T} = \mathbf{C}^* \otimes_{\mathbf{Z}} \tilde{N}$ the associated torus. Then $T = \mathbf{C}^* \otimes_{\mathbf{Z}} N$ embeds into $\tilde{T}$.

To any cone $\sigma \in \Sigma$, we associate the cone $\tilde{\sigma}$ in $\tilde{N}_{\mathbf{Q}}$ generated by the e_τ such that τ is an edge of σ. Then the family $(\tilde{\sigma})_{\sigma\in\Sigma}$ is a fan in $\tilde{N}_{\mathbf{Q}}$, contained in $\{0\} \times \mathbf{Q}^{\Sigma(1)}$. We denote by $\tilde{X}$ the associated toric variety. If $\tilde{T}$ is identified with $T \times (\mathbf{C}^*)^{\Sigma(1)}$, then $\tilde{X}$ is identified with the product of T by an open subset of $\mathbf{C}^{\Sigma(1)}$. In particular, $\tilde{X}$ is smooth.

The map

$$f : \tilde{N} \to N$$
$$(n, \sum x_\tau e_\tau) \to n + \sum x_\tau n_\tau$$

is surjective with kernel

$$N' := \{(-\sum x_\tau n_\tau, \sum x_\tau e_\tau)\} \simeq \mathbf{Z}^{\Sigma(1)}.$$

Therefore, f induces an exact sequence

$$1 \to T' \to \tilde{T} \to T \to 1$$

where $T' \simeq (\mathbf{C}^*)^{\Sigma(1)}$, and we have a T'-invariant morphism $f : \tilde{X} \to X$. Observe that $f^{-1}(X_\sigma) = \tilde{X}_{\tilde{\sigma}}$ for all $\sigma \in \Sigma$. It follows that f is affine.

1.2. *Proof of theorem 1.1 (continued)*

Let $\tilde{m}$ be a character of $\tilde{T}$. Denote by m its restriction to T, and by $\hat{m}$ the unique character of $\tilde{T} = T \times (\mathbf{C}^*)^{\Sigma(1)}$ such that $\hat{m}$ is trivial on $(\mathbf{C}^*)^{\Sigma(1)}$ and that $\hat{m}|_T = m$. Set $a_\tau = -\langle \tilde{m}, e_\tau \rangle$ for each $\tau \in \Sigma(1)$.

Lemma. *There is an isomorphism of $\tilde{T}$-linearized coherent sheaves:*

$$\mathcal{O}_{\tilde{X}} \otimes \tilde{m} \simeq \mathcal{O}_{\tilde{X}}(\sum_{\tau\in\Sigma(1)} a_\tau D_{\tilde{\tau}}) \otimes \hat{m} \ .$$

Moreover, there is an isomorphism of T-linearized coherent sheaves:

$$f_*^{T'}(\mathcal{O}_{\tilde{X}} \otimes \tilde{m}) \simeq \mathcal{O}_X(\sum_{\tau\in\Sigma(1)} a_\tau D_\tau) \otimes m \ .$$

In particular, $f_*^{T'}\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$, i.e. $f : \tilde{X} \to X$ is the universal quotient by T'.

Proof. Denote by (e_τ^*) the dual basis of (e_τ). Then we can consider e_τ^* as a character of $\tilde{T}$ and the divisor of this character in $\tilde{X}$ is $-D_{\tilde{\tau}}$. Writing $\tilde{m} = \hat{m} - \sum_{\tau\in\Sigma(1)} a_\tau e_\tau^*$, we obtain our first isomorphism. For the second isomorphism, observe that $f^{-1}(D_\tau) = D_{\tilde{\tau}}$ and hence we have a map $\mathcal{O}_X(\sum_{\tau\in\Sigma(1)} a_\tau D_\tau) \to f_*^{T'}\mathcal{O}_{\tilde{X}}(\sum_{\tau\in\Sigma(1)} a_\tau D_{\tilde{\tau}})$. We check that this map is an isomorphism over X_σ for a given $\sigma \in \Sigma$. Namely, the vector space $H^0(X_\sigma, \mathcal{O}_X(\sum_{\tau\in\Sigma(1)} a_\tau D_\tau))$ is generated by all $m \in M$ such that $\langle m, n_\tau\rangle + a_\tau \geq 0$ for all $\tau \in \sigma(1)$, whereas the space of T'-invariants in $H^0(\tilde{X}_{\tilde{\sigma}}, \mathcal{O}_{\tilde{X}}(\sum_{\tau\in\Sigma(1)} a_\tau D_{\tilde{\tau}}))$ is generated by all $\tilde{m}$ in $\mathrm{Hom}(\tilde{T}, \mathbf{C}^*)^{T'} = M$ such that $\langle \tilde{m}, n_{\tilde{\tau}}\rangle + a_\tau \geq 0$.

End of the proof of theorem 1.1. Let $\mathcal{F}$ be a coherent, T-linearized sheaf on X. Then $f^*\mathcal{F}$ is a coherent, $\tilde{T}$-linearized sheaf on $\tilde{X}$. Set $e := d + |\Sigma(1)|$ and embed $\tilde{X}$ into $\mathbf{C}^e$ as an open subset, invariant under the natural action of $\tilde{T} = (\mathbf{C}^*)^e$. Then $f^*\mathcal{F}$ extends to a coherent, $(\mathbf{C}^*)^e$-linearized sheaf on $\mathbf{C}^e$, see [T] 2.4. The latter corresponds to a finite, $\mathbf{Z}^e$-graded module over the polynomial ring $\mathbf{C}[x_1, \ldots, x_e]$. Using the theorem of Hilbert-Serre, it follows that there exists an exact sequence of $\tilde{T}$-linearized coherent sheaves:

$$0 \to \mathcal{O}_{\tilde{X}} \otimes V_e \to \cdots \to \mathcal{O}_{\tilde{X}} \otimes V_0 \to f^*\mathcal{F} \to 0$$

where each V_i is a finite dimensional module over $\tilde{T}$. Because f is affine, f_* is exact and satisfies to the projection formula. Moreover, taking invariants by the torus T' is exact. Therefore, we have an exact sequence of T-linearized coherent sheaves

$$0 \to f_*^{T'}(\mathcal{O}_{\tilde{X}} \otimes V_e) \to \cdots \to f_*^{T'}(\mathcal{O}_{\tilde{X}} \otimes V_0) \to \mathcal{F} \to 0$$

To finish the proof, decompose each V_i into a direct sum of one-dimensional modules over $\tilde{T}$. Such a module is associated to a character $\tilde{m}$ of $\tilde{T}$, and we conclude by the lemma above. In the case where $\mathcal{F}$ is an arbitrary coherent sheaf, $f^*\mathcal{F}$ is a T'-linearized coherent sheaf on $\tilde{X}$ and our arguments adapt easily.

Denote by $G(X)$ ($G^T(X)$) the Grothendieck group of (T-linearized) coherent sheaves on X, see [T]. Then theorem 1.1 imply readily the following

Corollary. *For any toric variety X, the group $G^T(X)$ is generated by the classes of $\mathcal{O}_X(D)\otimes m$, where D is a T-stable divisor in X, and m is a character of T. Moreover, the forgetful map $G^T(X) \to G(X)$ is surjective.*

1.3. *Euler characteristics*

Let $\mathbf{Z}[M]$ be the group ring over $\mathbf{Z}$ of the abelian group M. We denote by $\mathbf{Z}[[M]]$ the set of all formal power series $\sum_{m\in M} a_m e^m$ with integral coefficients. Then $\mathbf{Z}[[M]]$ is a module over $\mathbf{Z}[M]$, multiplication by e^m being defined by $e^m \sum_{\mu\in M} a_\mu e^\mu = \sum_{\mu\in M} a_{\mu-m} e^\mu$. We call $f \in \mathbf{Z}[[M]]$ *summable* if there exist $P \in \mathbf{Z}[M]$ and a finite sequence $(m_i)_{i\in I}$ of non-zero points in M, such that the following equality holds in $\mathbf{Z}[[M]]$:

$$f \prod_{i\in I} (1 - e^{m_i}) = P \ .$$

Then the *sum* of f is defined as the following element of $\mathbf{Q}(M)$ (the fraction field of $\mathbf{Z}[M]$):

$$\mathcal{S}(f) = P \prod_{i\in I} (1 - e^{m_i})^{-1} \ .$$

Clearly, $\mathcal{S}(f)$ does not depend of the choices of P and of the sequence $(m_i)_{i\in I}$.

To any coherent, T-linearized sheaf $\mathcal{F}$ on a toric variety X, and to any cone σ in the fan of X, we associate a formal power series $\chi_\sigma^T(\mathcal{F})$ as follows. The space $H^0(X_\sigma, \mathcal{F})$ is a rational T-module, and a finite module over $H^0(X_\sigma, \mathcal{O}_X)$ as well. Both structures are compatible; moreover, the multiplicity of any character of T in $H^0(X_\sigma, \mathcal{O}_X)$ is zero or one. It follows that the multiplicity of any $m \in M$ in $H^0(X_\sigma, \mathcal{F})$ is finite. Denote this multiplicity by $\operatorname{mult}(m, H^0(X_\sigma, \mathcal{F}))$ and set:

$$\chi_\sigma^T(\mathcal{F}) = \sum_{m\in M} \operatorname{mult}(m, H^0(X_\sigma, \mathcal{F})) e^m \ .$$

Proposition. *With the notation as above, the formal power series $\chi_\sigma^T(\mathcal{F})$ is summable. Moreover, its sum is zero if and only if $\dim(\sigma) < d$.*

Proof. We may assume that $X = X_\sigma$ is affine; we set $A := H^0(X_\sigma, \mathcal{O}_X)$. Then $F := H^0(X_\sigma, \mathcal{F})$ is a finite A-module with a compatible T-action.

If $\dim(\sigma) < d$ then we may choose $m_0 \in M$ such that $\langle m_0, n\rangle = 0$ for all $n \in \sigma$. We can consider m_0 as an invertible element of A; it follows that the multiplicity of m in F is invariant under translation by m_0. Therefore, we have $(1 - e^{m_0})\chi_\sigma^T(\mathcal{F}) = 0$, i.e. $\chi_\sigma^T(\mathcal{F})$ is summable with sum zero.

If $\dim(\sigma) = d$, choose an interior point n_0 of σ, and consider n_0 as a linear form on $M_{\mathbf{R}}$. Then n_0 takes positive values at all non-zero weights of A. Now the proof of existence of the Hilbert series of a finite, graded module over a finitely generated, graded algebra can be easily adapted, to yield summability of $\chi_\sigma^T(\mathcal{F})$. If its sum is zero, let $m_1, \ldots, m_r$ be non-zero elements of M such that $\chi_\sigma^T(\mathcal{F}) \prod_{i=1}^r (1 - e^{m_i}) = 0$. Changing m_i into $-m_i$ (which amounts to multiplication of $1 - e^{m_i}$ by $-e^{-m_i}$), we may assume that $\langle n_0, m_1 \rangle, \ldots, \langle n_0, m_r \rangle$ are non-negative. On the other hand, there exists a weight m_0 of F such that $\langle n_0, m_0 \rangle \leq \langle n_0, m \rangle$ for all weights m of F. Therefore, the coefficient of e^{m_0} in $\chi_\sigma^T(\mathcal{F}) \prod_{i=1}^r (1 - e^{m_i})$ cannot vanish, a contradiction.

Corollary. *The multiplicity of any $m \in M$ in any cohomology group $H^i(X, \mathcal{F})$ is finite. Moreover, the formal power series*

$$\chi^T(\mathcal{F}) := \sum_{m \in M} \sum_{i=0}^{d} (-1)^i \operatorname{mult}(m, H^i(X, \mathcal{F}))\, e^m$$

is summable, and we have

$$\mathcal{S}(\chi^T(\mathcal{F})) = \sum_{\sigma \in \Sigma(d)} \mathcal{S}(\chi_\sigma^T(\mathcal{F})) \ .$$

Proof. The T-module $H^i(X, \mathcal{F})$ is the i-th cohomology space of the Cech complex associated to the covering $(X_\sigma)_{\sigma \in \Sigma}$ of X; namely, each X_σ is affine and T-stable, and the family (X_σ) is stable under intersections. This observation, combined with the proposition above, implies readily our statements.

Remark. Both maps $\mathcal{F} \to \chi^T(\mathcal{F})$ and $\mathcal{F} \to \chi_\sigma^T(\mathcal{F})$ are additive on exact sequences. Therefore, these maps define $\chi^T, \chi_\sigma^T : G^T(X) \to \mathbf{Z}[[M]]$. Clearly, χ^T and χ_σ^T are morphisms of $\mathbf{Z}[M]$-modules.

1.4. *An exact sequence*

Consider a toric variety X and a closed orbit Ω_σ in X, associated to a (maximal) cone σ in Σ. Denote by $i_\sigma : \Omega_\sigma \to X$ the inclusion, and by $j_\sigma : X \setminus \Omega_\sigma \to X$ the inclusion of the complement of Ω_σ in X.

Proposition. *(i) The map $G^T(\Omega_\sigma) \to \mathbf{Z}[[M]] : [\mathcal{F}] \to \chi_\sigma^T(i_{\sigma*}\mathcal{F})$ is injective.*
(ii) The sequence

$$0 \to G^T(\Omega_\sigma) \to G^T(X) \to G^T(X \setminus \Omega_\sigma) \to 0$$

is exact.

Proof. (i) Recall that the isotropy group T_σ of Ω_σ is connected, with character group $M/M \cap \sigma^\perp$. Hence $G^T(\Omega_\sigma) = G^T(T/T_\sigma)$ identifies to $\mathbf{Z}[M/M \cap \sigma^\perp]$. Moreover, denoting by u_σ the image in $G^T(\Omega_\sigma)$ of the structure sheaf of Ω_σ, we have

$$\chi_\sigma^T(i_{\sigma*}u_\sigma) = \sum_{m \in M \cap \sigma^\perp} e^m \ .$$

Therefore, $\chi_\sigma^T \circ i_{\sigma*}$ identifies to the map

$$\mathbf{Z}[M/M \cap \sigma^\perp] \to \mathbf{Z}[[M]]$$

$$e^{\mu + (M \cap \sigma^\perp)} \mapsto \sum_{m \in M \cap \sigma^\perp} e^{\mu + m}$$

(where $\mu \in M$) and the latter is clearly injective.
(ii) By theorem 2.7 in [T] , it suffices to check that $i_{\sigma*} : G^T(\Omega_\sigma) \to G^T(X)$ is injective. But this follows from (i).

1.5. *Localization*

Denote by $i : X^T \to X$ the inclusion of the fixed point set (which coincides with the fixed point scheme in our case of a toric variety). Then i induces a morphism of $\mathbf{Z}[M]$-modules $i_* : G^T(X^T) \to G^T(X)$. Observe that the $\mathbf{Z}[M]$-module $G^T(X^T)$ is isomorphic to $\prod_{\sigma \in \Sigma(d)} \mathbf{Z}[M]$. By a general localization theorem in equivariant K-theory, the map i_* is an isomorphism after inverting all $1 - e^m$, where m is a non-zero point in M; see [Q]. For toric varieties, we obtain the following more precise statement.

Proposition. *The map $i_* : G^T(X^T) \to G^T(X)$ is injective. Moreover, the cokernel of i_* is killed by any product of $1 - e^{m_\sigma}$, where σ runs over all maximal cones of positive codimension, and where m_σ is any non-zero point in $\sigma^\perp$.*
Proof. Injectivity of i_* follows from 1.4 (i). So we have an exact sequence

$$0 \to G^T(X^T) \to G^T(X) \to G^T(X \setminus X^T) \to 0 \ .$$

If Ω is an orbit in $X \setminus X^T$, then $\Omega = \Omega_\sigma$ for some cone σ of positive codimension. In this case, the $\mathbf{Z}[M]$-module $G^T(\Omega) = \mathbf{Z}[M/M \cap \sigma^\perp]$ is killed by $1 - e^{m_\sigma}$ for any non-zero point m_σ in $\sigma^\perp$. Using 1.4 (ii), it follows that $G^T(X \setminus X^T)$ is killed by any product of such terms.

2. Linearized sheaves on simplicial toric varieties

2.1. *Preliminary computations*

Let $\sigma \in \Sigma$ be a simplicial cone, and let D be a T-stable divisor of X. We will compute $\chi_\sigma^T(\mathcal{O}_X(D))$. For this, denote by $\tau_1, \ldots, \tau_r$ the edges of σ, and by $n_1, \ldots, n_r$ the corresponding primitive vectors. Choose a decomposition $M = (M \cap \sigma^\perp) \oplus M^\sigma$. Then $n_1, \ldots, n_r$ generate the dual of M^σ over the rationals. Therefore, there exist uniquely defined primitive vectors $m_1, \ldots, m_r$ in M^σ such that $\langle m_i, n_j \rangle = 0$ for all $i \neq j$ and that $\langle m_i, n_i \rangle$ is a positive integer for all i. We set $q_i := \langle m_i, n_i \rangle$.

Define integers $a_1, \ldots, a_r$ by

$$D = \sum_{i=1}^{r} a_i D_{\tau_i} + \sum_{\tau \notin \sigma(1)} a_\tau D_\tau \ .$$

We set

$$Q_D^\sigma := \{\sum_{i=1}^{r} x_i m_i \mid x_i \in \mathbf{Q},\ 0 \leq x_i + q_i^{-1} a_i < 1\} \ .$$

In particular, we set

$$Q^\sigma := \{\sum_{i=1}^{r} x_i m_i \mid x_i \in \mathbf{Q},\ 0 \leq x_i < 1\} \ .$$

Proposition. *Notation being as above, we have in* $\mathbf{Z}[[M]]$*:*

$$\chi_\sigma^T(\mathcal{O}_X) \prod_{i=1}^{r} (1 - e^{m_i}) = (\sum_{m \in M \cap \sigma^\perp} e^m)(\sum_{m \in Q^\sigma \cap M^\sigma} e^m)$$

and also

$$\chi_\sigma^T(\mathcal{O}_X(D))(\sum_{m \in Q^\sigma \cap M^\sigma} e^m) = \chi_\sigma^T(\mathcal{O}_X)(\sum_{m \in Q_D^\sigma \cap M^\sigma} e^m) \ .$$

Then $Q_D^\sigma \cap M^\sigma$ and $Q^\sigma \cap M^\sigma$ are finite sets with the same cardinality: the index in M^σ of the subgroup generated by $m_1, \ldots, m_r$.

Proof. Clearly, we have

$$\chi_\sigma^T(\mathcal{O}_X(D)) = \sum_{\langle m, n_i \rangle + a_i \geq 0} e^m = (\sum_{m \in M \cap \sigma^\perp} e^m)(\sum_{m \in M^\sigma, \langle m, n_i \rangle + a_i \geq 0} e^m) \ .$$

Consider $m \in M^\sigma$. Then $\langle m, n_i \rangle + a_i \geq 0$ for all i, if and only if m can be written as $m' + \sum_{i=1}^r x_i m_i$ where $m' \in Q_D^\sigma \cap M^\sigma$ and where the x_i's are non-negative integers; such a representation is unique. It follows that

$$\chi_\sigma^T(\mathcal{O}_X(D)) = (\sum_{m \in M \cap \sigma^\perp} e^m)(\sum_{m \in Q_D^\sigma} e^m)(\sum_{x_i \geq 0} e^{x_1 m_1 + \cdots + x_r m_r}) \ .$$

Our statements follow at once from this identity.

2.2. *Localized Grothendieck groups*

Denote by S the multiplicative subset of $\mathbf{Z}[M]$ generated by all sums $\sum_{m \in E} e^m$ where E is a finite subset of M.

Proposition. *Let X be a simplicial toric variety. For any $\tau \in \Sigma(1)$, choose a positive integer a_τ such that the divisor $a_\tau D_\tau$ is Cartier. Then the $S^{-1}\mathbf{Z}[M]$-module $S^{-1}G^T(X)$ is generated by the elements $[\mathcal{O}_X(-\sum_{\tau \in \sigma(1)} a_\tau D_\tau)]$ where $\sigma \in \Sigma$.*

Proof: By induction over the number of orbits, the case of one orbit being trivial. For the general case, choose a closed orbit Ω_σ in X and consider the exact sequence of 1.4:

$$0 \to G^T(\Omega_\sigma) \to G^T(X) \to G^T(X \setminus \Omega_\sigma) \to 0 \ .$$

The $\mathbf{Z}[M]$-module $G^T(\Omega_\sigma)$ is generated by the class u_σ of the structure sheaf of Ω_σ. Therefore, it suffices to check that Su_σ contains

$$\sum_{I \subset \sigma(1)} (-1)^{|I|} [\mathcal{O}_X(-\sum_{\tau \in I} a_\tau D_\tau)] \ .$$

For this, we use the notation of 2.1, and we set $L_i := \mathcal{O}_X(-a_i D_i)$ for $1 \leq i \leq r$. Then, as L_i is invertible, each a_i is a multiple of q_i. Because X is simplicial, we have $\operatorname{codim}_X(D_{i_1} \cap \cdots \cap D_{i_s}) = s$ whenever $1 \leq i_1 < \cdots < i_s \leq r$. Hence, because X is Cohen-Macaulay, the Koszul complex

$$0 \to L_1 \otimes \cdots \otimes L_r \to \cdots \to \bigoplus_{1 \leq i < j \leq r} L_i \otimes L_j \to \bigoplus_{1 \leq i \leq r} L_i \to \mathcal{O}_X \to \mathcal{O}_X / \sum_{1 \leq i \leq r} L_i \to 0$$

is exact. Therefore, we have in $G^T(X)$:

$$[\mathcal{O}_X / \sum_{1 \leq i \leq r} L_i] = \sum_{I \subset \{1, \ldots, r\}} (-1)^{|I|} [\otimes_{i \in I} L_i] \ .$$

To conclude the proof, we observe that the sheaf $\mathcal{O}_X / \sum_{1 \leq i \leq r} L_i$ is coherent, T-linearized, and is supported in Ω_σ. It follows that this sheaf has a finite filtration by coherent, T-linearized sheaves, with various twists of the structure sheaf of Ω_σ as subquotients. So we can write in $G^T(X)$:

$$[\mathcal{O}_X / \sum_{1 \leq i \leq r} L_i] = P u_\sigma$$

for a unique $P \in \mathbf{Z}[M/M \cap \sigma^\perp]$. It follows that

$$\chi_\sigma^T(\mathcal{O}_X / \sum_{i=1}^{r} L_i) = P \chi_\sigma^T(u_\sigma) \ .$$

But $\chi_\sigma^T(u_\sigma) = \sum_{m \in M \cap \sigma^\perp} e^m$ and moreover, using 2.1, we obtain, setting $b_i = a_i q_i^{-1}$:

$$\chi_\sigma^T(\mathcal{O}_X / \sum_{1 \leq i \leq r} L_i) = (\sum_{m \in M \cap \sigma^\perp} e^m)(\sum_{m \in Q^\sigma \cap M^\sigma} e^m) \prod_{1 \leq i \leq r} \frac{1 - e^{b_i m_i}}{1 - e^{m_i}} \ .$$

Therefore, we have

$$P = (\sum_{m \in Q^\sigma \cap M^\sigma} e^m) \prod_{i=1}^{r} (1 + e^{m_i} + e^{2m_i} + \cdots + e^{(b_i - 1)m_i})$$

which shows that P is in S.

Denote by $K^T(X)$ ($K(X)$) the Grothendieck group of (T-linearized) locally free sheaves on X. Then $K^T(X)$ is a $\mathbf{Z}[M]$-algebra and moreover $G^T(X)$ is a module over $K^T(X)$, via the canonical map $K^T(X) \to G^T(X)$. Similarly, we have the canonical map $K(X) \to G(X)$.

Corollary. *If X is a simplicial toric variety, then the map $K^T(X) \to G^T(X)$ induces a surjective map $S^{-1}K^T(X) \to S^{-1}G^T(X)$. Moreover, the map $K(X) \to G(X)$ is surjective over the rationals.*

Remark. We ignore whether the maps $S^{-1}K^T(X) \to S^{-1}G^T(X)$ and $K(X)_{\mathbf{Q}} \to G(X)_{\mathbf{Q}}$ are isomorphisms for simplicial X. But it is easy to see that for any non-simplicial toric variety X, the map $K(X) \to G(X)$ is not surjective over the rationals. Namely, choose a non-simplicial cone σ in the fan of X. Then we have a commutative square

$$\begin{array}{ccc} K(X) & \to & G(X) \\ \downarrow & & \downarrow \\ K(X_\sigma) & \to & G(X_\sigma) \end{array}$$

where the vertical arrows are restrictions to X_σ. Moreover, the map $G(X) \to G(X_\sigma)$ is surjective. Therefore, surjectivity over the rationals of the map $K(X) \to G(X)$ would imply the corresponding statement for X_σ. So we may assume that X is affine.

In this case, $K(X)$ is isomorphic to $\mathbf{Z}$ via the rank. On the other hand, denoting by U the union of orbits of codimension at most one in X, the restriction map $G(X) \to G(U)$ is surjective, and moreover $G(U) = K(U)$ because U is smooth. Finally, the kernel of the rank map $K(U) \to \mathbf{Z}$ surjects onto the Picard group of U, and the latter is infinite (because σ is not simplicial). So the rank of the abelian group $K(U)$ is at least two, and hence the rank of $G(X)$ is at least two as well.

2.3. *Local Chern character*

Let $X = X_\sigma$ be a toric variety, and let $\mathcal{F}$ be a coherent, T-linearized sheaf on X. Choose a simplicial cone σ in Σ and denote by $\mathcal{F}_\sigma$ the restriction of $\mathcal{F}$ to the locally closed subvariety S_σ (the fiber of the equivariant retraction $X_\sigma \to \Omega_\sigma$). Then $\mathcal{F}_\sigma$ is T_σ-linearized. Moreover, the map $\mathcal{F} \mapsto \mathcal{F}_\sigma$ defines an isomorphism $G^T(X_\sigma) \to G^{T_\sigma}(S_\sigma)$, see [T].

Proposition. *(i) There exists a unique element $ch^T_\sigma(\mathcal{F})$ in $S^{-1}\mathbf{Z}[M/M\cap\sigma^\perp]$ such that*

$$\chi^{T_\sigma}(\mathcal{F}_\sigma) = ch^T_\sigma(\mathcal{F})\chi^{T_\sigma}(\mathcal{O}_{S_\sigma}) \ .$$

(ii) For any face τ of σ, the image of $ch^T_\sigma(\mathcal{F})$ under the natural map $\mathbf{Z}[M/M\cap \sigma^\perp] \to \mathbf{Z}[M/M\cap\tau^\perp]$ is $ch^T_\tau(\mathcal{F})$.
(iii) If moreover $\mathcal{F}$ is locally free, then $ch^T_\sigma(\mathcal{F})$ is in $\mathbf{Z}[M/M\cap\sigma^\perp]$.

Proof. (i) For existence of $ch^T_\sigma(\mathcal{F})$, we may assume (using 1.1) that $\mathcal{F} = \mathcal{O}_X(D)$. Then

$$ch^T_\sigma(\mathcal{F}) = (\sum_{m\in Q^\sigma\cap M^\sigma} e^m)^{-1}(\sum_{m\in Q^\sigma_D\cap M^\sigma} e^m)$$

with the notation of 2.1.

Unicity of $ch^T_\sigma(\mathcal{F})$ follows from the fact that $\chi^{T_\sigma}_\sigma$ is summable in $\mathbf{Q}(M/M\cap \sigma^\perp)$, and that its sum is non-zero; see 1.3.

(iii) If $\mathcal{F}$ is locally free, then $\mathcal{F}_\sigma \simeq \mathcal{O}_{S_\sigma} \otimes V$ for some T_σ-module V. In this case, we have

$$\chi^{T_\sigma}(\mathcal{F}_\sigma) = \chi^{T_\sigma}(\mathcal{O}_{S_\sigma})\chi^{T_\sigma}(V)$$

where $\chi^{T_\sigma}(V)$ denotes the character of the T_σ-module V. It follows that $ch^T_\sigma(\mathcal{F}) = \chi^{T_\sigma}(V)$ is in $\mathbf{Z}[M/M \cap \sigma^\perp]$.

(ii) If $\mathcal{F}$ is locally free, then our statement follows from the discussion above. In the general case, observe that ch^T_σ is additive on short exact sequences, and hence that we have a well-defined map

$$ch^T_\sigma : S^{-1}G^T(X) \to S^{-1}\mathbf{Z}[M/M \cap \sigma^\perp] \ .$$

Now we conclude by corollary 2.2.

2.4. *Chern character*

Denote by E_Σ the set of all families $(f_\sigma)_{\sigma\in\Sigma}$ such that $f_\sigma \in \mathbf{Z}[M/M \cap \sigma^\perp]$ and that, for all $\tau \subset \sigma$, the image of f_σ in $\mathbf{Z}[M/M \cap \tau^\perp]$ is f_τ. Then E_Σ is a ring for pointwise addition and multiplication: the *ring of continuous, piecewise exponential functions on* Σ. Moreover, $\mathbf{Z}[M]$ maps to a subring of E_Σ by $f \to (f)_{\sigma\in\Sigma}$. This gives E_Σ the structure of a $\mathbf{Z}[M]$-algebra.

By proposition 2.3, the map

$$\mathcal{F} \to (ch^T_\sigma(\mathcal{F}))_{\sigma\in\Sigma}$$

defines a map

$$ch^T : G^T(X) \to S^{-1}E_\Sigma \ .$$

Clearly, ch^T is a morphism of $\mathbf{Z}[M]$-modules. We will see in 3.6 below that ch^T is the equivariant Chern character.

Theorem. *The map $S^{-1}ch^T : S^{-1}G^T(X) \to S^{-1}E_\Sigma$ is an isomorphism.*

Proof. First we check that $S^{-1}ch^T$ is injective. We argue by induction over the number of orbits in X, the case of one orbit being obvious. Choose a closed orbit Ω_σ and consider the diagram

$$\begin{array}{ccccccc} 0 & \to & G^T(\Omega_\sigma) & \to & G^T(X) & \to & G^T(X) \setminus \Omega_\sigma) \to \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & S^{-1}E_{\sigma^0} & \to & S^{-1}E_\Sigma & \to & S^{-1}E_{\Sigma\setminus\{\sigma\}} \end{array}$$

where E_{σ^0} denotes the kernel of the restriction map $E_\Sigma \to E_{\Sigma\setminus\{\sigma\}}$ (i.e. E_{σ^0} is the space of piecewise exponential functions which vanish outside the relative interior of σ). This diagram commutes, and its rows are exact. Therefore, it defines a map $c_\sigma : S^{-1}G^T(\Omega_\sigma) \to S^{-1}E_{\sigma^0}$. Recall that the $\mathbf{Z}[M]$-module $G^T(\Omega_\sigma)$ is isomorphic to $\mathbf{Z}[M/M\cap\sigma^\perp]$. On the other hand, E_{σ^0} is a torsion-free module over $\mathbf{Z}[M/M\cap\sigma^\perp]$, and c_σ is $\mathbf{Z}[M]$-linear. So it is enough to check that c_σ is non-zero.

Notation being as in 2.1, we consider

$$v_\sigma := \sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|}[\mathcal{O}_X(-\sum_{i\in I} q_iD_i)] \ .$$

Then, as in the proof of 2.2, we see that v_σ is in $i_{\sigma *}G^T(\Omega_\sigma)$. On the other hand, restriction of q_iD_i to X_σ coincides with the divisor of m_i. It follows that v_σ is mapped by c_σ to $\prod_{i=1}^r(1-e^{m_i+\sigma^\perp})$. Because $m_i \notin \sigma^\perp$, we conclude that c_σ is non-zero.

To check surjectivity of $S^{-1}ch^T$, it is enough to show that the composition

$$S^{-1}K^T(X) \to S^{-1}G^T(X) \to S^{-1}E_\Sigma$$

is surjective, or even that $K^T(X)$ is mapped surjectively to E_Σ. For this, let $f = (f_\sigma)_{\sigma\in\Sigma}$ be in E_Σ. Then we may choose $g \in \mathbf{Z}[M]$ such that each $f_\sigma + g$ is a positive linear combination of e^m's, $m \in \mathbf{Z}[M/M\cap\sigma^\perp]$. In other words, $f_\sigma+g$ is the character of some T_σ-module V_σ. By definition of E_Σ, the restriction to T_τ of V_σ is isomorphic to V_τ whenever $\tau\subset\sigma$. Therefore, the T-linearized sheaves on $X_\sigma = T\times^{T_\sigma} S_\sigma$, induced by the T_σ-linearized sheaves $\mathcal{O}_{S_\sigma}\otimes_k V_\sigma$ on S_σ, can be glued together to a T-linearized, locally free sheaf $\mathcal{E}$, and we have $ch^T(\mathcal{E}) = f+g$. Hence $f = ch^T(\mathcal{E}) - ch^T(\mathcal{O}_X\otimes_k V)$ where V is a T-module with character g.

3. Equivariant cohomology of simplicial toric varieties

3.1. *Equivariant cohomology*

First we review some basic facts about equivariant cohomology; see [A-B] for more details. Choose a contractible topological space ET where T acts freely, and denote by $BT = (ET)/T$ the quotient. For any T-space Z, the quotient of $Z\times ET$ by the diagonal T-action exists; denote this quotient by $Z\times^T ET$. Then the equivariant cohomology ring of Z with rational coefficients, is defined by

$$H^*_T(Z) := H^*(Z\times^T ET, \mathbf{Q}) \ .$$

In particular, the equivariant cohomology ring of the point is $H_T^*(pt) = H^*(BT)$. If $T = (\mathbf{C}^*)^d$ then we may take $ET = (\mathbf{C}^\infty \setminus \{0\})^d$ where T acts by scalar multiplications; then $BT = (\mathbf{P}^\infty)^d$.

Any one-dimensional T-module $\mathbf{C}m$ (with character $m \in M$) defines a line bundle $\mathbf{C}m \times^T ET$ over BT. Denote by $c(m) \in H_T^2(pt)$ the first Chern class of this line bundle. Then the map $c : M \to H_T^2(pt)$ is additive, and it extends to an isomorphism (which multiplies degree by 2)

$$c : S^*(M_{\mathbf{Q}}) \to H_T^*(pt)$$

where $S^*(M_{\mathbf{Q}})$ is the symmetric algebra of $M_{\mathbf{Q}}$ over the rationals.

More generally, for any T-space Z, we have a fibration $Z \times^T ET \to BT$ which gives $H_T^*(Z)$ the structure of an algebra over $H^*(BT) = S^*(M_{\mathbf{Q}})$. Furthermore, restriction to fibers defines a homomorphism of graded rings $H_T^*(Z) \to H^*(Z)$. This homomorphism vanishes on $M_{\mathbf{Q}}$ and hence it factors through

$$H_T^*(Z)/M_{\mathbf{Q}} H_T^*(Z) \to H^*(Z) \ .$$

We will use the following observation: If a closed subgroup $G \subset T$ acts on X with finite isotropy groups, then $H_T^*(X)$ is naturally isomorphic to $H_{T/G}^*(X/G)$. Namely, choose a closed subgroup $T' \subset T$ such that $T = T'G$ and that $T' \cap G$ is finite. Then we can take $ET' = ET$. Now the fibers of all maps in the diagram

$$X \times^T ET \to X/G \times^{T'} ET \leftarrow X/G \times^{T'} (ET \times E(T/G)) \to X/G \times^{T/G} E(T/G)$$

are quotients of contractible spaces by finite groups. Therefore, these maps induce isomorphisms in cohomology.

We will study equivariant cohomology of simplicial toric varieties, generalizing results of [B1] concerning smooth toric varieties. Let Σ be a fan, and let $\sigma \in \Sigma$ be a simplicial cone. Recall that $r_\sigma : X_\sigma \to \Omega_\sigma$ denotes the T-equivariant retraction.

Proposition. *Notation being as above, the map $r_\sigma^* : H_T^*(X_\sigma) \to H_T^*(\Omega_\sigma) \simeq S^*(M_{\mathbf{Q}}/\sigma^\perp)$ is an isomorphism of graded algebas over $S^*(M_{\mathbf{Q}})$. Moreover, for any face τ of σ, the diagram*

$$\begin{array}{ccc} H_T^*(X_\sigma) & \to & S^*(M_{\mathbf{Q}}/\sigma^\perp) \\ \downarrow & & \downarrow \\ H_T^*(X_\tau) & \to & S^*(M_{\mathbf{Q}}/\tau^\perp) \end{array}$$

commutes, where the left (resp. right) vertical arrow is defined by inclusion of X_τ in X_σ (resp. by the map $M_{\mathbf{Q}}/\sigma^\perp \to M_{\mathbf{Q}}/\tau^\perp$).

Proof. Observe that $H^*_T(X_\sigma) \simeq H^*_{T_\sigma}(S_\sigma)$ and that $r^*_\sigma : H^*_T(X_\sigma) \to H^*_T(\Omega_\sigma)$ identifies to restriction $H^*_{T_\sigma}(S_\sigma) \to H^*_{T_\sigma}(x)$ where x denotes the T_σ-fixed point in S_σ. So we may assume that Ω_σ consists in one point x. Then σ is a d-dimensional cone with edges generated by $n_1, \ldots, n_d$. Denote by $\tilde{N}$ the subgroup of N generated by $n_1, \ldots, n_d$. Then $\tilde{N} \subset N$ corresponds to a torus $\tilde{T}$ mapping surjectively to T with a finite kernel G. Moreover, $\tilde{T}$ acts linearly on $\mathbf{A}^d$, and the quotient $\mathbf{A}^d/G$ is isomorphic to X, the preimage of $x \in X$ being the $\tilde{T}$-fixed point $0 \in \mathbf{A}^d$. Now restriction to 0 induces isomorphisms

$$H^*_{\tilde{T}}(\mathbf{A}^d) \simeq H^*_{\tilde{T}}(0) \simeq S^*(\tilde{M}_{\mathbf{Q}}) \simeq S^*(M_{\mathbf{Q}}) \ .$$

Moreover, $H^*_{\tilde{T}}(\mathbf{A}^d)$ is isomorphic to $H^*_T(X)$ by our observation, and this isomorphism is compatible with restriction to the fixed point. This proves our first statement. The commutativity of our diagram is easy, because all maps are homomorphisms of $S^*(M_{\mathbf{Q}})$-algebras.

3.2. *Piecewise polynomial functions*

Denote by R_Σ the set of all families $(f_\sigma)_{\sigma\in\Sigma}$ such that $f_\sigma \in S^*(M_{\mathbf{Q}}/\sigma^\perp)$ and that, for all $\tau \subset \sigma$, the image of f_σ in $S^*(M_{\mathbf{Q}}/\sigma^\perp)$ is equal to f_τ. Then R_Σ is an algebra over $S^*(M_{\mathbf{Q}})$: the algebra of continuous, piecewise polynomial functions on Σ.

For $f \in R_\Sigma$, decompose f_σ into the sum of its homogeneous components $f_{\sigma,n}$. Then for fixed n, the family $(f_{\sigma,n})_{\sigma\in\Sigma}$ is in R_Σ. This defines a grading $R_\Sigma = \oplus_{n=0}^\infty R_{\Sigma,n}$ of the algebra R_Σ.

Assume that the fan Σ is simplicial. For $\sigma \in \Sigma$, consider the restriction map $H^*_T(X) \to H^*_T(X_\sigma), u \to u_\sigma$. By 3.1, we can identify u_σ with an element of $S^*(M_{\mathbf{Q}}/\sigma^\perp)$, and moreover the family $(u_\sigma)_{\sigma\in\Sigma}$ is in R_Σ.

Proposition. *(i) For any simplicial toric variety $X = X_\Sigma$, the map*

$$\begin{array}{ccc} H^*_T(X) & \to & R_\Sigma \\ u & \to & (u_\sigma)_{\sigma\in\Sigma} \end{array}$$

is an isomorphism of graded algebras over $S^*(M_{\mathbf{Q}})$.
(ii) If moreover X is complete, then the map

$$H^*_T(X)/M_{\mathbf{Q}}H^*_T(X) \to H^*(X)$$

is an isomorphism.

Proof. (i) is proved in [B1] in the case where X is smooth. This proof can be adapted to the simplicial case; alternatively, we may reduce to the smooth case, following a method of [A].

Let $\tilde{N}$, N', T' and $f : \tilde{X} \to X$ as in the proof of 1.1. Because Σ is simplicial, f is the geometric quotient by T' acting with finite isotropy groups. Using our observation on 3.1, we see that $H^*_T(X) = H^*_{\tilde{T}/G}(\tilde{X}/G)$ is isomorphic to $H^*_{\tilde{T}}(\tilde{X})$. On the other hand, we have an isomorphism $R_\Sigma \simeq R_{\tilde{\Sigma}}$ compatible with maps from equivariant cohomology. In this way, we reduce to the case where $T = (\mathbf{C}^*)^e$ and where X is a T-stable open subset of $\mathbf{C}^e$. Then we conclude by proposition 2.2 in [B1].

(ii) follows easily from the Leray spectral sequence of the fibration $X \times_T ET \to BT$ because the fiber X has no odd cohomology, see [F2].

Remark. Denote by $A^*(X)_{\mathbf{Q}}$ the rational Chow group of the complete, simplicial toric variety X. Then the cycle map $cl_X : A^*(X)_{\mathbf{Q}} \to H^*(X)$ is an isomorphism, see [F2].

3.3. *Equivariant cohomology classes*

Let $X = X_\Sigma$ be a simplicial toric variety, and let $\sigma \in \Sigma$. Then the orbit closure $\overline{\Omega_\sigma}$ defines an equivariant cohomology class

$$F_\sigma \in H_T^{2\dim(\sigma)}(X)$$

as follows. Observe that $ET = (\mathbf{C}^\infty \setminus \{0\})^d$ is an increasing union of the smooth, T-stable algebraic varieties $(\mathbf{C}^n \setminus \{0\})^d$. Moreover, each space $X \times^T (\mathbf{C}^n \setminus \{0\})^d$ is locally a quotient of a smooth algebraic variety by a finite group of algebraic automorphisms. Therefore, this space satisfies to Poincaré duality over $\mathbf{Q}$, and we may define the cohomology class of $\overline{\Omega_\sigma} \times^T (\mathbf{C}^n \setminus \{0\})^d$. As n increases, these classes are compatible, and hence the cohomology class of $\overline{\Omega_\sigma} \times_T ET$ makes sense; we denote it by F_σ.

We will describe F_σ as an element of R_Σ. To this aim, denote by φ_τ the element of $R_{\Sigma,1}$ such that $\varphi_\tau(n_\tau) = 1$ and that $\varphi_\tau(n_{\tau'}) = 0$ for all $\tau' \in \Sigma(1)$, $\tau' \neq \tau$. Then φ_τ is called the *Courant function* associated to the edge τ, see [B].

For $\sigma \in \Sigma$, we denote by N_σ (resp. $N_{\sigma(1)}$) the subgroup of N generated by $N \cap \sigma$ (resp. by the n_τ's where $\tau \in \sigma(1)$). Then $N_{\sigma(1)}$ is a subgroup of finite index in N_σ. The index $[N_\sigma : N_{\sigma(1)}]$ is called the *multiplicity* of σ. We denote it by $mult(\sigma)$, and we set

$$\varphi_\sigma := mult(\sigma) \prod_{\tau\in\sigma(1)} \varphi_\tau \ .$$

Then φ_σ is a continuous, piecewise polynomial function on Σ, of degree $\dim(\sigma)$. Moreover, φ_σ vanishes identically on all cones which do not contain σ.

In particular, if σ is d-dimensional, then φ_σ vanishes identically outside its interior σ^0. Therefore, denoting by Φ_σ the unique polynomial function on $M_{\mathbf{R}}$ which restricts to φ_σ on σ, we see that Φ_σ is a constant multiple of the product of equations of facets of σ. More precisely, we have with notation as in 2.1:

$$\Phi_\sigma = |Q^\sigma \cap M|^{-1} \prod_{i=1}^{d} m_i \ .$$

Namely, $\Phi_\sigma = mult(\sigma)\prod_{i=1}^d q_i^{-1}m_i$ and moreover

$$\prod_{i=1}^d q_i = [\sum_{i=1}^d \mathbf{Z}q_i^{-1}m_i : \sum_{i=1}^d \mathbf{Z}m_i] = [N_{\sigma(1)} : N][M : \sum_{i=1}^d \mathbf{Z}m_i] = mult(\sigma)|Q^\sigma \cap M| \ .$$

Proposition. *Notation being as above, the image of F_γ in R_Σ is $(-1)^{\dim(\gamma)}\varphi_\gamma$.*

Proof. Because the map $H^*_T(X) \to \prod_{\sigma\in\Sigma} H^*_T(X_\sigma)$ is injective, and because F_γ is compatible with restriction, we may assume that $X = X_\sigma$ is affine.

First we consider the case where γ is an edge of Σ. If $\gamma \notin \sigma(1)$ then $F_\gamma = 0$; on the other hand, φ_γ vanishes on σ. So we may assume that $\gamma = \tau_i$ with notation as in 2.1. Then q_iD_i is the divisor of zeroes of the character m_i. It follows that $\mathcal{O}_X(q_iD_i) = \mathcal{O}_X \otimes (-m_i)$ and hence that $q_iF_\gamma = -m_i$. On the other hand, restriction to σ of φ_γ is $q_i^{-1}m_i$, and hence $F_\gamma = -\varphi_\gamma$.

In the general case, choose an edge τ of γ and denote by δ the unique facet of γ such that $\tau \notin \delta$. Clearly, F_γ is a rational multiple of the product $F_\tau F_\delta$. Using the map $H^*_T(X) \to H^*(X)$ and [F2] 5.1, we obtain

$$F_\gamma = mult(\gamma)mult(\delta)^{-1}F_\tau F_\delta \ .$$

We conclude by induction over $\dim(\gamma)$.

3.4. *Localization*

Let $X = X_\Sigma$ be a complete, simplicial toric variety. As in 1.5, denote by $i : X^T \to X$ the inclusion of the fixed point set, and consider the induced map $i_* : H_T^*(X^T) \to H_T^*(X)$ (defined via Poincaré duality as above). Then i_* is a morphism of $S^*(M_{\mathbf{Q}})$-modules, of degree $2d$. Moreover, $H_T^*(X^T)$ is isomorphic to $\prod_{\sigma\in\Sigma(d)} S^*(M_{\mathbf{Q}})$.

By a general localization theorem in equivariant cohomology, the morphism i_* is an isomorphism after inverting all elements of $S^*(M_{\mathbf{Q}})$ which do not vanish at the origin. We will obtain a sharper version of this result.

For each $w \in \Sigma(d-1)$, choose a non-zero $m_w \in M$ which vanishes identically on w. Set

$$\Phi_\Sigma := \prod_{w\in\Sigma(d-1)} m_w \ .$$

Then Φ_Σ is the least common multiple of all Φ_σ ($\sigma \in \Sigma(d)$).

Proposition. *The map $i_* : H_T^*(X^T) \to H_T^*(X)$ is injective. Moreover, the cokernel of i_* is killed by Φ_Σ.*

Proof. Let $f = (f_\sigma)_{\sigma\in\Sigma(d)}$ be in $H_T^*(X^T)$. Because i_* is $S^*(M_{\mathbf{Q}})$-linear, we have

$$i_*(f) = \sum_{\sigma\in\Sigma(d)} f_\sigma F_\sigma = (-1)^d \sum_{\sigma\in\Sigma(d)} f_\sigma \varphi_\sigma \ .$$

If $i_*(f) = 0$ then, evaluating at an interior point of σ, we obtain $f_\sigma = 0$. So i_* is injective.

On the other hand, for any $g \in R_\Sigma$, we have

$$\Phi_\Sigma g = \sum_{\sigma\in\Sigma(d)} \Phi_\Sigma \Phi_\sigma^{-1} \varphi_\sigma g_\sigma$$

(namely, this equation reduces to $(\Phi_\sigma g)_\sigma = \Phi_\sigma g_\sigma$ on a given $\sigma \in \Sigma(d)$). Because each $\Phi_\Sigma\Phi_\sigma^{-1}\varphi_\sigma$ is in $S^*(M_{\mathbf{Q}})$, it follows that $\Phi_\Sigma g$ is in the image of i_*.

3.5. *Equivariant push-forward*

Let $X = X_\Sigma$ be a complete, simplicial toric variety. Then the map $X \to pt$ induces a fibration $X \times^T ET \to BT$ with fiber X. Therefore, we have a push-forward map

$$\int_X : H_T^*(X) \to H_T^*(pt)$$

which is homogeneous of degree $-2d$. By the projection formula, $\int_X$ is a morphism of $H_T^*(pt)$-modules.

Proposition. *Via the isomorphisms $H_T^*(X) \simeq R_\Sigma$ and $H_T^*(pt) \simeq S^*(M_{\mathbf{Q}})$, the push-forward map $\int_X$ is given by*

$$\int_X f = (-1)^d \sum_{\sigma \in \Sigma(d)} f_\sigma \Phi_\sigma^{-1} \ .$$

Proof. Let $\sigma \in \Sigma(d)$. Then $F_\sigma \in H_T^*(X)$ is the cohomology class of a T-fixed point. Using 3.4, it follows that

$$\int_X \varphi_\sigma = (-1)^d \ .$$

For any $f \in R_\Sigma$, we have as in the proof of 3.4:

$$\Phi_\Sigma f = \sum_{\sigma \in \Sigma(d)} \Phi_\Sigma \Phi_\sigma^{-1} f_\sigma \varphi_\sigma \ .$$

Because $\int_X$ is $S^*(M_{\mathbf{Q}})$-linear, we obtain

$$\Phi_\Sigma \int_X f = \sum_{\sigma \in \Sigma(d)} \Phi_\Sigma \Phi_\sigma^{-1} f_\sigma (-1)^d \ .$$

This implies our formula.

The following result is an easy consequence of this explicit formula (see [B2] 2.4 for more details).

Corollary. *The $H^*(BT)$-bilinear map*

$$\begin{array}{ccc} H_T^*(X) \times H_T^*(X) & \to & H^*(BT) \\ (f,g) & \to & \int_X fg \end{array}$$

is a perfect pairing.

3.6. *Equivariant Chern character*

Let $X = X_\Sigma$ be a simplicial toric variety. Any T-linearized, locally free sheaf $\mathcal{E}$ on X defines a T-equivariant vector bundle E on X, and hence a T-equivariant vector bundle p^*E on $X \times ET$ where p denotes the projection

$X \times ET \to X$. Because T acts freely on $X \times ET$, we can push forward p^*E to a vector bundle E_T on $X \times^T ET$. The Chern character of E_T will be denoted by $Ch^T(\mathcal{E})$, an element of

$$\prod_{n=0}^{\infty} H^n(X \times^T ET) := \hat{H}_T(X)$$

(the completion of the graded algebra $H_T^*(X)$). Observe that the image of $Ch^T(\mathcal{E})$ in $\hat{H}_T(X)/M_{\mathbf{Q}}\hat{H}^T(X) = H^*(X)$ is the usual Chern character of E.

By 3.2, we can identify $\hat{H}_T(X)$ with $\hat{R}_\Sigma$. Moreover, by [B1], the algebra $\hat{R}_\Sigma$ consists in all families $(f_\sigma)_{\sigma\in\Sigma}$ with $f_\sigma \in \hat{S}(M_{\mathbf{Q}}/\sigma^\perp)$ and f_σ restricts to f_τ whenever τ is a face of σ. On the other hand, it is easily checked that E_Σ embeds into $\hat{R}_\Sigma$ by mapping each e^m to $\sum_{n=0}^{\infty} m^n/n!$. Therefore, we may consider $ch^T(\mathcal{E})$ (defined in 2.2) in $\hat{R}_\Sigma$.

Proposition. *With the notation as above, we have $Ch^T(\mathcal{E}) = ch^T(\mathcal{E})$ for any T-linearized locally free sheaf $\mathcal{E}$ on X.*

Proof. We may assume that X is affine. Then there exists a T-module V such that $\mathcal{E} \simeq \mathcal{O}_X \otimes V$. So E is the trivial bundle with fiber V, and $Ch^T(\mathcal{E})$ is the character of V. But the latter coincides with $ch^T(\mathcal{E})$ by definition, see 2.2.

Using [B2] 4.2, we derive now the following

Corollary. *The map $ch^T : G^T(X) \to \hat{H}_T(X)$ is injective, and its image is dense.*

4. The equivariant Todd class of complete, simplicial toric varieties

4.1. *Equivariant Riemann-Roch*

Theorem. *Let $X = X_\Sigma$ be a complete, simplicial toric variety. Then there exists a unique class $Td^T(X) \in \hat{H}_T(X)$ (the* equivariant Todd class *of X) such that*

$$\chi^T(\mathcal{F}) = \int_X ch^T(\mathcal{F})\, Td^T(X)$$

for any coherent, T-linearized sheaf $\mathcal{F}$ on X. Moreover, for any $\sigma \in \Sigma(d)$, *restriction to X_σ of $Td^T(X)$ is the following element of $\hat{H}_T(X_\sigma) = \hat{S}(M_{\mathbf{Q}})$:*

$$Td^T_\sigma(X) = |M \cap Q^\sigma|^{-1}(\sum_{m \in Q^\sigma \cap M} e^m) \prod_{i=1}^d \frac{-m_i}{1 - e^{m_i}}$$

with notation as in 2.1.

Proof. By 1.4, we have

$$\chi^T(\mathcal{F}) = \sum_{\sigma \in \Sigma(d)} \mathcal{S}(\chi^T_\sigma(\mathcal{F})) \ .$$

Moreover, we have by 2.1:

$$\mathcal{S}(\chi^T_\sigma(\mathcal{F})) = ch^T_\sigma(\mathcal{F}) \mathcal{S}(\chi^T_\sigma(\mathcal{O}_X)) = ch^T_\sigma(\mathcal{F})(\sum_{m \in M \cap Q^\sigma} e^m) \prod_{i=1}^d (1 - e^{m_i})^{-1} \ .$$

Define $Td^T_\sigma(X) \in \hat{S}(M_{\mathbf{Q}})$ by the formula of the theorem. Then the $Td^T_\sigma(X)$ ($\sigma \in \Sigma(d)$) glue together into $Td^T(X) \in \hat{R}_\Sigma$ (this can be checked directly; it will be a consequence of an alternative formula for $Td^T(X)$, proved in the next subsection). By 3.3, we have

$$Td^T_\sigma(X) = (-1)^d \, \Phi_\sigma \, \mathcal{S}(\chi^T_\sigma(\mathcal{O}_X))$$

and hence, by 3.5:

$$\int_X ch^T(X) \, Td^T(X) = \sum_{\sigma \in \Sigma(d)} ch^T_\sigma(\mathcal{F}) \, \mathcal{S}(\chi^T_\sigma(\mathcal{O}_X)) = \sum_{\sigma \in \Sigma(d)} \mathcal{S}(\chi^T_\sigma(\mathcal{F})) \ .$$

This proves existence of the class $Td^T(X)$. Unicity follows from corollaries 3.5 and 3.6.

Corollary. *(i) For any equivariant morphism $\pi : X' \to X$ between complete, simplicial toric varieties, we have $Td^T(X) = \pi_* Td^T(X')$.*
(ii) The image of $Td^T(X)$ in $H^*(X) = A^*(X)_{\mathbf{Q}}$ is the Todd class of X defined in [F1].

Proof. (i) follows from unicity of $Td^T(X)$ and from vanishing of $R^j \pi_* \mathcal{O}_{X'}$ for all $j \geq 1$.

(ii) follows from the fact that the ring $A^*(X)_{\mathbf{Q}}$ is generated by Chern characters of T-equivariant line bundles.

4.2. *A closed formula for the equivariant Todd class*

Let Σ be a simplicial fan. Define a homomorphism from the torus $(\mathbf{C}^*)^{\Sigma(1)}$ to T, by mapping $(t_\tau)_{\tau\in\Sigma(1)}$ to $\prod_{\tau\in\Sigma(1)} n_\tau(t_\tau)$ (recall that n_τ is a one-parameter subgroup of T). We denote by G the kernel of this homomorphism.

For any simplicial cone σ generated by elements n_τ where $\tau \in \Sigma(1)$, we denote by G_σ the intersection of G with the subgroup $(\mathbf{C}^*)^{\sigma(1)}$ of $(\mathbf{C}^*)^{\Sigma(1)}$. More concretely,

$$G_\sigma = \{(t_\tau)_{\tau\in\sigma(1)} \mid t_\tau \in \mathbf{C}^*, \prod_{\tau\in\sigma(1)} n_\tau(t_\tau) = 1\}$$

identifies to the quotient $N_\sigma/\sum_{\tau\in\sigma(1)} \mathbf{Z}n_\tau$. In particular, the order of G_σ is the multiplicity of σ.

We denote by $G_\Sigma \subset G$ the union of all subgroups G_σ ($\sigma \in \Sigma$). Similarly, we denote by $G_{\Sigma(1)}$ the union of all subgroups G_σ where σ ranges over simplicial cones generated by subsets of $\Sigma(1)$. For $\tau \in \Sigma(1)$, we denote by $a_\tau : G_{\Sigma(1)} \to \mathbf{C}^*$ the restriction to $G_{\Sigma(1)}$ of the τ-component $(\mathbf{C}^*)^{\Sigma(1)} \to \mathbf{C}^*$. Then restriction of a_τ to G_σ is a character, and this character is non-trivial if and only if τ is an edge of σ.

Finally, recall that the equivariant cohomology class of the divisor D_τ is denoted by F_τ.

Theorem. *Let $X = X_\Sigma$ be a complete, simplicial toric variety. Then, notation being as above, the equivariant Todd class of X is given by*

$$Td^T(X) = \sum_{g\in G_\Sigma} \prod_{\tau\in\Sigma(1)} \frac{F_\tau}{1 - a_\tau(g)e^{-F_\tau}} \ .$$

Moreover, we have

$$Td^T(X) = \sum_{g\in G_{\Sigma(1)}} \prod_{\tau\in\Sigma(1)} \frac{F_\tau}{1 - a_\tau(g)e^{-F_\tau}} \ .$$

In particular, $Td^T(X)$ can be expressed in terms of $\Sigma(1)$ only.

Proof. The first formula defines a class θ in $\hat{H}_T(X) \simeq \hat{R}_\Sigma$; in terms of piecewise formal power series, we have

$$\theta = \sum_{g\in G_\Sigma} \prod_{\tau\in\Sigma(1)} \frac{-\varphi_\tau}{1 - a_\tau(g)e^{\varphi_\tau}}$$

where φ_τ is defined in 3.3. We check that $\theta_\sigma = Td_\sigma^T(X)$ for all $\sigma \in \Sigma(d)$. This will imply that the $Td_\sigma^T(X)$ glue together into an element of $\hat{R}_\Sigma$.

Let $g \in G_\Sigma$ and let $\sigma \in \Sigma(d)$. If $g \notin G_\sigma$ then there exists an edge τ of Σ such that $a_\tau(g) \neq 1$, and such a τ is not an edge of σ. Then the formal power series expansion of

$$\frac{-\varphi_\tau}{1 - a_\tau(g) e^{\varphi_\tau}}$$

is divisible by φ_τ, and φ_τ vanishes identically on σ. It follows that

$$\theta_\sigma = \sum_{g \in G_\sigma} \prod_{\tau \in \Sigma(1)} \frac{-\varphi_\tau}{1 - a_\tau(g) e^{\varphi_\tau}} \ .$$

Moreover, for $g \in G_\sigma$ and $\tau \notin \sigma(1)$, restriction to σ of

$$\frac{-\varphi_\tau}{1 - a_\tau(g) e^{\varphi_\tau}}$$

is equal to 1, because $\chi_\tau(g) = 1$ and $(\varphi_\tau)_\sigma = 0$. It follows that

$$\theta_\sigma = \sum_{g \in G_\sigma} \prod_{\tau \in \sigma(1)} \frac{-\varphi_\tau}{1 - a_\tau(g) e^{\varphi_\tau}} \ .$$

Notation being as in 2.1, we obtain

$$\theta_\sigma = \sum_{g \in G_\sigma} \prod_{i=1}^{d} \frac{-q_i^{-1} m_i}{1 - a_i(g) e^{q_i^{-1} m_i}} \ .$$

Denote by $M_{\sigma(1)} \subset M_{\mathbf{Q}}$ the dual lattice to $N_{\sigma(1)}$. Then $M_{\sigma(1)}$ is generated by the $q_i^{-1} m_i$ $(1 \leq i \leq d)$. The group G_σ acts on the group algebra $\mathbf{C}[M_{\sigma(1)}]$ by

$$g \cdot e^{q_i^{-1} m_i} = a_i(g) e^{q_i^{-1} m_i}$$

and the algebra of invariants for this action is $\mathbf{C}[M]$. Consider the subalgebra of $\mathbf{C}[M_{\sigma(1)}]$ generated by $e^{q_1^{-1} m_1}, \ldots, e^{q_d^{-1} m_d}$. Then this subalgebra is stable under G_σ, and its algebra of invariants identifies to the algebra of regular functions on X_σ. By Molien's formula, it follows that

$$\chi_\sigma^T(\mathcal{O}_X) = |G_\sigma|^{-1} \sum_{g \in G_\sigma} \prod_{i=1}^{d} \frac{1}{1 - a_i(g) e^{q_i^{-1} m_i}} \ .$$

Therefore, we obtain (using 2.1)

$$\theta_\sigma = |G_\sigma| \chi_\sigma^T(\mathcal{O}_X) \prod_{i=1}^{d} (-q_i^{-1} m_i) = |G_\sigma| (\sum_{m \in M \cap Q^\sigma} e^m) \prod_{i=1}^{d} \frac{-q_i^{-1} m_i}{1 - e^{m_i}}$$

and hence

$$\theta_\sigma = |G_\sigma| \, |M \cap Q^\sigma| (\prod_{i=1}^{d} q_i^{-1}) Td_\sigma^T(X) = Td_\sigma^T(X) \ .$$

To obtain the second formula, we observe that, for $g \in G_{\Sigma(1)}$, the term

$$\prod_{\tau \in \Sigma(1)} \frac{F_\tau}{1 - a_\tau(g) e^{-F_\tau}}$$

vanishes when $g \notin G_\Sigma$.

Corollary *The equivariant Todd class of a complete, smooth toric variety* X_Σ *is given by*

$$Td^T(X) = \prod_{\tau \in \Sigma(1)} \frac{F_\tau}{1 - e^{-F_\tau}} \ .$$

Proof. Recall that X_Σ is smooth if and only if each cone σ is generated by part of a basis of N. This is equivalent to: $N_\sigma = \sum_{\tau \in \sigma(1)} \mathbf{Z} n_\tau$ for all σ, or to: G_Σ consists in one point.

4.3. *The combinatorics of the equivariant Todd class*

Let $X = X_\Sigma$ be a complete, simplicial toric variety. Following an idea of [P1], we define the *mock equivariant Todd class* of X by

$$TD^T(X) := \prod_{\tau \in \Sigma(1)} \frac{F_\tau}{1 - e^{-F_\tau}}$$

where F_τ denotes the equivariant cohomology class of D_τ. We will analyze the difference $Td^T(X) - TD^T(X)$. Let c be the largest integer such that each cone in $\Sigma(c-1)$ is generated by part of a basis of N; then c is the codimension in X of its singular locus.

Proposition. *Notation being as above, the lowest degree term in $Td^T(X) - TD^T(X)$ occurs in degree at least c. Moreover, its term of degree c equals*

$$\sum_{\sigma\in\Sigma(c)} t(\sigma)\, F_\sigma$$

where $t(\sigma)$ is given by

$$t(\sigma) = 2^{-c} mult(\sigma)^{-1} \sum_{g\in G_\sigma, g\neq 1} \prod_{j=1}^{c} (1 + icot(\pi x_j)) \ .$$

Here each g in $G_\sigma = N_\sigma/N_{\sigma(1)}$ is represented by $\sum_{i=1}^{c} x_j n_j$ where $n_1, \ldots, n_c$ are the primitive vectors on edges of σ.

Proof. By 4.2, we have

$$Td^T(X) - TD^T(X) = \sum_{g\in G_\Sigma, g\neq 1} \prod_{\tau\in\Sigma(1)} \frac{F_\tau}{1 - a_\tau(g) e^{-F_\tau}} \ .$$

Write G_Σ as the disjoint union of the sets G_σ^0 ($\sigma \in \Sigma$) where G_σ^0 denotes the complement in G_σ of the union of its subsets $G_{\sigma'}$ (σ' a face of σ). For $g \in G_\sigma^0$, observe that $a_\tau(g) = 1$ if and only if $\tau \notin \sigma(1)$. It follows that $Td^T(X) - TD^T(X)$ can be written as

$$\sum_{\sigma\in\Sigma, \sigma\neq 0} \Big(\sum_{g\in G_\sigma^0} \prod_{\tau\in\sigma(1)} \frac{1}{1 - a_\tau(g) e^{-F_\tau}}\Big)\Big(\prod_{\tau\in\sigma(1)} F_\tau\Big)\Big(\prod_{\tau\notin\sigma(1)} \frac{F_\tau}{1 - e^{-F_\tau}}\Big) \ .$$

Moreover, the set G_σ^0 is empty unless $\dim(\sigma) \geq c$, and $G_\sigma^0 = G_\sigma$ if $\dim(\sigma) = c$. It follows that all terms of degree less than c in $Td^T(X) - TD^T(X)$ vanish, and that its term of degree c equals

$$\sum_{\sigma\in\Sigma(c)} \Big(\sum_{g\in G_\sigma, g\neq 1} \prod_{\tau\in\sigma(1)} \frac{1}{1 - a_\tau(g)}\Big) \prod_{\tau\in\sigma(1)} F_\tau \ .$$

Now we have $\prod_{\tau\in\sigma(1)} F_\tau = mult(\sigma)^{-1} F_\sigma$ by 3.3, and moreover

$$\sum_{g\in G_\sigma, g\neq 1} \prod_{\tau\in\sigma(1)} \frac{1}{1 - a_\tau(g)} = \sum_{g\in G_\sigma, g\neq 1} \prod_{j=1}^{c} \frac{1}{1 - exp(2i\pi x_j)}$$

$$= \sum_{g \in G_\sigma, g \neq 1} \prod_{j=1}^{c} \frac{exp(i\pi x_j)}{2sin(\pi x_j)} .$$

This proves our formula.

4.4. *A connection with higher Dedekind sums*

Notation being as in 4.3, assume for simplicity that $c = d$, i.e. that any $(d-1)$-dimensional cone in Σ is generated by part of a basis of N. Then, for a fixed $\sigma \in \Sigma(d)$, we can find a basis $(e_1, \ldots, e_d)$ of N and integers $p_1, \ldots, p_{d-1}, q$ such that:
(i) σ is generated by $e_1, \ldots, e_{d-1}$ and $p_1 e_1 + \cdots + p_{d-1} e_{d-1} + q e_d$,
(ii) $0 \leq p_1, \ldots, p_{d-1} \leq q$ and $p_1, \ldots, p_{d-1}$ are prime to q.

Generalizing results of Pommersheim ([P1] Theorem 3 for $d = 2$, [P2] Theorem 4 for $d = 3$), we will express the rational number $t(\sigma)$ defined in 4.3, in terms of higher Dedekind sums. These sums are defined as follows by Zagier, see [Z].

Let n be an even positive integer; let $a_1, \ldots, a_n$ and q be integers such that $q > 0$ and that $a_1, \ldots, a_n$ are prime to q. Then set

$$s(q; a_1, \ldots, a_n) := (-1)^{n/2} \sum_{k=1}^{q-1} cot(\frac{\pi k a_1}{q}) \cdots cot(\frac{\pi k a_n}{q})$$

(Zagier's notation is $d(p; a_1, \ldots, a_n)$).

Proposition. *Notation being as above, we have*

$$t(\sigma) = \frac{1}{2^d q}(q - 1 + \sum_{1 \leq i_1 < \cdots < i_{2j} \leq d-1} s(q; p_{i_1}, \ldots, p_{i_{2j}}) - \sum_{1 \leq i_1 < \cdots < i_{2j-1} \leq d-1} s(q; p_{i_1}, \ldots, p_{i_{2j-1}}, 1))$$

and moreover

$$t(\sigma) = -\frac{1}{2^d q}(1-q)^d + (-1)^d \sum_{1 \leq k_1, \ldots, k_{d-1} \leq q-1} \frac{k_1 \cdots k_{d-1}}{q^{d-1}} \{\frac{p_1 k_1 + \cdots + p_{d-1} k_{d-1}}{q}\}$$

where $\{x\}$ denotes the fractional part of x.

Proof. We can identify $G_\sigma = N/N_{\sigma(1)}$ with the set

$$\{n = (x_1+p_1x_d)e_1+\cdots+(x_{d-1}+p_{d-1}x_d)e_{d-1}+qx_de_d \mid n \in N,\ 0 \le x_1,\ldots,x_d < 1\} .$$

Then $n \in G_\sigma$ if and only if: $x_d = kq^{-1}$ for some integer k with $0 \le k \le q-1$, and moreover $x_1 + p_1kq^{-1}, \ldots, x_{d-1} + p_{d-1}kq^{-1}$ are integers. It follows that $mult(\sigma) = q$ and, using 4.3, that

$$t(\sigma) = \frac{1}{2^dq}\sum_{k=1}^{q-1}(1 + icot(\pi kq^{-1}))\prod_{j=1}^{d-1}(1 - icot(\frac{\pi kp_j}{q})) .$$

This proves our first formula. For the second formula, remember that

$$Td_\sigma^T(X) = |M \cap Q^\sigma|^{-1}(\sum_{m\in M\cap Q^\sigma} e^m)\prod_{j=1}^{d}\frac{-m_j}{1-e^{m_j}}$$

and that

$$TD_\sigma^T(X) = \prod_{j=1}^{d}\frac{-q_j^{-1}m_j}{1-e^{q_j^{-1}m_j}}$$

with notation as in 2.1. Denoting by $(e_1^*,\ldots,e_d^*)$ the dual basis of $(e_1,\ldots,e_d)$, we have here $m_j = qe_j^* - p_je_d^*$ for $1 \le j \le d-1$, and $m_d = e_d^*$. So $q_1 = \ldots = q_d = q$. Moreover, $M \cap Q^\sigma$ consists in all points $(\sum_{j=1}^{d-1} k_iq^{-1}m_i) + x_dm_d$ where $k_1,\ldots,k_{d-1}$ are integers between 0 and $q-1$, and where $x_d = \{(p_1k_1 + \cdots + p_{d-1}k_{d-1})q^{-1}\}$.

By 3.3 and 4.3, the lowest degree term in $Td_\sigma^T(X) - TD_\sigma^T(X)$ is

$$(-1)^dt(\sigma)q^{1-d}\prod_{j=1}^{d} m_j .$$

Because the constant term in $TD_\sigma^T(X)$ is 1, it follows that $(-1)^dq^{1-d}t(\sigma)$ is the coefficient of $\prod_{j=1}^d m_j$ in the expansion of

$$q^{1-d}(\sum_{m\in M\cap Q^\sigma} e^m)\prod_{j=1}^{d}\frac{q(e^{q^{-1}m_j}-1)}{e^{m_j}-1}$$

into a power series in $m_1,\ldots,m_d$. Moreover, this expansion involves no term of degree $1, 2, \ldots, d-1$.

Let $u_1, \dots, u_d, v_1, \dots, v_d$ be homogeneous elements in $S^*(M_{\mathbf{Q}})$ (of degree equal to their index) such that

$$q^{1-d} \sum_{m \in M \cap Q^\sigma} e^m = 1 + u_1 + \cdots + u_d$$

and that

$$\prod_{j=1}^{d} \frac{e^{m_j} - 1}{q(e^{q^{-1} m_j} - 1)} = 1 + v_1 + \cdots + v_d$$

up to terms of degree at least $d+1$. Then we must have $u_j = v_j$ for $1 \le j \le d-1$, and hence $(-1)^d q^{1-d} t(\sigma)$ is the coefficient of $\prod_{j=1}^{d} m_j$ in $u_d - v_d$. This implies our second formula.

Remark. For $d = 2$, we obtain

$$t(\sigma) = \frac{1}{4q}(q - 1 + \sum_{k=1}^{q-1} cot(\frac{\pi}{q}) cot(\frac{k\pi}{q})) = -\frac{(q-1)^2}{4q} + \sum_{k=1}^{q-1} \frac{k}{q} \{\frac{pk}{q}\} \ .$$

This amounts to the classical identity

$$\sum_{k=1}^{q-1} (\frac{k}{q} - \frac{1}{2})(\{\frac{pk}{q}\} - \frac{1}{2}) = \frac{1}{4q} \sum_{k=1}^{q-1} cot(\frac{\pi}{q}) cot(\frac{k\pi}{q})$$

(see e.g. [Z] p. 151).

4.5. *Euler-MacLaurin formula for convex lattice polytopes*

Let $P \subset M_{\mathbf{R}}$ be a convex lattice polytope (i.e. the convex hull of finitely many points of M) of dimension d. For each facet F of P, there exists a unique primitive vector $(n_F, \lambda_F) \in N \times \mathbf{Z}$ such that the affine form $x \to \langle n_F, x \rangle + \lambda_F$ is identically zero on F, and is positive on $P \setminus F$. To each face G of P, we associate the cone σ_G in $\mathbf{N_Q}$ generated by the n_F such that F is a facet of P which contains G. This defines a complete fan Σ_P. We identify the set $\Sigma_P(1)$ with the set of all facets of P, and we denote it by $\mathcal{F}$ for simplicity.

For $h = (h_\tau)_{\tau \in \mathcal{F}} \in \mathbf{R}^{\mathcal{F}}$, we define a convex polytope $P(h)$ in $\mathbf{M_R}$ by the inequalities

$$\langle n_\tau, x \rangle + \lambda_\tau + h_\tau \ge 0 \ .$$

In particular, $P(0) = P$. On the other hand, we define a differential operator of infinite order $Todd_{\mathcal{F}}(\partial/\partial h)$ (the *Todd operator associated to* $\mathcal{F}$) by

$$Todd_{\mathcal{F}}(\partial/\partial h) := \sum_{g \in G_{\mathcal{F}}} \prod_{\tau \in \mathcal{F}} \frac{\partial/\partial h_\tau}{1 - a_\tau(g) e^{-\partial/\partial h_\tau}}$$

with notation as in 4.2. Then $Todd_{\mathcal{F}}(\partial/\partial h)$ acts for example on the space of polynomial functions in h.

Theorem. *For any polynomial function φ on $M_{\mathbf{R}}$, the function $h \to \int_{P(h)} \varphi(x)\,dx$ is continuous and piecewise polynomial. Moreover, we have*

$$\sum_{m \in P \cap M} \varphi(m) = (Todd_{\mathcal{F}}(\partial/\partial h) \int_{P(h)} \varphi(x)\,dx)_{h=0}\ .$$

Proof. There exists a neighborhood U of the origin in $\mathbf{R}^{\mathcal{F}}$, and a complete fan Φ in $\mathbf{R}^{\mathcal{F}}$ such that, for $h \in U$, the fan $\Sigma_{P(h)}$ only depends on the smallest cone of Φ which contains h. If moreover h is in the interior of some maximal cone C in Φ, then $\Sigma_{P(h)}$ is a simplicial subdivision of Σ, with the same set of edges. We fix such a maximal cone C, and we set $\Sigma := \Sigma_{P(h)}$and $X := X_\Sigma$.

For any $h \in \mathbf{R}^{\mathcal{F}}$, define a T-stable divisor (with real coefficients) on X by

$$D(h) := \sum_{\tau \in \Sigma(1)} (\lambda_\tau + h_\tau) D_\tau\ .$$

If $h \in C \cap U$ and if moreover $kh \in \mathbf{Z}^{\mathcal{F}}$ for some non-zero integer k, then the divisor (with integral coefficients) $kD(h)$ is Cartier and generated by its global sections, see [O] 2.1 and 2.7. Moreover, we have:

$$\chi^T(\mathcal{O}_X(kD(h))) = \sum_{m \in M \cap kP(h)} e^m\ .$$

By theorem 4.1, we have

$$\int_X e^{kc_1(D(h))} Td^T(X) = \sum_{m \in M \cap kP(h)} e^m$$

where $c_1(D(h)) \in H^2_T(X)$ denotes the equivariant cohomology class of $D(h)$. Viewing $c_1(D(h))$ as a piecewise linear function, we have by 3.3:

$$c_1(D(h)) = -\sum_{\tau \in \mathcal{F}} (\lambda_\tau + h_\tau) \varphi_\tau\ .$$

For $\sigma \in \Sigma(d)$, denote by $c_\sigma(h)$ the element of $M_{\mathbf{R}}$ such that $c_1(D(h))$ coincides with $c_\sigma(h)$ on σ. Then we have, using 3.5:

$$\sum_{m \in M \cap kP(h)} e^m = \sum_{\sigma \in \Sigma(d)} e^{-kc_\sigma(h)} \Phi_\sigma^{-1} Td_\sigma^T(X) \ .$$

Evaluating both sides at a point $k^{-1}y$ of $N_{\mathbf{R}}$ where y does not belong to any hyperplane generated by some $(d-1)$-dimensional cone of Σ, and dividing by k^d, we obtain

$$k^{-d} \sum_{m \in P(h) \cap k^{-1}M} exp\langle m, y\rangle = \sum_{\sigma \in \Sigma(d)} exp(-c_\sigma(h)(y)) \Phi_\sigma(y)^{-1} Td_\sigma^T(X)(k^{-1}y) \ .$$

This equation holds for all k such that kh is integral. Therefore, letting $k \to \infty$ and remembering that the constant term of $Td_\sigma^T(X)$ is 1, we obtain

$$\int_{P(h)} exp\langle x, y\rangle \, dx = \sum_{\sigma \in \Sigma(d)} exp(-c_\sigma(h)(y)) \Phi_\sigma(y)^{-1} \ .$$

This holds for all rational points h in $C^0 \cap U$ and hence (by continuity) for all points in $C^0 \cap U$. Therefore, we obtain using 3.5:

$$\int_{P(h)} exp(x) \, dx = \int_X exp(- \sum_{\tau \in \Sigma(1)} (\lambda_\tau + h_\tau)\varphi_\tau)$$

for any $h \in C^0 \cap U$. ¿From this formula, we deduce that for any polynomial function f on $\mathbf{R}^{\mathcal{F}}$, we have

$$(f(\partial/\partial h) \int_{P(h)} exp(x) \, dx)_{h=0} = \int_X exp(- \sum_{\tau \in \mathcal{F}} \lambda_\tau \varphi_\tau) f(-\varphi_\tau) \ .$$

Using the second formula in theorem 4.2, it follows that

$$(Todd_{\mathcal{F}}(\partial/\partial h) \int_{P(h)} exp(x) \, dx)_{h=0} = \int_X e^{c_1(D)} Td^T(X) = \sum_{m \in M \cap P} e^m \ .$$

Expanding both sides into power series and observing that any polynomial function is a linear combination of powers of linear forms, we obtain our formula.

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