VOLUME COMPUTATION FOR POLYTOPES AND PARTITION FUNCTIONS FOR CLASSICAL ROOT SYSTEMS

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ABSTRACT. This paper presents an algorithm to compute the value of the inverse Laplace transforms of rational functions with poles on arrangements of hyperplanes. As an application, we present an efficient computation of the partition function for classical root systems.

1. INTRODUCTION

The ultimate goal of this work is to present an algorithm for a fast computation of the partition function of classical root systems. We achieve this goal in somewhat more general terms, namely we develop algorithms to compute the volume of a polytope and its discrete analog, the number of integer points in the polytope. These formulas, in turn, are inverse Laplace transforms of certain rational functions, and our work can be viewed in these general terms.

Let U be a finite-dimensional real vector space of dimension r. Denote its dual vector space U^* by V. Consider a set of elements

$$\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$$

of non-zero vectors of V. We assume that the convex cone $\mathcal{C}(\mathcal{A})$ generated by non-negative linear combinations of the elements α_i is an acute convex cone in V with non-empty interior.

The elements ℓ in V produce linear functions $u \mapsto \ell(u)$ on the complexified vector space $U_{\mathbb{C}}$. In particular, to the set $\mathcal{A}$ we associate the arrangement

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of hyperplanes

$$\mathcal{H}_{\mathbb{C}}(\mathcal{A}) := \bigcup_{i=1}^{N} \{u \in U_{\mathbb{C}} \mid \alpha_i(u) = 0\}$$

in $U_{\mathbb{C}}$ and its complement

$$U_{\mathbb{C}}(\mathcal{A}) := \left\{ u \in U_{\mathbb{C}} \;\middle|\; \prod_{i=1}^{N} \alpha_i(u) \neq 0 \right\}.$$

We denote by $\mathcal{R}_{\mathcal{A}}$ the ring of rational functions on $U_{\mathbb{C}}(\mathcal{A})$ with poles along $\mathcal{H}_{\mathbb{C}}(\mathcal{A})$. Then each element $\phi \in \mathcal{R}_{\mathcal{A}}$ can be written as P/Q where P is a polynomial function on r complex variables and Q is a product of elements, not necessarily distinct, of $\mathcal{A}$.

Our first aim is to present an algorithm to compute the value of the inverse Laplace transform of functions in $\mathcal{R}_{\mathcal{A}}$ at a point $h \in V$. In other words, we study the value at a point $h \in V$ of convolutions of a number of Heaviside distributions $\phi \mapsto \int_0^\infty \phi(t\alpha_i)dt$. The first theoretical ingredient is the notion of Jeffrey-Kirwan residues [14]. Going a step further, DeConcini-Procesi [12] proved that one can compute Jeffrey-Kirwan residues using maximal nested sets (in short MNS), a combinatorial tool related to no-broken-circuit bases of the set of vectors $\mathcal{A}$.

The applications in view are volume computation for polytopes, enumeration of integral points in polytopes and, more generally, discrete or continuous integration of polynomial functions over polytopes. Indeed, Szenes-Vergne [21], refining a formula of Brion-Vergne [7], stated formulae for the volume and number of integral points in polytopes involving Jeffrey-Kirwan residues.

Consider the polytope

$$\Pi_{\mathcal{A}}(h) := \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{N} x_i\, \alpha_i = h,\ x_i \geq 0 \right\}.$$

As a function of h, the volume of $\Pi_{\mathcal{A}}(h)$ is a piecewise-defined polynomial. The chambers of polynomiality in the parameter space V are polyhedral cones.

Our programs are extremely efficient for computing the volume of the polytope $\Pi_{\mathcal{A}}(h)$ when $\mathcal{A}$ is a classical root system. An important fact is that our algorithm can work with formal parameters, thus giving the polynomial volume formula for $\Pi_{\mathcal{A}}(h)$ when h runs over a particular chamber.

For an analogous theory for integral-point enumeration, we have to assume that the α_i are vectors in a lattice $V_{\mathbb{Z}}$. For $h \in V_{\mathbb{Z}}$, the function $N_{\mathcal{A}}(h)$ which associates to the vector h the number of integral points in $\Pi_{\mathcal{A}}(h)$, that is the number of ways to represents the vector h as a sum of a certain number of vectors α_i, is called the (vector)-partition function of $\mathcal{A}$. For example for B_2, given a vector (h_1, h_2) with integral coordinates we would like to compute the number $N_{B_2}(h)$ of vectors $(x_i) \in \mathbb{Z}_+^4$ such that

$$x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

As a function of h, the number $N_{\mathcal{A}}(h)$ of integral points in $\Pi_{\mathcal{A}}(h)$ is a piecewise-defined quasipolynomial, and again the chambers of quasipolynomiality are polyhedra in V [19, 20].

In this paper, we describe an efficient algorithm for MNS computation for classical root systems. This algorithm for MNS gives rise to programs for Kostant partition function for the classical root systems A_n, B_n, C_n, and D_n. Again, our algorithm works with a formal parameter h that is assumed to be confined to a particular chamber.

These calculations are valuable because partition functions play a fundamental role also in representation theory of semisimple Lie algebras $\mathfrak{g}$. Indeed, partition functions arise naturally when we want to compute the multiplicity of a weight in a finite-dimensional representation or the tensor-product decomposition of two representations, both being basic problems to understand characters of representations. Cochet [9] has obtained very efficient algorithms for both these problems in the case of A_n, implementing results of [2]. See also a forthcoming paper [10] for multiplicities computation in all the classical Lie algebras using the results obtained in this paper. There is also a class of infinite-dimensional representations, the discrete-series representations, whose understanding is central for the general theory of admissible irreducible representations. The decomposition of such representations to a maximal compact subgroup of $\mathfrak{g}$ is predicted by Blattner's formula, which is a partition function in which the roots involved are the so-called noncompact roots.

We conclude by describing the way the paper is organized. Section 2 introduces Laplace transforms and polytopes. In Section 3, we recall Jeffrey-Kirwan residues and its link with counting formulae. DeConcini-Procesi's maximal nested sets are described in Section 4, as well as how they are related to Jeffrey-Kirwan residues. Section 5 describes our general algorithm for MNS computations. Details of particular cases of the algorithm for the root systems A_n, B_n, C_n and D_n are examined in Sections 7–10. Finally comparative tests of our programs with existing softwares are performed in Section 11.

A number of theoretical results on the function $N_{\mathcal{A}}(h)$ when $\mathcal{A}$ is a subset of the system A_n can be found in Baldoni-Vergne [1] (as, for example, the computation of the volume of the Chan-Robbins polytope).

Computer programs for volume computation/integral-point enumeration in polytopes have only been implemented in the very recent past, most notably `LattE` [16, 17] and `barvinok` [6], both of which are implementations of Barvinok's algorithm [3]. To the best of our knowledge, these two are the only general programs for volume computation/integral-point enumeration in polytopes. More specialized programs include algorithms of Baldoni-DeLoera-Vergne for flow polytopes [2] and Beck-Pixton for the Birkhoff polytope [4].

Our programs have been especially designed for classical root systems, are faster than all actual existing softwares and can compute new examples that were not reachable by previous algorithms. Note in particular that our programs can perform computations for $N_{\mathcal{A}}(h)$ for $\mathcal{A}_n$ at least up to $n = 10$ (11 coordinates vector). For $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$ the algorithms are efficient at least up to $n = 6$. For our methods (as well as for `LattE`), the size of the vector h affects only little on the computation time. Recall that our methods can also calculate the multivariate quasi-polynomials $h \mapsto N_{\mathcal{A}}(h)$ when h varies on a chamber, and as a particular case for a fixed h the function $k \mapsto N_{\mathcal{A}}(kh)$ which is the Ehrhart quasipolynomial in k.

2. Laplace transform and polytopes

We start by briefly recalling the notations of the introduction, aiming to relate the inverse of the Laplace transform with various *counting formulae* for a polytope. A good introduction on this theme is the survey article [23].

2.1. **Laplace transform.** Let U be a finite-dimensional real vector space of dimension r with dual space V. We fix the choice of a Lebesgue measure dh on V. Consider a set

$$\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

of non-zero vectors of V. We assume that the set of vectors α_i spans V. For any subset S of V, we denote by $\mathcal{C}(S)$ the convex cone generated by non-negative linear combinations of elements of S. We assume that the convex cone $\mathcal{C}(\mathcal{A})$ is acute in V with non-empty interior.

Let $\mathcal{V}_{sing}(\mathcal{A})$ be the union of the boundaries of the cones $\mathcal{C}(S)$, where S ranges over all the subsets of $\mathcal{A}$. The complement of $\mathcal{V}_{sing}(\mathcal{A})$ in $\mathcal{C}(\mathcal{A})$ is by definition the open set $\mathcal{C}_{reg}(\mathcal{A})$ of *regular* elements. A connected component $\mathfrak{c}$ of $\mathcal{C}_{reg}(\mathcal{A})$ is called a *chamber* of $\mathcal{C}(\mathcal{A})$. Figures 1 and 2 represent slices of the cones $\mathcal{C}(A_3)$ and $\mathcal{C}(B_3)$, where the dots represent the intersection of a slice with a ray $\mathbb{R}_{\geq 0}$ hence showing the chambers. Note that the chambers for B_r and C_r are the same (as roots in B_r and C_r are proportional). In dimension 3, the root system A_3 is isomorphic to D_3. See [2] for the computation of chambers. Very little is known about the total number of chambers. On the other hand, given a vector h, it is easy to compute the equations of the chamber containing h. This was done in [2, 11]. We have incorporated this small part of the corresponding program in our programs for classical root systems.

Table 3 represents the only numbers of chambers that have been computed (and the computation time).

Consider now a cone $\mathcal{C}(S)$ spanned by a subset S of $\mathcal{A}$ and let p be a function on $\mathcal{C}(S)$. We assume that p is the restriction to $\mathcal{C}(S)$ of a polynomial function on V. By superposing such functions p, we obtain a space $\mathcal{LP}(V, \mathcal{A})$

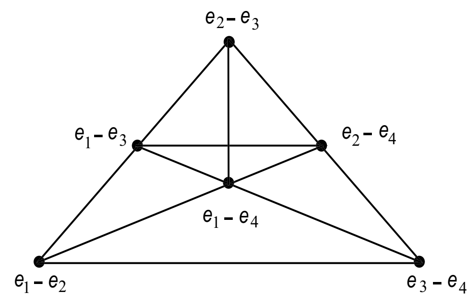

Figure 1. The 7 chambers for A_3

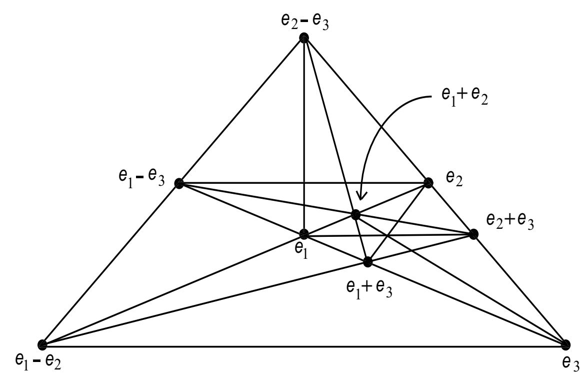

Figure 2. The 23 chambers for B_3

	A	B	C	D	F	G
1	1 (0s)	1 (0s)	1 (0s)			
2	2 (0s)	3 (0s)	3 (0s)	1 (0s)		5 (0s)
3	7 (1s)	23 (8s)	23 (8s)	7 (1s)		
4	48 (23s)	695 (11m)	695 (11m)	133 (90s)	12946 (3d16h)	
5	820 (19m)	>26905 ?	>26905 ?	12926 (1d5h)		
6	44288 (24d18h)	?	?	?		

Figure 3. Number of chambers and computation time

of locally polynomial functions on $\mathcal{C}(\mathcal{A})$. For $f \in \mathcal{LP}(V, \mathcal{A})$, the restriction of f to any chamber $\mathfrak{c}$ of $\mathcal{C}(\mathcal{A})$ is given by a polynomial function.

The Laplace transform $L(f)$ of such a function f is defined as follows. Consider the dual cone $\mathcal{C}(\mathcal{A})^* \subset U$ of $\mathcal{C}(\mathcal{A})$ defined by:

$$\mathcal{C}(\mathcal{A})^* = \{u \in U \mid \langle h, u\rangle \geq 0 \text{ for all } h \in \mathcal{C}(\mathcal{A})\}\,.$$

Then for u in the interior of the cone $\mathcal{C}(\mathcal{A})^*$, the integral

$$L(f)(u) = \int_{\mathcal{C}(\mathcal{A})} e^{-\langle h,u\rangle} f(h)dh$$

is convergent. It is easy to see that the function $L(f)$ is the restriction to $\mathcal{C}(\mathcal{A})^*$ of a function in $\mathcal{R}_{\mathcal{A}}$. (Recall that $\mathcal{R}_{\mathcal{A}}$ is the ring of rational functions P/Q on U where P is a polynomial function on U and Q is a product of elements of $\mathcal{A}$.) It is easy [7] to characterize the functions $L(f)$ on U arising this way.

Let ν be a subset of $\{1, 2, \ldots, n\}$. We will say that ν is *generating* (respectively *basic*) if the set $\{\alpha_i \mid i \in \nu\}$ generates (respectively is a basis of) the vector space V.

Every basic subset is of cardinality r and we write $\mathrm{Bases}(\mathcal{A})$ for the set of basic subsets. Given $\sigma \in \mathrm{Bases}(\mathcal{A})$, the associated *basic fraction* is

$$f_\sigma = \frac{1}{\prod_{i\in\sigma} \alpha_i}. \tag{1}$$

In a system of coordinates (depending on σ) on U where $\alpha_i(u) = u_i$ (for $i \in \sigma$), such a basic fraction is simply of the form

$$\frac{1}{u_1 u_2 \cdots u_r}.$$

Define $\mathcal{G}(U, \mathcal{A}) \subset \mathcal{R}_{\mathcal{A}}$ as the linear span of functions $\frac{1}{\prod_{i\in\nu}\alpha_i^{n_i}}$, where ν is generating and n_i are positive integers. The following proposition gives the characterization we were speaking of and is easy to prove:

Proposition 2.1. [7] *If f is a locally polynomial function on $\mathcal{C}(\mathcal{A})$, the Laplace transform $L(f)$ of f is the restriction to $\mathcal{C}(\mathcal{A})^*$ of a function in $\mathcal{G}(U, \mathcal{A})$. Reciprocally, for any generating set ν and every set of positive integers $n_i > 0$, there exists a locally polynomial function f on V such that*

$$\frac{1}{\prod_{i\in\nu}\alpha_i(u)^{n_i}} = \int_{\mathcal{C}(\mathcal{A})} e^{-\langle h,u\rangle} f(h)dh$$

for any u in the interior of $\mathcal{C}(\mathcal{A})^*$.

We define the inverse Laplace transform $L^{-1} : \mathcal{G}(U, \mathcal{A}) \to \mathcal{LP}(V, \mathcal{A})$ as follows. For $\phi \in \mathcal{G}(U, \mathcal{A})$, the function $L^{-1}\phi$ is the unique locally polynomial function that satisfies

$$\phi(u) = \int_{\mathcal{C}(\mathcal{A})} e^{-\langle h,u\rangle} (L^{-1}\phi)(h)dh$$

for any $u \in \mathcal{C}(\mathcal{A})^*$.

In the next sections, we will explain the relation between Laplace transforms and the enumeration of integral points of families of polytopes. We will see in Section 4 that one can write efficient formulae for the inversion of Laplace transforms in terms of residues, whose algorithmic implementation is working in a quite impressive way, at least for low dimension.

2.2. **Volume and number of integral points of a polytope.** In this subsection we consider a sequence

$$\mathcal{A}^+ = [\alpha_1, \alpha_2, \ldots, \alpha_N]$$

of non-zero elements of $\mathcal{A}$. We assume that each element $\alpha \in \mathcal{A}$ occurs in the sequence; in particular $N \geq n$ and the set $\mathcal{A}^+$ spans V.

Remark 2.2. In all our examples, the sequence $\mathcal{A}^+$ will not have multiplicities, so that we will freely identify $\mathcal{A}^+$ and $\mathcal{A}$.

We introduce now the notion of a partition polytope.

We consider the space $\mathbb{R}^N$ with its standard basis ω_i and Lebesgue measure dx.

If $x = \sum_{i=1}^N x_i\omega_i \in \mathbb{R}^N$ with $x_i \geq 0$ $(1 \leq i \leq N)$ then we will simply write $x \geq 0$.

Consider the surjective map $A : \mathbb{R}^N \to V$ defined by $A(\omega_i) = \alpha_i$ and denote by K its kernel. Then K is a vector space of dimension $d = N - r$ equipped with the quotient Lebesgue measure dx/dh.

If $h \in V$, we define

$$\Pi_{\mathcal{A}^+}(h) = \left\{x \in \mathbb{R}^N \mid Ax = h; x \geq 0\right\}.$$

The set $\Pi_{\mathcal{A}^+}(h)$ is a convex polytope. It is the intersection of the non-negative quadrant in $\mathbb{R}^N$ with an affine translate of the vector space K. This polytope consists of all non-negative solutions of the system of r linear equations

$$\sum_{i=1}^N x_i\alpha_i = h.$$

Remark 2.3. It might be appropriate to recall that any full dimensional convex polytope P in a vector space E of dimension d, defined by a system of N linear inequations

$$P = \{y \in E \mid \langle u_i, y\rangle + \lambda_i \geq 0\}$$

(where $u_i \in E^*$ and λ_i are real numbers), can be canonically realized as a partition polytope $\Pi_{\mathcal{A}^+}(h)$. Here $\mathcal{A}^+$ is a sequence of N elements in a vector space of dimension $r = N - d$. Indeed, consider the diagram

$$E \xrightarrow{i} \mathbb{R}^N \xrightarrow{A} V = \mathbb{R}^N/i(E)$$

where $i : y \mapsto \sum_{i=1}^N \langle u_i, y\rangle\omega_i$ and A is the projection map $\mathbb{R}^N \longrightarrow V$. Let α_i be the images of the canonical basis ω_i of $\mathbb{R}^N$. Define $\mathcal{A}^+ = [\alpha_1, \ldots, \alpha_N]$ and consider the point $h := A(\sum_{i=1}^N \lambda_i\omega_i)$. Then the polytope $\Pi_{\mathcal{A}^+}(h)$ is isomorphic to P. Indeed, the points in $\Pi_{\mathcal{A}^+}(h)$ are exactly the points x_i such that $\sum_{i=1}^N (x_i - \lambda_i)A(\omega_i) = 0$ with $x_i \geq 0$. By definition of the space

$V = R^N/i(E)$, there exists $y \in E$ such that $x_i - \lambda_i = \langle u_i, y\rangle$. As $x_i \geq 0$, this means exactly that $\langle u_i, y\rangle + \lambda_i \geq 0$, so that the point y is in P.

More concretely, to determine the partition polytope $Ax = b$ starting from a polytope P given by $Qy^T \geq \lambda$ (where Q is a $N \times d$ matrix whose i^{th} row is given by a vector $u_i \in E^*$ and $\lambda \in \mathbb{R}^N$) we choose among the elements u_i a basis of E^*. Thus after relabeling the indices and doing an appropriate translation, we may assume the inequations of the polytope P are given in the form

$$\begin{cases} y_1 \geq 0 \\ y_2 \geq 0 \\ \dots \\ y_d \geq 0 \\ Cy^T + \lambda^T \geq 0 \end{cases} \quad \text{where} \quad \begin{array}{l} C \text{ is a } r \times d \text{ matrix}, \\ \lambda \in \mathbb{R}^r, \\ \text{and } y = (y_1, \dots, y_d) \in \mathbb{R}^d. \end{array}$$

Then the polytope P is isomorphic to the polytope defined by

$$\left\{ x \geq 0 \,|\, Ax^{\mathrm{T}} = \lambda^{\mathrm{T}} \right\}$$

where A is the $r \times N$ matrix given by

$$A = \left(\underbrace{-C}_{r \times d} \quad \underbrace{I_r}_{r \times r} \right), \quad I_r \text{ being the identity matrix.}$$

Example 2.4. Let $P \subset \mathbb{R}^2$ be the polytope defined by the system of inequalities:

$$\begin{cases} -x_1 + 1 & \geq & 0, \\ -x_2 + 2 & \geq & 0, \\ -x_1 - x_2 + 2 & \geq & 0, \\ 2x_1 + x_2 - 1 & \geq & 0. \end{cases}$$

Choosing the basis $u_1 = (1, 0)$, $u_2 = (0, 1)$ and using the translation $y_1 = -x_1 + 1$ and $y_2 = -x_2 + 2$ we can rewrite the system as:

$$\begin{cases} y_1 \geq 0 \\ y_2 \geq 0 \\ C \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \geq 0 \end{cases} \quad \text{where } C = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Therefore P is isomorphic to

$$\Pi_{\mathcal{A}^+}(h) = \left\{ y = (y_1, \dots, y_4) \in \mathbb{R}^4, y \geq 0 \;\middle|\; \begin{array}{rcl} -y_1 - y_2 + y_3 & = & -1 \\ 2y_1 + y_2 + y_4 & = & 3 \end{array} \right\}$$

with $h = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

We continue with our review. If h is in the interior of the cone $\mathcal{C}(\mathcal{A})$, then the polytope $\Pi_{\mathcal{A}^+}(h)$ is of dimension d. It lies in a translate of the vector space K, and this translated space is provided with the quotient measure dx/dh.

Definition 2.5. We write $\mathrm{vol}_{\mathcal{A}^+}(h)$ for the volume of $\Pi_{\mathcal{A}^+}(h)$ computed with respect to this measure.

Suppose further that V is provided with a lattice $V_{\mathbb{Z}}$ and that

$$\mathcal{A}^+ := [\alpha_1, \alpha_2, \dots, \alpha_N]$$

is a sequence of non-zero elements of $V_{\mathbb{Z}}$ spanning $V_{\mathbb{Z}}$, that is, $V_{\mathbb{Z}} = \sum_{i=1}^N \mathbb{Z}\alpha_i$.

In this case, the lattice $V_{\mathbb{Z}}$ determines a measure $d_{\mathbb{Z}}h$ on V so that the fundamental domain of the lattice $V_{\mathbb{Z}}$ is of measure 1 for $d_{\mathbb{Z}}h$. However, for reasons which will be clear later on, we keep our initial measure dh. We introduce the normalized volume.

Definition 2.6. The normalized volume $\mathrm{vol}_{\mathbb{Z},\mathcal{A}^+}(h)$ is the volume of $\Pi_{\mathcal{A}^+}(h)$ computed with respect to the measure $dx/d_{\mathbb{Z}}h$.

Remark 2.7. The reason for keeping our initial dh is that the root systems B_r, C_r, D_r live on the same standard vector space $V = \mathbb{R}^r$, where the most natural measure is the standard one. This measure is twice the measure given by the root lattice in the case of C_r and D_r.

If $\mathrm{vol}(V/V_{\mathbb{Z}}, dh)$ is the volume of a fundamental domain of $V_{\mathbb{Z}}$ for dh, clearly $\mathrm{vol}_{\mathbb{Z},\mathcal{A}^+}(h) = \mathrm{vol}(V/V_{\mathbb{Z}}, dh)\mathrm{vol}_{\mathcal{A}^+}(h)$.

Let now $h \in V_{\mathbb{Z}}$. A discrete analogue of the normalized volume of $\Pi_{\mathcal{A}^+}(h)$ is the number of integral points inside this polytope.

Definition 2.8. Let $N_{\mathcal{A}^+}(h)$ be the number of integral points in $\Pi_{\mathcal{A}^+}(h)$, that is the number of solutions $x = (x_1, \dots, x_N)$ of the equation $\sum_{i=1}^N x_i\alpha_i = h$ where x_i are non-negative integers. The function $h \mapsto N_{\mathcal{A}^+}(h)$ is called the partition function of $\mathcal{A}^+$.

We will see after stating Theorem 3.3 that the functions $h \mapsto \mathrm{vol}_{\mathcal{A}^+}(h)$ and $h \mapsto N_{\mathcal{A}^+}(h)$ are respectively polynomial and quasipolynomial on each chamber of $\mathcal{C}(\mathcal{A})$.

The following formulae (see, for example, [23]) compute the Laplace transform of the locally polynomial function $\mathrm{vol}_{\mathcal{A}^+}(h)$ and the discrete Laplace transform of the quasipolynomial function $N_{\mathcal{A}^+}(h)$.

Proposition 2.9. *Let $u \in \mathcal{C}(\mathcal{A})^*$. Then:*

(1) $$\int_{\mathcal{C}(\mathcal{A})} e^{-\langle h,u\rangle}\mathrm{vol}_{\mathcal{A}^+}(h)dh = \frac{1}{\prod_{i=1}^N \alpha_i(u)}.$$

(2) $$\sum_{h\in V_{\mathbb{Z}}\cap\mathcal{C}(\mathcal{A})} e^{-\langle h,u\rangle}N_{\mathcal{A}^+}(h) = \frac{1}{\prod_{i=1}^N (1 - e^{-\langle\alpha_i,u\rangle})}.$$

3. Jeffrey-Kirwan residue

The aim of this section is to explain some theoretical results due to Jeffrey and Kirwan which are fundamental for our work. They described an efficient scheme for computing the inverse Laplace transforms in the context of hyperplane arrangements.

Let's go back to the space of rational functions $\mathcal{R}_{\mathcal{A}}$. It is $\mathbb{Z}$-graded by degree. Of great importance for our exposition will be certain functions in $\mathcal{R}_{\mathcal{A}}$ of degree $-r$. Every function in $\mathcal{R}_{\mathcal{A}}$ of degree $-r$ may be decomposed into a sum of basic fractions f_σ (see Equation (1)) and degenerate fractions; degenerate fractions are those for which the linear forms in the denominator do not span V. Given $\sigma \in \mathrm{Bases}(\mathcal{A})$, we write $\mathcal{C}(\sigma)$ for the cone generated by α_i ($i \in \sigma$) and by $\mathrm{vol}(\sigma) > 0$ for the volume of the parallelotope $\sum_{i=1}^{r}[0,1]\alpha_i$ computed for the measure dh. Observe that $\mathrm{vol}(\sigma) = |\det(\sigma)|$, where σ is the matrix which columns are the α_i's. Now having fixed a chamber $\mathfrak{c}$, we define a functional $\mathrm{JK}_{\mathfrak{c}}(\phi)$ on $\mathcal{R}_{\mathcal{A}}$ called the Jeffrey-Kirwan residue (or *JK residue*) as follows. Let

$$\mathrm{JK}_{\mathfrak{c}}(f_\sigma) = \begin{cases} \mathrm{vol}(\sigma)^{-1}, & \text{if } \mathfrak{c} \subset \mathcal{C}(\sigma), \\ 0, & \text{if } \mathfrak{c} \cap \mathcal{C}(\sigma) = \emptyset. \end{cases} \tag{2}$$

By setting the value of the JK residue of a degenerate fraction or that of a rational function of pure degree different from $-r$ equal to zero, we have defined the JK residue on $\mathcal{R}_{\mathcal{A}}$.

We may go further and extend the definition to the space $\widehat{\mathcal{R}}_{\mathcal{A}}$ which is the space consisting of functions P/Q where Q is a product of powers of the linear forms α_i and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series. Indeed suppose that $P/Q \in \widehat{\mathcal{R}}_{\mathcal{A}}$ where we may assume that Q is of degree q, and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with P_k of degree k. Then we just define

$$\mathrm{JK}_{\mathfrak{c}}(P/Q) = \mathrm{JK}_{\mathfrak{c}}(P_{q-r}/Q)$$

as the JK residue of the component of degree $-r$ of P/Q. In particular if $\phi \in \mathcal{R}_{\mathcal{A}}$ and $h \in V$, the function

$$e^{\langle h,u\rangle}\phi(u) = \sum_{k=0}^{\infty} \frac{\langle h,u\rangle^k}{k!}\phi(u)$$

is in $\widehat{\mathcal{R}}_{\mathcal{A}}$ and we may compute its JK residue. Observe that the JK residue depends on the measure dh.

Let's now make a short digression that should clarify why JK residues compute inverse Laplace transforms. For $u \in \mathcal{C}(\mathcal{A})^*$ we have:

$$\frac{1}{\mathrm{vol}(\sigma)} \int_{\mathcal{C}(\sigma)} e^{-\langle h,u\rangle} dh = f_\sigma(u).$$

In other words the inverse Laplace transform of f_σ computed at the point $h \in \mathcal{C}(\mathcal{A})$ is $\frac{1}{\mathrm{vol}(\sigma)}\chi_\sigma(h)$, where χ_σ is the characteristic function of the cone $\mathcal{C}(\sigma)$. We state this as a formula:

$$\frac{1}{\mathrm{vol}(\sigma)}\chi_\sigma(h) = L^{-1}(f_\sigma)(h).$$

Since the JK residue can be written in terms of basic fractions, the following theorem [14] is not surprising:

Theorem 3.1 (Jeffrey-Kirwan)**.** *If $\phi \in \mathcal{R}_{\mathcal{A}}$, then for any $h \in \mathfrak{c}$ we have:*

$$(L^{-1}\phi)(h) = \mathrm{JK}_{\mathfrak{c}}\left(e^{\langle h,\cdot\rangle}\phi\right).$$

Assume that $\Psi : U \to U$ is a holomorphic transformation defined on a neighborhood of 0 in U and invertible. We also assume that $\alpha_j(F(u)) = \alpha_j(u)f_j(u)$, where $f_j(u)$ is holomorphic in a neighborhood of 0 and $f_j(0) \neq 0$.

If ϕ is a function in $\widehat{\mathcal{R}}_{\mathcal{A}}$, the function $\Psi^*\phi(u) = \phi(\Psi(u))$ is again in $\widehat{\mathcal{R}}_{\mathcal{A}}$. Let $\mathrm{Jac}(\Psi)$ be the Jacobian of the map Ψ. The function $\mathrm{Jac}(\Psi)$ is calculated as follows: write $\Psi(u) = (\Psi_1(u_1, u_2, \ldots, u_r), \ldots, \Psi_r(u_1, u_2, \ldots, u_r))$. Then $\mathrm{Jac}(\Psi)(u) = \det((\frac{\partial}{\partial u_i}\Psi_j)_{i,j})$. We assume that $\mathrm{Jac}(\Psi)(u)$ does not vanish at $u = 0$. For any ϕ in $\widehat{\mathcal{R}}_{\mathcal{A}}$ the following change of variable formula [1], Theorem 45, which will be useful in our calculations later on, holds:

Proposition 3.2. *The Jeffrey-Kirwan residue obeys the rule of change of variables:*

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \mathrm{JK}_{\mathfrak{c}}(\mathrm{Jac}(\Psi)(\Psi^*\phi)).$$

We conclude this section by recalling the formula for $N_{\mathcal{A}^+}(h)$.

Consider the dual lattice $U_{\mathbb{Z}} = \{u \in U \,|\, \langle u, V_{\mathbb{Z}}\rangle \subset \mathbb{Z}\}$ and the torus $T = U/U_{\mathbb{Z}}$. Choosing a basis $\{u_1, \ldots, u_r\}$ of $U_{\mathbb{Z}}$ we may identify T with the subset of U defined by the fundamental domain for translation by $U_{\mathbb{Z}}$:

$$\left\{\sum_{j=1}^{r} t_j u_j\right\}$$

with $0 \leq t_j < 1$.

Every element g in $T = U/U_{\mathbb{Z}}$ produces a function on $V_{\mathbb{Z}}$ by $h \mapsto e^{\langle h, 2\pi\sqrt{-1}G\rangle}$, where we denote by G a representative of $g \in U/U_{\mathbb{Z}}$.[1] For $\sigma \in \mathrm{Bases}(\mathcal{A})$ we denote by $T(\sigma)$ the subset of T defined by

$$T(\sigma) = \left\{g \in T \;\middle|\; e^{\langle \alpha, 2\pi\sqrt{-1}G\rangle} = 1 \text{ for all}\, \alpha \in \sigma\right\}.$$

This is a finite subset of T. In particular if σ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}$, then $T(\sigma)$ is reduced to the identity. More generally, consider the lattice $\mathbb{Z}\sigma$ generated by the elements α in σ. If p is an integer such that $\mathbb{Z}\sigma \subset pV_{\mathbb{Z}}$, then all elements of $T(\sigma)$ are of order p.

For $g \in T$ and $h \in V_{\mathbb{Z}}$, consider the *Kostant function* $F(g, h)$ on U defined by

$$F(g,h)(u) = \frac{e^{\langle h, 2\pi\sqrt{-1}G + u\rangle}}{\prod_{i=1}^{N}(1 - e^{-\langle \alpha_i, 2\pi\sqrt{-1}G + u\rangle})}. \tag{3}$$

For example when $g = 0$,

$$F(0,h)(u) = \frac{e^{\langle h, u\rangle}}{\prod_{i=1}^{N}(1 - e^{-\langle \alpha_i, u\rangle})}.$$

[1] We prefer to denote the complex number i by $\sqrt{-1}$ because we use i for many indices.

The function $F(g,h)(u)$ is an element of $\widehat{\mathcal{R}}_{\mathcal{A}}$. Indeed if we write

$$I(g) = \left\{ i \,\middle|\, 1 \le i \le N, e^{-\langle \alpha_i, 2\pi\sqrt{-1}G\rangle} = 1 \right\},$$

then

$$F(g,h)(u) = e^{\langle h, 2\pi\sqrt{-1}G\rangle} \frac{e^{\langle h,u\rangle}}{\prod_{i\in I(g)} \langle \alpha_i, u\rangle} \psi^g(u) \tag{4}$$

where $\psi^g(u)$ is the holomorphic function of u (in a neighborhood of zero) defined by

$$\psi^g(u) = \prod_{i\in I(g)} \frac{\langle \alpha_i, u\rangle}{(1 - e^{-\langle \alpha_i, u\rangle})} \times \prod_{i\notin I(g)} \frac{1}{(1 - e^{-\langle \alpha_i, 2\pi\sqrt{-1}G + u\rangle})}.$$

If $\mathfrak{c}$ is a chamber of $\mathcal{C}(\mathcal{A})$, the Jeffrey-Kirwan residue $\mathrm{JK}_{\mathfrak{c}}(F(g,h))$ is well defined.

The following theorem is due to Szenes-Vergne [21]. If the set $\mathcal{A}$ is unimodular (that is, each $\sigma \in \mathrm{Bases}(\mathcal{A})$ is a $\mathbb{Z}$-basis of $V_{\mathbb{Z}}$), it is a reformulation of Khovanskii-Pukhlikhov Riemann-Roch calculus on simple polytopes [15]. For a general set $\mathcal{A}$, this refines the formula of Brion-Vergne [7].

Theorem 3.3. *Let $\mathfrak{c}$ be a chamber of the cone $\mathcal{C}(\mathcal{A})$ and $\overline{\mathfrak{c}}$ its closure. Then:*

(1) *For $h \in \overline{\mathfrak{c}}$ we have*

$$\mathrm{vol}_{\mathbb{Z},\mathcal{A}^+}(h) = \mathrm{vol}(V/V_{\mathbb{Z}}, dh)\, \mathrm{JK}_{\mathfrak{c}} \left(\frac{e^{\langle h,\cdot\rangle}}{\prod_{i=1}^N \alpha_i} \right).$$

(2) *Assume that F is a finite subset of T such that for any $\sigma \in \mathrm{Bases}(\mathcal{A})$, we have $T(\sigma) \subset F$. Then for $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$, we have*

$$N_{\mathcal{A}^+}(h) = \mathrm{vol}(V/V_{\mathbb{Z}}, dh) \sum_{g\in F} \mathrm{JK}_{\mathfrak{c}}(F(g,h)).$$

Observe that the right-hand side of (2) does not depend on the measure dh, as it should be.

Let us explain the behavior of these functions on a chamber $\mathfrak{c}$. By definition, a quasipolynomial function on a lattice L is a linear combination of products of polynomial functions and of periodic functions (functions constants on cosets $h + pL$ where p is an integer). We now show that the normalized volume $\mathrm{vol}_{\mathbb{Z},\mathcal{A}^+}(h)$ is given by a polynomial formula, when h varies in a chamber $\overline{\mathfrak{c}}$, while $N_{\mathcal{A}^+}(h)$ is given by a quasipolynomial formula when h varies in $V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$.

The residue vanishes except on degree $-r$, so that

$$\mathrm{JK}_{\mathfrak{c}} \left(\frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N \langle \alpha_i, u\rangle} \right) = \frac{1}{(N-r)!}\, \mathrm{JK}_{\mathfrak{c}} \left(\frac{\langle h,u\rangle^{N-r}}{\prod_{i=1}^N \langle \alpha_i, u\rangle} \right),$$

and as expected the normalized volume is a polynomial homogeneous function of h of degree $N - r$ on each chamber.

Now if $\mathcal{A}$ is unimodular then the above defined set F is $\{0\}$. Hence the number of integral points in the polytope $\Pi_{\mathcal{A}^+}(h)$ satisfies

$$N_{\mathcal{A}^+}(h) = \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh)\,\mathrm{JK}_{\mathfrak{c}}\left(\frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N (1-e^{-\langle \alpha_i,u\rangle})}\right)$$

and is a polynomial of degree $N-r$ whose homogeneous component of degree $N-r$ is the normalized volume. More precisely, write

$$\frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N (1-e^{-\langle \alpha_i,u\rangle})} = \frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N \langle \alpha_i,u\rangle} \times \frac{\prod_{i=1}^N \langle \alpha_i,u\rangle}{\prod_{i=1}^N (1-e^{-\langle \alpha_i,u\rangle})}$$

where

$$\frac{\prod_{i=1}^N \langle \alpha_i,u\rangle}{\prod_{i=1}^N (1-e^{-\langle \alpha_i,u\rangle})} = \sum_{k=0}^{+\infty} \psi_k(u)$$

is a holomorphic function of u in a neighborhood of 0 with $\psi_0(u) = 1$. Consequently

$$\begin{aligned} N_{\mathcal{A}^+}(h) &= \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh)\,\mathrm{JK}_{\mathfrak{c}}\left(\frac{e^{\langle h,u\rangle}}{\prod_{i=1}^N \langle \alpha_i,u\rangle} \times \sum_{k=0}^{+\infty} \psi_k(u)\right) \\ &= \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh) \sum_{k=0}^{N-r} \frac{1}{(N-r-k)!}\,\mathrm{JK}_{\mathfrak{c}}\left(\frac{\langle h,u\rangle^{N-r-k}\psi_k(u)}{\prod_{i=1}^N \langle \alpha_i,u\rangle}\right). \qquad (5) \end{aligned}$$

The unimodular case applies to the root system A_r.

Finally in the non-unimodular case (for example for the root systems B_r, C_r, D_r) the set F is no longer reduced to $\{0\}$. Let us denote by $\psi^g(u) = \sum_{k=0}^{+\infty} \psi_k^g(u)$ the series development of the holomorphic function ψ^g appearing in formula (4). Then we see that $\mathrm{JK}_{\mathfrak{c}}(F(g,h))$ equals

$$\begin{aligned} (6) \qquad & \mathrm{JK}_{\mathfrak{c}}\left(e^{\langle h,2\pi\sqrt{-1}G\rangle}\frac{e^{\langle h,u\rangle}}{\prod_{i\in I(g)} \langle \alpha_i,u\rangle}\psi^g(u)\right) \\ &= e^{\langle h,2\pi\sqrt{-1}G\rangle} \sum_{k=0}^{|I(g)|-r} \frac{1}{(|I(g)|-r-k)!}\,\mathrm{JK}_{\mathfrak{c}}\left(\frac{\langle h,u\rangle^{|I(g)|-r-k}}{\prod_{i\in I(g)} \langle \alpha_i,u\rangle}\psi_k^g(u)\right). \end{aligned}$$

If g is of order p, the function $h \mapsto e^{\langle h,2\pi\sqrt{-1}G\rangle}$ is constant on each coset $h + pV_{\mathbb{Z}}$ of the lattice $pV_{\mathbb{Z}}$, while the function $h \mapsto \mathrm{JK}_{\mathfrak{c}}\left(\frac{\langle h,u\rangle^{|I(g)|-r-k}}{\prod_{i\in I(g)} \langle \alpha_i,u\rangle}\psi_k^g(u)\right)$ is a polynomial function of h of degree $|I(g)|-r-k$. Thus the function

$$N_{\mathcal{A}^+}(h) = \mathrm{vol}\,(V/V_{\mathbb{Z}}, dh) \sum_{g\in F} \mathrm{JK}_{\mathfrak{c}}(F(g,h)) \qquad (7)$$

is given by a quasipolynomial formula when h varies in the closure of a chamber. Note that its highest degree component is polynomial and is the normalized volume as expected.

Example 3.4. Let us compute the normalized volume and number of integral points for the root system B_2, that is for $\mathcal{A}^+ = \mathcal{B}_2 = \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$. Fix a chamber $\mathfrak{c}$ and an integral vector $h = (h_1, h_2)$ in the cone $\mathcal{C}(\mathcal{B}_2)$. Observe that the root lattice is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $\mathrm{vol}\,(V/V_{\mathbb{Z}}, dh) = 1$ for the measure $dh = dh_1 dh_2$. Then the normalized volume equals

$$\frac{1}{2!}\,\mathrm{JK}_{\mathfrak{c}}\left(\frac{(h_1u_1 + h_2u_2)^2}{u_1u_2(u_1+u_2)(u_1-u_2)}\right).$$

Note that

$$\frac{u_1^2}{u_1u_2(u_1+u_2)(u_1-u_2)} = \frac{1}{u_2(u_1+u_2)} + \frac{1}{(u_1+u_2)(u_1-u_2)}$$

(and similar quotients of u_2^2 and u_1u_2), so that the normalized volume is

$$\frac{1}{2}\,\mathrm{JK}_{\mathfrak{c}}\left(\frac{h_1^2}{u_2(u_1+u_2)} + \frac{h_1^2 + 2h_1h_2 + h_2^2}{(u_1+u_2)(u_1-u_2)} - \frac{h_2^2}{u_1(u_1+u_2)}\right).$$

There are three chambers, namely $\mathfrak{c}_1 = \mathcal{C}(\{e_2, e_1+e_2\})$, $\mathfrak{c}_2 = \mathcal{C}(\{e_1, e_1+e_2\})$, $\mathfrak{c}_3 = \mathcal{C}(\{e_1 - e_2, e_1\})$ (see Figure 4). Now let us compute the Jeffrey-Kirwan residues on the chambers. As

$$\begin{aligned}
\mathrm{JK}_{\mathfrak{c}_1}\left(\tfrac{1}{u_2(u_1+u_2)}\right) &= 1, \quad \mathrm{JK}_{\mathfrak{c}_2}\left(\tfrac{1}{(u_1+u_2)(u_1-u_2)}\right) = \tfrac{1}{2},\\
\mathrm{JK}_{\mathfrak{c}_2}\left(\tfrac{1}{u_1(u_1-u_2)}\right) &= 1, \quad \mathrm{JK}_{\mathfrak{c}_3}\left(\tfrac{1}{(u_1+u_2)(u_1-u_2)}\right) = \tfrac{1}{2},
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathrm{vol}(\Pi_{\mathcal{B}_2}(h)) &= \frac{1}{2}h_1^2 \quad \text{if } h \in \mathfrak{c}_1,\\
\mathrm{vol}(\Pi_{\mathcal{B}_2}(h)) &= \frac{1}{4}(h_1+h_2)^2 - \frac{1}{2}h_2^2 \quad \text{if } h \in \mathfrak{c}_2,\\
\mathrm{vol}(\Pi_{\mathcal{B}_2}(h)) &= \frac{1}{4}(h_1+h_2)^2 \quad \text{if } h \in \mathfrak{c}_3.
\end{aligned}$$

Note that the formulae agree on walls $\mathfrak{c}_1 \cap \mathfrak{c}_2$ and $\mathfrak{c}_2 \cap \mathfrak{c}_3$.

For the number of integral points, we first note that $F = \{(0,0), (1/2,1/2)\}$. Consequently $N_{\mathcal{B}_2}(h)$ is equal to the Jeffrey-Kirwan residue of $f_1 = F((0,0), h)$ plus $f_2 = F((1/2,1/2), h)$. We rewrite the series f_j ($j = 1,\ 2$) as $f_j = f_j' \times e^{u_1h_1+u_2h_2}/u_1u_2(u_1+u_2)(u_1-u_2)$ where

$$\begin{aligned}
f_1' &= \frac{u_1}{1-e^{-u_1}} \times \frac{u_2}{1-e^{-u_2}} \times \frac{u_1+u_2}{1-e^{-(u_1+u_2)}} \times \frac{u_1-u_2}{1-e^{-(u_1-u_2)}},\\
f_2' &= \frac{u_1}{1+e^{-u_1}} \times \frac{u_2}{1+e^{-u_2}} \times \frac{u_1+u_2}{1-e^{-(u_1+u_2)}} \times \frac{u_1-u_2}{1-e^{-(u_1-u_2)}} \times (-1)^{h_1+h_2}.
\end{aligned}$$

Using the series expansions $\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + O(x^3)$ and $\frac{x}{1+e^{-x}} = \frac{1}{2}x + O(x^2)$, we obtain that the number of integral points is the JK residue

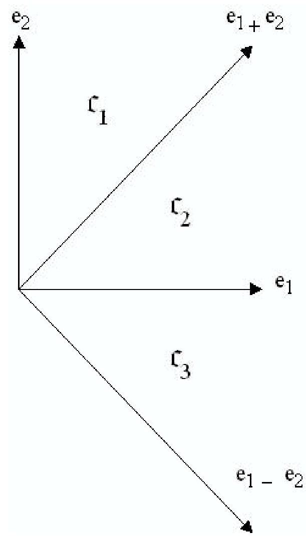

FIGURE 4. The 3 chambers for B_2

of

$$\frac{u_1(1+\frac{3}{2}h_1+\frac{1}{2}h_1^2)}{u_2(u_1-u_2)(u_1+u_2)}+\frac{\frac{3}{4}+h_1h_2+\frac{3}{2}h_2+\frac{1}{2}h_1}{(u_1-u_2)(u_1+u_2)}+\frac{u_2(\frac{1}{2}h_2^2+\frac{1}{2}h_2)}{u_1(u_1-u_2)(u_1+u_2)}$$
$$+(-1)^{h_1+h_2}\frac{\frac{1}{4}}{(u_1+u_2)(u_1-u_2)}$$

$$=\frac{(1+\frac{3}{2}h_1+\frac{1}{2}h_1^2)}{u_2(u_1+u_2)}+\frac{\frac{7}{4}+2(h_1+h_2)+\frac{1}{2}h_2^2+h_1h_2+\frac{1}{2}h_1^2}{(u_1-u_2)(u_1+u_2)}-\frac{(\frac{1}{2}h_2^2+\frac{1}{2}h_2)}{u_1(u_1+u_2)}$$
$$+(-1)^{h_1+h_2}\frac{\frac{1}{4}}{(u_1+u_2)(u_1-u_2)}.$$

We then obtain:

$$\begin{aligned}
N_{\mathcal{B}_2}(h) &= 1+\frac{3}{2}h_1+\frac{1}{2}h_1^2 \\
&\quad \text{if } h\in\mathfrak{c}_1, \\
N_{\mathcal{B}_2}(h) &= \frac{1}{4}h_1^2+\frac{1}{2}h_1h_2-\frac{1}{4}h_2^2+h_1+\frac{1}{2}h_2+\frac{7}{8}+(-1)^{h_1+h_2}\frac{1}{8} \\
&\quad \text{if } h\in\mathfrak{c}_2, \\
N_{\mathcal{B}_2}(h) &= \frac{1}{4}h_1^2+\frac{1}{2}h_1h_2+\frac{1}{4}h_2^2+h_1+h_2+\frac{7}{8}+(-1)^{h_1+h_2}\frac{1}{8} \\
&\quad \text{if } h\in\mathfrak{c}_3.
\end{aligned}$$

Note that the functions $N_{\mathcal{B}_2}$ agree on walls, and the formulae above are valid on the closures of the chambers.

Our general method to implement Theorem 3.3 for root systems is more systematic and will be explained in the course of this article.

Remark 3.5. Combining (6) and (7), we can see that the quasipolynomial character of the integral-point counting functions $N_{\mathcal{A}}^{+}$ stems precisely from the root of unity in (6). Furthermore, we will see in Lemmas 8.2, 9.1, and 10.1 that for root systems of type B, C, and D, these roots of unity are of order 2, as in the above example for $\mathcal{B}_2$. (For root systems of type A, (5) shows that $N_{\mathcal{A}}^{+}$ is always a polynomial.) Let us record the following immediate consequence:

Corollary 3.6. *The integral-point counting functions $N_{\mathcal{B}_r}, N_{\mathcal{C}_r}, N_{\mathcal{D}_r}$ are quasipolynomials with period* 2.

Remark 3.7. The partition functions $N_{\mathcal{A}_r}, N_{\mathcal{B}_r}, N_{\mathcal{C}_r}, N_{\mathcal{D}_r}$ can be interpreted as (weak) flow quasipolynomials on certain signed graphs [5]. The polynomiality of $N_{\mathcal{A}_r}$ follows immediately from this interpretation and a unimodularity argument; the fact that the quasipolynomials $N_{\mathcal{B}_r}, N_{\mathcal{C}_r}, N_{\mathcal{D}_r}$ have period 2 follows from a half-integrality result of Lee [18].

Remark 3.8. In the case where $\mathcal{A}^+$ is an arbitrary sequence of vectors in $V_{\mathbb{Z}}$, the straightforward implementation of Theorem 3.3 above is of exponential complexity. Indeed we make a summation on the set F, which can become arbitrarily large. Barvinok uses a signed cone decomposition to obtain an algorithm of polynomial complexity, when the number of elements of $\mathcal{A}^+$ is fixed, to compute the number $N_{\mathcal{A}^+}(h)$; the `LattE` team implemented Barvinok's algorithm [16, 17] in the language C. Our work will be dealing either with volumes of polytopes, where the set F does not enter, or with partition function of classical root systems, where the set F is reasonably small. Then we obtain a fast algorithm, implemented for the moment in the formal calculation software MAPLE. This algorithm for these particular cases can reach examples not obtainable by the `LattE` program.

In next Section 4 we will give the basic formula for $\mathrm{JK}_{\mathfrak{c}}$ involving maximal proper nested sets, as developed in [12], and iterated residues. These formulae are implemented in our algorithms.

4. A formula for the Jeffrey-Kirwan residue

If f is a meromorphic function of one variable z with a pole of order less than or equal to h at $z = 0$ then we can write $f(z) = Q(z)/z^h$, where $Q(z)$ is a holomorphic function near $z = 0$. If the Taylor series of Q is given by $Q(z) = \sum_{k=0}^{\infty} q_k z^k$, then as usual the residue at $z = 0$ of the function $f(z) = \sum_{k=0}^{\infty} q_k z^{k-h}$ is the coefficient of $1/z$, that is, q_{h-1}. We will denote it by $\mathrm{res}_{z=0} f(z)$. To compute this residue we can either expand Q into a power series and search for the coefficient of z^{-1}, or employ the formula

$$\mathrm{res}_{z=0} f(z) = \frac{1}{(h-1)!} (\partial_z)^{h-1} \left(z^h f(z) \right) \Big|_{z=0}. \tag{8}$$

We now introduce the notion of iterated residue on the space $\mathcal{R}_{\mathcal{A}}$.

Let $\vec{\nu} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ be an ordered basis of V consisting of elements of $\mathcal{A}$ (here we have implicitly renumbered the elements of $\mathcal{A}$ in order that the elements of our basis are listed first). We choose a system of coordinates on U such that $\alpha_i(u) = u_i$. A function $\phi \in \mathcal{R}_{\mathcal{A}}$ is thus written as a rational fraction $\phi(u_1, u_2, \ldots, u_r) = \frac{P(u_1,u_2,\ldots,u_r)}{Q(u_1,u_2,\ldots,u_r)}$ where the denominator Q is a product of linear forms.

Definition 4.1. If $\phi \in \mathcal{R}_{\mathcal{A}}$, the iterated residue $\mathrm{Ires}_{\vec{\nu}}(\phi)$ of ϕ for $\vec{\nu}$ is the scalar

$$\mathrm{Ires}_{\vec{\nu}}(\phi) = \mathrm{res}_{u_r=0}\mathrm{res}_{u_{r-1}=0} \cdots \mathrm{res}_{u_1=0}\phi(u_1, u_2, \ldots, u_r)$$

where each residue is taken assuming that the variables with higher indices are considered constants.

Keep in mind that at each step the residue operation augments the homogeneous degree of a rational function by $+1$ (as for example $\mathrm{res}_{x=0}(1/xy) = 1/y$) so that the iterated residue vanishes on homogeneous elements $\phi \in \mathcal{R}_{\mathcal{A}}$, if the homogeneous degree of ϕ is different from $-r$.

Observe that the value of $\mathrm{Ires}_{\vec{\nu}}(\phi)$ depends on the order of $\vec{\nu}$. For example, for $f = 1/(x(y-x))$ we have $\mathrm{res}_{x=0}\mathrm{res}_{y=0}(f) = 0$ and $\mathrm{res}_{y=0}\mathrm{res}_{x=0}(f) = 1$.

Remark 4.2. Choose any basis $\gamma_1, \gamma_2, \ldots, \gamma_r$ of V such that $\oplus_{k=1}^{j}\alpha_j = \oplus_{k=1}^{j}\gamma_j$ for every $1 \leq j \leq r$ and such that $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_r = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r$. Then, by induction, it is easy to see that for $\phi \in \mathcal{R}_{\mathcal{A}}$

$$\mathrm{res}_{\alpha_r=0} \cdots \mathrm{res}_{\alpha_1=0}\phi = \mathrm{res}_{\gamma_r=0} \cdots \mathrm{res}_{\gamma_1=0}\phi.$$

Thus given an ordered basis, we may modify α_2 by $\alpha_2 + c\alpha_1, \ldots,$ with the purpose of getting easier computations.

The following lemma will be useful later on.

Lemma 4.3. *Let $\vec{\nu} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ and $f_\beta = \frac{1}{\prod_{i=1}^{r}\beta_i}$ be a basic fraction. Then the iterated residue $\mathrm{Ires}_{\vec{\nu}}(f_\beta)$ is non zero if and only if there exists a permutation w of $\{1, 2, \ldots, r\}$ such that:*

$$\begin{array}{rcl} \beta_{w(1)} & \in & \mathbb{R}\alpha_1, \\ \beta_{w(2)} & \in & \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2, \\ & \vdots & \\ \beta_{w(r)} & \in & \mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_r. \end{array}$$

Definition 4.4. Let $\vec{\nu} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ and let $u_j = \alpha_j(u)$. Choose a sequence of real numbers: $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_r$. Then define the torus

$$T(\vec{\nu}) = \{u \in U_{\mathbb{C}} \mid |u_j| = \epsilon_j,\ j = 1, \ldots, r\}. \tag{9}$$

The torus $T(\vec{\nu})$ is identified via the basis α_j with the product of r circles oriented counterclockwise. The sequence $[\epsilon_1, \epsilon_2, \ldots, \epsilon_r]$ is chosen so that elements α_q not in $\oplus_{k=1}^{j}\mathbb{R}\alpha_j$ do not vanish on the domain $\{u \in U_{\mathbb{C}} \,|\, |u_k| \leq \epsilon_k, 1 \leq k \leq j\,;\, |u_i| = \epsilon_i, i = j+1, \ldots, r\}$. This is achieved by choosing the ratios $\epsilon_j/\epsilon_{j+1}$ very small. The torus $T(\vec{\nu})$ is contained in $U_{\mathbb{C}}(\mathcal{A})$ and the homology class $[T(\vec{\nu})]$ of this torus is independent of the choice of the sequence of the ordered ϵ_j [22].

Choose an ordered basis $e_1, e_2, \ldots, e_r$ of V of volume 1 with respect to the measure dh. For $z \in U_{\mathbb{C}}$, define $z_j = \langle z, e_j\rangle$ and $dz = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_r$. Denote by $\det(\vec{\nu})$ the determinant of the basis $\alpha_1, \alpha_2, \ldots, \alpha_r$ with respect to the basis $e_1, e_2, \ldots, e_r$.

Lemma 4.5. *For $\phi \in \mathcal{R}_{\mathcal{A}}$, we have*

$$\frac{1}{\det(\vec{\nu})}\mathrm{res}_{\alpha_r=0}\cdots\mathrm{res}_{\alpha_1=0}\phi = \frac{1}{(2\pi\sqrt{-1})^r}\int_{T(\vec{\nu})}\phi(z)dz.$$

Thus, as for the usual residue, the iterated residue can be expressed as an integral.

We now introduce the notion of maximal proper nested set, MPNS in short.

De Concini-Procesi [12] prove that the set of MPNS is in bijection with the so-called no broken circuits bases of $\mathcal{A}$ (with respect to a order to be specified). This is helpful as the JK residue can be computed in terms of iterated residues with respect to these bases.

If S is a subset of $\mathcal{A}$, we denote by $\langle S\rangle$ the vector space spanned by S. More generally if $M = \{S_i\}$ is a set of subsets of $\mathcal{A}$, we denote by $\langle M\rangle$ the vector space spanned by all elements of the sets S_i. We say that a subset S of $\mathcal{A}$ is *complete* if $S = \langle S\rangle \cap \mathcal{A}$ or in other words if any linear combination of elements of S belongs to S. A complete subset S is called *reducible* if we can find a decomposition $V = V_1 \oplus V_2$ such that $S = S_1 \cup S_2$ with $S_1 \subset V_1$ and $S_2 \subset V_2$. Otherwise S is said to be *irreducible*.

Definition 4.6. Let $\mathcal{I}$ be the set of irreducible subsets of $\mathcal{A}$. A set $M = \{I_1, I_2, \ldots, I_k\}$ of irreducible subsets of $\mathcal{A}$ is called *nested* if, given any subfamily $\{I_1, \ldots, I_m\}$ of M such that there exists no i, j with $I_i \subset I_j$, then the set $I_1 \cup \cdots \cup I_m$ is **complete** and the elements I_j are the irreducible components of $I_1 \cup I_2 \cup \cdots \cup I_m$.

Example 4.7. Let E be an $r+1$-dimensional vector space with basis e_i ($i = 1, \ldots, r$). We consider the set

$$\mathcal{K}_{r+1} = \{e_i - e_j \,|\, 1 \leq i < j \leq r+1\}.$$

These are the positive roots for the system A_r. The irreducible subsets of $\mathcal{K}_{r+1}$ are indexed by subsets S of $\{1, 2, \ldots, r+1\}$, the corresponding irreducible subset being $\{e_i - e_j \,|\, i, j \in S, i < j\}$. For instance the set $S = \{1, 2, 4\}$ parametrizes the set of roots given by $\{e_1 - e_2, e_2 - e_4, e_1 - e_4\}$.

A nested set is represented by a collection $M = \{S_1, S_2, \ldots, S_k\}$ of subsets of $\{1, 2, \ldots, r+1\}$ such that if S_i, $S_j \in M$ then either $S_i \cap S_j$ is empty, or one of them is contained in another.

Definition 4.8. A maximal nested set (in short MNS) M is a nested set such that for every irreducible set $\mathcal{I}$ of $\mathcal{A}$ the set $M \cup \{\mathcal{I}\}$ is no longer nested.

A maximal nested set has exactly r elements [12].

Assume now that $\mathcal{A}$ is irreducible, otherwise just take the irreducible components. Then every maximal nested set M contains $\mathcal{A}$. Let I_1, I_2, $\dots$, I_k be the maximal elements of the set $M \setminus \mathcal{A}$. We see that the vector space spanned by $\langle I_1\rangle \oplus \langle I_2\rangle \oplus \cdots \oplus \langle I_k\rangle$ is of codimension 1 [12, Proposition 1.3].

Definition 4.9. A hyperplane H in V is $\mathcal{A}$-*admissible* if it is spanned by a set of vectors of $\mathcal{A}$.

Thus if M is a MNS, the vector space $\langle M \setminus \mathcal{A}\rangle$ is an admissible hyperplane H.

Definition 4.10. Let $\mathcal{A}$ be irreducible and let H be a $\mathcal{A}$-admissible hyperplane. All MNPS's such that $\langle M \setminus \mathcal{A}\rangle = H$ are said attached to H.

Therefore to classify maximal nested sets (MNS) for an irreducible set $\mathcal{A}$ we proceed by running over the set of $\mathcal{A}$-admissible hyperplanes, as described in Figure 5.

- Take a hyperplane H spanned by a set of vectors of $\mathcal{A}$.
- Break $\mathcal{A} \cap H$ into irreducible subsets $I_1 \cup I_2 \cup \cdots \cup I_k$.
- For each irreducible I_i construct the set $\{M_1^i, \dots, M_{k_i}^i\}$ of maximal nested sets for I_i.
- Set $C_i = \{1, \dots, k_i\}$.
- A maximal nested set is then given by the union $M_{c_1}^1 \cup M_{c_2}^2 \cup \cdots \cup M_{c_k}^k \cup \{\mathcal{A}\}$ where $c_1 \in C_1$, $\dots$, $c_k \in C_k$, and all of them are obtained by letting c_i vary.

FIGURE 5. Building of all MNSs attached to an $\mathcal{A}$-admissible hyperplane H

The whole algorithm will be described in detail in Figure 7, Section 5.

We describe now the notion of maximal proper nested set of $\mathcal{A}$.

Fix a total order on the set $\mathcal{A}$. For example, we can choose a linear functional ht on V so that the values $\mathrm{ht}(\alpha_i)$ are all distinct and positive. Thus the value $\mathrm{ht}(\alpha)$ is larger if α is deeper in the interior of the cone.

Let $M = \{S_1, S_2, \dots, S_k\}$ be a set of subsets of $\mathcal{A}$. In each S_j we choose the element α_j maximal for the order given by ht. This defines a map θ from M to $\mathcal{A}$.

Definition 4.11. A maximal nested set M is called *proper* if $\theta(M)$ is a basis of V. We denote by $\mathcal{P}(\mathcal{A})$ the set of maximal proper nested sets, in short MPNS.

If $M = \{I_1, I_2, \dots, I_r\}$ is a maximal nested set, we associate to M the list $[\theta(I_{i_1}), \dots, \theta(I_{i_r})]$ using the total order on the elements $\theta(M)$; that is we have $\mathrm{ht}(\theta(I_{i_1})) < \mathrm{ht}(\theta(I_{i_2})) < \cdots < \mathrm{ht}(\theta(I_{i_r}))$. Observe that, if $\mathcal{A}$ is irreducible, for every maximal nested set, I_{i_r} is always equal to $\mathcal{A}$ and $\theta(I_{i_r})$ is the highest element of $\mathcal{A}$. We will often implicitly renumber our elements in M such that $\mathrm{ht}(\theta(I_1)) < \mathrm{ht}(\theta(I_2)) < \cdots < \mathrm{ht}(\theta(I_r))$.

So we have associated to every maximal proper nested set M an ordered basis $\overrightarrow{\theta(M)} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ of elements of $\mathcal{A}$. In all implementations, we calculate $\overrightarrow{\theta(M)}$ from a MNS M with the procedure `ThetaMNS(M)`. We denote by $\mathrm{vol}(M) > 0$ the volume of the parallelepiped $\sum_{i=1}^{r}[0,1]\alpha_i$ with respect to our measure, and by $\mathcal{C}(M) = \sum_{i=1}^{r} \mathbb{R}_{\geq 0}\alpha_i \subset \mathcal{C}(\mathcal{A})$ the cone generated by $\theta(M)$.

If v is a regular element of V, let

$$\mathcal{P}(v, \mathcal{A}) = \{M \in \mathcal{P}(\mathcal{A}) \,|\, v \in \mathcal{C}(M)\}. \tag{10}$$

The set $\mathcal{P}(v, \mathcal{A})$ depends only of the chamber $\mathfrak{c}$ where v belongs. We are now ready to state the basic formula for our calculations.

Theorem 4.12 (DeConcini-Procesi, [12])**.** *Let $\mathfrak{c}$ be a chamber and let $v \in \mathfrak{c}$. Then, for $\phi \in \mathcal{R}_{\mathcal{A}}$, we have*

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \sum_{M \in \mathcal{P}(v,\mathcal{A})} \frac{1}{\mathrm{vol}(M)} \mathrm{Ires}_{\overrightarrow{\theta(M)}} \phi.$$

We will use also the corresponding integration formula.

Each maximal proper nested set $M \in \mathcal{P}(v, \mathcal{A})$ determines an oriented cycle $\left[T\left(\overrightarrow{\theta(M)}\right)\right]$ contained in the open set $U_{\mathbb{C}}(\mathcal{A})$, as described in Definition 4.4.

Definition 4.13. Let $\mathfrak{c}$ be a chamber. Define the oriented cycle:

$$H(\mathfrak{c}) = \sum_{M \in \mathcal{P}(v,\mathcal{A})} \mathrm{sign}\left(\det\left(\overrightarrow{\theta(M)}\right)\right) \left[T\left(\overrightarrow{\theta(M)}\right)\right].$$

The following integral version of Theorem 4.12 will be useful.

Theorem 4.14. *Let $\mathfrak{c}$ be a chamber. Then for $\phi \in \mathcal{R}_{\mathcal{A}}$ we have*

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \frac{1}{(2\pi\sqrt{-1})^r} \int_{H(\mathfrak{c})} \phi(z)dz.$$

The following example should help clarifying the notions introduced.

Example 4.15. We consider the set $\mathcal{K}_4$ of positive roots for A_3 (see Figure 6) defined by

$$\mathcal{K}_4 = \{e_i - e_j \,|\, 1 \leq i < j \leq 4\}.$$

We let V be the vector space generated by the elements in $\mathcal{K}_4$. Then V has dimension 3 and we write an element of V as

$$a = a_1e_1 + a_2e_2 + a_3e_3 - (a_1 + a_2 + a_3)e_4.$$

We consider the height function defined by

$$\mathrm{ht}(e_1 - e_2) = 10, \quad \mathrm{ht}(e_2 - e_3) = 11, \quad \mathrm{ht}(e_3 - e_4) = 12.$$

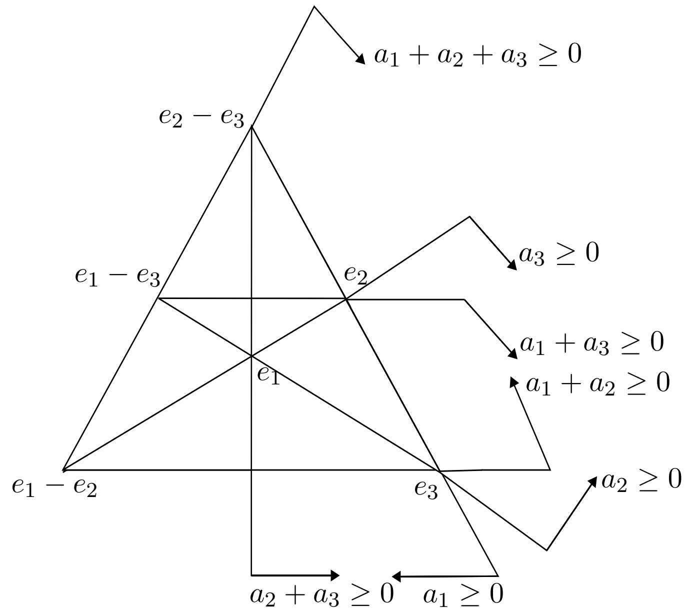

FIGURE 6. Hyperplanes for A_3 with $a = a_1e_1 + a_2e_2 + a_3e_3 - (a_1 + a_2 + a_3)e_4$

This choice gives the following order on the roots:

$$e_1 - e_2 < e_2 - e_3 < e_3 - e_4 < e_1 - e_3 < e_2 - e_4 < e_1 - e_4.$$

Take a hyperplane H in V spanned by two linearly independent elements of $\mathcal{K}_4$. Therefore it is the kernel of a linear form $\sum_{i \in I_H} a_i$, where I_H is a proper subset of $\{1, 2, 3, 4\}$. The set of complementary indices gives the same hyperplane. Thus each admissible hyperplane partitions the set of indices $\{1, 2, 3, 4\}$ in two sets Z_1 and Z_2, where $Z_1 := \{i \in I_H\}$ and Z_2 is the set of complementary indices. In our example we have 7 choices of admissible hyperplanes corresponding to the following partitions:

$$\begin{aligned} &H_1 = \{[1,2,3],[4]\}, \quad H_2 = \{[1,2,4],[3]\}, \quad H_3 = \{[1,3,4],[2]\}, \\ &H_4 = \{[2,3,4],[1]\}, \quad H_5 = \{[1,2],[3,4]\}, \quad H_6 = \{[1,3],[2,4]\}, \\ &H_7 = \{[1,4],[2,3]\}. \end{aligned}$$

Now observe that if the hyperplane H_i already contains the highest root $e_1 - e_4$ then it cannot lead to a maximal proper nested set. Indeed we must get a basis if we add the highest root to a set of vectors contained in H_i. Thus H_2, H_3, H_7 can be excluded. It remains to consider the hyperplanes H_1, H_4, H_5, H_6.

Hyperplanes H_1 and H_4 give rise to two MPNSs each, while H_5 and H_6 give rise to only one. So we obtain a list of 6 maximal nested sets (as described in Example 4.7, we identify an irreducible subset I with a subset

S of $[1,2,3,4]$):

$$\begin{array}{ll} M_1=\{[1,2],[1,2,3],[1,2,3,4]\}, & M_2=\{[2,3],[1,2,3],[1,2,3,4]\}, \\ M_3=\{[2,3],[2,3,4],[1,2,3,4]\}, & M_4=\{[3,4],[2,3,4],[1,2,3,4]\}, \\ M_5=\{[1,3],[2,4],[1,2,3,4]\}, & M_6=\{[1,2],[3,4],[1,2,3,4]\}. \end{array}$$

5. Search for maximal proper nested sets adapted to a vector: the general case

Given a vector v in the cone $\mathcal{C}(\mathcal{A})$, we describe how to search for all maximal proper nested sets belonging to $\mathcal{P}(v,\mathcal{A})$, without enumerating all MPNS.

We use as height function a linear form that is positive and that takes different values on all elements α_i, and consider the total order it induces. Let H be an $\mathcal{A}$-admissible hyperplane in V, that is, a hyperplane spanned by a set of vectors of $\mathcal{A}$. Then the cone $\mathcal{C}(\mathcal{A}\cap H)$ generated by the elements of $\mathcal{A}$ belonging to H is a cone with non-empty interior in H.

We have already seen that to list all the MPNS, we have to first list all admissible hyperplanes H and then find the irreducible components J_1, J_2, $\ldots$, J_s of $\mathcal{A}\cap H$. Then we choose a MPNS $M_i := \{I_i^a\}$ for J_i, and define $M = M_1 \cup M_2 \cup M_3 \cup \cdots \cup M_s \cup \{\mathcal{A}\}$.

As we have seen in Example 4.15 we can discard some of the hyperplanes *a priori*, because they cannot lead to a maximal proper nested set. The next lemma examines the general situation. Let θ be the highest element in $\mathcal{A}$ and H a hyperplane of $\mathcal{A}$.

Lemma 5.1. *There exists a maximal proper nested set $M \in \mathcal{P}(v,\mathcal{A})$ attached to H, if and only if θ does not belong to H and if v belongs to the cone generated by θ and $\mathcal{A}\cap H$.*

Proof. The condition is necessary. Indeed v must belong to the cone generated by the elements $\theta(I_i^a)$ and θ, and all the elements $\theta(I_i^a)$ are in $\mathcal{A}\cap H$. Reciprocally consider the projection $v - \frac{\langle u,v\rangle}{\langle u,\theta\rangle}\theta$, where u is the equation of the hyperplane H. This can be written as $v_1 \oplus v_2 \oplus \cdots \oplus v_s$, where each v_i is in the cone $\mathcal{C}(J_i)$. Let now $M_i \in \mathcal{P}(v_i, J_i)$ be a MPNS in J_i. The element v_i belongs to $\mathcal{C}(\theta(M_i))$. We can write

$$v = t\theta + \sum_{i=1}^{s} \sum_{I_i^a \in M_i} t_i^a \theta(I_i^a)$$

with $t_i^a > 0$. Thus we see that the collection $M_1 \cup \cdots \cup M_s \cup \mathcal{A}$ is a maximal proper nested set in $\mathcal{P}(v,\mathcal{A})$. Moreover in this way we list all elements of $\mathcal{P}(v,\mathcal{A})$. □

Our search for maximal proper nested sets in $\mathcal{P}(v,\mathcal{A})$ will then be pursued by constructing all possible admissible hyperplanes H for which v is in the convex hull of $\mathcal{C}(\mathcal{A}\cap H)$ and θ. We denote by $\mathrm{Hyp}(v,\mathcal{A})$ the set of such $\mathcal{A}$-admissible hyperplanes.

The following easy lemma lists some obvious conditions for the set $\mathrm{Hyp}(v, \mathcal{A})$. Let $u_H \in U$ be the normal vector to an $\mathcal{A}$-admissible hyperplane, meaning that $H := \{h \in V \mid \langle u_H, v\rangle = 0\}$.

Lemma 5.2. *If $H \in \mathrm{Hyp}(v, \mathcal{A})$ then H satisfies the following conditions:*

(1) $\langle u_H, \theta\rangle \neq 0$.
(2) $\langle u_H, v\rangle \times \langle u_H, \theta\rangle \geq 0$.

Thus if a hyperplane H satisfies the above conditions we define

$$\mathrm{proj}_H(v) = v - \frac{\langle u_H, v\rangle}{\langle u_H, \theta\rangle}\theta.$$

Hence to decide if $H \in \mathrm{Hyp}(v, \mathcal{A})$ we simply have to test if $\mathrm{proj}_H(v)$ is in the cone generated by $\mathcal{A} \cap H$, which is done by standard methods. Our search for the hyperplanes $H \in \mathrm{Hyp}(v, \mathcal{A})$ will also be considerably sped up by the following remark.

Proposition 5.3. *Let H be an $\mathcal{A}$-admissible hyperplane. Let $u \in U$ be a linear form on V which is non negative on $\mathcal{A} \cap H$ and on θ. If $\langle u, v\rangle < 0$, then H is not in* $\mathrm{Hyp}(v, \mathcal{A})$.

Proof. Indeed if v was in the cone generated by $\mathcal{A} \cap H$ and θ, the value of u would be non negative on v. □

The point of this remark is that in classical examples of root systems, an *a priori* description of the $\mathcal{A}$-admissible hyperplanes is available, together with the defining equations of the cone $\mathcal{C}(\mathcal{A} \cap H)$. This condition will allow us to disregard right away many $\mathcal{A}$-admissible hyperplanes.

Let us summarize the scheme of the algorithm in Figure 7. Recall that we have as input a vector v, and as output the list of all MPNS's belonging to $\mathcal{P}(v, \mathcal{A})$.

We will explain our algorithm in more details for each classical root system (see Sections 7–10).

6. Trees and order of poles

Let M be a maximal nested proper set for the system $\mathcal{A} := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. In our algorithms, we will need to take an iterated residue with respect to a basis $\overrightarrow{\theta(M)}$ of a function of the form $\phi = \frac{P}{\prod_{i=1}^{N} \alpha_i}$, where P is a polynomial function on U. It is thus important to understand the order of the poles of the function obtained after performing a certain number of residues. We also prove that the iterated residue associated to M depends only on the tree associated to M.

We associate to a maximal nested set M a tree T as follows. Let $M = \{I_1, \ldots, I_r\}$ be a maximal nested set. The vertices of T are the elements of M and the oriented edges are determined from the reverse order relation by inclusion: the ends of the tree are irreducible sets with just one element and if $\mathcal{A}$ is irreducible, the base is the set $\mathcal{A}$. A subset N of M will be called

check if $v \in \mathcal{C}(\mathcal{A})$
for each hyperplane H do
check if v and θ are on the same side of H
if not, then skip this hyperplane
define the projection $\mathrm{proj}_H(v)$ of v on H along θ
check if $\mathrm{proj}_H(v)$ belongs to $\mathcal{C}(\mathcal{A} \cap H)$; if not then skip this hyperplane
write $\mathcal{A} \cap H$ as the union of its irreducible components $I_1 \cup \cdots \cup I_k$
write v as $v_1 \oplus \cdots \oplus v_k$ according to the previous decomposition
for each I_j do
compute all MPNS's for v_j and I_j
collect all these MPNS's for v_j and I_j
end of loop running across I_j's
collect all MPNS's for the hyperplane H
end of loop running across H's
return the set of all MPNS's for all hyperplanes

FIGURE 7. Algorithm for MPNS's computation (general case)

saturated if it contains all elements above elements of N in the tree order. Thus if N contains an element S, it contains all the elements S' of M which are contained in S.

Example 6.1. The two MNSs named M_1 and M_5 described in Example 4.15 can be rewritten respectively as

$$\begin{tikzcd} {[1,2]} \\ {[1,2,3]} \arrow[u] \\ {[1,2,3,4]} \arrow[u] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} {[1,3]} & & {[2,4]} \\ \\ & {[1,2,3,4]} \arrow[uul] \arrow[uur] & \end{tikzcd}$$

Lemmas 7.3, 8.6, and 10.4 describe the decomposition of $\mathcal{A} \cap H$ in irreducible nested sets and lead to the following result:

Proposition 6.2. *Let T be the tree associated to an irreducible classical root system. Then T is a connected tree for which every vertex is adjacent to at most two other vertices.*

Lemma 6.3. *Let $M = \{I_1, I_2, \ldots, I_r\}$ be a MPNS. Here we have numbered our irreducible sets such that $\theta(I_1) < \theta(I_2) < \cdots < \theta(I_r)$. Let k be an integer smaller than or equal to r. Then the set $\{I_1, I_2, \ldots, I_k\}$ is saturated.*

Indeed if two sets I, J belongs to M and $I \subset J$, then $\theta(I) < \theta(J)$.

Proposition 6.4. *Let $M = [I_1, I_2, \ldots, I_r]$ be a maximal nested proper family. Let $[I'_1, I'_2, \ldots, I'_r]$ be a reordering of the sequence $[I_1, I_2, \ldots, I_r]$. We assume that this reordering is compatible with the partial order given by inclusion: if $I'_j \subset I'_k$ then $j < k$. Let*

$$\vec{\nu} = [\theta(I_1), \theta(I_2), \ldots, \theta(I_r)]$$

and

$$\vec{\nu'} = [\theta(I'_1), \theta(I'_2), \ldots, \theta(I'_r)].$$

Then we have $\mathrm{Ires}_{\vec{\nu}} = \mathrm{Ires}_{\vec{\nu'}}$.

Proof. We prove this proposition by induction on r.

If $\mathcal{A}$ is irreducible, then necessarily $I_r = I'_r = \mathcal{A}$ and $[I'_1, I'_2, \ldots, I'_{r-1}]$ is a reordering of the sequence $[I_1, I_2, \ldots, I_{r-1}]$. Furthermore the families $\{I_1, I_2, \ldots, I_{r-1}\}$ and $\{I'_1, I'_2, \ldots, I'_{r-1}\}$ are maximal proper nested sets for $\mathcal{A}_0 = \cup_{j=1}^{r-1} I_j$. The set $\mathcal{A}_0$ spans a codimension 1 vector space in V.

To prove that $\mathrm{Ires}_{\vec{\nu}} = \mathrm{Ires}_{\vec{\nu'}}$, it suffices to test it on basic fractions f_σ. Let $\sigma = \{\beta_1, \beta_2, \ldots, \beta_r\}$ be a basic subset of $\mathcal{A}$. By Lemma 4.3, if $\mathrm{Ires}_{\vec{\nu}} f_\sigma \neq 0$, then the set $\sigma \cap \langle \mathcal{A}_0 \rangle$ is of cardinality $r-1$, and there exists an element of σ, say β_r, of the form $c\theta + \xi$ where ξ belongs to $\langle \mathcal{A}_0 \rangle$, c is a non-zero constant and θ is the highest element of $\mathcal{A}$. Let

$$\vec{\nu_0} = [\theta(I_1), \theta(I_2), \ldots, \theta(I_{r-1})]$$

and

$$\vec{\nu'_0} = [\theta(I'_1), \theta(I'_2), \ldots, \theta(I'_{r-1})].$$

Then we have

$$\mathrm{Ires}_{\vec{\nu}} f_\sigma = \frac{1}{c} \mathrm{Ires}_{\vec{\nu_0}} f_{\sigma \cap \langle \mathcal{A}_0 \rangle}$$

and

$$\mathrm{Ires}_{\vec{\nu'}} f_\sigma = \frac{1}{c} \mathrm{Ires}_{\vec{\nu'_0}} f_{\sigma \cap \langle \mathcal{A}_0 \rangle}.$$

We conclude by induction.

When $\mathcal{A}$ is not irreducible, we write $\mathcal{A} = \cup_{a=1}^{s} J_a$ where J_a are irreducibles. We have $V = \oplus_{a=1}^{s} \langle J_a \rangle$. Every basic subset σ of $\mathcal{A}$ is the union of basic subsets for the irreducible sets J_a. Define

$$\vec{\nu}_a = [\theta(I_a^{i_1}), \theta(I_a^{i_2}), \ldots, \theta(J_a)]$$

where $[I_a^{i_1}, I_a^{i_2}, \ldots, J_a]$ is the subsequence of irreducible sets contained in J_a extracted (with conserving order) from the sequence $[I_1, I_2, \ldots, I_r]$. Similarly let

$$\vec{\nu'}_a = [\theta({I'_a}^{i_1}), \theta({I'_a}^{i_2}), \ldots, \theta(J_a)]$$

where $[{I'_a}^{i_1}, {I'_a}^{i_2}, \ldots, I'_a]$ is the subsequence of irreducible sets contained in J_a extracted from the sequence $[I'_1, I'_2, \ldots, I'_r]$. Then, as the calculation takes place with respect to independent variables, we have

$$\begin{aligned}\mathrm{Ires}_{\vec{\nu}}(f_\sigma) &= \prod_{a=1}^{s}(\mathrm{Ires}_{\vec{\nu}_a} f_{\sigma\cap\langle J_a\rangle}),\\ \mathrm{Ires}_{\vec{\nu'}}(f_\sigma) &= \prod_{a=1}^{s}(\mathrm{Ires}_{\vec{\nu'}_a} f_{\sigma\cap\langle J_a\rangle}).\end{aligned}$$

Each of the vector space $\langle J_a\rangle$ is of dimension less than r, so that by induction hypothesis $\mathrm{Ires}_{\vec{\nu}_a} = \mathrm{Ires}_{\vec{\nu'}_a}$. This concludes the proof. □

Let us now consider partial iterated residues. To a set ν of elements of $\mathcal{A}$, we associate the vector space

$$H_\nu := \{u \in U \mid \langle \alpha, u\rangle = 0 \text{ for all } \alpha \in \nu\}.$$

A linear function $\alpha \in \mathcal{A}$ produces a linear function on H_ν by restriction. If $\vec{\nu} := [\alpha_1, \alpha_2, \ldots, \alpha_k]$ is a sequence of elements of $\mathcal{A}$, the partial iterated residue

$$\mathrm{Ires}_{\vec{\nu}}\phi := \mathrm{res}_{\alpha_k=0}\cdots\mathrm{res}_{\alpha_1=0}\phi$$

associates to a rational function ϕ in $R_{\mathcal{A}}$ a rational function on H_ν of the form

$$\frac{G}{\prod_{i=1,\ldots,n\,;\,\overline{\alpha_i}\neq 0}\overline{\alpha}_i^{\,n_i}},$$

where G is a polynomial function on H_ν and $\overline{\alpha}$ is the restriction of α to H_ν. Let M be a MPNS and consider the tree associated to M. Given a saturated subset S of M, we can define the iterated residue with respect to this saturated set: we choose any order $S := [I_1, I_2, \ldots, I_k]$ on S compatible with the inclusion relation and define $\mathrm{Ires}_S := \mathrm{Ires}_{\vec{\nu}}$ with $\vec{\nu} = [\theta(I_1), \theta(I_2), \ldots, \theta(I_k)]$. With the same proof as for Proposition 6.4, this partial residue depends only on the set S. We denote by H_S the intersection of the kernels of the elements α for $\alpha \in S$. It is also the intersection of the kernels of the elements $\theta(I_k)$, as the set ν is a basic sequence in S.

Let ϕ be a function in $R_{\mathcal{A}}$ of the form

$$\phi = \frac{P}{\prod_{i=1}^{n}\alpha_i}.$$

Let M be a MPNS and $J_1, J_2, \ldots, J_s$ be elements of M. We consider the saturated subset S of M consisting of the elements of the tree *strictly above* $J_1, J_2, \ldots, J_s$. The iterated residue $\mathrm{Ires}_S\phi$ is a function on H_S. Denote by u_a the restriction of the function $\theta(J_a)$ to H_S.

Proposition 6.5. *The pole of the linear function u_a in the iterated residue $\mathrm{Ires}_S\phi$ is of order less than or equal to $|J_a| - \dim\langle J_a\rangle + 1$.*

See Figures 8 and 9 for an application of the proposition.

Proof. Choose a vector space E such that
$$V = \langle J_1\rangle \oplus \cdots \oplus \langle J_s\rangle \oplus E.$$
Let $B = \bigcup_{a=1}^{s} J_a$ and $C = \mathcal{A}\setminus B$. Write $C := \{\beta_1, \beta_2, \ldots, \beta_q\}$ and
$$\phi = P \times \phi_1 \times \phi_2 \times \cdots \times \phi_s \times Q$$
with $\phi_a = \frac{1}{\prod_{\alpha\in J_a}\alpha}$ and $Q = \frac{1}{\prod_{j=1}^{q}\beta_j}$.

For $\beta_j \in C$, we write $\beta_j = \sum_{i=1}^{s}\beta_j^i + \gamma_j$ with $\beta_j^i \in \langle J_i\rangle$ and $\gamma_j \in E$. The element γ_j is necessarily non zero, as the set B is complete. Thus we write
$$\frac{1}{\beta_j} = \frac{1}{\gamma_j\left(1 + \frac{\sum_{i=1}^{s}\beta_j^i}{\gamma_j}\right)}$$
and the iterated residue is by definition
$$\mathrm{Ires}_S(\phi) = \mathrm{Ires}_S\left(P \times (\phi_1\cdots\phi_s) \times \prod_{j=1}^{q}\frac{1}{\gamma_j}\sum_{k=0}^{\infty}\left((-1)^k\frac{\sum_{i=1}^{s}\beta_j^i}{\gamma_j}\right)^k\right).$$
Here, when taking the residue, the elements γ_j are considered as constants and this sum is finite.

Consider the subset M_a of elements of M contained in J_a. This is a MPNS for the set J_a. Let J_a^+ be the saturated subset of M_a consisting of all elements of M_a different from J_a. Then J_a^+ has $\dim\langle J_a\rangle - 1$ elements. If $g = \frac{P_a}{\prod_{\alpha\in J_a}\alpha^{n_\alpha}}$, the iterated residue $\mathrm{Ires}_{J_a^+} g$ is a Laurent polynomial in u_a.

Now $\mathrm{Ires}_S\phi$ is a sum of products of residues of the form $\mathrm{Ires}_{J_a^+} g_a$ where $g_a = \frac{P_a}{\prod_{\alpha\in J_a}\alpha}$ and P_a is a polynomial. Thus we obtain a Laurent polynomial in $u_a, a = 1, \ldots, s$ (with coefficients rational functions on the vector space E^*). Now the homogeneous degree of g_a is greater than or equal to $-|J_a|$. The number of residues we are taking is equal to $\dim\langle J_a\rangle - 1$. So we obtain a function of u_a of homogeneous degree greater than or equal to $-|J_a| + \dim\langle J_a\rangle - 1$. This means that the pole in u_a is of order less than or equal to $|J_a| - \dim\langle J_a\rangle + 1$. $\square$

Let us consider the MPNS whose tree representation is given by Figure 8. The orders of the poles of its nodes are given in Figure 9.

Remark 6.6. In our program for calculating iterated residues for root systems of type A_r, we will reorder roots according to the tree order: we take the residue first with respect to the elements $\theta(I_k)$ appearing at the end of the tree in arbitrary order, and we remove these variables. Then we take the variables appearing at the end of the tree when we have removed these irreducible sets. Here an irreducible set I is indexed by a subset S of $\{1, 2, \ldots, r+1\}$. A subset S of cardinality 2, for example $[1, 3]$, corresponds to the irreducible set with one element (here $e_1 - e_3$). Thus given a MNS M represented as $M = \{S_1, S_2, \ldots, S_r\}$ we will first take the residues with respect to the roots $\theta(I_k)$, for sets S_k of cardinality 2, in arbitrary order,

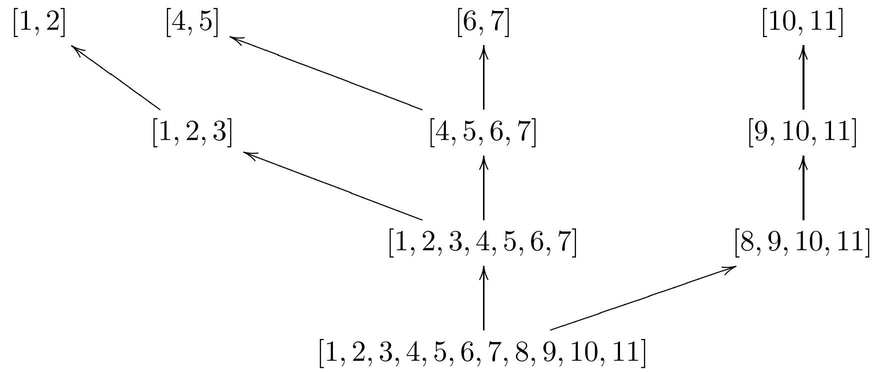

FIGURE 8. Irreducible components of a MPNS in A_{10}

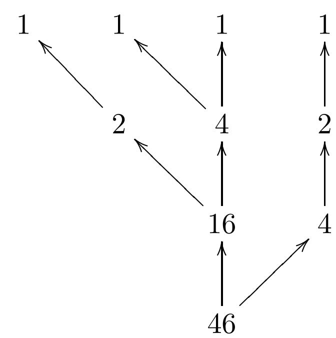

FIGURE 9. Order of nodes in the tree represented in Figure 8, according to Proposition 6.5

then with respect to irreducible sets associated to sets S_k of cardinality 3, *etc.* The procedure of ordering roots coming from a MNS $M = \{S_j\}$ according to the cardinality of the set S_k is called `OrderThetas`. Furthermore we will at the same time keep track of the order of the pole for calculating an iterated residue of a function $\phi = P/\prod_{i=1}^{N} \alpha_i$ in the procedure `FormalPathAwithOrders`.

7. Volume and partition function for the system A_{n-1}

7.1. **The formulae to be implemented.** Let E be an n-dimensional vector space with basis e_i ($i = 1, \ldots, n$) and consider the set

$$\mathcal{K}_n = \{e_i - e_j \,|\, 1 \leq i < j \leq n\}.$$

These are the positive roots for a system of type A_{n-1}. The number of elements in $\mathcal{K}_n$ is $N = n(n-1)/2$. Note that $\mathcal{K}_n$ is also the set of vectors in a complete graph with n nodes.

We let V be the vector space generated by the elements in $\mathcal{K}_n$. Then V has dimension $n-1$ and it is defined by:

$$V = \left\{ v = \sum_{i=1}^{n} v_i e_i \in E \,\Big|\, \sum_{i=1}^{n} v_i = 0 \right\}.$$

In our procedures, a vector v of length n such that $\sum_{i=1}^{n} v_i = 0$ will be called an *A-vector* and written as $v = [v_1, v_2, \dots, v_n]$. The lattice spanned by $\mathcal{K}_n$ is simply

$$V_{\mathbb{Z}} = \left\{ h = \sum_{i=1}^{n} h_i e_i \in \mathbb{Z}^n \,\Big|\, \sum_{i=1}^{n} h_i = 0 \right\}.$$

It is well known and easy to prove that $\mathcal{K}_n$ is unimodular. The cone $\mathcal{C}(\mathcal{K}_n)$ generated by $\mathcal{K}_n$ is simplicial with generators the $n-1$ simple roots $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$. This cone is described as:

$$\mathcal{C}(\mathcal{K}_n) = \{A\text{-vector } v = [v_1, v_2, \dots, v_n] \,|\, v_1 + v_2 + \cdots + v_i \geq 0 \text{ for all } i\}.$$

Keep in mind that our vector v satisfies the condition

$$v_1 + v_2 + \cdots + v_{n-1} + v_n = 0.$$

We choose on V the measure dh determined by $V_{\mathbb{Z}}$. Let v be in the cone $\mathcal{C}(\mathcal{K}_n)$. We are interested to compute the volume $\mathrm{vol}_{\mathbb{Z},\mathcal{K}_n}(v)$ of the polytope

$$\Pi_{\mathcal{K}_n}(v) = \left\{ (x_\alpha)_\alpha \in \mathbb{R}^N \,\Big|\, x \geq 0, \sum_{\alpha \in \mathcal{K}_n} x_\alpha \alpha = v \right\}.$$

If h is a point in V with integral coordinates then we are also interested in computing the number $N_{\mathcal{K}_n}(h)$ of integral points in $\Pi_{\mathcal{K}_n}(h)$.

We apply the formulae of Theorem 3.3. Since $\mathcal{K}_n$ is unimodular, the set F can be taken as $F := \{0\}$ (Remark 3.8).

Since V is contained in E, then we have a canonical map $E^* \longrightarrow V^*$ given by restriction. Define $U = V^*$ as in the general setting. We identify U with $\mathbb{R}^{n-1}$ by sending $u \in \mathbb{R}^{n-1}$ to $u = \sum_{i=1}^{n-1} u_i e^i \in E^*$, where e^i is the dual basis to e_i. Thus the root $e_i - e_j$ $(1 \leq i < j < n)$ produces the linear function $u_i - u_j$ on U, while the root $e_i - e_n$ produces the linear function u_i.

Definition 7.1. Let $v = \sum_{i=1}^{n} v_i e_i \in V$ be a vector with real coordinates. Let $h = \sum_{i=1}^{n} h_i e_i \in V$ be a vector with integral coordinates. Then for $u \in U$ define:

- $J_A(v)(u) = \dfrac{e^{\sum_{i=1}^{n-1} u_i v_i}}{\prod_{i=1}^{n-1} u_i \prod_{1 \leq i < j \leq n-1} (u_i - u_j)}$
- $\mathcal{F}_A(h)(u) = \dfrac{\prod_{i=1}^{n-1} (1 + u_i)^{h_i + n - 1 - i}}{\prod_{i=1}^{n-1} u_i \prod_{1 \leq i < j \leq n-1} (u_i - u_j)}$.

Theorem 7.2. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{K}_n)$.*

- *For $v \in \overline{\mathfrak{c}}$, we have*

$$\mathrm{vol}_{\mathbb{Z},\mathcal{K}_n}(v) = \mathrm{JK}_{\mathfrak{c}}(J_A(v)).$$

- *For $h \in \mathbb{Z}^n \cap \overline{\mathfrak{c}}$, we have*

$$N_{\mathcal{K}_n}(h) = \mathrm{JK}_{\mathfrak{c}}\left(\mathcal{F}_A(h)\right).$$

Proof. The first assertion is the general formula.

The function $F(0,h)(u) = e^{\langle h,u\rangle}/\prod_{\alpha\in\mathcal{A}}(1-e^{-\langle\alpha,u\rangle})$ for the system $\mathcal{K}_n$ is

$$F(0,h)(u) = \frac{e^{\sum_{i=1}^{n-1} u_i v_i}}{\prod_{i=1}^{n-1}(1-e^{-u_i})\prod_{1\le i<j\le n-1}(1-e^{-(u_i-u_j)})}.$$

Note that the change of variable $1+z_i = e^{u_i}$ preserves the hyperplanes $u_i = 0$ and $u_i = u_j$. After the change of variable, we get

$$F(0,h)(u) = \frac{\prod_{i=1}^{n-1}(1+z_i)^{h_i+n-i}}{\prod_{1\le i<j\le n-1}(z_i-z_j)\times\prod_{i=1}^{n-1} z_i}. \tag{11}$$

But $z_i = e^{u_i}-1$ leads to $dz_i = e^{u_i}du_i = (1+z_i)du_i$ and hence we obtain the desired exponent $h_i+n-i-1$ thanks to the formula involving Jacobians in Proposition 3.2. □

In order to implement these formulae, we first have to describe the set $\mathcal{P}(v,\mathcal{K}_n)$ (Section 7.2), then calculate the iterated residue formulae associated to these paths (Section 7.3). Below we explain how these computations fit together to get a global procedure for the Kostant partition function for A_{n-1} (Section 7.4). As a short digression, we will explain how we adapted our program to deal with formal parameters (Section 7.5).

7.2. **The search for maximal proper nested sets adapted to a vector.** We now look for maximal proper nested sets adapted to a vector following the general method as outlined in Figure 7: we will begin by listing all possible $\mathcal{K}_n$-admissible hyperplanes. The usual height function is

$$\mathrm{ht}(v) = \sum_{i=1}^{n-1}(n-i)v_i$$

which takes the value 1 on all the simple roots, and hence the value $j-i$ on e_i-e_j. We deform ht slightly in order to have a function taking different values on all roots: If two elements e_i-e_j and e_k-e_ℓ are such that $j-i = \ell-k$, we decide that $\mathrm{ht}(e_i-e_j) < \mathrm{ht}(e_k-e_\ell)$ if $i<k$.

If P is a proper subset of $\{1,2,\ldots,n\}$ and v is an A-vector, we denote by $\langle u_P, v\rangle$ the linear form $\sum_{i\in P} v_i$, and by H_P the hyperplane

$$H_P := \{v \in V \mid \langle u_P, v\rangle = 0\}.$$

We will see shortly that all $\mathcal{K}_n$-admissible hyperplanes are obtained in this way, that is giving a proper subset P of $\{1,\ldots,n\}$. Observe that the hyperplane H_P is equal to the hyperplane H_Q determined by the complement Q of P. We denote

$$\mathcal{K}(P) := \{e_i - e_j \mid 1 \leq i < j \leq n\,;\, i,j \in P\} \subset \mathcal{K}_n.$$

Note that $\mathcal{K}(P)$ is the positive system $A_{|P|-1}$, where the positivity is induced by the lexicographic order.

Lemma 7.3.
- *The hyperplane H_P is a $\mathcal{K}_n$-admissible hyperplane.*
- *The set $\mathcal{K}_n \cap H_P$ is the union of $\mathcal{K}(P)$ and $\mathcal{K}(Q)$, where Q is the complement of P in $\{1, 2, \dots, n\}$.*
- *Every $\mathcal{K}_n$-admissible hyperplane is of this form.*

Proof. The first two assumptions are easy to see. We prove the third by induction on n, the case $n = 2$ being trivial. Let H be a $\mathcal{K}_n$-admissible hyperplane. Let α be a root in H. Renumbering the roots, we may assume that $\alpha = e_{n-1} - e_n$. The map q sending e_i to e_i if $i < n$ and e_n to e_{n-1} sends the set $\mathcal{K}_n \setminus \{\alpha\}$ to $\mathcal{K}_{n-1}$. The space $H/\mathbb{R}\alpha$ becomes a $\mathcal{K}_{n-1}$-admissible hyperplane. It is thus determined by a subset P' of $\{1, 2, \dots, n-1\}$. If P' does not contain $n-1$, the hyperplane H is equal to the hyperplane determined by the subset P' of $\{1, 2, \dots, n-1\}$. If P' does contain $n-1$, then the hyperplane H is equal to the hyperplane determined by $P = P' \cup \{n\}$. □

We now proceed to the detailed description of our algorithm. Recall our description of an A-vector as an array $v = [v_1, v_2, \dots, v_n]$ with $\sum_{i=1}^n v_i = 0$. Referring to Figure 7 we need to check if the vector is in the cone $\mathcal{C}(\mathcal{K}_n)$, that is, $\sum_{j=1}^i v_j \geq 0$ for $1 \leq i \leq n-1$. This is done by using the procedure `CheckVector(v)`, which gives an answer true or false.

For the system $\mathcal{K}_n$ the highest root θ is equal to

$$\theta = [1, 0, \dots, 0, -1] \in \mathbb{R}^n$$

and computed with the procedure `theta`.

At this point, we need to list all hyperplanes that are in $\mathrm{Hyp}(v, \mathcal{K}_n)$. This is done in the procedure `TwoSets(v)`, that we are about to describe. As explained in Lemma 7.3, each hyperplane is determined by an equation $\sum_{i \in I} a_i = 0$. It therefore produces a set of two lists P, Q, where $P = [i \in I]$ and $Q = [i \notin I \,|\, 1 \leq i \leq n]$. Note that P and Q are sorted. To verify that such a hyperplane is in $\mathrm{Hyp}(v, \mathcal{K}_n)$, we need to test if $\langle u_P, \theta \rangle$ is not zero (θ is not in the hyperplane) and if $\langle u_P, v \rangle \times \langle u_P, \theta \rangle$ is non-negative (θ and v are on the same side of the hyperplane).

Furthermore, the procedure `ProjH(v,H)` constructs the vector

$$\mathrm{proj}_H(v) = v - \frac{\langle u_P, v \rangle}{\langle u_P, \theta \rangle} \theta$$

that we represent as $\{[v_1, P], [v_2, Q]\}$. Each of the vectors v_1, v_2 is an A-vector (sum of coordinates equal to zero). So the last condition for H being in $\mathrm{Hyp}(v, \mathcal{K}_n)$ is that $v_1 \in \mathcal{C}(\mathcal{K}(P))$ and $v_2 \in \mathcal{C}(\mathcal{K}(Q))$.

Hence a hyperplane H is in $\mathrm{Hyp}(v, \mathcal{K}_n)$ if it satisfies the series of conditions:

$$\begin{aligned} \langle u_P, \theta \rangle \neq 0 \quad &\text{with} \quad \texttt{Hvalue(theta(n),P)} \neq 0, \\ \langle u_P, v \rangle \times \langle u_P, \theta \rangle \geq 0 \quad &\text{with} \quad \texttt{CheckSide(v,P)} = \texttt{true}, \\ v_1 \in \mathcal{C}(\mathcal{K}_{|P|-1}) \quad &\text{with} \quad \texttt{CheckVector}(\texttt{v}_1) = \texttt{true}, \\ v_2 \in \mathcal{C}(\mathcal{K}_{|Q|-1}) \quad &\text{with} \quad \texttt{CheckVector}(\texttt{v}_2) = \texttt{true}. \end{aligned}$$

The procedure `CheckList(v,H)` implements all these sub-routines. It is used in the procedure `TwoSets(v)`, computing all elements of $\mathrm{Hyp}(v, \mathcal{K}_n)$. We combine `TwoSets` with a procedure named `TwoVector` to finally get the procedure `TwoVectors(v)` determining all hyperplanes in $\mathrm{Hyp}(v, \mathcal{K}_n)$ and projections of v on these hyperplanes.

We now have to perform the next step of our algorithm. Let $\{[v_1, K_1], [v_2, K_2]\}$ be the output of `TwoVectors(v)`. Then we construct the MNSs for $[v_1, K_1]$ and $[v_2, K_2]$, and go on recursively until the procedure stops. These iterated steps are done by the procedure `Splits`.

Finally the procedure `MNSs(v)`, computing all MNSs for a given vector v, works as follows. We begin by building the first seed of MNSs with the procedure `MNS1`, containing the regularization of the result of `TwoVectors`. We then call repeatedly the procedure `AllNewMNSs`, which performs the regularization of the output of `Splits`.

7.3. **Residues associated to maximal proper nested sets.** An element M in $\mathcal{P}(v, \mathcal{K}_n)$ is represented as a collection $M = \{K_1, K_2, \ldots, K_{n-1}\}$ of $(n-1)$ subsets of $[1, 2, \ldots, n]$. As we have said in Remark 6.6, given a maximal proper nested set $M := \{K_1, K_2, \ldots, K_{n-1}\}$ we associate to it an ordered basis $\overrightarrow{\theta(M)}$ of V (procedure `OrderThetas`). If $p = [\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ is the list of roots singled out by our procedure, then α_1 is an element associated to a set K_i of cardinality 2 and $\alpha_{n-1} = \theta$. We identify the root $e_i - e_n$ to the linear function z_i on $\mathbb{C}^{n-1}$ and the root $e_i - e_j$ to $z_i - z_j$.

Let h be a A-vector with integral coordinates. Let us consider the Kostant function

$$\mathcal{F}_A(h)(z_1, z_2, \ldots, z_{n-1}) = \frac{\prod_{i=1}^{n-1} (1 + z_i)^{h_i + n - 1 - i}}{\prod_{1 \leq i < j \leq n-1} (z_i - z_j) \times \prod_{i=1}^{n-1} z_i}$$

(Definition 7.1). To compute $N_{\mathcal{K}_n}(h)$, we will have to compute

$$\mathrm{res}_M \phi := \mathrm{res}_{\alpha_{n-1}=0} \mathrm{res}_{\alpha_2=0} \cdots \mathrm{res}_{\alpha_1=0} \phi$$

with $\phi = \mathcal{F}_A(h)$. Using Proposition 6.5, we know in advance the order of the pole in $\alpha_k = 0$ of the function obtained after taking the first $(k-1)$ residues. These orders are recorded in the procedure `FormalPathAwithOrders`.

If $\alpha_1 = z_i - z_j$, we can replace — after taking the residue at $z_i = z_j$ — the variable z_i by the variable z_j in all the other roots. Thus we get rid of the variable z_i. The procedure `NewR` produces the ordered path resulting from all these substitutions.

Recursively, we will have to compute the residue at $z_{i_0} - z_{j_0} = 0$ of an expression

$$f = \frac{A(z_i, i \in L)}{\prod_{i,j \in L; i<j} (z_i - z_j)^{m_{i,j}} \prod_{i \in L} z_i^{m_i}}, \tag{12}$$

where L is a list of indices taken in $\{1, \ldots, n-1\}$. Denote by $maxi$ the order m_{i_0,j_0} of the root $z_{i_0} - z_{j_0}$ (the exponent $maxi$ is recorded in the procedure `FormalPathAwithOrders`). Note that computing the residue is exactly the same as computing the coefficient of z of degree $maxi - 1$ of the expansion of $f \times (z_{i_0} - z_{j_0})^{maxi}$ at $z_{i_0} = z + z_{j_0}$. Let us describe in detail the procedure `ComputeRes`, performing this task.

For $j \in L \setminus \{i_0, j_0\}$, let et $e_j = m_{i_0,j}$ if $i_0 < j$ and $e_j = m_{j,i_0}$ if $i_0 > j$. Then $f \times (z_{i_0} - z_{j_0})^{maxi}$ can be written as $g \times h \times R$ where

$$g = \frac{A(z_i, i \in L)}{\prod_{i,j \in L; i,j \neq i_0; i<j} (z_i - z_j)^{m_{i,j}}} \times \frac{1}{\prod_{i \in L; i \neq i_0} z_i^{m_i}}, \tag{13}$$

$$h = \frac{1}{\prod_{j \in L; j \neq i_0, j_0} (z_j - z_{i_0})^{e_j}} \times \frac{1}{z_{i_0}^{e_0}}, \tag{14}$$

$$R = (-1)^{\sum_{j > i_0} e_j}. \tag{15}$$

Let $z_{i_0} = z + z_{j_0}$. So to get the desired residue, we need to calculate the expansion of g and h at $z = 0$. More precisely if $g = \sum_{i=0}^{maxi-1} g_i z^i$ and $h = \sum_{i=0}^{maxi-1} h_i z^i$, then the coefficient of degree $maxi-1$ of $f \times (z_{i_0} - z_{j_0})^{maxi}$ is simply $R \times \sum_{i=0}^{maxi-1} g_i h_{maxi-1-i}$. Let us describe how the procedure `ComputeRes` performs this task.

Rewrite the fraction h defined in Equation (14) as $B \times \tilde{h}$, where

$$\begin{aligned} B &= \frac{1}{\prod_{j \in L; j \neq i_0, j_0} (z_j - z_{j_0})^{e_j}} \times \frac{1}{z_{j_0}^{e_0}}, \\ \tilde{h} &= \frac{1}{\prod_{j \in L; j \neq i_0, j_0} \left(1 - \frac{z}{z_j - z_{j_0}}\right)^{e_j}} \times \frac{1}{\left(1 + \frac{z}{z_{j_0}}\right)^{e_0}}. \end{aligned}$$

Consequently, to expand h as a function of z, we only need to expand $\tilde{h}$. This is done in the procedure `CoeffBin` using the binomial coefficients. In the procedure `CoeffFun` we calculate the expansion at $z_{i_0} - z_{j_0} = 0$ of the fraction

$$\begin{aligned} & f \times (z_{i_0} - z_{j_0})^{maxi} \times z_{i_0}^{e_0} \times \prod_{j \in L; j \neq i_0, j_0} (z_{i_0} - z_j)^{e_j} \\ = \; & g \times R \times (-1)^{\sum_{j \neq i_0, j_0} e_j}. \end{aligned} \tag{16}$$

Finally the procedure `ComputeRes` performs the sum over i ranging from 0 to $maxi$ of

$$\begin{aligned} & (S := (-1)^{\sum_{j\neq i_0,j_0} e_j}) \times B \\ \times\quad & (\text{the component of degree } i \text{ of } \texttt{CoeffFun}) \\ \times\quad & (\text{the component of degree } maxi - 1 - i \text{ of } \texttt{CoeffBin}). \end{aligned}$$

Rewrite this as the sum over i of

$$\begin{aligned} & R \\ \times\quad & (\text{the component of degree } i \text{ of } \texttt{CoeffFun}) \times R \times S \\ \times\quad & B \times (\text{the component of degree } maxi - 1 - i \text{ of } \texttt{CoeffBin}), \end{aligned}$$

or, equivalently, as $\sum_{i=0}^{maxi-1} R \times g_i \times h_{maxi-1-i}$: this is exactly the desired coefficient.

Remark 7.4. For residues along roots of type z_{i_0} instead of $z_{i_0} - z_{j_0}$ the procedure `ComputeRes` also calls procedures `srCoeffFun` and `srCoeffBin`, similar to `CoeffFun` and `CoeffBin`.

7.4. **The procedure `MNS_KostantA`.** We finish the section dedicated to A_{n-1} by giving the global outline of the procedure `MNS_KostantA(v)` computing the Kostant partition number of a vector v lying in the root lattice. We begin by slightly deforming v so that it lies on no admissible hyperplanes, with the command `v' := DefVector(v, n)`. We compute all MPNSs for v' with the procedure `MNSs(v')`.

Given such a MPNS $M = \{S_k\}$, we extract the highest roots of its irreducible components with the call `R := ThetaMNS(M)`. We obtain a set R where each element of R is a root represented as $[i, j]$ together with the cardinality of the set S_k it comes from. We then transform this set R into a path p keeping track of order of poles by setting `p := FormalPathAwithOrders(R)`.

Finally we compute the residue associated to this path with the command `OneIteratedResidue(p, v, n)`. Summing all these residues over the set of MNSs, we obtain thanks to Theorem 7.2, the desired partition number for v.

Let us describe in detail the procedure `OneIteratedResidue(p, v, n)` computing the iterated residue along a path p for a vector v lying in the root lattice for A_{n-1}. We first compute the Kostant fraction (second item of Definition 7.1, procedure `KostantFunctionA`). Then we replace in the path all roots $z_i - z_n$ by z_i (with `Kpath`). We also build a upper bound for the orders of the roots m_{ij} (with `Multiplicity`). Keep in mind that the exponent $maxi$ needed in the residue calculation is computed *a priori*, and our computation seems quite optimal. Then we compute iteratively the residues, using the procedure `ComputeRes` (Section 7.3). Note that at each step we have to update the list of orders (with `MultRoots`) and the list of remaining variables (with `ListOfVariables`).

7.5. **Parametrized version of the algorithm.** Our algorithm can work with formal parameters, only needing slight modifications of procedures. We are then able to compute directly the polynomial $h \mapsto N_{\mathcal{K}_n}(h)$ giving the number of integral points in the polytope $\Pi_{\mathcal{K}_n}(h)$, on the chamber determined by h (this chamber is easily computed). As a consequence we can easily get the Ehrhart polynomial $t \mapsto N_{\mathcal{K}_n}(th_1, \dots, th_n)$ of the polytope. See [13].

Now let us outline how this modified program works. Given an element $h = (h_1, \dots, h_n)$ of the root lattice for A_{n-1}, we want to compute the Kostant partition function for the vector $(h_1, \dots, h_n)$, when h varies in $\mathbb{Z}^n \cap \bar{\mathfrak{c}}$.

Recall that we have to perform the residue at $z_{i_0} - z_{j_0} = 0$ of the fraction defined in Equations (11) and (12). Note that the numerator of the fraction $\mathcal{F}_A(h)$ contains terms of the form $(1+z_i)^{h_i+n-1-i}$ that we must formally expand. For any parameter b, we write at $z_{i_0} = z + z_{j_0}$,

$$(1+z_{i_0})^b = (1+z_{j_0})^b \left(1 + \sum_{j=1}^{maxi-1} \binom{b}{j} \left(\frac{z}{1+z_{j_0}}\right)^j\right) + O(z^{maxi}).$$

Hence, for any parameter b, the procedure `CoeffFun` (see Equation (16)) now computes the expansion at $z_{i_0} - z_{j_0} = 0$ of the fraction

$$\begin{aligned} f \quad &\times \quad \left(z_{i_0}^{e_0} \times \prod_{j \in L; j \neq i_0, j_0} (z_{i_0} - z_j)^{e_j}\right) \\ &\times \quad \left(1 + \sum_{j=1}^{maxi-1} \binom{b}{j} \left(\frac{z_{i_0} - z_{j_0}}{1+z_{j_0}}\right)^j\right) \times (1+z_{j_0})^{maxi-1}. \end{aligned}$$

For the residue along a root of type z_{i_0} instead of $z_{i_0} - z_{j_0}$, the procedure `srCoeffFun` has been modified in a similar way.

8. The type B_n

8.1. **The formulae to be implemented.** Consider a vector space V with basis $e_1, e_2, \dots, e_n$. We choose on V the standard Lebesgue measure dh. Let

$$\mathcal{B}_n = \{e_i \,|\, 1 \le i \le n\} \cup \{e_i - e_j \,|\, 1 \le i < j \le n\} \cup \{e_i + e_j \,|\, 1 \le i < j \le n\}.$$

Then $\mathcal{B}_n$ is a positive roots system of type B_n and generates V. The number of elements in $\mathcal{B}_n$ is $N = n^2$. We denote by U the dual of V. The lattice $V_{\mathbb{Z}}$ generated by roots is equal to $\mathbb{Z}^n$, so the constant $\operatorname{vol}(V/V_{\mathbb{Z}}, dh) = 1$.

The cone $\mathcal{C}(\mathcal{B}_n)$ is simplicial and spanned by the n simple roots $e_1 - e_2$, $e_2 - e_3$, $\dots$, $e_{n-1} - e_n$, e_n. A vector $v = [v_1, v_2, \dots, v_n]$ is in $\mathcal{C}(\mathcal{B}_n)$ if and only if it satisfies the inequations $v_1 + \cdots + v_i \ge 0$ for all $i = 1, \dots, n$.

Let v be in the cone $\mathcal{C}(\mathcal{B}_n)$. Consider the polytope

$$\Pi_{\mathcal{B}_n}(v) = \left\{ (x_\alpha)_\alpha \geq 0 \,\Big|\, \sum_{\alpha \in \mathcal{B}_n} x_\alpha \alpha = v \right\}.$$

If h is a point in V with integral coordinates, we are interested in computing the number $N_{\mathcal{B}_n}(h)$ of integral points in $\Pi_{\mathcal{B}_n}(h)$.

Let $U_{\mathbb{Z}}$ be the lattice dual to $V_{\mathbb{Z}}$. We identify the torus $T = U/U_{\mathbb{Z}} = \mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$ by

$$(u_1, u_2, \ldots, u_n) \mapsto \left(e^{2\pi\sqrt{-1}u_1}, \ldots, e^{2\pi\sqrt{-1}u_n} \right).$$

If G is a representative of $g = (g_1, g_2, \ldots, g_n) \in T$, and $h = \sum_{i=1}^n h_i e_i$ in $V_{\mathbb{Z}}$, then $e^{\langle h, 2\pi\sqrt{-1}G\rangle}$ is equal to $\prod_{i=1}^n g_i^{h_i} = g^h$. As the set $\mathcal{B}_n$ is not unimodular, the sets $T(\sigma)$ are not reduced to 1.

Example 8.1. Let σ be the basic set $\{e_1 + e_2, e_1 - e_2\}$ for B_2. Then $T(\sigma) = \{(1, 1), (-1, -1)\}$.

We now determine a set F containing all sets $T(\sigma)$.

Lemma 8.2. *Let σ be a basic subset of $\mathcal{B}_n$. Assume $g \in T(\sigma)$. Then all the coordinates of g are equal to ± 1. Furthermore, if g is not 1, there are at least two coordinates of g which are equal to -1.*

Proof. We prove this by induction on n. For $\mathcal{B}_2$, we have seen this by direct computation.

Let σ be a basic subset of $\mathcal{B}_n$. Assume first that σ contains a root e_i. Up to renumbering, we may assume that this root is e_n. Then the basis σ produces a basis σ' of $\mathcal{B}_{n-1}$ by putting $e_n = 0$. Let $g = (g_1, g_2, \ldots, g_n)$ in $T(\sigma)$. We see that $g' = (g_1, g_2, \ldots, g_{n-1})$ is in $T(\sigma')$. Thus, by induction the first $n-1$ coordinates of g' are equal to ± 1. But since e_n is in σ we get $1 = g_n$. Note that $g \neq 1$ if and only if $g' \neq 1$, hence by induction hypothesis g' has at least two coordinates not equal to 1.

Consider now the case where σ does not contain any root e_i. Up to renumbering, it contains a root $e_{n-1} - e_n$ or $e_{n-1} + e_n$.

Let us examine first the case where σ contains the root $\alpha = e_{n-1} - e_n$. Let $g = (g_1, g_2, \ldots, g_{n-1}, g_n)$ in $T(\sigma)$. This implies $g_{n-1} = g_n$. Consider the map q sending e_i to e_i if $i < n$ and e_n to e_{n-1}. Then q sends $\sigma \setminus \{e_{n-1} - e_n\}$ to a basis σ' of $\mathcal{B}_{n-1}$. The element $g' = (g_1, g_2, \ldots, g_{n-1})$ is easily seen to belong to $T(\sigma')$. Indeed if α equals $e_i \pm e_j$ with $1 \leq i < j < n$, this is by definition. On the other hand $q(e_i \pm e_n) = e_i \pm e_{n-1}$ and $g_{n-1} = g_n$ imply that $g_i g_{n-1}^{\pm 1}$ coincides with the value of $g_i g_n^{\pm 1}$. By induction hypothesis, all coordinates of g' are equal to ± 1. Moreover $g \neq 1$ if and only if $g' \neq 1$, so that g is of the desired form.

Finally, the same argument works if σ contains $\alpha = e_{n-1} + e_n$, by considering the map q sending e_i to e_i if $i < n$, and e_n to $-e_{n-1}$. $\square$

Definition 8.3. If I is a subset of $\{1,2,\dots,n\}$ with at least two elements, we consider the set $F(I) := \{(g_1,g_2,\dots,g_n)\,|\,g_i=-1, i\in I; g_j=1, j\notin I\}$.

We define $F\subset T$ to be the finite subset of T union of such sets $F(I)$ together with the identity $(1,1,\dots,1)$.

Let $v=\sum_{i=1}^n v_ie_i \in V$ be a vector with real coordinates and $h=\sum_{i=1}^n h_ie_i\in V$ a vector with integral coordinates. We will compute the normalized volume of $\Pi_{\mathcal{B}_n}(v)$ and the number of integral points in $\Pi_{\mathcal{B}_n}(h)$ using Theorem 3.3. Thus we introduce the function $J_B(v)$ on U defined by:

$$J_B(v)(u)=\frac{e^{\sum_{i=1}^n u_iv_i}}{\prod_{i=1}^n u_i\prod_{1\le i<j\le n}(u_i-u_j)\prod_{1\le i<j\le n}(u_i+u_j)}.$$

For $g=(g_1,g_2,\dots,g_n)\in F$ and $h\in V_{\mathbb{Z}}\cap\mathcal{C}(\mathcal{B}_n)$ the Kostant fraction (3) is the function on U defined by:

$$\begin{aligned}F_B(g,h)(u) &= \frac{\prod_{i=1}^n g_i^{h_i}e^{\sum_{i=1}^n u_ih_i}}{\prod_{i=1}^n(1-g_i^{-1}e^{-u_i})\times\prod_{1\le i<j\le n}(1-g_i^{-1}g_je^{-(u_i-u_j)})}\\ &\quad\times\frac{1}{\prod_{1\le i<j\le n}(1-g_i^{-1}g_j^{-1}e^{-(u_i+u_j)})}.\end{aligned}$$

We have then

Theorem 8.4. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{B}_n)$.*

- *For any $v\in\overline{\mathfrak{c}}$, we have*
$$\mathrm{vol}_{\mathbb{Z},\mathcal{B}_n}(v)=\mathrm{JK}_{\mathfrak{c}}\left(J_B(v)\right).$$
- *For any $h\in V_{\mathbb{Z}}\cap\overline{\mathfrak{c}}$, the value of the partition function is given by:*
$$N_{\mathcal{B}_n}(h)=\sum_{g\in F}\mathrm{JK}_{\mathfrak{c}}(F_B(g,h)).$$

As in the case of A_n, we will use the change of variable $1+z_i=e^{u_i}$ to compute more easily $N_{\mathcal{B}_n}(h)$. However, let us note that this transformation does not leave the hyperplane $u_i+u_j=0$ fixed. This hypersurface is transformed into the hypersurface $z_i+z_j+z_iz_j=0$. So we use the expression of $\mathrm{JK}_{\mathfrak{c}}$ as an integral over the cycle $H(\mathfrak{c})$ defined in Theorem 4.14. This cycle (its homology class) is stable by the transformation $e^{u_i}=1+z_i$ which is close to the identity. Thus define the following function on U:

$$\begin{aligned}\mathcal{F}_B(g,h)(z) &= \frac{\prod_{i=1}^n(1+z_i)^{h_i+2n-i-1}\times\prod_{i=1}^n g_i^{h_i}}{\prod_{i=1}^n(1+z_i-g_i)\times\prod_{1\le i<j\le n}(1+z_i-g_ig_j(1+z_j))}\\ &\quad\times\frac{1}{\prod_{1\le i<j\le n}(1+z_i)(1+z_j)-g_ig_j}.\end{aligned}$$

Performing the change of variables $e^{u_i}=1+z_i$ on the function $F_B(g,h)(u)$ and computing the Jacobian, Theorem 3.3 becomes:

Theorem 8.5. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{B}_n)$.*

- *For any $v \in \overline{\mathfrak{c}}$, we have*
$$\mathrm{vol}_{\mathbb{Z},\mathcal{B}_n}(v) = \mathrm{JK}_{\mathfrak{c}}\left(J_B(v)\right).$$
- *For any $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$, the value of the partition function is given by:*
$$N_{\mathcal{B}_n}(h) = \sum_{g\in F} \frac{1}{(2\pi\sqrt{-1})^n} \int_{H(\mathfrak{c})} \mathcal{F}_B(g,h)(z)dz.$$

As in the case of type A, in order to implement these formulae we first have to describe the set $\mathcal{P}(v, \mathcal{B}_n)$ (Section 8.2), then we will explain how the integral over the cycle $H(\mathfrak{c})$ is calculated similarly to an iterated residue formula associated to these paths (Section 8.3), using an estimate of the order of poles. Finally we explain how these computations fit together to get a global procedure for Kostant partition function for B_n (Section 8.4).

8.2. **The search for maximal proper nested sets.** A height function is
$$\mathrm{ht}(v) = \sum_{i=1}^{n}(n+1-i)v_i$$
which takes value 1 on all simple roots. We will deform ht later on in order to have a function taking different values on roots.

We now proceed to describe hyperplanes for $\mathcal{B}_n$. If $P = [P^+, P^-]$ are two disjoints subsets of $\{1, 2, \ldots, n\}$, we denote by $\langle u_P, v\rangle$ the linear form $\sum_{i\in P^+} v_i - \sum_{j\in P^-} v_j$. Consider the hyperplane
$$H_P = \{v \in V, \langle u_P, v\rangle = 0\}$$
in V. It is equal to the hyperplane determined by the reverse list $[P^-, P^+]$. Thus to each set $P = \{P^+, P^-\}$ of two disjoint sets P^+, P^- such that at least one is non empty, we associate a hyperplane H_P.

We denote by Z the complement of $P^+ \cup P^-$ in $\{1, 2, \ldots, n\}$ and by $\mathcal{B}(Z)$ the subset of $\mathcal{B}_n$ defined by
$$\mathcal{B}(Z) = \{e_i \,|\, i \in Z\} \cup \{e_i \pm e_j \,|\, 1 \le i < j \le n\,;\, i, j \in Z\}.$$
This is the positive root system $B_{|Z|}$, with the positivity induced by the lexicographic order.

Let $\mathcal{K}(P^+, P^-)$ be the subset of $\mathcal{B}_n$ defined by
$$\begin{aligned} &\{e_i - e_j \,|\, 1 \le i < j \le n\,;\, i, j \in P^+\} \\ \cup\ &\{e_i + e_k \,|\, i \in P^+, k \in P^-\} \\ \cup\ &\{e_k - e_\ell \,|\, 1 \le k < \ell \le n;\, k, \ell \in P^-\}. \end{aligned}$$
Note that by defining $f_i = e_i$ if $i \in P^+$ and $f_k = -e_{|P^-|-k+1}$ if $k \in P^-$, the set $\mathcal{K}(P^+, P^-)$ coincides with

$$\begin{array}{rl} & \{f_i - f_j \,|\, 1 \leq i < j \leq n;\ i,j \in P^+\} \\ \cup & \{f_i - f_k |\ i \in P^+, k \in P^-\} \\ \cup & \{f_k - f_\ell \,|\, 1 \leq k < \ell \leq n;\ k,\ell \in P^-\}. \end{array}$$

Thus the set $\mathcal{K}(P^+,P^-)$ is a positive root system of type $A_{|P^+|+|P^-|-1}$. However the positivity is induced by the lexicographic order on P^+ and the reverse lexicographic order on P^-. Observe also that H_P is the vector space spanned by $\mathcal{K}(P^+,P^-) \cup \mathcal{B}(Z)$.

Lemma 8.6.
- *The hyperplane H_P is a $\mathcal{B}_n$-admissible hyperplane.*
- *The set $\mathcal{B}_n \cap H_P$ is the union of $\mathcal{B}(Z)$ and $\mathcal{K}(P^+,P^-)$.*
- *Every $\mathcal{B}_n$-admissible hyperplane is of this form.*

Proof. The first two assumptions are easy to see. We prove the third assumption by induction on n, the case $n = 2$ being trivial. Let H be a $\mathcal{B}_n$-admissible hyperplane and let α be a root in H. There are 3 possibilities for α: up to renumbering roots, we can consider the cases $\alpha = e_n$, $\alpha = e_{n-1} - e_n$ and $\alpha = e_{n-1} + e_n$.

In the first case, the map q sending e_i to e_i if $i < n$ and e_n to 0 maps the set $\mathcal{B}_n \setminus \{\alpha\}$ to $\mathcal{B}_{n-1}$. The space $H/\mathbb{R}\alpha$ becomes a $\mathcal{B}_{n-1}$-admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$, where P'^+ and P'^- are two disjoint sets contained in $\{1,2,\ldots,n-1\}$. Then the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$.

In the second case, the map q sending e_i to e_i if $i < n$ and e_n to e_{n-1} sends the set $\mathcal{B}_n \setminus \{\alpha\}$ to $\mathcal{B}_{n-1}$. The space $H/\mathbb{R}\alpha$ becomes a $\mathcal{B}_{n-1}$-admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$. If neither P'^+ nor P'^- contain $n-1$, the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$. Otherwise assume that for example P'^+ contains $n-1$. Then the hyperplane H is equal to the hyperplane determined by $[P^+, P^-]$, where $P^+ = P'^+ \cup \{n\}$ and $P^- = P'^-$.

In the third case, the map q sending e_i to e_i if $i < n$ and e_n to $-e_{n-1}$ sends the set $\mathcal{B}_n \setminus \{\alpha\}$ to $\mathcal{B}_{n-1}$. The space $H/\mathbb{R}\alpha$ becomes a $\mathcal{B}_{n-1}$-admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$. If neither P'^+ nor P'^- contains $n-1$, the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$. Assume that P'^+ contains $n-1$. Then the hyperplane H is equal to the the hyperplane determined by $[P^+, P^-]$, where $P^+ = P'^+$ and $P^- = P'^- \cup \{n\}$. ☐

We now give a detailed description of our algorithm computing maximal nested sets. We describe a vector as an array $v = [v_1, v_2, \ldots, v_n]$. To check if v is in the cone $\mathcal{C}(\mathcal{B}_n)$, we need to verify if $\sum_{j=1}^{i} v_j \geq 0$ for $1 \leq i \leq n$. This is done by the procedure `CheckBvector`, which returns the answer true or false.

For the system $\mathcal{B}_n$ the highest root $\theta^B(n)$ is equal to

$$\theta^B(n) = [1, 1, 0, 0, 0, \dots, 0].$$

We recall here that P is divided in two sets $P^+ \cup P^-$, one of them being non empty. The first task is to list the hyperplanes in $\mathrm{Hyp}(v, \mathcal{B}_n)$. This set of hyperplanes is obtained by the command line `AllPossibleBwalls(v)`. The input of this procedure is the vector v. The output is a set of elements $P = \{P^+, P^-\}$, where $P^+ = [i_1, i_2, \dots i_p]$ and $P^- = [j_1, j_2, \dots, j_q]$ are two ordered disjoint lists made from indices taken in $\{1, \dots, n\}$, with at least one of P^+ or P^- being non empty. Let $\langle u_P, v\rangle = \sum_{i\in P^+} v_i - \sum_{j\in P^-} v_j$ be the normal vector to H_P. Then as stated in Lemma 5.2 we need to test if $\langle u_P, \theta^B(n)\rangle$ is not zero and if $\langle u_P, v\rangle \times \langle u_P, \theta^B(n)\rangle$ is non negative.

We then construct the vector

$$\mathrm{proj}_H(v) = v - \frac{\langle u_P, v\rangle}{\langle u_P, \theta^B(n)\rangle}\theta^B(n).$$

This vector is represented as $\{[v_1, P^+], [v_2, P^-], [w, Z]\}$. The sum of coordinates of v_1 is equal to the sum of the coordinates of v_2. Now Z is the ordered list $[k_1, k_2, \dots, k_\ell]$ of complementary indices to $P^+ \cup P^-$ and

$$w = [\mathrm{proj}_H(v)[k_1], \dots, \mathrm{proj}_H(v)[k_\ell]].$$

Note that the equations of the cone $\mathcal{C}(\mathcal{K}(P^+, P^-))$ can be given in the convenient form $v_1 \oplus v_2 \in \mathcal{C}(\mathcal{K}(P^+, P^-))$ if and only if $\mathtt{CheckBvector(v_1)}$ and $\mathtt{CheckBvector(v_2)}$ are true. Equations of the cone $\mathcal{C}(\mathcal{B}(Z))$ are given in the form $w \in \mathcal{C}(\mathcal{B}(Z))$ if and only if `CheckBvector(w)` is true.

Thus the condition that H is in $\mathrm{Hyp}(v, \mathcal{B}_n)$ is equivalent to the series of conditions:

$$\begin{array}{rcl} \langle u_P, \theta^B(n)\rangle & \neq & 0, \\ \langle u_P, v\rangle \times \langle u_P, \theta^B(n)\rangle & \geq & 0, \\ \mathtt{CheckBvector(v_1)} & = & \mathtt{true}, \\ \mathtt{CheckBvector(v_2)} & = & \mathtt{true}, \\ \mathtt{CheckBvector(w)} & = & \mathtt{true}. \end{array}$$

Those five conditions are checked by the command line `CheckBwall(v,H)`, that gives an answer true or false.

Remark 8.7. We can first construct all disjoint subsets P^+, P^- of $\{1, 2, \dots, n\}$ and test these five conditions successively on all of them. However it is highly desirable to throw away *a priori* a great number of these partitions by noticing the following restrictive conditions on the possible lists to be considered.

Let $\{P^+, P^-\} = \{[i_1, i_2, \dots i_p], [j_1, j_2, \dots, j_q]\}$ be a set of two disjoint subset of $\{1, 2, \dots n\}$ represented as lists with strictly increasing indices. Let $Z = [k_1, k_2 \dots, k_\ell]$ be the list of complementary indices to $P^+ \cup P^-$ in

$\{1, \ldots, n\}$. The following linear forms are positive on the cone $\mathcal{C}(\mathcal{K}(P^+, P^-))$ generated by $\mathcal{K}(P^+)$ and $\mathcal{K}(P^-)$:

$$\begin{array}{rcll} v_{i_1} + v_{i_2} + \cdots + v_{i_s} & \geq & 0 & \text{for all } 1 \leq s \leq p, \\ v_{j_1} + v_{j_2} + \cdots + v_{j_t} & \geq & 0 & \text{for all } 1 \leq t \leq q, \\ v_{k_1} + v_{k_2} + \cdots + v_{k_s} & \geq & 0 & \text{for all } 1 \leq s \leq \ell. \end{array}$$

Note that all the above linear forms take positive values on $\theta^B(n)$. We employ Lemma 8.6. Thus if $v[i_1] < 0$, the index i_1 cannot start the list P^+ of an element $\{P^+, P^-\}$ in `AllPossibleBwalls(v)` and we reject all such $\{P^+, P^-\}$.

Similarly assume that we have constructed a list of indices $[i_1, i_2]$ satisfying conditions $v[i_1] \geq 0$ and $v[i_1] + v[i_2] \geq 0$. Then if $v[i_1] + v[i_2] + v[i_3] < 0$, a list starting with $[i_1, i_2, i_3]$ cannot be the first three indices of the component P^+ of an element $\{P^+, P^-\}$ in the set `AllPossibleBwalls(v)` and we skip it right away.

This achieves the description of the procedure `AllPossibleBwalls`. We now have to perform the next step of our algorithm. As for type A we build MNSs iteratively. At each step we get a set of partial MNSs, to which we will apply recursively our algorithm. Note that after Lemma 8.6 the intersection of a $\mathcal{B}_n$-admissible hyperplane H_P with $\mathcal{B}_n$ is the union of a system of type A and a system of type B.

The part of the MNS coming from the subsystem of type A is computed with the procedure `AddAnests`. It performs a reordering of the result of a call to the procedure `MNSs` described in Section 7.2.

The part of the MNS coming from the subsystem of type B is computed with the procedure `Bsplits`, calling the previously described procedure `AllPossibleBwalls`.

Procedures `AddAnests` and `Bsplits` are enclosed in `MoreNSs`, thus giving a new iteration of the process. After regularization of the result we hence get a procedure named `AllNewNSs`, performing a new step in the building of MNSs.

Finally the procedure `B_MNSs`, computing MNSs for a given vector v for type B, is the following. First, we use a procedure named `B_NS1` to calculate the first seed of all MNSs. After, repeated calls to the procedure `AllNewNSs` build the desired MNSs.

8.3. **Residues associated to maximal proper nested sets.** A proper maximal nested set M gives rise to an ordered basis α_i, and a cycle $H(M)$. We need to compute

$$\int_{H(M)} \mathcal{F}_B(g, h)(z) dz$$

where

$$H(M) := \{z, |\langle \alpha_i, z\rangle| = \epsilon_i\}.$$

The function $z \mapsto \mathcal{F}_B(g,h)(z)$ is deduced from the function $F_B(g,h)(u)$ in the space $\widehat{R}_{\mathcal{A}}$ by the change of variable $e^{u_i} = 1+z_i$. Thus its denominator is a product of factors, either of the form z_i corresponding to the root u_i, or of the form $z_i - z_j$ corresponding to the root $u_i - u_j$ or $z_i + z_j + z_i z_j$ corresponding to the root $u_i + u_j$. We denote by $u(z)$ the point with coordinates u_i satisfying $e^{u_i} = 1 + z_i$.

We start integrating our function $\mathcal{F}_B(g,h)(z)$ over the smaller circle $|\langle \alpha_1, z\rangle| = \epsilon_1$ keeping the other variables fixed. By our condition on the cycle, the function we integrate has poles on the domain $|\langle \alpha_1, z\rangle| \leq \epsilon_1$ only when $\alpha_1(u(z)) = 0$. If $\alpha_1(u(z)) = u_i - u_j$ or $\alpha_1(u(z)) = u_i$, the poles are obtained for $z_i = z_j$ or $z_i = 0$. If $\alpha_1(u(z)) = u_i + u_j$, the pole on the domain $|\langle \alpha_1, z\rangle| \leq \epsilon_1$ is obtained for $z_i = -z_j/(1+z_j)$. Thus we compute the integral over the circle by the residue theorem in one variable, and proceed. From the general theory, the poles of the function we obtain, replacing z_i by one of the values above are again of the same form with respect to the remaining variables, as is easily checked.

As in case A_{n-1}, for a root $\alpha = u_i$ (resp. $\alpha = u_i \pm u_j$) we can replace after taking the residue at $\alpha = 0$ the variable z_i by 0 (resp. by $\mp z_j$) in all other roots. Thus we get rid of the variable z_i. The procedure `FormalPathB` produces the ordered path resulting from all these substitutions.

In the case of type B we compute the residue by directly checking the order of the pole at $\alpha = 0$, and then using differentiation. The program works in the same way with parameters. The function obtained is locally polynomial with polynomial coefficients depending of the parity of the integers h_i.

8.4. **The procedure `MNS_KostantB`.** We finish the section dedicated to B_n by giving the global outline of the procedure `MNS_KostantB(v)` computing the Kostant partition number of a vector v lying in the root lattice of $\mathcal{B}_n$. We begin by slightly deforming v so that it lies on no wall, by setting `v' := DefVectorB(v, n)`. We then compute all MNSs for v' with the call `B_MNSs(v')` (Section 8.2). For every MNS M, we extract the list R of highest roots of its irreducible components by setting `R := BthetaMNS(M)`. We sort these roots by their height with the command line `R' := BorderThetas(R, n)`. We then transform the list of roots R' into a path p by setting `p := FormalPathB(R')`.

Now remark that our procedures are designed to take residues along positive roots, using the fact that $\mathrm{res}_{-\alpha} = -\mathrm{res}_{\alpha}$ for any root α. The sign that appears (more precisely -1 to the power the number of negative roots in the path p) is computed with the procedure `PathSign(p, n)`.

Then for every g in F we do the following. The iterated residue along the path p and for g is obtained by the command line `OneIteratedBresidue(p, g, v, n)`. Let us briefly describe its implementation. We first compute the Kostant fraction (second item of Definition 7.1, procedure `KostantFunctionB`). Then for every root of the path we apply the procedure `ComputeOneResidue`

(Section 8.3) and update the order of the pole with a procedure named `OrderPoleB`.

Finally summing all products $\mathtt{PathSign}(\mathtt{p},\mathtt{n})\times\mathtt{OneIteratedBresidue}(\mathtt{p},\mathtt{g},\mathtt{v},\mathtt{n})$ over the sets of g's and of M's, we get the desired result.

Remark 8.8. Let us fix a list $R' = [\alpha_1, \dots, \alpha_n]$ of ordered roots coming from a MNS, and an element g. We say that R' and g are *compatible* if the following condition is satisfied. If indices of monomial(s) of α_k have not yet occured among indices of roots α_ℓ with $\ell < k$, then g must satisfy $g^{\alpha_k} = 1$ (that is $g_i g_j^{\pm 1} = 1$ if $\alpha_k = e_i \pm e_j$ and $g_i = 1$ if $\alpha_k = e_i$). Note that the iterated residue for g and for the path p associated to R' is zero if g and R' are not compatible. Hence summing only over g's that are compatible with a given list R' saves useless computations. The check of compatibility is performed by the procedure $\mathtt{ListAndGAreCompatible}(\mathtt{R}',\mathtt{g},\mathtt{n})$.

9. The type C_n

Consider a vector space V with basis $e_1, e_2, \dots, e_n$. We choose on V the standard Lebesgue measure dh. Let

$$\mathcal{C}_n = \{2e_i \,|\, 1 \leq i \leq n\} \cup \{e_i - e_j \,|\, 1 \leq i < j \leq n\} \cup \{e_i + e_j \,|\, 1 \leq i < j \leq n\}.$$

Then $\mathcal{C}_n$ is a positive roots system of type C_n, and generates V. The number of elements in $\mathcal{C}_n$ is $N = n^2$. Note that elements of $\mathcal{C}_n$ and $\mathcal{B}_n$ are proportional, so they determine the same hyperplane arrangement and the same chambers.

Let L be the lattice defined by $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$. We remark that the lattice $V_{\mathbb{Z}}$ generated by $\mathcal{C}_n$ is the sublattice of index 2 in L consisting of all elements $v = [v_1, v_2, \dots, v_n]$ with integral coordinates and such that the sum $\sum_{i=1}^n v_i$ is an even integer. A $\mathbb{Z}$-basis of $V_{\mathbb{Z}}$ is, for example,

$$\mathbb{Z}(e_1 - e_n) \oplus \mathbb{Z}(e_2 - e_n) \oplus \cdots \oplus \mathbb{Z}(e_{n-1} - e_n) \oplus \mathbb{Z}(2e_n)\ ,$$

so $\mathrm{vol}\,(V/V_{\mathbb{Z}}) = 2$.

The dual lattice $U_{\mathbb{Z}}$ is the lattice of vectors $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ such that γ_i are half integers and such that $\gamma_i + \gamma_j$ is an integer for all i, j. The set $U_{\mathbb{Z}}/\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ is of cardinality 2 with representative elements $(0, 0, \dots, 0, 0)$ and $(1/2, \dots, 1/2)$.

As before, we identify the torus $\tilde{T} = U/(\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n) = \mathbb{R}^n/\mathbb{Z}^n$ with $(S^1)^n$ by

$$(u_1, u_2, \dots, u_n) \mapsto \left(e^{2\pi\sqrt{-1}u_1}, \dots, e^{2\pi\sqrt{-1}u_n}\right).$$

Then

$$T = \tilde{T}/\{\pm 1\} = U/U_{\mathbb{Z}}.$$

Let G be a representative of $g = (g_1, g_2, \dots, g_n) \in \tilde{T}$ and $h = \sum_{i=1}^n h_i e_i$ in $V_{\mathbb{Z}}$. Then $e^{\langle h, 2\pi\sqrt{-1}G\rangle}$ is equal to $\prod_{i=1}^n g_i^{h_i} = g^h$. This function is well defined on $T = \tilde{T}/\{\pm 1\}$ since $\sum_{i=1}^n h_i$ is even.

For σ a basic subset of $\mathcal{C}_n$, define

$$\tilde{T}(\sigma) = \left\{ g \in \tilde{T} \,\middle|\, e^{\langle \alpha, 2\pi\sqrt{-1}G\rangle} = 1 \text{ for all } \alpha \in \sigma \right\}.$$

As the set $\mathcal{C}_n$ is not unimodular, sets $\tilde{T}(\sigma)$ are not reduced to 1.

Lemma 9.1. *Let σ be a basic subset of $\mathcal{C}_n$. Then $\tilde{T}(\sigma) \subset \{\pm 1\}^n$.*

Proof. We prove by induction on n that if σ is basic then the condition $g = (g_1, \ldots, g_n) \in \tilde{T}(\sigma)$ forces $g_i^2 = 1$ $(1 \le i \le n)$. In other words $g^\alpha = 1$ for all long roots α. If so then $g_i = \pm 1$ for all i. The base of the induction, that is $\mathcal{C}_2$, is straightforward and we omit it. We thus proceed considering various possibilities for our σ.

If there exists a long root in σ we may assume that this long root is $2e_n$. We embed the system $\mathcal{C}_{n-1}$ in $\mathcal{C}_n$ via the first $(n-1)$ coordinates. Then the basis σ of $\mathcal{C}_n$ produces a basis σ' of $\mathcal{C}_{n-1}$ consisting of roots $\{e_i \pm e_j \in \sigma \,|\, 1 \le i < j \le n-1\}$, of roots $\{2e_i \in \sigma \,|\, 1 \le i \le n-1\}$, and of roots $\{2e_i \,|\, e_i \pm e_n \in \sigma; i \ne n\}$. It is easy to see that the elements $(g_1, g_2, \ldots, g_{n-1})$ are in $\tilde{T}(\sigma')$. Indeed $g_i^2 = 1$ if $e_i \pm e_n \in \sigma$ as $g_i g_n^{\pm 1} = 1$ and $g_n^2 = 1$; and similarly $g_i^2 = 1$ if $2e_i \in \sigma$. Thus by induction we obtain $g_i^2 = 1$ for every i.

Now assume that there is no long root in σ. We may assume that there is a root of the form $e_{n-1} - e_n$ or $e_{n-1} + e_n$.

In the first case, consider the basis σ' of $\mathcal{C}_{n-1}$ consisting of the roots $\{e_i \pm e_j \in \sigma \,|\, 1 \le i < j \le n-1\}$ and of the roots $\{e_i \pm e_{n-1} \,|\, e_i \pm e_n \in \sigma\}$. It is easy to see that the elements $(g_1, g_2, \ldots, g_{n-1})$ are in $\tilde{T}(\sigma')$. Indeed, for example, $g_i g_{n-1}^{\pm 1} = 1$ if $e_i \pm e_{n-1} \in \sigma'$, as $g_i g_n^{\pm 1} = 1$ and $g_n = g_{n-1}$. Thus by the induction hypothesis we obtain $g_i^2 = 1$ for all $i \ne n$. Since $g_n = g_{n-1}$, we also obtain $g_n^2 = 1$.

The second case is similar. $\square$

Let $v = \sum_{i=1}^n v_i e_i \in V$ be a vector with real coordinates and $h = \sum_{i=1}^n h_i e_i \in V$ a vector with integral coordinates and such that $\sum_{i=1}^n h_i$ is even. We will compute the normalized volume of $\Pi_{\mathcal{C}_n}(v)$ and the number of integral points in $\Pi_{\mathcal{C}_n}(h)$ using Theorem 3.3. We will use the JK residue with respect to the measure dh associated to the basis $e_1, e_2, \ldots, e_n$. However, the normalized volume $\mathrm{vol}_{\mathbb{Z},\mathcal{C}_n}(h)$ is computed for the measure determined by the lattice spanned by $\mathcal{C}_n$ which is of index 2 in $\oplus_{i=1}^n \mathbb{Z} e_i$.

We introduce the function $J_C(v)$ on U defined by:

$$J_C(v)(u) = \frac{e^{\sum_{i=1}^n u_i v_i}}{\prod_{i=1}^n 2u_i \prod_{1 \le i < j \le n}(u_i - u_j) \prod_{1 \le i < j \le n}(u_i + u_j)}.$$

For $g = (g_1, g_2, \ldots, g_n) \in \{\pm 1\}^n$ the Kostant fraction (3) is the function on U defined by:

$$F_C(g,h)(u) = \frac{\prod_{i=1}^n g_i^{h_i} e^{\sum_{i=1}^n u_i h_i}}{\prod_{i=1}^n (1-e^{-2u_i}) \times \prod_{1\leq i<j\leq n}(1-g_i^{-1}g_j e^{-(u_i-u_j)})} \times \frac{1}{\prod_{1\leq i<j\leq n}(1-g_i^{-1}g_j^{-1}e^{-(u_i+u_j)})}.$$

Theorem 9.2. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{C}_n)$.*

- *For any $v \in \overline{\mathfrak{c}}$, we have*

$$\mathrm{vol}_{\mathbb{Z},\mathcal{C}_n}(v) = 2\ \mathrm{JK}_{\mathfrak{c}}\left(J_C(v)\right).$$

- *For any vector $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$ with integral coordinates such that $\sum_{i=1}^n h_i$ is even, the value of the partition function is given by:*

$$N_{\mathcal{C}_n}(h) = \sum_{g\in\{\pm1\}^n} \mathrm{JK}_{\mathfrak{c}}(F_C(g,h)).$$

In the second formula, there should be a multiplication by a factor 2 as the volume of the fundamental domain of the lattice spanned by $\mathcal{C}_n$ is 2. However, we should sum only on $T = \tilde{T}/\{\pm1\}$. Thus the two factors of 2 compensate each other. In fact, we will indeed sum over T represented as $\{\pm1\}^{n-1} \times \{1\}$ and multiply the result by the constant 2.

As in the case of B_n, we will use the change of variable $1 + z_i = e^{u_i}$ to compute more easily the formula for $N_{\mathcal{C}_n}(h)$. As explained in the case of B_n we need to use the integral formulation of the Jeffrey-Kirwan residue. Thus define

$$\mathcal{F}_C(g,h)(z) = \frac{\prod_{i=1}^n (1+z_i)^{h_i+2n-i} \times \prod_{i=1}^n g_i^{h_i}}{\prod_{i=1}^n ((1+z_i)^2-1) \times \prod_{1\leq i<j\leq n}(1+z_i - g_i g_j(1+z_j))} \times \frac{1}{\prod_{1\leq i<j\leq n}(1+z_i)(1+z_j) - g_i g_j}.$$

Performing the change of variables $e^{u_i} = 1 + z_i$ on the function $F_C(g,h)(u)$ and computing the Jacobian, Theorem 3.3 becomes:

Theorem 9.3. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{C}_n)$.*

- *For any $v \in \overline{\mathfrak{c}}$, we have*

$$\mathrm{vol}_{\mathbb{Z},\mathcal{C}_n}(v) = 2\ \mathrm{JK}_{\mathfrak{c}}\left(J_C(v)\right).$$

- *For any vector $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$ with integral coordinates h_i with $\sum_{i=1}^n h_i$ even, the value of the partition function is given by:*

$$N_{\mathcal{C}_n}(h) = \sum_{g\in\{\pm1\}^n} \frac{1}{(2\pi\sqrt{-1})^n} \int_{H(\mathfrak{c})} \mathcal{F}_C(g,h)(z)dz.$$

Similarly we will sum over T represented as $\{\pm 1\}^{n-1} \times \{1\}$ and multiply the result by the constant 2.

The cycle $H(\mathfrak{c})$ associated to a chamber $\mathfrak{c}$ containing a regular element $v = \sum_{i=1}^{n} v_i e_i$ is the same cycle that we computed in the preceding section for B_n . Hence we can reuse most of procedures from the type B_n. Paths are the same, and the residue calculations are the same. More precisely, the only two changes are in the computation of the set G (procedure `GC(n)`) and in the computation of the Kostant function (procedure `UCKostant`). This terminates the case of C_n.

10. The type D_n

10.1. **The formulae to be implemented.** Consider a vector space V with basis $e_1, e_2, \ldots, e_n$. We choose the standard Lebesgue measure dh. Let

$$\mathcal{D}_n = \{e_i - e_j \,|\, 1 \leq i < j \leq n\} \cup \{e_i + e_j \,|\, 1 \leq i < j \leq n\}.$$

Then $\mathcal{D}_n$ is a positive roots system of type D_n, and generates V. The number of elements in $\mathcal{D}_n$ is $N = n^2 - n$.

We remark that the lattice $V_{\mathbb{Z}}$ generated by roots of $\mathcal{D}_n$ is the same lattice as the one generated by the roots of $\mathcal{C}_n$. It is of index 2 in $L := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$ and consists of elements $v = [v_1, v_2, \ldots, v_n]$ with integral coordinates such that the sum $\sum_{i=1}^{n} v_i$ is an even integer. The group $T = U/U_{\mathbb{Z}}$ is thus the quotient of $\tilde{T} = U/\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, obtained by identifying g and $-g$, that is $T = \tilde{T}/\{\pm 1\}$. As in Section 9, we identify the torus $\tilde{T} = U/(\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n) = \mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$ by

$$(u_1, u_2, \ldots, u_n) \mapsto \left(e^{2\pi\sqrt{-1}u_1}, \ldots, e^{2\pi\sqrt{-1}u_n}\right).$$

Consider the set $F = \{\pm 1\}^n \subset (S^1)^n$. For σ a basic subset of $\mathcal{D}_n$, define

$$\tilde{T}(\sigma) = \left\{ g \in \tilde{T} \,\middle|\, e^{\langle \alpha, 2\pi\sqrt{-1}G \rangle} = 1 \text{ for all } \alpha \in \sigma \right\}.$$

Lemma 10.1. *Let σ be a basic subset of $\mathcal{D}_n$. Then $\tilde{T}(\sigma)$ is contained in F.*

Proof. Basic subsets of $\mathcal{D}_n$ are basic subsets of $\mathcal{C}_n$ so that we can choose the same set $F = \{\pm 1\}^n$. □

Let $v = \sum_{i=1}^{n} v_i e_i \in V$ be a vector with real coordinates and $h = \sum_{i=1}^{n} h_i e_i \in V$ a vector with integral coordinates and such that $\sum_{i=1}^{n} h_i$ is even. We will compute the normalized volume of $\Pi_{\mathcal{D}_n}(v)$ and the number of integral points in $\Pi_{\mathcal{D}_n}(h)$ using Theorem 3.3.

Thus we introduce the function $J_D(v)$ on U defined by:

$$J_D(v)(u) = \frac{e^{\sum_{i=1}^{n} u_i v_i}}{\prod_{1 \leq i < j \leq n}(u_i - u_j) \prod_{1 \leq i < j \leq n}(u_i + u_j)}.$$

For $g = (g_1, g_2, \dots, g_n) \in \{\pm 1\}^n$ the Kostant fraction (3) is the function on U defined by:

$$\begin{aligned} F_D(g,h)(u) &= \frac{\prod_{i=1}^n g_i^{h_i} \times e^{\sum_{i=1}^n u_i h_i}}{\prod_{1\le i<j\le n}(1 - g_i^{-1} g_j e^{-(u_i-u_j)})} \\ &\quad \times \frac{1}{\prod_{1\le i<j\le n}(1 - g_i^{-1} g_j^{-1} e^{-(u_i+u_j)})}. \end{aligned}$$

We have then

Theorem 10.2. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{D}_n)$.*

- *For any $v \in \overline{\mathfrak{c}}$, we have*

$$\mathrm{vol}_{\mathbb{Z},\mathcal{D}_n}(v) = 2\ \mathrm{JK}_{\mathfrak{c}}\left(J_D(v)\right).$$

- *For any vector $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$ with integral coordinates such that $\sum_{i=1}^n h_i$ is even, the value of the partition function is given by:*

$$N_{\mathcal{D}_n}(h) = \sum_{g\in\{\pm1\}^n} \mathrm{JK}_{\mathfrak{c}}(F_D(g,h)).$$

We use the change of variable $1 + z_i = e^{u_i}$ to compute more easily the formula for $N_{\mathcal{D}_n}(h)$ and thus introduce integration over a cycle. Thus define

$$\begin{aligned} \mathcal{F}_D(g,h)(z) &= \frac{\prod_{i=1}^n (1+z_i)^{h_i+2n-i-2} \times \prod_{i=1}^n g_i^{h_i}}{\prod_{1\le i<j\le n}(1 + z_i - g_i g_j (1+z_j))} \\ &\quad \times \frac{1}{\prod_{1\le i<j\le n}(1+z_i)(1+z_j) - g_i g_j}. \end{aligned}$$

After performing the change of variables $e^{u_i} = 1 + z_i$ on the function $F_D(g,h)(u)$ and after computing the Jacobian, Theorem 3.3 becomes:

Theorem 10.3. *Let $\mathfrak{c}$ be a chamber of $\mathcal{C}(\mathcal{D}_n)$.*

- *For any $v \in \overline{\mathfrak{c}}$, we have*

$$\mathrm{vol}_{\mathbb{Z},\mathcal{D}_n}(v) = 2\ \mathrm{JK}_{\mathfrak{c}}\left(J_D(v)\right).$$

- *For any vector $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$ with integral coordinates h_i such that $\sum_{i=1}^n h_i$ is even, the value of the partition function is given by:*

$$N(\mathcal{D}_n, h) = \sum_{g\in\{\pm1\}^n} \frac{1}{(2\pi\sqrt{-1})^n} \int_{H(\mathfrak{c})} \mathcal{F}_D(g,h)(z)dz.$$

As for types A and B, in order to implement these formulae we first have to describe the set $\mathcal{P}(v, \mathcal{D}_n)$ (Section 10.2). We finish to explain the implementation of case D in Section 10.3, using the fact that types B and D are similar.

10.2. **The search for maximal proper nested sets.** A height function is
$$\mathrm{ht}(v) = \sum_{i=1}^{n} (n-i) v_i$$
which takes value 1 on all simple roots. We will deform it later on in order to have a function taking different values on roots.

We now proceed to describe hyperplanes for D_n. If $P = [P^+, P^-]$ are two disjoints subsets of $\{1, 2, \ldots, n\}$, we denote by $\langle u_P, v\rangle$ the linear form $\sum_{i \in P^+} v_i - \sum_{j \in P^-} v_j$. Consider the hyperplane in V defined by
$$H_P = \{v \in V, \langle u_P, v\rangle = 0\}$$
and remark that it is equal to the hyperplane determined by the reverse list $[P^-, P^+]$. Thus to each set $P = \{P^+, P^-\}$ of two disjoint sets P^+, P^- such that at least one is non empty, is associated a hyperplane H_P.

We denote by Z the complement of $P^+ \cup P^-$ in $\{1, 2, \ldots, n\}$ and by $\mathcal{D}(Z)$ the subset of $\mathcal{D}_n$ defined by
$$\mathcal{D}(Z) = \{e_i \pm e_j \mid 1 \le i < j \le n;\ i, j \in Z\}.$$
This is the positive roots system of type $D_{|Z|}$, with the positivity induced by the lexicographic order.

Let $\mathcal{K}(P^+, P^-)$ be the subset of $\mathcal{D}_n$ defined by
$$\begin{aligned} & \{e_i - e_j \mid 1 \le i < j \le n;\ i, j \in P^+\} \\ \cup\ & \{e_i + e_k \mid i \in P^+, k \in P^-\} \\ \cup\ & \{e_k - e_\ell \mid 1 \le k < \ell \le n;\ k, \ell \in P^-\}. \end{aligned}$$
As we observed in Section 8.2 for B_n, by defining $f_i = e_i$ if $i \in P^+$ and $f_k = -e_{|P|-k+1}$ if $k \in P^-$, the set $\mathcal{K}(P^+, P^-)$ is a positive roots system of type $A_{|P^+|+|P^-|-1}$. Here the positivity is induced by the lexicographic order on P^+ and the reverse lexicographic order on P^-.

Observe also that H_P is the vector space spanned by $\mathcal{K}(P^+, P^-) \cup \mathcal{D}(Z)$.

Lemma 10.4.
- *The hyperplane H_P is a $\mathcal{D}_n$-admissible hyperplane.*
- *The set $\mathcal{D}_n \cap H_P$ is the union of $\mathcal{D}(Z)$ and $\mathcal{K}(P^+, P^-)$.*
- *Every $\mathcal{D}_n$-admissible hyperplane is of this form.*

Proof. The first two assumptions are easy to see. Now as $\mathcal{D}_n$ is contained in $\mathcal{B}_n$, a $\mathcal{D}_n$-admissible hyperplane is $\mathcal{B}_n$ admissible, so is of this form. □

10.3. **The procedure `MNS_KostantD`.** Most of procedures from type B_n are kept unchanged. More precisely, the iterated residue calculation, the estimate of the order of poles and the global procedures coordinating computations are exactly the same as for type B_n.

The only serious adaptations to the case of D_n appears in the procedure `CheckDvector(n,v)`. In fact now we check that

$$\begin{aligned} v_1 + \cdots + v_i &\geq 0 \quad \text{for } 1 \leq i \leq n-1, \\ v_1 + \cdots + v_{n-1} + v_n &\geq 0 \quad \text{and is even}, \\ v_1 + \cdots + v_{n-1} - v_n &\geq 0 \quad \text{and is even}. \end{aligned}$$

Other modifications are in procedures that are parent of `CheckDvector`. For example the procedure `CheckDwall` works exactly as `CheckBwall`, but now calls `CheckDvector` instead of `CheckBvector`. See Section 8.2.

11. Performance of the programs

In this Section, we describe several tests of our programs implementing the above MNS algorithms for types A_n, B_n, C_n, D_n. The algorithm implementation is made with Maple. We compare our results with the ones obtained by two previous algorithms:

- The Sp (for *special permutations*) algorithm by Baldoni-DeLoera-Vergne [2], only for A_n.
- The implementation `LattE` of Barvinok's algorithm [17], for every classical algebra;

These two methods also helped us to test our algorithms on various examples.

Note that for our programs most of computation time is spent while computing iterated residues. Indeed MNS computation is fast and efficient. Note also that most of memory used by our programs serves to store all fractions that occur in the iterated residue process. The number of MNSs has a great influence on computation time, since we sum over all MNSs. In any case it seems that the deeper a vector is in the cone generated by positive roots, the higher the number of MNSs is. This is morally bound to the fact that there are more simplicial cones that might contain the vector. In Figure 10, we attach to every chamber $\mathfrak{c}$ for B_3 the number of MNSs associated to any vector $v \in \mathfrak{c}$.

Recall that the Sp algorithm relies on sums over a set $\mathrm{Sp}(a)$ of *special permutations* for a vector a. The main advantage of our algorithms is that we compute fewer iterated residues. In fact the number of MNS seems to be smaller than the number of special permutations that occur, for a given generic example. But, examples at the end of Table 11 show that a number of MNSs considerably smaller than those of Sp's doesn't lead to a better performance in time computation, even in the extreme case of just one MNS. Indeed this one residue computation can be very time consuming due to the substitutions $z_i = z_j$, which takes more time that the substitutions $z_i = 0$ used in the $\mathrm{Sp}(a)$ algorithm. In the near future, we will improve this minor point. The MNSs method should be better and is better in general.

During comparative tests, we figured out that one example in [2] has not been correctly copied from draft. More precisely in their Table 2 for complete graph K_n, for the vector

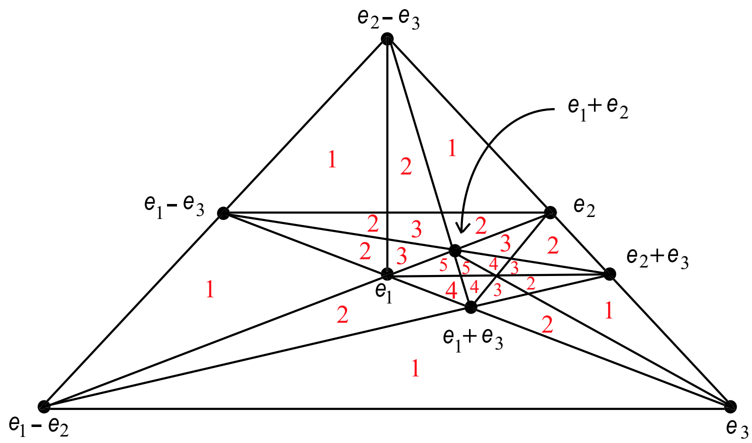

FIGURE 10. Number of MNS containing any vector in a given chamber for B_3

$$a = (82275, 33212, 91868, -57457, 47254, -64616, 94854, -227390)$$

in the root lattice for A_7, the correct Kostant partition number is the 103-digits integer

$$\begin{gathered}226040494681135377722281761934040091356424181 \\ \qquad 2426694976148018460580929729751205803349614264 97\end{gathered}$$

and not only the first line of 45 digits. The Kostant number and Ehrhart polynomials for this a were computed on a 1GHz computer in $2,14$ s and $18,54$ s respectively, using 26 special permutations. Now with our programs running on a $1,13$GHz computer these times drop to $1,38$ s and $2,50$ s respectively, using 14 maximal nested sets. Similarly for the biggest example examined in [2], that is for the vector

$$\begin{aligned}a = (&46398, 36794, 92409, -16156, 29524, -68385, \\ &93335, 50738, 75167, -54015, -285809)\end{aligned}$$

in the root lattice for A_{10}, the 189-digits answer was obtained in 2193 s using 322 special permutations, whereas now we get the same result in 308 s using 109 maximal nested sets.

Table 11 contains respective performances for A_n of `LattE`, Sp algorithms and our programs, a part the last four examples that compare only the last program with ours. Tables 12–14 contain respective performances for B_n, C_n and D_n of `LattE` and our programs. We also indicated the number of special permutations (Sp) and maximal nested sets (MNS).

Tests were performed on Pentium IV 1,13GHz computers with 1500 or 2000 mega-octets (Mo) of RAM memory. We stopped several computations with `LattE` when we figured out that they would overcome computers' memory or take too much time with respects to the other algorithms; in this case we indicate the time spent and the number of mega-octets used by the computer.

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Root lattice element	LattE	Sp	MNS
$(2215, 571, 4553, -600, -6739)$	$1,6\,\mathrm{s}$	$< 0,1\,\mathrm{s}, 4$ Sp	$< 0,1\,\mathrm{s}, 3$ MNS
$(6440, -4866, 6174, -5683, 7112, -9177)$	$2,0\,\mathrm{s}$	$< 0,1\,\mathrm{s}, 4$ Sp	$0,1\,\mathrm{s}, 1$ MNS
$(5067, 3639, -3103, 435, -729, 2267, -7576)$	$61,5\,\mathrm{s}$	$0,1\,\mathrm{s}, 12$ Sp	$0,3\,\mathrm{s}, 8$ MNS
$(2232, -1656, 7452, 99, 601, -2870, -2908, -2950)$	$808,8\,\mathrm{s}$	$1,6\,\mathrm{s}, 56$ Sp	$1,2\,\mathrm{s}, 9$ MNS
$(4060, 183, -4211, 5914, 2790, -5360, -1730, 3916, -5562)$	$1646,2\,\mathrm{s}$	$4,8\,\mathrm{s}, 40$ Sp	$0,3\,\mathrm{s}, 2$ MNS
$(4058, -1343, -2236, 7114, 1909,$ $-5696, 193, 5298, -689, -8608)$	–	$42,5\,\mathrm{s}$ 64 Sp	$2,3\,\mathrm{s}$ 8 MNS
$(1388, 4024, -1586, -1135, 5998, -6067,$ $3562, -4599, 7818, -2542, -6861)$	–	$1162,9\,\mathrm{s}$ 256 Sp	$12,1\,\mathrm{s}$ 6 MNS
$(1094, -11, -75, 1, -1009)$	$0,6\,\mathrm{s}$	$< 0,1\,\mathrm{s}, 4$ Sp	$< 0,1\,\mathrm{s}, 1$ MNS
$(1034, 49, -75, 25, -33, -1000)$	$7,1\,\mathrm{s}$	$< 0,1\,\mathrm{s}, 16$ Sp	$0,3\,\mathrm{s}, 6$ MNS
$(1022, 36, 33, -53, -21, -1, -1016)$	$182,1\,\mathrm{s}$	$0,3\,\mathrm{s}, 40$ Sp	$1,4\,\mathrm{s}, 20$ MNS
$(1099, -99, 77, -15, -29, 24, 36, -1093)$	$337,0\,\mathrm{s}$	$0,3\,\mathrm{s}, 8$ Sp	$0,3\,\mathrm{s}, 4$ MNS
$(1050, -36, 5, -130, -16, 43, 20, 91, -1027)$	$3764,1\,\mathrm{s}$	$1,6\,\mathrm{s}, 20$ Sp	$0,7\,\mathrm{s}, 3$ MNS
$(1079, -64, 28, 11, -48, 5, -4, 25, 20, -1052)$	–	$23,8\,\mathrm{s}, 40$ Sp	$5,0\,\mathrm{s}, 12$ MNS
$(1052, -46, -52, 25, -21, 69, -26, 25, -43, 24, -1007)$	–	$896,4\,\mathrm{s}, 216$ Sp	$41,6\,\mathrm{s}, 32$ MNS
$(31011, 1000, 600, 500, -500, -600, -1000, -31011)$	$12832,8\,\mathrm{s}$ $1500\,\mathrm{Mo}$	$3,1\,\mathrm{s}$ 206 Sp	$18,0\,\mathrm{s}$ 137 MNS
$(31011, 10000, 6000, 5000, 0, -5000, -6000, -10000, -31011)$	$> 23000\,\mathrm{s}$	$60,4\,\mathrm{s}$ 898 Sp	$1865,8\,\mathrm{s}$ 548 MNS
$(46398, 36794, 92409, -16156, 29524, -68385,$ $93335, 50738, 75167, -54015, -285809)$	–	$2193,2\,\mathrm{s}$ 322 Sp	$308,5\,\mathrm{s}$ 109 MNS
$(37, -9, -7, -6, -5, -4, -3, -2, -1)$	$> 12000\,\mathrm{s}$ $> 2400\,\mathrm{Mo}$	$7,0\,\mathrm{s}$ 128 Sp	$213,6\,\mathrm{s}$ 1 MNS

FIGURE 11. Computation time for LattE, B-DL-V and our programs, for A_n

Root lattice element	LattE	MNS
$(1388, 4024, 3826)$	$0,8\,\mathrm{s}$	$< 0,1\,\mathrm{s}$ 3 MNS
$(2691, 5998, -6067, 6184)$	$2,6\,\mathrm{s}$	$0,1\,\mathrm{s}$ 1 MNS
$(1585, 7818, -2542, -2803, 2715)$	$214,9\,\mathrm{s}$	$3,0\,\mathrm{s}$ 2 MNS
$(479, 7114, 1909, -5696, 193, 9297)$	$16369,6\,\mathrm{s}$	$27,5\,\mathrm{s}$ 8 MNS
$(1070, 1006, -37)$	$0,9\,\mathrm{s}$	$0,1\,\mathrm{s}$ 3 MNS
$(1082, 947, 27, 42)$	$22,9\,\mathrm{s}$	$1,2\,\mathrm{s}$ 15 MNS
$(1047, 974, 20, 44, -35)$	$1939,9\,\mathrm{s}$	$21,7\,\mathrm{s}$ 51 MNS
$(1015, 1082, -37, -21, -28, 14)$	$> 7000\,\mathrm{s}$ $> 1500\,\mathrm{Mo}$	$378,0\,\mathrm{s}$ 26 MNS

FIGURE 12. Computation time for LattE and our programs, for B_n

Root lattice element	LattE	MNS
$(1388, 4024, 7652)$	$0,8\,\mathrm{s}$	$< 0,1\,\mathrm{s}$ 1 MNS
$(2691, 5998, -6067, 12368)$	$2,8\,\mathrm{s}$	$0,1\,\mathrm{s}$ 1 MNS
$(1585, 7818, -2542, -2803, 5430)$	$163,0\,\mathrm{s}$	$1,4\,\mathrm{s}$ 1 MNS
$(479, 7114, 1909, -5696, 192, 18594)$	$> 5400\,\mathrm{s}$ $> 900\,\mathrm{Mo}$	$65,3\,\mathrm{s}$ 8 MNS
$(1038, 22, -2)$	$0,8\,\mathrm{s}$	$0,1\,\mathrm{s}$ 3 MNS
$(1021, 37, -40, 178)$	$12,2\,\mathrm{s}$	$0,5\,\mathrm{s}$ 4 MNS
$(1051, -45, 26, -5, -131)$	$195,4\,\mathrm{s}$	$2,8\,\mathrm{s}$ 6 MNS
$(1024, 6, 60, -6, -42, 52)$	$> 10800\,\mathrm{s}$ $> 2000\,\mathrm{Mo}$	$1292,4\,\mathrm{s}$ 42 MNS

FIGURE 13. Computation time for LattE and our programs, for C_n

Root lattice element	LattE	MNS
$(8608, -305, 183)$	$0,3\,\mathrm{s}$	$< 0,1\,\mathrm{s}$ 1 MNS
$(32, 5914, 6166, -5360)$	$1,5\,\mathrm{s}$	$< 0,1\,\mathrm{s}$ 1 MNS
$(1646, 3916, -3330, 6372, 7452)$	$18,0\,\mathrm{s}$	$0,5\,\mathrm{s}$ 2 MNS
$(8127, 601, -2870, -2908, 10823, 3639)$	$313,5\,\mathrm{s}$	$3,1\,\mathrm{s}$ 2 MNS
$(1009, 1106, -9)$	$0,2\,\mathrm{s}$	$< 0,1\,\mathrm{s}$ 1 MNS
$(1074, 959, 64, 77)$	$3,0\,\mathrm{s}$	$0,3\,\mathrm{s}$ 6 MNS
$(1100, 973, 2, -1, -60)$	$100,2\,\mathrm{s}$	$3,1\,\mathrm{s}$ 18 MNS
$(1096, 965, -54, 68, -34, -1)$	$7076,7\,\mathrm{s}$	$763,3\,\mathrm{s}$ 47 MNS

FIGURE 14. Computation time for LattE and our programs, for D_n