1. Lecture 1: Symplectic linear algebra

Let $V$ be a real vector space.

**Definition 1.1.** A symplectic form on $V$ is a skew-symmetric bilinear nondegenerate form:

1. $\omega(x, y) = -\omega(y, x) \implies \omega(x, x) = 0$;
2. $\forall x, \exists y$ such that $\omega(x, y) \neq 0$.

For a general 2-form $\omega$ on a vector space, $V$, we denote $\ker(\omega)$ the subspace given by

$$\ker(\omega) = \{ v \in V \mid \forall w \in V \omega(v, w) = 0 \}$$

The second condition implies that $\ker(\omega)$ reduces to zero, so when $\omega$ is symplectic, there are no “preferred directions” in $V$.

There are special types of subspaces in symplectic manifolds. For a vector subspace $F$, we denote by

$$F^\omega = \{ v \in V \mid \forall w \in F, \omega(v, w) = 0 \}$$

From Grassmann’s formula it follows that $\dim(F^\omega) = \text{codim}(F) = \dim(V) - \dim(F)$.

Also we have

**Proposition 1.2.**

$$(F^\omega)\omega = F$$

$$(F_1 + F_2)^\omega = F_1^\omega \cap F_2^\omega$$

**Definition 1.3.** A subspace $F$ of $V$, $\omega$ is

- isotropic if $F \subset F^\omega \iff \omega|_F = 0$;
- coisotropic if $F^\omega \subset F$;
- Lagrangian if it is maximal isotropic.

**Proposition 1.4.**

1. Any symplectic vector space has even dimension.
2. Any isotropic subspace is contained in a Lagrangian subspace and Lagrangians have dimension equal to half the dimension of the total space.
3. If $(V_1, \omega_1)$, $(V_2, \omega_2)$ are symplectic vector spaces with $L_1, L_2$ Lagrangian subspaces, and if $\dim(V_1) = \dim(V_2)$, then there is a linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi^* \omega_2 = \omega_1$ and $\varphi(L_1) = L_2$.
Proof. We first prove that if \( I \) is an isotropic subspace it is contained in a Lagrangian subspace. Indeed, \( I \) is contained in a maximal isotropic subspace. We denote it again by \( I \) and we just have to prove \( 2 \dim(I) = \dim(V) \).

Since \( I \subset I^\omega \) we have \( \dim(I) \leq \dim(I^\omega) = \dim(V) - \dim(I) \) so that \( 2 \dim(I) \leq \dim(V) \). Now assume the inequality is strict. Then there exist a non zero vector, \( e \), in \( I^\omega \setminus I \), and \( I \oplus \mathbb{R} e \) is isotropic and contains \( I \). Therefore \( I \) was not maximal, a contradiction.

We thus proved that \( I = I^\omega \) and \( 2 \dim(I) = \dim(V) \), and \( \dim(V) \) is even.

Applying the above result to \( \{0\} \), an obviously isotropic subspace, we conclude that we may always find a Lagrangian subspace, and \( V \) is always even-dimensional. This proves (1) and (2).

Let us prove (3).

We shall consider a standard symplectic vector space \( (\mathbb{R}^2, \sigma) \) with canonical base \( e_x, e_y \) and the symplectic form given by

\[
\sigma(x_1 e_x + y_1 e_y, x_2 e_x + y_2 e_y) = x_1 y_2 - y_1 x_2.
\]

Similarly by orthogonal direct sum, we get the symplectic space \( (\mathbb{R}^{2n}, \sigma_n) \)

\[
\sigma((x_1, \ldots, x_n, y_1, \ldots, y_n), (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)) = \sum_{j=1}^{n} x_j y'_j - x'_j y_j
\]

It contains an obvious Lagrangian subspace,

\[
Z_n = \mathbb{R}^n \oplus 0 = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid \forall 1 \leq j \leq n, y_j = 0\}
\]

Let \( (V, \omega) \) be a symplectic vector space and \( L \) a Lagrangian. Pick any \( e_1 \in L \). Since \( \omega \) is nondegenerate, there exists an \( f_1 \in V \) such that \( \omega(e_1, f_1) = 1 \). Then \( f_1 \notin L \). Define

\[
V_2 = \text{Vect}(e_1, f_1)^\omega = \{x \in V \mid \omega(x, e_1) = \omega(x, f_1) = 0\}.
\]

It is easy to see that \( (V_2, \omega|_{V_2}) \) is symplectic since only non-degeneracy is an issue, and it follows from the fact that

\[
\ker(\omega|_{V_2}) = V_2 \cap V_2^\omega = \{0\}
\]

We now claim that \( L_2 = L \cap V_2 \) is a Lagrangian in \( V_2 \). First, since \( \omega|_{L_2} \) is the restriction of \( \omega|_{L} \), it is clearly isotropic. It is maximal isotropic, since otherwise, there would be an isotropic \( V_2 \supset W \supseteq L_2 \), and then \( W \oplus \mathbb{R} e_1 \) would be a strictly larger isotropic subspace than \( L \), which is impossible.

Now we claim by induction that there is a symplectic map, \( \varphi_{n-1} \) from \( (\mathbb{R}^{2n-2}, \sigma) \) to \( (V_2, \omega) \) sending \( Z_{n-1} \) to \( L_2 \). Now the map

\[
\varphi_n : (\mathbb{R}^2, \sigma_2) \oplus (\mathbb{R}^{2n-2}, \sigma) \longrightarrow (V_2, \omega)
\]

\[
(x_1, y_1; z) \longrightarrow x_1 e_1 + y_1 f_1 + \varphi_{n-1}(z)
\]

is symplectic and sends \( Z_n \) to \( L \).

Now given two symplectic manifolds, \( (V_1, \omega_1), (V_2, \omega_2) \) of dimension \( 2n \), and two lagrangians \( L_1, L_2 \), we get two symplectic maps

\[
\psi_j(\mathbb{R}^{2n}, \sigma_n) \longrightarrow (V_j, \omega_j)
\]

sending \( Z_n \) to \( L_j \). Then the map \( \psi_2 \circ \psi_1^{-1} \) is a symplectic map from \( (V_1, \omega_1) \) to \( (V_2, \omega_2) \) sending \( L_1 \) to \( L_2 \).
We now give a better description of the set of lagrangians

**Proposition 1.5.**

1. The action of $Sp(n) = \{ \varphi \in GL(V) | \varphi^* \omega = \omega \}$ acts bi-transitively in the set of Lagrangians;
2. $\{ \Lambda | \text{Lagrangian and } \Lambda \cap L = \{0\} \} \longleftrightarrow \{ \text{quadratic forms on } L^* \}$.

**Proof.** The first statement is a rephrasing of (3) of proposition 1.4 applied to $V_1 = V_2 = V$.

For (2), we notice that $W = L \oplus L^*$ with the symplectic form

$$
\sigma((e, f), (e', f')) = \langle e', f \rangle - \langle e, f' \rangle
$$

is a symplectic vector space and that $L \oplus 0$ is a Lagrangian subspace.

According to the previous proposition there is a symplectic map $\psi : V \rightarrow W$ such that $\psi(L) = L \oplus 0$, so we can work in $W$.

Let $\Lambda$ be a Lagrangian in $W$ with $\Lambda \cap L = \{0\}$. Then $\Lambda$ is the graph of $A : L^* \rightarrow L$ more precisely

$$
\Lambda = \{(Ay^*, y^*) | y^* \in L^* \}.
$$

The subspace $\Lambda$ is Lagrangian if and only if

$$
\sigma((Ay_1^*, y_1^*), (Ay_2^*, y_2^*)) = 0, \text{ for all } y_1, y_2
$$

i.e. if and only if

$$
\langle y_1^*, Ay_2^* \rangle = \langle y_2^*, Ay_1^* \rangle
$$

that is if $\langle , A \cdot \rangle$ is a bilinear symmetric form on $L^*$. But such bilinear form are in 1-1 correspondence with quadratic forms.

**Exercice 1:** (Witt’s Theorem) Let $V_1$ and $V_2$ be two symplectic vector spaces with the same dimension and $F_i \subset (V_i, \omega_i)$, $i = 1, 2$. Assume that there exists $\varphi : F_1 \cong F_2$, i.e. $\varphi^*(\omega_2)|_{F_2} = (\omega_1)|_{F_1}$. Then $\varphi$ extends to a symplectic map $\tilde{\varphi} : (V_1, \omega_1) \rightarrow (V_2, \omega_2)$.

**Exercice 2:** Prove that the above results are valid over any field of any characteristic, but for the last statement, where in characteristic 2, quadratic forms and bilinear symmetric forms are not equivalent.

### 2. Complex structure

Let $h$ be a hermitian form on a complex vector space $V$ in the sense:

1) $h(z, z') = \bar{h}(z', z)$;
2) $h(\lambda z, z') = \lambda h(z, z')$ for $\lambda \in \mathbb{C}$;
3) $h(z, \lambda z') = \bar{\lambda} h(z, z')$ for $\lambda \in \mathbb{C}$;
4) $h(z, z) > 0$ for all $z \neq 0$.

Then

$$
h(z, z') = g(z, z') + i \omega(z, z'),
$$

where $g$ is a scalar product and $\omega$ is symplectic.

**Example:** On $\mathbb{C}^n$, define

$$
h((z_1, \cdots, z_n), (z'_1, \cdots, z'_n)) = \sum_{j=1}^{n} z_j \bar{z}'_j \in \mathbb{C}.
$$

Then the symmetric part is the usual scalar product on $\mathbb{R}^{2n}$ while $\omega$ is the standard symplectic form.
Denote by $J$ the multiplication by $i = \sqrt{-1}$.

Theorem 2.1.

\[
\begin{cases}
g(Jz, z') = -\omega(z, z') \\
\omega(z, Jz') = -g(z, z')
\end{cases}
\]

Remark 2.2. $\omega$ is nondegenerate because $\omega(z, Jz) = -g(z, z) < 0$ for all $z \neq 0$.

Conclusion: Any hermitian space $V$ has a canonical symplectic form.

We will now answer the following question: Can a symplectic vector space be made into a hermitian space? In how many ways?

Proposition 2.3. Let $(V, \omega)$ be a symplectic vector space. Then there is a complex structure on $V$ such that $\omega(J\xi, \xi)$ is a scalar product. Moreover, the set of such $J$ is contractible.

Proof. Let $(\cdot, \cdot)$ be any fixed scalar product on $V$. Then there exists $A$ such that

\[
\omega(x, y) = (Ax, y).
\]

Since $\omega$ is skew-symmetric, $A^* = -A$ where $A^*$ is the adjoint of $A$ with respect to $(\cdot, \cdot)$. Since any other scalar product can be given by a positive definite symmetric matrix $M$, we look for $J$ such that $J^2 = -I$ and $M$ such that $M^* = M$ and $(x, y)_M = (Ax, y)$ and $\omega(Jx, y) = (x, y)_M$. The last equality is

\[
(AJx, y) = (Mx, y) \text{ for all } x, y.
\]

It’s easy to check that $M = (AA^*)^{1/2}$ and $J = A^{-1}M$ solves $AJ = M$, $J^2 = -I$ and $M^* = M$.

In summary, for any fixed scalar product $(\cdot, \cdot)$, we can find a pair $(J_0, M_0)$ such that $\omega(J_0x, y)$ is the scalar product $(M_0, \cdot)$. If we know $(J_0, M_0)$ is such a pair and we start from the scalar product $(M_0, \cdot)$, then we get the pair $(J_0, id)$.

Define $X$ to be the set of all $J$’s such that $\omega(J, \cdot)$ is a scalar product. Define $Y$ to be the set of all scalar product. By previous discussion, there is continuous map

\[
\Psi: Y \to X.
\]

Moreover, if $J$ is in $X$, $\Psi$ maps $\omega(J, \cdot)$ to $J$. On the other hand, we have a continuous embedding $i$ from $X$ to $Y$ which maps $J$ to $\omega(J, \cdot)$. Let $p \in Y$ be in the image. Since we know $Y$ is contractible, there is a continuous family

\[
F_t: Y \to Y
\]

such that $F_0 = id$ and $F_1$ maps anything to $p$. Consider

\[
\tilde{F}_t: X \to X
\]

given by

\[
\tilde{F}_t = \Psi \circ F_t \circ i.
\]

By the definition of $\Psi$, we know $\tilde{F}_0 = id$ and $\tilde{F}_1 = J_p$. This shows that $X$ is contractible.

Exercise: Let $L$ be a Lagrangian subspace, show that $JL$ is also a Lagrangian and $L \cap JL = \{0\}$. 