Functors and Computations in Floer homology with Applications
Part I

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October 20, 1996
(revised on May 31, 1999).

Abstract

This paper is concerned with Floer cohomology of manifolds with contact type boundary. In this case, there is no conjecture on this ring, as opposed to the compact case, where it is isomorphic to the usual cohomology (with the quantum product). We construct two mappings in Floer cohomology and prove some functorial properties of these two mappings. The first one is a map from the Floer cohomology of M to the relative cohomology of M modulo its boundary. The other is associated to a codimension zero embedding, and may be considered as a cohomological transfer. These maps are used to define some properties of symplectic manifolds with contact type boundary. These are algebraic versions of the Weinstein conjecture, asserting existence of closed characteristics on ∂M. This is proved for many cases, Euclidean space and subcritical Stein manifolds, vector bundles, products, cotangent bundles. It is also proved that the above property implies some restrictions on Lagrangian embeddings, and also yields in certain cases, existence results for holomorphic curves bounded by the Lagrange submanifold. The last section is devoted to applications of this existence result, to real forms of Stein manifolds

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and obstructions to polynomial convexity in Stein manifolds. Many of our applications rely on the computation of the Floer cohomology of a cotangent bundle, that is the subject of Part II.

Contents

Introduction

Let (M, ω) be a symplectic manifold. One of the main tools in symplectic topology is the Floer cohomology $FH^*(M)$ of (M, ω). This is a group, defined by Andreas Floer in his fundamental papers [Fl1], [Fl2] for a closed manifold, satisfying some topological assumption (namely $[\omega]\pi_2(M) = 0$). This was later extended by Floer, and then Hofer-Salamon ([HS 1]), Ono ([Ono]), Fukaya-Ono, Hofer-Salamon and Liu-Tian ([F-O, HS 2, LT]) on one hand to remove this topological assumption, and on the other hand by Floer, Hofer et als. ([FH, CFH] to the case of manifolds with contact type boundary.

While to the closed case corresponds a standard guess, the isomorphism of $FH^*(M)$ with $H^*(M)$, the main difficulty being to prove that $FH^*(M)$ is well defined and that this isomorphism holds, there is no obvious guess in the case of manifolds with contact type boundary. In fact, in the latter case, it is useful to have some tools to understand better these groups (in fact as shown in [V8], they are modules over $H^*(\Lambda N)$, where ΛN is the free loop space of N, or rings using the "quantum product").

The aim of this paper will be to develop two such tools, and give numerous applications.

The first is the construction of natural maps in Floer cohomology.

In section 1 we define the map $c_M^* : FH^*(M) \to H^*(M, \partial M)$, and in section 2 a transfer map $(Fj)^! : FH^*(W) \to FH^*(M)$ associated to a codimension zero embedding $W \to M$, satisfying some assumption (called (A) in the text).

Our main theorem proves that we have a commutative diagram

$$\begin{array}{ccc} FH^*(W) & \xrightarrow{(Fj)^!} & FH^*(M) \\ {\scriptstyle c_W^*}\downarrow & & \downarrow{\scriptstyle c_M^*} \\ H^*(W) & \xrightarrow[j^!]{} & H^*(M) \end{array}$$

We also show the compatibility of the above maps with the product structure defined in [V1] (see prop. 1.5).

Section 5 gives an equivariant version of Floer cohomology and extends the above constructions to this setting.

Section 4 is concerned with applications of these results. The map c_M^* may be used to state an algebraic version of Weinstein's conjecture. It should be noticed that most proofs of the ordinary Weinstein conjecture (except for 3-dimensional cases due to [Ho] and the compact case of [HV2]) actually prove the stronger form we state. The advantages of this stronger form are that they easily extend from one manifold to related ones, like those obtained by product or fibrations; and it also implies other ”symplectic rigidity” statements, like non-existence of exact compact Lagrange embeddings.

We prove some new cases of Weinstein's conjecture. In particular that any subcritical Stein manifold satisfies the algebraic Weinstein conjecture, as well as products of such manifolds with a compact manifold. We also get as a corollary that the dynamical system of a particle in a compact space of positions, subject to a force and magnetic field derived from a potential has a periodic solution.

Many of these applications rely on the other tool, developed in the second and third part of this paper: computations. The main result of part II, of crucial importance in the applications, is the isomorphism $FH^*(DT^*N) = H^*(\Lambda N)$ (DT^*N is the unit disc cotangent bundle) and its equivariant version. Part III will deal with a version of Leray-Serre spectral sequence.

Finally section 6 deals with construction of holomorphic curves using Floer cohomology. We apply these results to give some restrictions, on the real part of complex manifolds, and to the geometry of polynomially convex totally real submanifolds. This is only a first ”sampling” of the theorems one can get. More applications will appear in a forthcoming paper, where we shall deal with closed manifolds.

Note that some of the results presented here had been proved previously, in particular in [V1], other results from this paper may be rephrased in the language given here.

Finally, I wish to thank the students of the DEA of Paris-Sud, Orsay, who heard parts of this paper, as well as the audience of the Séminaire de Géométrie Symplectique of Ecole Polytechnique. I also wish to thank David Hermann for improving and correcting many proofs, Paul Seidel for useful comments, Julien Duval, Viktor Ginzburg, Viatcheslav Kharlamov Yong-Geun Oh and Nessim Sibony for many useful conversations as well as for their encouragement concerning section 6.

Part I

Floer cohomology pushforward for symplectic manifolds with contact type boundary.

1 Floer cohomology for manifolds with contact type boundary. The case $[\omega]\pi_2(M) = 0$

1.1 Basic definitions and results

Throughout this paper (M, ω) will be a *compact symplectic manifold* such that $[\omega]\pi_2(M) = 0$, and with contact type boundary ∂M (possibly empty). This means that there is a one-form α on ∂M such that $d\alpha = \omega$ and $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing. The form α is called a *contact form.*

An *almost complex structure* on M will be called *admissible* if it satisfies the following two conditions

$$(i) \qquad \omega(J\xi, \xi) \text{ is positive definite.}$$

$$(ii) \qquad J \text{ preserves the } \textit{contact hyperplanes} \{\alpha = 0\} \text{ in } \partial M.$$

To such a manifold (M, ω) we may associate a noncompact symplectic manifold, denoted $(\widehat{M}, \widehat{\omega})$ such that

$$\widehat{M} = M \cup_{\partial M \times \{1\}} \partial M \times [1, +\infty[$$

$$\widehat{\omega} = \begin{cases} \omega & \text{on} \quad M \\ d(z\alpha) & \text{on} \quad \partial M \times [1, +\infty[\end{cases}$$

where z is the second coordinate.

Since a neighborhood of ∂M in M may be identified with $(\partial M \times [1 - \varepsilon, 1], d(z\alpha))$, $\widehat{\omega}$ is smooth.

We then extend J to $\widehat{J}$, so that J is constant on the $\partial M \times \{z\}$.

It will be useful to notice that $\widehat{J}$ makes the submanifolds $\partial M \times \{z\}$ pseudoconvex. More generally if H is a time dependent Hamiltonian on $\widehat{M}$ such that $H(t,x,z) = h(z)$ on $\mathbb{R} \times \partial M \times [1,+\infty[$, with h convex in z, then according to [McD] a solution of $\overline{\partial}u(t,\theta) = \nabla H(t,u(t,\theta))$ may not have an interior tangency with the $\partial M \times \{z\}$.(Here $\overline{\partial}u$ means $\frac{\partial}{\partial t}u + \widehat{J}(u)\frac{\partial}{\partial \theta}u$).

Given a smooth function $H(t,x)$ on $\mathbb{R} \times M$, we denote by X_H the associated *Hamiltonian vector field.* It is defined by the formula

$$\omega(X_H, \xi) = \frac{\partial}{\partial x}H(t,x).\xi \quad \forall \xi \in T_xM.$$

Given a Hamiltonian H on $\mathbb{R} \times \widehat{M}$ such that $H(t,x,z) = h(z)$ with h convex increasing for z large (we call such a Hamiltonian *admissible*), we may define the Floer cohomology $FH^*(H;a,b)$ for any two real numbers $a < b$, as follows.

First we need to assume that all contractible 1- periodic solutions of $\dot{v} = X_H(t,v)$ are *nondegenerate* (i.e. 1 is not an eigenvalue of the linearized Poincaré map) and contained in a compact subset of $\widehat{M}$ (this happens provided the set of z such that $h'(z)$ equals the action of a closed characteristic of ∂M is bounded). We then define $C^k(H;a,b)$ as $\oplus_v R$ (R is some ring) where v varies in the set of contractible 1- periodic solutions of $\dot{v} = X_H(t,v)$ such that

- $i_{cz}(v)$, the Conley-Zehnder index of v equals k (see [Co-Z], [S-Z])
- $A_H(v) = \int_{S^1}[\overline{v}^*(\omega) - H(t,v)]dt$ is in [a,b], where $\overline{v}$ is a map from the 2-disk to M, whose restriction to the boundary equals v. Note that A_H is well defined because $[\omega]\pi_2(M) = 0$.

One then defines a boundary operator

$$d : C^k(H;a,b) \to C^{k+1}(H;a,b)$$

such that the matrix element $< dv, w >$ is equal to the algebraic number of solutions of the system

$$\left\{ \begin{array}{ll} u : \mathbb{R} \times S^1 \to \widehat{M} & \overline{\partial}u = \nabla H(t,u) \\ \lim_{s\to\infty} u(s,t) = w(t) & \lim_{S\to-\infty} u(s,t) = v(t) \end{array} \right.$$

We denote by $\mathcal{M}(v,w)$ the set of such solutions, and notice that a shift in the s variable induces a free $\mathbb{R}$ action on $\mathcal{M}(v,w)$. Thus $< dv, w >$ is the number of points in $\overline{\mathcal{M}}(v,w) = \mathcal{M}(v,w)/\mathbb{R}$.

For a generic J, the dimension of $\mathcal{M}(v,w)$ is given by the difference $i_{cz}(w)-i_{cz}(v)$.

The crucial property $d^2=0$ follows from the fact that the set of points in $\bigcup_u \overline{\mathcal{M}}(x,u)\times\overline{\mathcal{M}}(u,v)$ where $i_{cz}(u)=i_{cz}(x)+1=i_{cz}(v)-1$ may be identified with the set of ends of the 1-dimensional manifold $\overline{\mathcal{M}}(x,v)$.

The assertion that the only non compactness in $\overline{\mathcal{M}}(x,v)$ may be caused by the breaking of a trajectory requires two properties. The first is the non-existence of holomorphic spheres and follows from the assumption $[\omega]\pi_2(M)=0$.

The second one is that the image of u, solution of $\overline{\partial}u=\nabla H(t,u)$ $\lim_{s\to-\infty}u(s,t)=x_-(t)$, $\lim_{s\to+\infty}u(s,t)=x_+(t)$ stays in a bounded region of $\widehat{M}$. This follows from the assumption that the 1-periodic solutions of $\dot{v}=X_H(t,v)$ stay in a bounded region, and from the pseudoconvexity of $\partial M\times\{z\}$.

Remark. Note that it is not hard to show that the standard transversality results for solutions of $\overline{\partial}_J u=0$ or $\overline{\partial}_J u=-\nabla H(t,u)$ (see for example [McD-S], pp. 34-35) still hold if we assume that $\widehat{J}$ is admissible. The proof in[McD-S] works by showing that $\{(u,J)|\overline{\partial}_J u=0\}$ is a manifold. For this one has to prove that if η is in the L^2 orthogonal of the range of the linearized operator, it must vanish on the set of points of the curve where $du(z)\neq 0$ and $u^{-1}(u(z))=\{z\}$. By unique continuation, since η satisfies an elliptic equation (adjoint of the linearization of $\overline{\partial}$) it vanishes everywhere. In our case, we may prove that η vanishes at these points where, in addition to $du(z)=0$ and $u^{-1}(u(z))=\{z\}$, u is *not tangent* to $\ker\alpha$. But for a $\widehat{J}$- holomorphic (or plurisubharmonic) u, the set of such points is isolated, and the same argument as in [McD-S] applies to show that η is identically zero.

Now we may define $FH^*(H;a,b)$ to be cohomology of the complex $(C^*(H;a,b),d)$.

A standard argument shows that $FH^*(H;a,b)$ does not depend on the choice of J, since the set of almost complex structures satisfying (i) and (ii) is contractible.

Note that this is *also independent from the choice of the contact structure* defined by $\{\alpha=0\}$ on ∂M.

Indeed, we may identify α to the vector field ξ given by $i_\xi\omega=\alpha$. Then α is a contact form compatible with ω if and only if ξ is transverse to ∂M and $L_\xi\omega=\omega$. We may assume that ξ points outward, and then the set of such vector fields is clearly convex, hence the set of contact structures compatible with ω is contractible.

We now shall establish certain more or less obvious properties summarized in prop 1.1,1.3,1.4.

Let $H_\lambda(\lambda \in [0,1])$ be a continuous 1-parameter family of Hamiltonians, and a_λ, b_λ be real numbers such that neither is the action of a 1-periodic orbit for X_{H_λ}.

Proposition 1.1. *Under the above assumptions, we have a natural isomorphism $FH^*(H_0; a_0, b_0) \to FH^*(H_1; a_1, b_1)$.*

The proof is rather straightforward using the results from [CFH]. $\square$

Finally for $a < b < c$ we have a short exact sequence.

$$0 \to C^*(H; b, c) \to C^*(H; a, c) \to C^*(H; a, b) \to 0$$

and we get a long exact sequence

$$\to FH^*(H; b, c) \to FH^*(H; a, c) \to FH^*(H; a, b) \xrightarrow{\partial^*} \tag{1}$$

Remark. 1.2

From (1), it follows that if H is a C^0 function we may define $FH^*(H; a, b)$ as the

$$\lim_{\substack{K \to H \\ K \in C^\infty}} FH^*(K; a, b) \tag{2}$$

Of course, we have to prove that this definition coincide with the standard one when H is smooth and "generic". This is proved as follows:

Let H_0 and H_1 be C^0 close, and H_λ be a smooth homotopy such that $|H_\lambda - H_\mu|_{C^0} < \varepsilon|\lambda - \mu|$ for all λ, μ in $[0,1]$.

Consider the bifurcation diagram of the set of actions of 1-periodic solutions of X_{H_λ}. This is a union of curves in the plane with cusps (called a Cerf diagram, see figure 1). It is easy to see that each smooth portion of such a diagram has its slope bounded by $\sup_{\lambda \in [0,1]} |\frac{\partial}{\partial \lambda} H_\lambda| \leq \varepsilon$.

As a result, given $a < b$ such that the distance between the action of a 1-periodic solution of X_{H_0} and the boundary of the interval $[a, b]$ is greater than 2ε, we have for any H_1 such that $|H_1 - H_0|_{C^\circ} \leq \varepsilon$ an isomorphism $FH^*(H_0; a, b) \to FH^*(H_1; a, b)$.

As a consequence, the definition of $FH^*(H; a, b)$ for H a "C^0 Hamiltonian" coincides with the standard one in case H is C^∞.

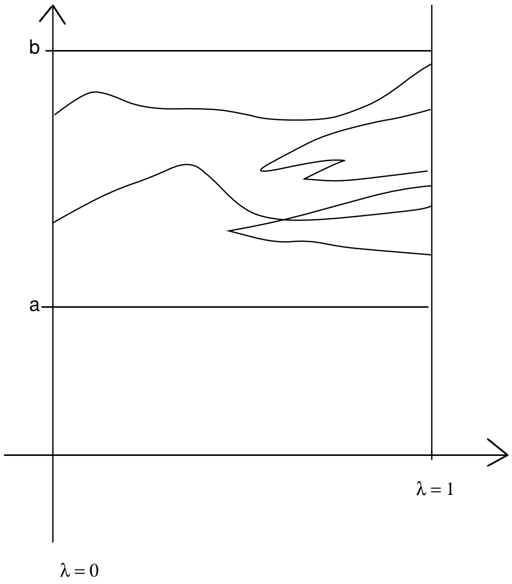

Figure 1: Cerf diagram

1.2 Definition of $FH^*(M)$.

We are now going to consider a Hamiltonian on $\widehat{M}$ such that $H(t, x, z) = h(z)$ for $z \geq 1$, with $h'(z) = h_\infty$, a constant, for z large enough.

For $z \geq 1$, the 1-periodic orbits of X_H are related to the closed characteristics on ∂M. More precisely if X is the Reeb vector field for α, that is $i_X d\alpha = 0, i_X\alpha = 1$, we have $X_H(t, x, z) = h'(z)X$ for $z \geq 1$. Thus, if $T = h'(z)$ is the area of a contractible closed characteristic, γ, on ∂M, then $(\gamma(t.T), z)$ is a 1-periodic solution of X_H, with action $\int_0^1 z\alpha - h(z) = zh'(z) - h(z)$.

We now investigate what happens to $FH^*(H; a, b)$ when we modify h, while H is unchanged inside M. Let $h_\infty^0 < h_\infty^1$ and assume that $h_0(z) = h_\infty^o(z-2)$ for $z \geq 2$. Set $h_1(z) = h_0(z)$ for $z \leq 2$, and $h_1(z)$ is convex on $2 \leq z \leq 3$ and $h_1'(z) = h_\infty^1$ for $z \geq 3$.

To understand the changes in $FH^*(H_\lambda; a, b)$ where $H_\lambda = (1-\lambda)H_0 + \lambda H_1$ (and $H_i = H$ on $M, H_i(t, x, z) = h_i(z)$ for $i = 0, 1$), we must find the 1-periodic orbits of X_{H_λ} with action crossing a or b. In other words, we have to solve $zh_\lambda'(z) - h_\lambda(z) = a$ (resp.b) with $h'(z)$ the area a closed characteristic on ∂M.

To simplify our problem, we shall assume that

$(i) \qquad z \to zh'_\lambda(z) - h_\lambda(z)$is monotone increasing for each λ.

$(ii) \qquad \lambda \to zh'_\lambda(z) - h_\lambda(z)$ is monotone increasing for each z.

since for $z = 2$, $\lambda = 0$ we have $zh'(z) - h(z) = 2h^0_\infty - h^0_\infty.0 = 2h^0_\infty$, we get that if $a < b < 2h^0_\infty$ all the actions of 1-periodic trajectories for X_{H_λ} in $\partial M \times [2, +\infty[$ will be outside $[a, b]$. As a consequence $FH^*(H_\lambda; a, b)$ does not depend on λ.

Let now K_λ be a function equal to 0 in M and to $k_\lambda(z) = \lambda.(z-1)$ for $z \geq 1$. Clearly, K_λ is neither smooth nor "generic ", but by remark 1.1, we may still define $FH^*(K_\lambda; a, b)$ as a limit. In this case, the limit is rather simple to estimate.

Indeed, let $h_{\lambda,\varepsilon}(z)$ be as in figure 2, so that assuming λ is not the area of a closed characteristic on ∂M, we have $zh'_{\lambda,\varepsilon}(z) - h_{\lambda,\varepsilon}(z) = T + 0(\varepsilon)$ if $h'_{\lambda,\varepsilon}(z) = T$ is such an area.

Set now $K_{\lambda,\varepsilon}$ to be $h_{\lambda,\varepsilon}(z)$ for $z \geq 1 - \varepsilon$ and equal zero elsewhere.

Thus $C_*(H_{\lambda,\varepsilon}; a, b)$ is generated by the constant periodic solutions inside M, and the closed characteristics on ∂M with area in $[a, b] \cap [0, \lambda]$. (for ε small enough). Then, for ε small enough, $zh'{}_{\lambda,\varepsilon}(z)$ is distinct from a or b, hence we get isomorphism.

$$FH^*(H_{\lambda,\varepsilon}; a, b) \overset{\sim}{\to} FH^*(H_{\lambda,\varepsilon'}; a, b)$$

We may now set

Definition.

$$(3) \qquad FH^*(M; a, b) = \lim_{\lambda \to +\infty} FH^*(K_\lambda; a, b).$$

$$(4) \qquad FH^*(M) = \lim_{\substack{b \to +\infty \\ a < 0}} FH^*(M; a, b).$$

It may be useful to notice that for $a < 0, FH^*(K_\lambda; a, b)$ does not depend on a , since $H_{\lambda,\varepsilon}$ has no 1-periodic orbit with negative action.

Note that here $FH^*(M)$ is defined as $\lim_{b \to +\infty} \lim_{\lambda \to +\infty} FH^*(K_\lambda, a, b)$ for (a nonpositive). We shall sometimes need to revert the limits. Thus we want to prove:

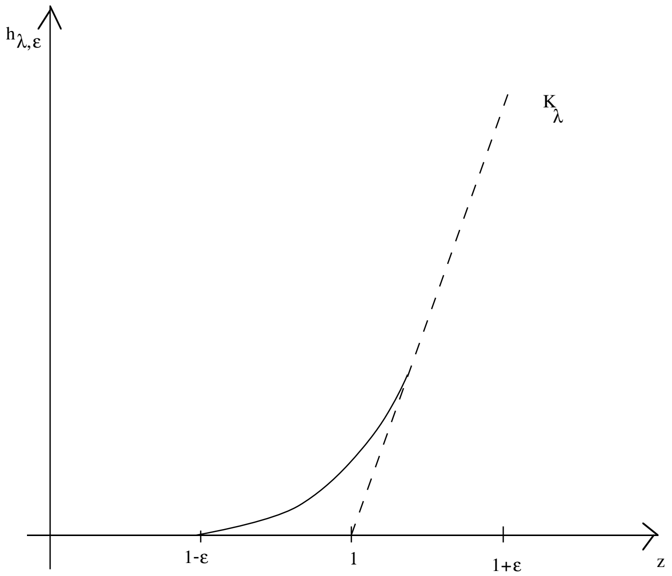

Figure 2: The Hamiltonians

Lemma 1.2. *We have* $FH^*(M) = \lim_{\lambda\to+\infty} \lim_{b\to+\infty} FH^*(K_\lambda, a, b)$.

Proof. It will be enough to prove this, when λ is in the complement of the set of areas of periodic orbits of ∂M. Because, according to a remark by Sikorav, this set is open and dense, we may limit λ to this subset. Now for each λ in this subset, let η_λ be such that $]\lambda - \eta_\lambda, \lambda - \eta_\lambda[$ does not contain the area of a closed characteristic of ∂M.

Let K_∞ be such that K_∞ is zero inside M while $K_\infty = (z-1)^2$ outside M. Again, one should think of the approximation of K_∞ since it is not smooth.

Then as we deform from K_λ to K_∞, we only get periodic orbits of area larger than λ. Thus for $b < \lambda + \eta_\lambda$ we have

$FH^*(K_\lambda, a, b) = FH^*(K_\infty, a, b)$

As a result, for any a, b *finite*, we have

$$\lim_{\lambda\to\infty} FH^*(K_\lambda, a, b) = FH^*(K_\infty, a, b)$$

Thus we have

$$\lim_{b\to\infty} FH^*(K_\infty, a, b) = \lim_{b\to\infty}\lim_{\lambda\to\infty} FH^*(K_\lambda, a, b) = FH^*(M)$$

On the other hand, for $b > \lambda - \eta_\lambda$,

$FH^*(K_\lambda, a, b) = FH^*(K_\infty, a, b)$, thus

$$\lim_{\lambda\to\infty}\lim_{b\to\infty} FH^*(K_\lambda, a, b) = \lim_{\lambda\to\infty} FH^*(K_\lambda, a, b(\lambda))$$

where $b(\lambda) \in]\lambda - \eta_\lambda, \lambda - \eta_\lambda[$.

But then $FH^*(K_\lambda, a, b(\lambda)) = FH^*(K_\infty, a, b(\lambda))$

and

$$\lim_{\lambda\to\infty}\lim_{b\to\infty} FH^*(K_\lambda, a, b) = \lim_{\lambda\to\infty} FH^*(K_\lambda, a, b(\lambda)) = \lim_{\lambda\to\infty} FH^*(K_\infty, a, b(\lambda)) = FH^*(M)$$

□

With the above definition, let us prove the

Proposition 1.3. *For δ small enough, we have*

$$FH^*(M, -\delta, \delta) \simeq H^*(M, \partial M)$$

Proof. Let $\widetilde{H}_{\lambda,\varepsilon}$ be a "generic" perturbation of $H_{\lambda,\varepsilon}$. Then its 1-periodic orbits are

(i) constants with action near 0

(ii) perturbation of 1- periodic orbits of $H_{\lambda,\varepsilon}$, with action bounded away from zero (since the area of closed characteristics of ∂M are bounded away from zero).

Thus for δ small enough, $C_*(H_{\lambda,\varepsilon}; -\delta, \delta)$ is generated by the constant solutions, that is critical points of $H_{\lambda,\varepsilon}$. Moreover, if v, w are such constant solutions, $\mathcal{M}(v, w)$ coincides with the set of trajectories for the gradient flow of $H_{\lambda,\varepsilon}$ (these are solutions $u(s,t)$ independent of t, see [S-Z] for the proof), so that $(C_*(H_{\lambda,\varepsilon}; -\delta, \delta), d)$ is isomorphic to the Thom-Smale-Witten complex of $-H_{\lambda,\varepsilon}$, its (co)homology is equal to $H^*(M, \partial M)$ (this is an easy extension of the classical result for the case $\partial M = \emptyset$ of Thom-Smale- Witten).

□

Now the exact sequence 1 yields

$$FH^*(M; \delta, +\infty) \to FH^*(M) \to FH^*(M, -\delta, \delta) \stackrel{\partial^*}{\to} .$$

where $FH^*(M; \delta, +\infty) = \lim\limits_{b\to+\infty} FH^*(M; \delta, b)$.

Using proposition 1.3, we get

Proposition 1.4. *There is an exact sequence*

$$\ldots \to FH^*(M;\delta,+\infty) \to FH^*(M) \to H^*(M,\partial M) \xrightarrow{\partial^*}$$
$$\to FH^{*+1}(M;\delta,+\infty) \to FH^{*+1}(M) \to \ldots$$

Of particular relevance will be the map

$$c_M^* : FH^*(M) \to H^*(M,\partial M).$$

The following result will also be useful.

Lemma 1.5.

$$FH^*(K_\lambda;-\infty,+\infty) \simeq FH^*(M;-\infty,\lambda)$$

Proof. First K_λ has no 1-periodic orbit with action outside $[0,\lambda]$. Thus we have

$$FH^*(K_\lambda;-\infty,+\infty) \simeq FH^*(K_\lambda;-\infty,\lambda).$$

Now as μ goes from λ to infinity, let us consider $FH^*(K_\mu;-\infty,\lambda)$. Since the new critical points of A_{K_μ} that appear are at a higher level μ, we have that for $\mu > \lambda, FH^*(K_\mu;-\infty,\lambda)$ does not depend on μ. Keeping in mind that $\lim\limits_{\mu\to+\infty} FH^*(K_\mu;-\infty,\lambda) = FH^*(M;-\infty,\lambda)$ concludes our proof. $\square$

1.3 Invariance of Floer cohomology by deformation

Theorem 1.6 (Floer homology deformation invariance). *Let ω_s be a continuous family of symplectic forms on W such that ∂W is of contact type for each of them. Then we have an isomorphism*

$$FH^*(W,\omega_0) \longrightarrow FH^*(W,\omega_1)$$

We first need

Lemma 1.7 (Generalized maximum principle). *Let Ω be a domain with contact type boundary, z be the second coordinate on $\partial\Omega \times [1,+\infty[$). Let $J(s,t)$ be an almost complex structure depending on s,t, and u be a solution on some plane domain D of*

$$\partial_s u + J(s,t)\partial_t u = -\nabla H(s,t,u)$$

where $H(s,t,u) = h(z)$ for $z \geq 1$ is monotone increasing, and has $\partial\Omega \times \{1\}$ as a regular level.

Then $z \circ u$ satisfies the maximum principle in $\partial\Omega \times [1, +\infty[)$ and in particular, if the boundary of u is in Ω, we have $z \circ u \leq 1$

Proof. Indeed, we remind the reader that $d^c = J^*d$ and use the fact that $d^c z = \lambda$ is the contact form, with $d\lambda = \omega$. Also $H(s,t,u) = h(z)$ for $z \geq 1$.

For $f = z \circ u$, $dd^c f = \Delta(f) ds \wedge dt$. Now

$$Jdu - du \circ i = -\overline{\partial} u dt - J\overline{\partial} u ds$$

Then

$$d^c f = dz(Jdu) = dz(u) du \circ i - dz(\overline{\partial} u) dt - dz(J\overline{\partial} u) ds$$
$$= u^*(\lambda) - dz(\nabla H(s,t,u)) dt - dz(J \nabla H(u)) ds$$
$$dd^c f = \omega(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}) - d(h'(z) dt)$$
$$= \omega(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s} + Jh'(z)\frac{\partial}{\partial z}) ds \wedge dt - h''(z)\frac{\partial f}{\partial s} ds \wedge dt$$
$$= |\frac{\partial u}{\partial s}|^2 + h'(z)\frac{\partial f}{\partial s} - h''(z)\frac{\partial f}{\partial s}$$

From all this follows

$$\Delta f + (h''(z) - h'(z))\frac{\partial f}{\partial s}) = |\frac{\partial u}{\partial s}|^2 > 0$$

This is an elliptic equation of order two, without zero order term, hence satisifes the maximum principle: a solution has no local maximum on an interior point of the domain.

□

Proof of 1.6. The proof is classical, and goes as follows.

Let J_s be an admissible almost complex structure for ω_s, ∂W. We extend J_s to $s \in \mathbb{R}$, so that $J_s = J_0$ for $s \leq 0$ and $J_s = J_1$ for $s \geq 1$. We similarly extend ω_s (so that $\omega_s = \omega_0$ for $s \leq 0$, and $\omega_s = \omega_1$

for $s \geq 1$) and denote by X_H^s the Hamiltonian vector field of H for the symplectic form ω_s.

Now consider the equation

$$\partial_s u + J(s,u)(\partial_t u - X_H^s(t,u)) = 0$$

where $(s,t) \in \mathbb{R} \times S^1$.

We then define a map

$$FC^*(H,\omega_0) \longrightarrow FC^*(H,\omega_1)$$

by counting the number of solutions of the above equation (modulo the action of $\mathbb{R} \times S^1$), such that

$$\lim_{s\to-\infty} u(s,t) = x(t) \qquad \lim_{s\to+\infty} u(s,t) = y(t)$$

where x, y are 1-periodic solutions of X_H^0 and X_H^1 respectively, with the same index.

That this map commutes with the coboundary, is due to the fact that, to any solution of the above equation, we may glue a solution of

$$\partial_s u + J_0(u)(\partial_t u - X_H^0(t,u)) = 0$$

at $s = -\infty$, or a solution of

$$\partial_s u + J_1(u)(\partial_t u - X_H^1(t,u)) = 0$$

at $s = +\infty$.

We only have to show that the set of such solutions stays in a compact set, and for this, check that the maximum principle applies. The ordinary maximum principle tells us that if Ω is a domain with pseudo-convex boundary, no holomorphic curve in Ω may be tangent to $\partial\Omega$. In other words, if f is plurisubharmonic, and $u : D \to \Omega$ is holomorphic, $f \circ u$ satisifes the maximum principle, since a simple computation shows that $\Delta(z \circ u) = u^*(dd^c f) \geq 0$. As D. McDuff noticed in [McD], a similar computation applies to the case of H having $\partial\Omega$ as a regular pseudoconvex level, and u a solution of $\partial_J u = -\nabla H(u)$. Here we need a slight extension of this, for J depending on s.

As a result, we get a map

$$FH^*(H,\omega_0) \longrightarrow FH^*(H,\omega_1)$$

This map is easily seen to be independent of the choices of J, and of the family ω_s, it only depends on its homotopy class with fixed endpoints.

The map is easily seen to be an isomorphism, by following the inverse path.

Now as we consider the family H_λ such that $\lim_{\lambda\to\infty} FH^*(H_\lambda, \omega_s)$, then the isomorphisms for each λ yield the sought after isomorphism in the limit.

□

1.4 Product structure

There are several "product structures" defined on $FH^*(H; a, b)$. Apart from the one we are describing here, there is also the so called "pant product" (see [Wi], [Sch], [Piu],[P-S-S] [RT]) or "quantum product".

It is rather clear that studying the functorial properties of the "pant product" with respect to the map we defined or shall define would be extremely useful. Here we shall only use the product described in [V8] that gives $FH^*(H; a, b)$ the structure of an $H^*(\Lambda M)$module (ΛM is the free loop space of M).

This structure was only defined for real coefficients (this is sufficient for our purpose), and for a compact symplectic manifold, but this last condition may be removed without difficulty, due to the pseudoconvexity of $\partial M \times \{z\}$.

It is not difficult and rather useful to notice that the maps defined above are $H^*(\Lambda M)$-module morphisms. Note in particular that the $H^*(\Lambda M)$ module structure on $H^*(M, \partial M)$ is just the one induced by the ordinary cup product of $H^*(M)$, through the map $\gamma^* : H^*(\Lambda M) \to H^*(M)$ (γ is the inclusion of constant loops $M \to \Lambda M$). That is

Lemma 1.8. *We have, for $\alpha \in FH^*(M), \varphi \in H^*(\Lambda M)$*

$$c^*(\alpha \cup \varphi) = c^*(\alpha) \cup \gamma^*(\varphi).$$

where c^* is the previously defined map $c^*_M : FH^*(M) \longrightarrow H^*(M, \partial M)$.

Proof. In the exact sequence of prop 1.4, let α be a cochain in $FH^*(M)$ that is $\alpha = \sum_x \alpha_x[x]$ where x is a closed orbit of the Hamilton H.

Then

$$c^*(\alpha) = \sum_{x\in M} \alpha_x[x], \quad \text{and}$$

$$
\begin{aligned}
\alpha \cup \varphi &= \sum_{y} \alpha_x \Big(\int_{\mathcal{M}(x,y)} \varphi \Big)[y], \quad \text{and} \quad \text{thus} \\
c^*(\alpha \cup \varphi) &= \sum_{y \in M} \alpha_x \Big(\int_{\mathcal{M}(x,y)} \varphi \Big)[y] \\
c^*(\alpha) \cup \gamma^*(\varphi) &= \sum_{\substack{y \in M \\ x \in M}} \alpha_x \Big(\int_{\Gamma(x,y)} \gamma^*(\varphi) \Big) y
\end{aligned}
$$

where $\Gamma(x,y) \subset \mathcal{M}(x,y)$ is the set of trajectories independent on $\theta \in S^1$.

But H being C^2 small on M, we have that $\Gamma(x,y) = \mathcal{M}(x,y)$(see e.g. [S-Z]). ◻

2 The transfer morphism

Our main goal now is to associate to a symplectic codimension 0 inclusion $j : W \to M$ of symplectic manifolds with contact type boundary, a map, denoted $(Fj)^!$ from $FH^*(W)$ to $FH^*(M)$ and to establish its main properties. We wish to warn the reader that some assumption (see assumption (A)) is necessary in order to construct $(Fj)^!$, and this assumption will be crucial in establishing our main results.

Our main tool will be a Hamiltonian H on M such that, apart from the constants, its 1-periodic solutions are either closed characteristics on ∂W-each of them appearing twice, once with positive action, and once with negative action - or closed characteristics on ∂M - each appearing once with negative action.

Then, the exact sequence 1 associated to such an H, for $a = -\infty,\ b = 0,\ c = +\infty$ yields a map $FH^*(H; b, c) \to FH^*(M)$.

It remains to identify $FH^*(H; b, c)$ to $FH^*(W)$, and this will require that the connecting trajectories between closed characteristics of ∂W remain inside W.

We first slightly adapt our construction of $FH^*(W, a, b)$ for $a = -\delta, b > 0$.

Assume there is a symplectic embedding of $(\partial W \times [\varepsilon, 1], d(t\alpha))$ into W, sending $\partial W \times \{1\}$ to ∂W,

and let $K_{\lambda,\mu}^{\varepsilon,\alpha}$ be the Hamiltonian of figure 3 (note that $K_\lambda = K_{\lambda,\lambda}^{1,\alpha} = K_{\lambda,\mu}^{1,+\infty}$ for all α, μ). Then we claim that if η_λ is such that there is no closed characteristic for ∂W with area in $[\lambda - \eta_\lambda, \lambda + \eta_\lambda]$, and $\alpha(\lambda - \eta_\lambda) < (\alpha - \varepsilon)\lambda$

that is $\varepsilon < \alpha\eta_\lambda/\lambda$, then the only closed characteristics occurring as periodic orbits for the Hamiltonian vector field of $K_\lambda^{\varepsilon,\alpha}$ yielding a positive action will be those on $\{z = \varepsilon\} = \partial W \times \{\varepsilon\}$, while these on $\partial W \times \{\alpha\}$ have negative action.

Those with positive action will have this action in the interval $[0, \varepsilon\lambda]$, corresponding to closed characteristics of ∂M with area in $[0, \lambda]$. Thus as μ varies between 0 and λ, $FH^*(K_{\lambda,\mu}^{\varepsilon,\alpha}, -\delta, a)$ does not change (according to Prop.1.3), and we get the isomorphism

$$FH^*(K_{\lambda,0}^{\varepsilon,\alpha}, -\delta, a) \simeq FH^*(K_{\lambda,\lambda}^{\varepsilon,\alpha}, -\delta, a) \simeq FH^*(K_\lambda, -\delta, a).$$

Definition. Let W be a symplectic manifold with contact type ∂W. We shall say that W has *restricted contact type boundary* if the contact form α on ∂W extends to a global solution of $d\alpha = \omega$ on W

Now assume $W \subset M$ has restricted contact type boundary in W. Note that in general the form α above does not extend to M. But, if φ_t is a conformal flow on W such that $\varphi_t^*\omega = e^{-t}\omega$, we may use this flow to find a symplectic embedding of $(\partial W \times [\varepsilon, 1], d(t\alpha))$ into W (sending $\partial W \times \{1\}$ on ∂W)

For example if L is a Lagrangian submanifold in M, and W is a neighborhood of L that, according to Weinstein's theorem, we may identify to a neighborhood of the zero section in T^*L, and our assumption is satisfied, even when L is not exact (so that ∂W is not, in general, of restricted contact type in the usual sense).

More generally, until the end of this section , we only require the existence of a family $\rho_\varepsilon : \partial W \times [\varepsilon, 1] \to M$ of symplectic embedding, such that for $\varepsilon' < \varepsilon$, $\rho_{\varepsilon'}$ restricted to $\partial W \times [\varepsilon, 1]$ is Hamiltonianly isotopic to ρ_ε.

Definition. We set $K_{\lambda,W}^\varepsilon$ to be the function $K_{\lambda,0}^{\varepsilon,1}$ on $\partial W \times [\varepsilon, 1]$, and $K_{\mu,W} = K_{\lambda,\mu}^{1,1}$ (which does not depend on λ!)

Now let H be a hamiltonian on $\widehat{M}$, depending on the set of parameters $\varepsilon, \lambda, \mu$ such that

$$\begin{aligned} H &= K_{\lambda,W}^\varepsilon \quad \text{on} \quad \partial W \times [\varepsilon, 1] \\ H &= 0 \quad \text{inside} \quad \partial W \times \{\varepsilon\} \\ H &= \lambda(1-\varepsilon) \quad \text{inside} \quad M - W \\ H &= K_{\mu,M} + \lambda(1-\varepsilon) \quad \text{on} \quad \widehat{M} - M \end{aligned}$$

We have the following isomorphisms

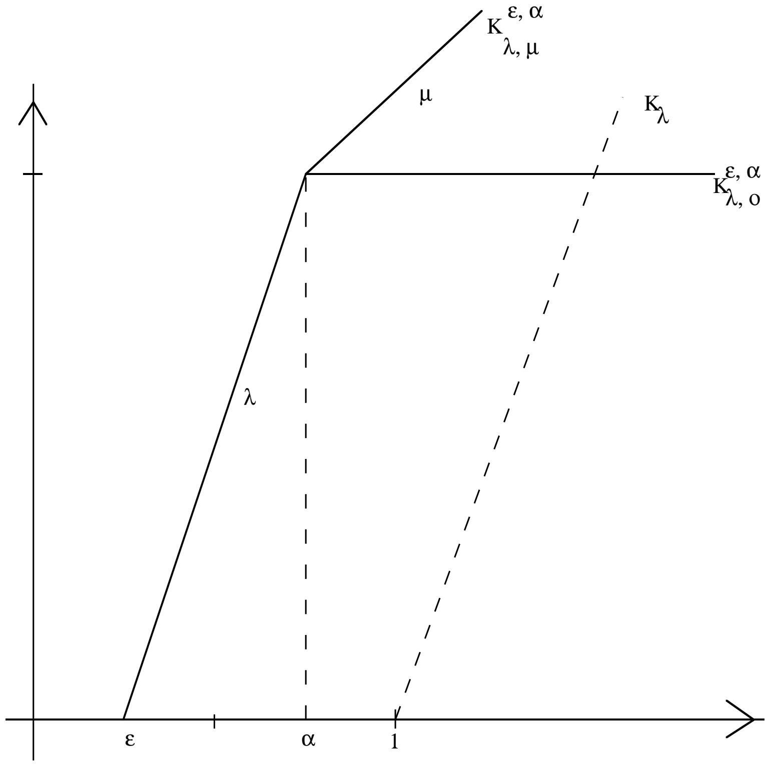

Figure 3: The family $K^{\varepsilon,\alpha}_{\lambda,\mu}$

$$FH^*(H;a,b) \simeq FH^*(K_{\mu,M};a+\lambda(1-\varepsilon),b+\lambda(1-\varepsilon))$$

for $a < a(\varepsilon,\lambda), b > b(\varepsilon,\lambda)$, because $H = K_{\mu,M} + \lambda(1-\varepsilon)$ outside a compact set.

Now the chain complex $C*(H;a,b)$ decomposes in two factors. The first one is generated by the set of 1-periodic orbits of H inside $\partial W \times \{2\varepsilon\}$ the second one by all other 1-periodic orbits. It is important to notice that the only ones with nonnegative action are those in the first factor, since we show that those near $\partial W \times \{1\}$ have negative action, and the same holds for the constants in $M - W$ (with action $-\lambda(1-\varepsilon)$) and the periodic orbits near $\partial M \times \{1\}$ with action $\mu - \lambda(1-\varepsilon)$, that will be negative for λ large enough.

We then consider the differential d_M on each factor. Note that since $< dx, y >$ may be nonzero only if the action of x is less than the action of y

it must send the first summand $C^*(K^\varepsilon_{\lambda,W}, -\delta, b)$ into itself. However on this summand d_M will coincide with the differential d_W only under the following extra assumption

$$(A)\begin{cases} \text{Any solution} & u:\mathbb{R}\times S^1 \to M \text{ of} \\ \overline{\partial} u = \nabla H(u) & \lim_{t\to\pm\infty} u(t,\theta) = x_\pm(\theta) \\ \text{where} \quad x_\pm(\theta) \quad \text{solves} & \dot{x}_\pm(\theta) = X_{\lambda,W}(x_\pm(\theta)) \\ \text{has its image inW} & \end{cases}$$

Note that we only need (A) to hold for a given complex structure, and not for all complex structures. However, in this case $(Fj)^!$ may depend on J. We have almost proved

Proposition 2.1. *Let ∂W be of restricted contact type in W and $j: W \to M$ be a symplectic embedding. Then, assuming (A) holds, there is a morphism, denoted $(Fj)!$, from $FH^*(W)$ to $FH^*(M)$.*

Proof. : Since $d_W = d_M$ on $C^*(K^\varepsilon_{\lambda,W}, -\delta, b)$, we get a map $FH^*(K^\varepsilon_{\lambda,W}, -\delta, b)$ into $FH^*(H; a, b)$.

Now $FH^*(K^\varepsilon_{\lambda,W}; -\delta, b) \simeq FH^*(K_{\lambda,W}; -\delta, b/\varepsilon)$ so that

$$\lim_{\lambda\to+\infty}\lim_{\varepsilon\to 0} FH^*(K^\varepsilon_{\lambda,W}; -\delta, b) \simeq FH^*(W)$$

while

$$lim_{\lambda\to+\infty}\lim_{\varepsilon\to 0} FH^*(H; -\infty, b) \simeq FH^*(K_{\mu,M}).$$

Now taking the limit for μ going to $+\infty$, we get our map $(Fj)^!$.

□

We now wish to prove some basic properties of $(Fj)!$. A first obvious result is

Theorem 2.2. *Assume (A) holds, and denote by $(\Lambda j)^*: H^*(\Lambda M) \to H^*(\Lambda W)$ the map induced by the inclusion j. Then we have*

$$(Fj)^!(\alpha \cup (\Lambda j)^*(\beta)) = (Fj)^!(\alpha) \cup \beta$$

for

$$\alpha \in FH^*(W), \beta \in H^*(\Lambda M).$$

The proof is straightforward, using the definition of the cup-product as in [V8].

3 The main theorem

Now we wish to prove

Theorem 3.1. (Main theorem) *Let $j^! : H^*(W, \partial W) \to H^*(M, \partial M)$ be the usual push-forward map in cohomology. Then, under assumption (A) we have a commutative diagram*

$$\begin{array}{ccc} FH^*(W) & \xrightarrow{(F_j)^!} & FH^*(M) \\ {\scriptstyle c^*}\downarrow & & \downarrow{\scriptstyle c^*} \\ H^*(W, \partial W) & \xrightarrow{j^!} & H^*(M, \partial M) \end{array}$$

We first need

Lemma 3.2. *Let H_λ be a family of Hamiltonians. Let $a < b < c$, and consider the long exact sequence (1.6).*

$$FH^*(H_\lambda; b, c) \to FH^*(H_\lambda; a, c) \to FH^*(H_\lambda; a, b) \stackrel{\partial^*}{\to}$$

If for λ in $[0, 1]$ no 1-periodic orbit of X_{H_λ} has action in $\{a, b, c\}$, then we have a commutative diagram

$$\begin{array}{ccc} FH^*(W) & \longrightarrow & FH^*(H_1; b, c) \\ \downarrow & & \downarrow \\ FH^*(H_0; a, c) & \longrightarrow & FH^*(H_1; a, c) \\ \downarrow & & \downarrow \\ FH^*(H_0; a, b) & \longrightarrow & FH^*(H_1; a, b) \\ \downarrow & & \downarrow \end{array}$$

where the horizontal arrows are the isomorphisms of Proposition 1.1 and the vertical arrows the maps in the exact sequence (1).

The proof is analogous to 1.1.
We must identify

Lemma 3.3. *Let $H = K^{\varepsilon}_{\lambda,W}$ be as in definition 2. Then the map $H^*(M^b, M^{-\delta}) \to H^*(M^b, M^a)$ (a, b, δ are determined as in the definition of $K^{\varepsilon}_{\lambda,W}$) may be identified to the map $j^! : H^*(W, \partial W) \to H^*(M, \partial M)$*

Proof. Once we notice that $A_H = -H$ on the set of constant loops, we may identify, modulo excision the pairs $(W^b, W^{-\delta})$ to $(W, \partial W)$ as well as $(M^b, M^{-\delta})$ to $(M, \partial M)$.

Note that the push-forward commutes with excision.

Now we claim that the natural map $H^*(f^c, f^b) \to H^*(f^c, f^a)$ corresponds to the map j^* induced by the inclusion $(f^c, f^b) \to (f^c, f^a)$. Indeed, for a cochain on f^c vanishing on f^b, we may consider it as a cochain vanishing on f^a, and this defines both maps. □

We shall also need:

Lemma 3.4. *The natural maps (from the triple exact sequence)*

$$FH^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty) \to FH^*(K^{\varepsilon,1}_{\lambda,\mu}; -\infty, +\infty)$$

and

$$FH^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty) \to FH^*(K^{\varepsilon,1}_{\lambda,\mu}, a, b)$$

may both be identified to the same map

$$FH^*(W; -\infty, \lambda) \to H^*(W, \partial W).$$

Proof. Let $K^{\varepsilon,\alpha}_{\lambda,\mu}$ be the Hamiltonian from section 1,

Note that $FH^*(K^{\varepsilon,\alpha}_{\lambda,\mu}; -\infty, +\infty) \simeq FH^*(K_{\mu}; -\infty, +\infty)$. Denoting by $\boxed{\partial^- W}$, $\boxed{\partial^+ W}$ the set of periodic orbits near $\partial W \times \{\alpha\}$ and $\partial W \times \{\varepsilon\}$, we may represent the situation, as a α goes from ε to 1, on figure 4.

Here $\boxed{\partial W}$ is the set of periodic orbits for $K^{\varepsilon,\varepsilon}_{\lambda,\mu}$, that corresponds to the set of solutions of X_H of area less than μ. Taking μ small enough, makes $\boxed{\partial W}$ empty, and this is the only case we really need. We shall make this assumption from now on.

Let us denote by $\boxed{W}$ the set of constant critical points of H, located on W. The correspond to zero the critical set for A_H. Now for $\alpha = 1$, let a, b be levels separating $\boxed{\partial^- W}$ from $\boxed{W}$ and $\boxed{W}$ from $\boxed{\partial^+ W}$.

Then $FH^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty) \simeq FH^*(W; -\infty, \lambda)$ and $FH^*(K^{\varepsilon,1}_{\lambda,\mu}, -\infty, +\infty) \simeq H^*(W, \partial W)$.

Thus we have a map

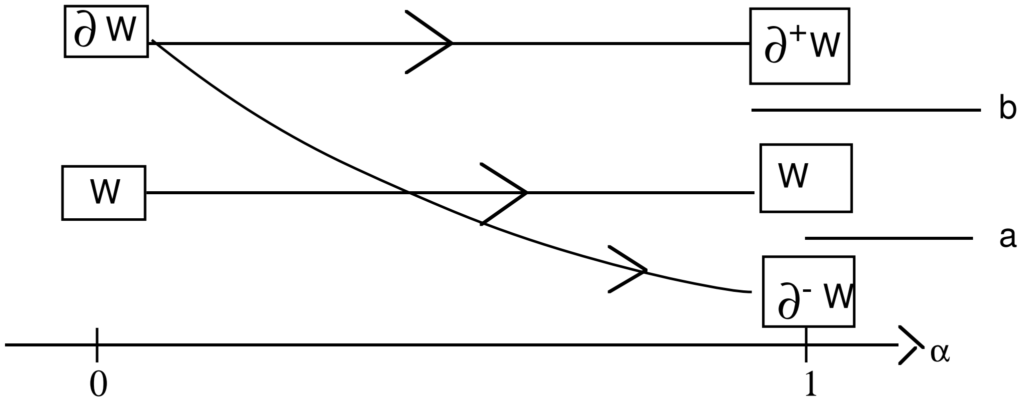

Figure 4: Variational picture of the homotopy, $\alpha \in [\varepsilon, 1]$

$$\phi : FH^*(W; -\infty, \lambda) \to H^*(W, \partial W).$$

But also, since $FH(K^{\varepsilon,1}_{\lambda,\mu}; a, b) \simeq H^*(W, \partial W)$, we get a second map

$$\psi : FH^*(W; -\infty, \lambda) \to H^*(W, \partial W)$$

We claim that $\phi = \psi$ in cohomology.

Indeed, let $c = \sum_x c_x[x]$ be a cochain in $C^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty)$, then $\phi(c)$ consists in the same cochain, but considered in $C^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty)$, while $\psi(c) = \sum_{x \in W} c_x[x]$.

We claim that $\phi(c) - \psi(c)$ is a coboundary in $C^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty)$. Since the chain map χ induced by the homotopy $\alpha \to K^{\varepsilon,\alpha}_{\lambda,\mu}$ sends $[x]$ to $[x]$ for x in W and $[x]$ to zero otherwise, (see figure 4).

Thus χ sends $\varphi(c) - \psi(c)$ to zero and being a cohomology isomorphism, this implies that $\varphi(c) = \psi(c)$ in $FH^*(K^{\varepsilon,1}_{\lambda,\mu}; a, +\infty)$ as announced. □

Proof of Main theorem. We may now conclude the proof of our main theorem. Let us consider the following Hamiltonian depending on the parameters λ, μ.

As usual we identify (symplectically) a subset of W with $\partial W \times [\varepsilon, 1]$.

We denote by α the contact form on ∂W and by ϕ the contact form on ∂M.

We have
- on $W:\ H = K^{\varepsilon}_{\lambda,W}$ with ε small enough,
- on $M - W:\quad H = \lambda(1-\varepsilon)$
- on $\widehat{M} - M:\ H = K^{\sigma,t}_{\mu,0} + \lambda(1-\varepsilon)$

We then have several families of 1-periodic solutions for X_H:
(I) constants in W, on level 0
(II) closed characteristics on $\partial W \times \{\varepsilon\}$, on level $\varepsilon \int_\gamma \alpha > 0$.
(III) closed characteristics on $\partial W \times \{1\}$, on level $-\lambda(1-\varepsilon) < \int_\gamma \alpha - \lambda(1-\varepsilon) < 0$.
(IV) constants in $M - W$ on level $-\lambda(1-\varepsilon)$
(V) closed characteristics in $\partial M \times \{\sigma\}$, on level $\sigma \int_\gamma \varphi - \lambda(1-\varepsilon)$.
(VI) closed characteristics on $\partial M \times \{\tau\}$, on level $\tau \int_\gamma \varphi - \lambda(1-\varepsilon) - \mu(\tau - 1)$, which is arbitrarily small with τ large. This Hamiltonian is represented on figure 5.

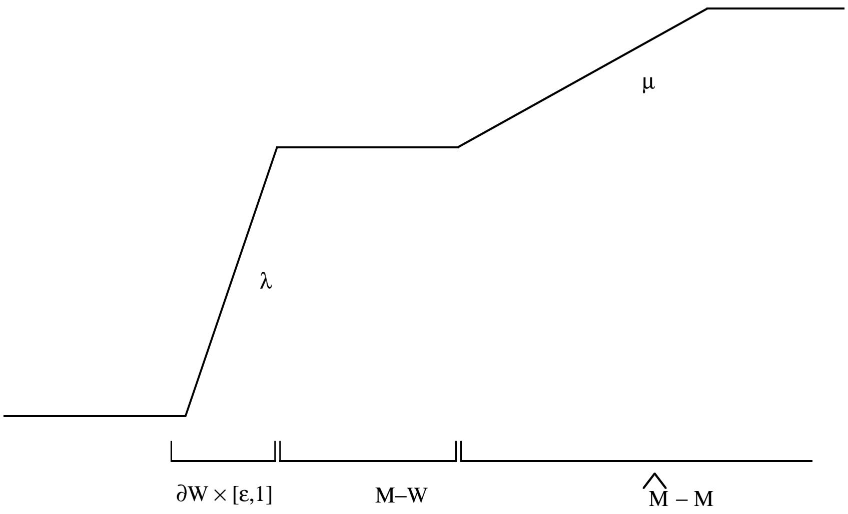

Figure 5: The Hamiltonian H defining $(Fj)^!$

On figure 6 we represented the different sets of critical points, the sets (I), (II), (III), (IV), (V), and (VI) are respectively denoted by $\boxed{W}$, $\boxed{\partial^+ W}$,

$\boxed{\partial^- W}$, $\boxed{M-W}$, $\boxed{\partial^+ M}$, $\boxed{\partial^- M}$ (for obvious reasons). The height of the set represents the level of the set of critical points. It is useful to notice that:
- for $\boxed{\partial^+ M}$ (i.e.(V)) since $\int_\gamma \varphi$ is in $]0, \mu]$, the level is in the interval $\sigma\delta - \lambda(1-\varepsilon)$ and $\sigma\mu - \lambda(1-\varepsilon)$ (δ is the smallest area of a closed characteristic of ∂M). By taking σ large enough, we can make $\sigma\delta - \lambda(1-\varepsilon)$ large, and the critical set $\boxed{\partial^+ M}$ is above all others, while taking $\sigma = 1$ and λ large will make $\boxed{\partial^+ M}$ to be below level zero.

In other words, we are free to move $\boxed{\partial^+ M}$ between, somewhere below the level of $\boxed{W}$ and an arbitrary high level.

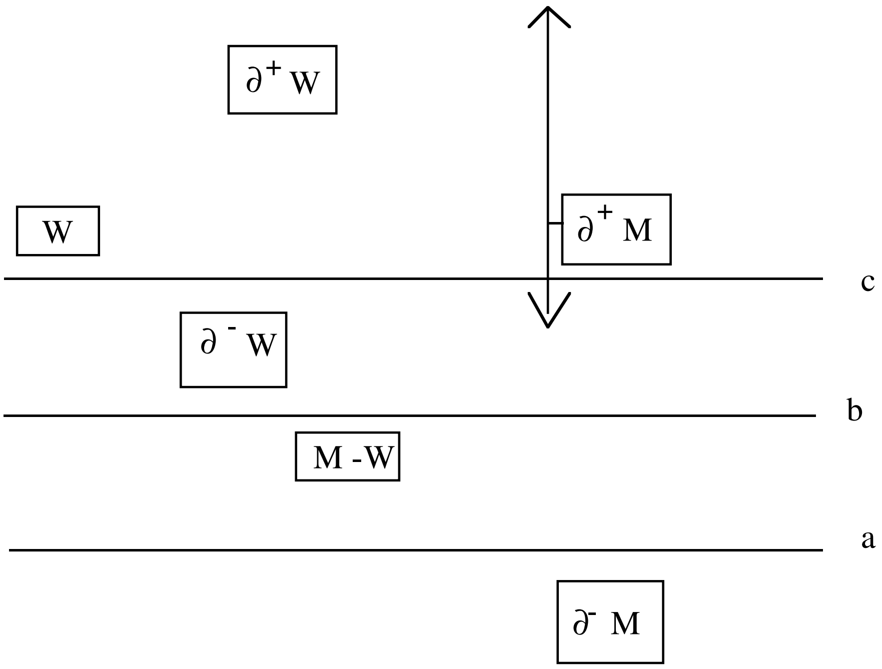

Figure 6:

Now let a, b, c be three levels as in figure 6. We have:
-$FH^*(H; c, +\infty) \simeq FH^*(W, -\infty, \lambda)$ according to lemma 1.5, provided we make $\boxed{\partial^+ M}$ to be below level c.
$FH^*(H; a, +\infty) \simeq FH^*(M, -\infty, \mu)$, since without crossing level a, we may deform H until H is constant inside M. Then the deformed Hamiltonian will be the one defining $FH^*(M; -\infty, \mu)$.

- $FH^*(H;-\infty,+\infty) \simeq H^*(M,\partial M)$ as we easily see by deforming H to a $K_{\delta,M}$ for δ small enough.

The exact sequence associated to $(-\infty,a,c,+\infty) \to FH^*(H;a,+\infty) \to FH^*(H,-\infty,+\infty)$ and yields, using the above identifications

$$FH^*(W,-\infty,\lambda) \stackrel{(F_j)^!}{\to} FH^*(M;-\infty,\mu) \stackrel{c^*}{\to} H^*(W,\partial W)$$

The identification of the second map with c^* follows from lemma 3.4.

Now since $\boxed{\partial^+ M}$ is below level c, we may deform H, without anything crossing level c, until the new Hamiltonian, H_1 is equal to $K_{\delta,M}$ on $\widehat{M}-M$.

According to lemma 3.2, we may identify

$$FH^*(H;c,+\infty) \stackrel{c^*\circ(F_j)^!}{\to} FH^*(H;-\infty,+\infty)\,\text{with}$$

$$FH^*(H_1;c,+\infty) \to FH^*(H_1;-\infty,+\infty)$$

Then $\boxed{\partial^- M}$ and $\boxed{\partial^+ M}$ disappeared (by " killing each other ").

We may now consider the map

$FH^*(H_1;c,+\infty) \to FH^*(H_1;-\infty,+\infty)$ as the composition of

$$FH^*(H_1;c,+\infty) \to FH^*(H_1;b,+\infty)$$

and of $FH^*(H_1;b,+\infty) \to FH^*(H_1,-\infty,+\infty)$.

According to lemma 3.4 again, the first map may be identified to

$$FH^*(W;-\infty,\lambda) \stackrel{c^*}{\to} H^*(W,\partial W).$$

To understand the second one, we only need one more deformation, from H_1 to H_2, by making λ, equal to some very small positive η.

Then $\boxed{\partial^+ W}$ and $\boxed{\partial^- W}$ disappear, and the only periodic orbits left-are the constants in $\boxed{W}$ and $\boxed{M-W}$. Since the only Floer trajectories are then gradient lines of H_2, the map

$$FH^*(H_2;b,+\infty) \to FH^*(H_2;-\infty,+\infty)$$

may be identified to the map

$$j^! : H^*(W,\partial W) \to H^*(M,\partial M)$$

according to the lemma 3.3.

We may then conclude that $c^* \circ (Fj)^! = j^! \circ c^*$ as required. $\square$

4 Applications to Weinstein's conjecture and obstructions to Lagrange embeddings

Consider a compact symplectic manifold with contact type boundary,$(M, \partial M)$. From proposition 1.4, we see that if the map $FH^*(M) \to H^*(M, \partial M)$ fails to be an isomorphism, then there must be closed characteristics on ∂M, hence ∂M satisfies Weinstein's conjecture.

Conjecture. Let Σ be a contact type hypersurface. Then Σ carries at least one closed characteristics.

Definition. We shall say that $(M, \partial M)$ satisfies the *Algebraic Weinstein conjecture* if either

$$(a) \qquad FH^*(M) \to H^*(M, \partial M) \text{is not injective} .$$

or

$$(b) \qquad FH^*(M) \to H^*(M, \partial M) \text{is not onto} .$$

We shall even be more precise, and state that $(M, \partial M)$ satisfies case (a) (resp.(b)) of the Algebraic Weinstein Conjecture (AWC) in degree k when the map $FH^k(M) \to H^k(M, \partial M)$ is not injective (resp. onto).

Finally, if M satisfies case (b) of the AWC in degree $2n$, we shall say that M is an $SAWC$ (Strong Algebraic Weinstein) manifold.

Clearly the AWC for $(M, \partial M)$ implies Weinstein's conjecture for ∂M, as will be proved in Proposition 4.1

Examples. 1. $M = D^{2n}, \partial M = S^{2n-1}$ with the canonical symplectic structure satisfies case (b) of the AWC in degree $2n$, since $FH^*(D^{2n}) = 0$. This is proved by considering the following Hamiltonian on $\mathbb{R}^{2n}$: $H(x) = c \cdot |x|^2$, with c a large constant different from an integer multiple of 2π. Then A_H has a single critical point, corresponding to the constant orbit at the origin, since there is no non trivial solution of $\dot{x} = Jcx$. Moreover such an orbit has Conley-Zehnder index equal to the integer part of $\frac{c}{2\pi}$. Thus we may prove that $FH^k(D^{2n}) = 0$ for $k \leq \frac{c}{2\pi}$. Since c is arbitrarily large, we proved that $FH^k(D^{2n}) = 0$ for all k.

This implies that any hypersurface in $\mathbb{R}^{2n}$ of restricted contact type (see definition 2 has a closed characteristic (see[V3], where this is proved for the general contact type case, and [Gi], [He] for counterexamples if this assumption is dropped).

2. $M = DT^*N$ the unit disk bundle in T^*N does *not* satisfy case (b) of the AWC as we define it. Indeed, according to Part II, we have a commutative diagram

$$\begin{array}{ccc} FH^*(DT^*N) & \longrightarrow & H^*(DT^*N, ST^*N) \\ {\scriptstyle\text{isomorphism}}\downarrow & & \downarrow{\scriptstyle\text{Thom isomorphism}} \\ H^{*n}(\Lambda N) & \underset{c^*}{\longrightarrow} & H^{*-n}(N) \end{array}$$

(c^* is induced by inclusion of constants $c : N \to \Lambda N$). but c^* is onto, having as right inverse $(ev)^*$, induced by the evaluation map, $ev(\gamma) = \gamma(1)$.

However the equivariant version of this works for N simply connected, as it is shown in subsection 5.2 (and was essentially proved in [V4])

3. We shall prove below that any subcritical Stein manifold satisfies case (b) of the AWC. Of course Example 1 is just a special case of this. And Example 2 shows that this cannot be improved, since T^*N is a Stein manifold, for some complex structure compatible with its canonical symplectic structure.

The following result shows how the $SAWC$ property may be inherited.

Theorem 4.1. *If $(M, \partial M)$ is an $SAWC$ manifold, then*

- *the same holds for any codimension 0 submanifold W, with restricted contact type boundary.*
- *Any contact type hypersurface in M bounding a compact region has a closed characteristics.*

PROOF: We have the following commutative diagram

$$\begin{array}{ccc}
FH^*(M) & \xrightarrow{c^*} & H^*(M,\partial M) \\
{\scriptstyle (Fj)^!}\downarrow & & \downarrow{\scriptstyle j^!} \\
FH^*(W) & \xrightarrow[c^*]{} & H^*(W,\partial W)
\end{array}$$

For $* = 2n$ the map $j^!$ is an isomorphism, so that if $H^{2n}(M,\partial M)$ is not in the image of $FH^*(M)$, then $H^{2n}(W,\partial W)$ cannot be in the image of $FH^*(W)$.

Now we still need to prove that (A) is satisfied. First, remark that assumption (A) may be replaced by the following slightly weaker hypothesis.

For any given finite set of closed characteristics of ∂W, there is an admissible almost complex structure, J, such that any solution of $\overline{\partial}u = \nabla H(u)$ converging at each end to a closed characteristic in the above set is contained in W.

Now for each positive ε, we have an embedding of $\partial W \times [\varepsilon, 1]$ such that the action of a closed characteristic on $\partial W \times \{t\}$ is t times the action of the corresponding characteristic on $\partial W \times \{1\}$ (here it is crucial that ∂W should be of restricted contact type in M).

Now let J be such that each $\partial W \times \{t\}$ is pseudoconvex. Then a solution of $\overline{\partial}u = \nabla H(u)$ as in (A) must exit W, according to the maximum principle.

Indeed, let us consider a Floer trajectory with its ends in $\partial W \times \{\varepsilon\}$. According to Lemma 1.7 if the solution is in W, it must stay in $\partial W \times \{\varepsilon\}$. But then the solution does not violate assumption (A).

We will prove that its area is bounded from below by some constant, hence the difference of the action of the two closed characteristics connected by this solution is also bounded from below. But as ε goes to zero this difference will be bounded by $C\varepsilon$, hence goes to zero, a contradiction.

Let us now prove this claim for $H = K_{\lambda,0}^{\varepsilon,\alpha}$ (see Figure 3).

First, let $\Gamma = \mathbb{R}\times S^1 = \Gamma_+ \cup \Gamma_-$ where Γ_- is the interior of $u^{-1}(\partial W \times \{\alpha\})$, and Γ_+ its complement.

Let x, y be the two periodic solutions. Then

$$A_H(y) - A_H(x) = \int_\Gamma |\frac{\partial u}{\partial s}|^2 dsdt \geq \int_{\Gamma_+} |\frac{\partial u}{\partial s}|^2 dsdt$$

Since H is constant on $u(\Gamma_+)$, so that u is J-holomorphic on Γ_+, and we have:

$$\int_{\Gamma_+} |\frac{\partial u}{\partial s}|^2 ds dt = \int_{\Gamma_+} u^*\omega$$

As a result, $A_H(y) - A_H(x) \geq \int_{\Gamma_+} u^*\omega + 0(\alpha)$. as α goes to zero.

Let us prove that $u(\Gamma_+)$ has its area bounded from below by a constant depending only on J and W.

Lemma 4.2. *The area of a curve C in $\partial W \times [1/2, 1]$ intersecting $\partial W \times \{1/2\}$ has its area bounded from below by a constant depending only on J and W.*

Proof. It is enough to consider a connected component of C that intersects $\partial W \times \{\alpha\}$. Note that C must also intersect $\partial W \times \{\alpha\}$, otherwise the second coordinate, z, would have a maximum on C, which is impossible by the maximum principle. Let $\Sigma = \partial W \times \{\frac{1+\alpha}{2}\}$. There is a constant r such that for $x \in \Sigma$ we have that $d(x, \partial W \times \{\alpha\})$ and $d(x, \partial W \times \{1\})$ are greater than r. Then, since $\Sigma \cap C \neq \emptyset$ we have that for $x_0 \in \Sigma \cap C$, $B(x_0, r) \cap C$ is a closed J holomorphic curve through the origin, and such a curve has its area bounded from below as a function of r and the sectional curvatures of W (see the Appendix of Part II for a detailed proof).

We now prove the second claim. Let us notice that from the above proof, we may also infer that condition (A) may not be violated by a Floer trajectory connecting two constant solutions, otherwise we would have a sphere with positive area in M, contrary to our assumptions. Thus if W has contact type in M, and M is $SAWC$, then either condition (A) is violated, or the map $(Fj)^!$ is defined, and in both cases there must be a non-constant closed characteristics on ∂W. This concludes our proof. ∎

Theorem 4.3. *If W is an $SAWC$ manifold , then:*

1. *There is no exact embedded Lagrange submanifold in W.*
2. *If L is a Lagrange submanifold in W, there exists a curve γ on L such that*
   (a) *γ is contractible.*
   (b) $\mathrm{area}(\gamma) > 0$, *where* $\mathrm{area}(\gamma) = \int_D u^*(\omega)$ *for any map u from the disc D to W bounding γ.*
   (c) *$<\mu, \gamma> \leq n+1$, where μ is the Maslov class.*

Proof. The first assertion follows from the above theorem, and example 2, since an exact embedding would yield according to Weinstein's theorem, an embedding of DT^*L in W of restricted contact type. Then DT^*L would be $SAWC$, contrary to what we claim. We refer to the appendix to prove that the condition $[\omega]\pi_2(M)$ is sufficient for the theorem.

As for 2, it follows from the ideas of [V4, V1]. Of course this statement about the Maslov number is only meant modulo $2c_1(M)$, since the Maslov class is otherwise undefined. □

We now prove:

Theorem 4.4. *Let W be a subcritical Stein manifold. That is W has boundary of contact type, and there is a negative strictly plurisubharmonic function on W, vanishing on ∂W, having all its critical points of index strictly less than $n = 1/2\dim(W)$. Then W is $SAWC$.*

Proof. For the basic properties of symplectic Stein manifolds (called Weinstein manifolds in [El-G]) we refer to [El], [El-G].

Let ϕ be a plurisubharmonic function on W, and X be the union of the unstable manifolds of the critical points of ϕ. Note that $\nabla\phi$ is a Liouville vector field, hence X is a union of isotropic submanifolds, of dimension at most $n-1$.

Let ψ_t be the flow of $Y = -\nabla\phi$. Then the $W_{1/t} = \psi_t(W)$ for $t \geq 0$ make a fundamental family of neighborhoods of X^{n-1}. Moreover they are all conformally equivalent. We claim that there is a family U_ε of neighborhoods of X^{n-1}, such that for ε small enough, $FH^{2n}(U_\varepsilon, -\delta, \varepsilon^{3/2}) = 0$. Since the map $FH^{2n}(U_\varepsilon) \to H^{2n}(U_\varepsilon, \partial U_\varepsilon)$ factors through $FH^{2n}(U_\varepsilon, -\delta, \varepsilon^{3/2}) = 0$, this proves that the map vanishes. Therefore U_ε is $SAWC$. Since for some α, ϵ we have the inclusion $W_\alpha \subset U_\varepsilon$, the same holds for W.

Now the assertion about U_ε may be proved by induction over the number l of critical points of ϕ. This is clear for $l = 0$ since we may then choose U_ε to be a ball (and in this case the statement is proved as Example 1 above).

Now let X^k be the union of the unstable manifolds of the first k critical points of ϕ. Assume we have a neighbourhood U_ε^{l-1} of X^{l-1} such that $FH^*(U_\varepsilon^{l-1}, -\delta, \varepsilon^{3/2}) = 0$. We shall then construct U_ε^l with the same property.

Since X^l is obtained from X^{l-1} by attaching an isotropic k-dimensional disc, D with $k < n$, U_ϵ^l will be a neighbourhhod of $U_\varepsilon^{l-1} \cup D$.

Note that since $k < n$, and the handle D is attached along a $k-1$-dimensional sphere ∂D, we may assume that for any given real number T, the attachment sphere is such that no trajectory of length less than T of the characteristic flow on $\partial U_\varepsilon^{l-1}$ crosses ∂D twice.

Therefore, since the handle intersects $\partial U_\varepsilon^{l-1}$ on an arbitrary small neighborhood of ∂D, we may assume that a trajectory on $\partial U_\varepsilon^{l-1}$ crossing twice this neighborhood has action larger than T. Thus the new periodic trajectories we may have introduced are either completely contained in the handle, or must have action larger than T.

Let us now take coordinates (q,p,z) in the neighborhood of the handle, where q is on the handle, p is the dual coordinate, and z is a normal symplectic direction. Then on $|p|^2+|z|^2=\varepsilon$ there are two types of periodic trajectories. Those that have (q,p) constant, and such that $z(t)=\exp(Jt)z(0)$ and action $|z(0)|^2=\varepsilon^2$ and the long ones, such that $q(t)$ will cross the disc, and thus the action is of the order of $\ell\varepsilon$ where ℓ is the diameter of the disc. So we see that $FH^{2n}(U_\varepsilon,-\delta,\varepsilon^{3/2})=0$ only involves the first type of small action solution. Moreover a Floer trajectory connecting two such solutions will be contained in a neighbourhood of the handle, since it as energy bounded by a constant times ε^2.

From the maximum principle, it is easy to see that any trajectory exiting from U_ε, connecting two periodic orbits contained in U_ε, has energy larger than some $\delta>0$ (independent of ε). The proof is the same as in Lemma 4.2

Thus we may localize the complex near the critical points, and it will coincide with the complex of the function $c\cdot|z|^2$ in $\mathbb{C}^k$, which is known to have zero cohomology in dimension $k\leq\frac{c}{2\pi}$.

Finally, as we add the l-th critical point, the Floer complex $FH^*(U_\varepsilon^l,-\delta,\varepsilon^{3/2})=0$ is obtained from the Floer complex of $CH^*(U_\varepsilon^l,-\delta,\varepsilon^{3/2})$ by adding at a lower level, some generators with partial homology equal to zero. Therefore $FC^*(U_\varepsilon^l,-\delta,\varepsilon^{3/2})=0$ has zero cohomology.

Note that U_ϵ is obtained by adding thinner and thinner handles, but since we prove that for α small enough some $W_\alpha\subset U_\epsilon$ and since U_ϵ is AWC and being AWC is inherited by subsets, this concludes our proof.

□

Remarks. :

1. Note that if W is only of contact type in M, and M is $SAWC$, we may not infer that W is also $SAWC$. The distinction is not meaningless, as we may not conclude that W has no exact Lagrange submanifolds. This happens for example if we have a Lagrange embedding of L in

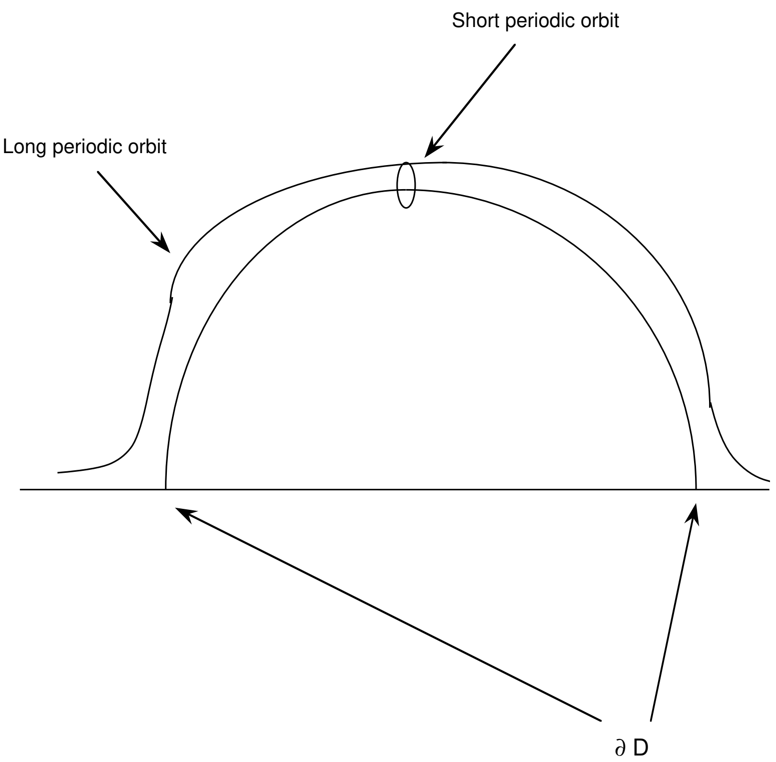

Figure 7: The periodic orbits near the handle attached to W^{l-1}

$\mathbb{R}^{2n}$. Then if W is a tubular neighborhood of L, ∂W carries periodic orbits, but obviously, L is exact in W, while it may not be exact in $\mathbb{R}^{2n}$.

2. Of course, the same proof shows that if $(M, \partial M)$ satisfies case (b) of (AWC) in degree k *and* $j^!$ is an isomorphism in degree k, then W satisfies the $SAWC$ (again, case (b) in degree k).

3. Also if W is a codimension zero submanifold with *restricted* contact type boundary in M, then if W satisfies case (a) of the AWC in degree k, and $(Fj)^!$ is injective, then $(M, \partial M)$ satisfies the AWC.

   However, no general method is known to prove injectivity of $(Fj)^!$.

4. As F. Laudenbach pointed out, the non existence of exact Lagrange embedding in the case of a subcritical Stein submanifold may be also

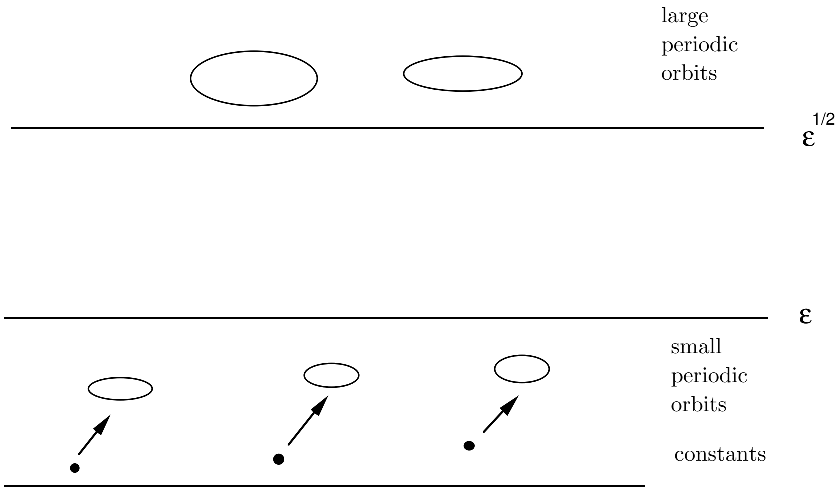

Figure 8: The critical orbits of the action for a subcritical Stein manifold

recovered by using his engulfing method (see [La]) provided the manifold has $\pi_1 = \pi_2 = 0$. Indeed engulfing can be done on such a manifold, therefore any exact Lagrange submanifold will intersect its image by a conformal map (as conformal maps restrict to Hamiltonian maps on exact Lagrangians). But the gradient flow of a plurisubharmonic function with all its critical points of index strictly less than n, will send an n-dimensional manifold to infinity (since it generically avoids the unstable manifolds of the critical points).

One could also use the results of Chekanov [Che], see section 6

5. As Helmut Hofer informed me, Michel Andenmatten has proved independently that the boundary of W, ∂W carries at least one closed characteristics (however, we prove here a stronger result, namely that this is the case of any contact type hypersurface in W, bounding a domain U, regardless of whether U is Stein subcritical or not.)

6. From example (3) we may deduce that there is no closed exact Lagrange submanifold in $\mathbb{R}^{2n}$, since otherwise the boundary ST^*N of such a manifold would be of restricted contact type, and thus DT^*N would be $SAWC$, a contradiction with the above result. Of course this proof is more complicated than the original one by Gromov (see [Gr1]. On the other hand it lends it self to generalizations. It is not clear what are the

symplectic manifolds without exact lagrange submanifolds. Gromov proved that any product $P \times \mathbb{C}$ or (see [LS]) any T^*M with M open satisfies this condition.

7. We shall explain in the next section an equivariant version of the $SAWC$ which holds for DT^*L with L simply connected.

Using yet unpublished results from [V10], we get that

If M, N are symplectic manifolds, with restricted contact type boundary, we have (Künneth formula).

$$FH^*(M \times N) \simeq FH^*(M) \otimes FH^*(N).$$

Note that $M \times N$ has corners, but if ξ (resp. η) is a conformal vector field on M (resp N), transverse to ∂N), then $\zeta = \pi_M^v(\xi) + \pi_N^*(\eta)$ is a conformal vector field on $M \times N$, "transverse" to $\partial(M \times N) = \partial M \times N \cup M \times \partial N$.

Thus any smoothing of the corners, will be transverse to ζ, and thus will yield a manifold with contact type boundary (still denoted for simplicity by $M \times N$).

The map

$$\begin{array}{ccc} FH^*(M \times N) & \longrightarrow & H^*(M \times N) \\ \downarrow & & \downarrow \\ FH^*(M) \otimes FH^*(N) & \longrightarrow & H^*(M) \otimes H^*(N) \end{array}$$

is then the tensor product of c_M^* and c_N^*.

Thus, if one of the two maps is not bijective, $c_{M\times N}^* = c_M^* \otimes c_N^*$ will not be bijective either. We just proved

Proposition 4.5. *Let M, N be symplectic manifolds with restricted contact type boundary. If M (or N) satisfies the $SAWC$, then so does $M \times N$.*

Similarly, if we have a *strong symplectic fibration* that is the following:

(i) a fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$

(ii) symplectic structures α, σ, ω on F, E, B respectively, such that

- σ restricted to the fibers coincides with α
- for each positive t, we have that $\sigma + t\pi^*(\omega)$ is symplectic

This is in particular the case for a holomorphic fibration with Kähler fibers and base.

Then according to [V10], there is a spectral sequence starting with $E_2^{p,q} = FH^p(B) \otimes FH^q(F)$ and converging, under mild conditions, to $FH^{p+q}(E)$. More over the map c_E^* is induced by $c_B^* \otimes c_F^*$.

We thus we have:

Theorem 4.6. *Let us consider $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$, a strong symplectic fibration, such that the above spectral sequence converges. Then if B or F satisfies the $SAWC$, then the same holds for E.*

This yields many classical, as well as new cases of Weinstein's conjecture, e.g. the total space E of a negative (i.e. such taht E^* is ample), holomorphic vector bundle over a compact manifold P (with $[\omega]\pi_2(P) = 0$ and $n > 0$) (see [FHV] for P compact, [Ma] for $P = T^*N$). Here the negativity of E guarantees that the boundary of the unit disc bundle is pseudo-convex, hence of contact type.

Now let $\widehat{M}$ be the union of M with $\partial M \times [1, +\infty[$ as defined at the beginning of this paper.

Proposition 4.7. *Let W be a codimension 0 submanifold of $\widehat{M}$ with restricted contact type boundary, such that ∂W is contained in $\widehat{M} - M$. Then if M satisfies that $FH^k(M) \longrightarrow H^k(M, \partial M)$ is not injective for some degree k (i.e. satisifes case (a) of the AWC ,in degree k), the same holds for W.*

Proof. : Remember that case (a) of the AWC means that $FH^*(M) \to H^*(M, \partial M)$ is not injective.

The inclusions $M \hookrightarrow W \hookrightarrow \widehat{M}$ induce a diagram

$$\begin{array}{ccccc}
FH^*(M) & \longrightarrow & FH^*(W) & \longrightarrow & FH^*(\widehat{M}) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(M, \partial M)) & \longrightarrow & H^*(W, \partial W) & \longrightarrow & H^*(M, \partial M)
\end{array}$$

Since W is compact, it is contained in some $\widehat{M_a} = M \cup \partial M \times [1, a]$ and $(\widehat{M_a}, \partial\widehat{M_a}) \simeq (M, \partial M)$.

Also the inclusion $M \hookrightarrow \widehat{M_a}$ induces an isomorphism in Floer cohomology.

Thus if ξ in $FH^*(M, \partial M)$ goes to 0 in $H^*(M, \partial M)$, the image of ξ in $FH^*(W)$ is nonzero (since pushing this image to $FH^*(\widehat{M})$ yields ξ again), and has zero image in $H^*(W, \partial W)$. $\square$

We also have

Proposition 4.8. *If M satisfies case (a) of the AWC, then for W with boundary of contact type contained in $\widehat{M} - M$, ∂W carries a closed characteristic.*

Example. $M = DT^*N$, then it is easy to see that case (a) of the AWC always holds, by distinguishing the cases $\pi_1(N) = 0$ and $\pi_1(N) \neq 0$, and we recover the result by Hofer and the author [HV1] granting the existence of a closed characteristic for any hypersurface of contact type in T^*N surrounding the zero section.

Remark. All the results of existence of a closed characteristics may be extended to results of "quasiexistence", that is for a given Hamiltonian, existence of a periodic orbit on almost any energy level [Ho-Z, St].

5 Equivariant Floer cohomology

The results in the previous section may be adapted to the following more general set-up:

(i) by taking into account the S^1 symmetry of A_H for autonomous H.

(ii) by extending our results to the case $[\omega]\pi_2(M) \neq 0$.

These is no "interaction" between (i) and (ii): once we know how to do (i) for $[\omega]\pi_2(M) = 0$, it is easy to extend it to the case $[\omega]\pi_2(M) \neq 0$ (once we know how to do (ii) in the non equivariant case).

Here we shall explain (i), and state the definition and main properties of the equivariant Floer cohomology, denoted by $FH^*_{S^1}$, more detailed proofs are postponed to [V10].

This will allow us to state and prove some equivariant versions of the Algebraic Weinstein conjecture, yielding several new applications.

As for (ii), we will postpone it to some further publication, and will not use it here anyway.

Let M be, as in section 1, a symplectic manifold with contact type boundary and such that $[\omega]\pi_2(M)$ vanishes.

Let $H_\lambda(t, x)$ be a family of time dependent Hamiltonians on M, parameterized by $\lambda \in S^{2N-1}$, 1-periodic in time.

Let $\theta \in S^1 \simeq \mathbb{R}/\mathbb{Z}$, and denote by $\theta.\lambda$ the Hopf action of S^1 over S^{2N-1}. We shall assume that the family H_λ satisfies

$$H_{\theta\cdot\lambda}(t,x) = H_\lambda(t-\theta,x).$$

We shall then say that such a family is equivariant. To a time independent Hamiltonian H, we associate the constant family defined by $H_\lambda(t,x) = H(x)$.

Such a family is obviously equivariant.

Now let $X_\lambda(t,x)$ be the Hamiltonian vector field of H_λ. Then we have that the set of solutions of

$$(5.1_\lambda)\begin{cases} \dot{x}_\lambda(t) = X_\lambda(t,x_\lambda(t)) \\ x_\lambda(1) = x_\lambda(0) \\ \int_{S^1}\frac{\partial H}{\partial\lambda}(t,x_\lambda(t))dt = 0. \end{cases}$$

is invariant by the S^1 action. In other words, the set $P = \{(x,\lambda)|x : S^1 \to M$ satisfies $5_\lambda\}$ is S^1 invariant: if $(x,\lambda) \in P$ then $(\theta * x, \theta\cdot\lambda)$ is also in P, where $(\theta * x)(t) = x(t-\theta)$. Indeed,

$$\begin{aligned}(\theta * x(t) = \dot{x}(t-\theta) \quad &= X_\lambda(t-\theta, x(t-\theta)) \\ &= X_{\theta\cdot\lambda}(t,(\theta * x)(t)).\end{aligned}$$

Now consider two S^1 orbits in S, that contain $(x_+,\lambda_+),(x_-,\lambda_-)$. We denote by $\mathcal{M}((x_-,\lambda_-)(x_+,\lambda_+))$ the set of solutions of

$$(5.2)\begin{cases} \overline{\partial}u(s,t) = \nabla H_{\lambda(s)}(t,u(s,t)) \\ \frac{d}{ds}\lambda(s) = -\int_{S^1}\frac{\partial}{\partial\lambda}H_{\lambda(s)}(t,u(s,t))dt \\ \lim\limits_{s\to\pm\infty} y(s,t) = x_\pm(t) \\ \lim\limits_{s\to\pm\infty} \lambda(s) = \lambda_\pm \end{cases}$$

Note that equation 5 represents the L^2 gradient of the parameterized action functional

$$A_H(x,\lambda) = \int_{D^2} z^*\omega - \int_{S^1} H_\lambda(t,x(t))dt.$$

where $z : D^2 \to M$ is such that $z_{|S^1} = x$. We claim that the system (5) has the same properties as in the absence of the parameter λ, with the extra feature of the S^1 action:

if $(u(s,t),\lambda(s))$ is a solution of (5) then $\theta * (u,\lambda)$ defined by $\theta * (u,\lambda)(s,t) = (u(s,t-\theta),\theta\cdot\lambda(s))$ is also a solution of 5.

Note that the S^1 action is free on $\mathcal{M}(x_-, \lambda_-), (x_+, \lambda_+))$, and we denote by $\mathcal{M}_{S^1}((x_-, \lambda_-), (x_+, \lambda_+))$ the quotient of $\mathcal{M}(x_-, \lambda_-), (x_+, \lambda_+))$ by the circle action. There is also, as in the non equivariant case, an $\mathbb{R}$ action induced by translation on the s-variable, and the quotient of $\mathcal{M}_{S^1}$ by this action is denoted by $\widehat{\mathcal{M}}_{S^1}$.

Now we may define a Floer complex associated to H_λ as follows $C^*_{S^1}(H_\lambda)$ is the sum of one copy of R for each element in P/S^1, d_{S^1} is defined by its matrix $< d_{S^1}(x, \lambda), (y, \mu) >= \#_{\text{alg}} \widehat{\mathcal{M}}_{S^1}((x, \lambda), (y, \mu))$. It is understood that this number is nonzero only if $\widehat{\mathcal{M}}_{S^1}((x, \lambda), (y, \mu))$ is 0-dimensional. This happens when, defining $\mu(x, H_\lambda)$ in an obvious way, we have $\mu(y, H_\lambda) - \mu(x, H_\lambda) = 1$.

Also μ induces a grading on $C^*_{S^1}(H_\lambda)$, as in the nonequivariant case.

If $2c_1(M).\pi_2(M) = d.\mathbb{Z}$, then i_{cz} and thus the grading is only defined modulo d, but this difficulty is not very important, and would be be removed by the use of Novikov coefficients.

We denote by $FH^*_{S^1}(H_\lambda)$ the cohomology of the complex $(C^*_{S^1}(H_\lambda), d_{S^1})$.

Of course, as in the non equivariant case, we may consider $C^*_{S^1}(H_\lambda; a, b)$ generated by the solutions of (2.2) such that $A_H(x, \lambda) \in [a, b]$. We then get the cohomology $FH^*_{S^1}(H_\lambda; a, b)$.

Now, let H be a time independent Hamiltonian on M. For each N, we get an equivariant family parameterized by S^{2N-1}, given by $H_\lambda(t, x) = H(x)$.

Such a family is of course non generic, but if $H_{\lambda,\varepsilon}(t, x)$ is a perturbation of $H_\lambda(t, x) = H_{\lambda,0}(t, x)$, then, for ε small enough, $FH^*_{S^1}(H_{\lambda,\varepsilon}; a, b)$ does not depend on ε.

We denote this group by $FH^*_{S^1,N}(H; a, b)$.

Finally it is not very difficult to see that as N goes to infinity $FH^*_{S^1,N}(H; a, b)$ stabilizes (there are obvious maps $FH^*_{S^1,N} \to FH^*_{S^1,N}$ for $N > N'$ induced by the inclusion $S^{2N'-1} \hookrightarrow S^{2N-1}$) and we denote the corresponding group by $FH^*_{S^1}(H; a, b)$.

Now, clearly we may set $FH^*_{S^1}(H) = \lim_{\substack{a \to -\infty \\ b \to +\infty}} FH^*_{S^1}(H; a, b)$, but one should be cautious about the fact that the number, $N(a, b)$ such that for N greater than $N(a, b)$ all the $FH^*_{S^1,N}(H; a, b)$ are isomorphic may well go to infinity as $a \to -\infty, b \to +\infty$. Taking the same families of Hamiltonians K_λ as in the definition of $FH^*(M; a, b)$ in the non-equivariant case, we may define $FH^*_{S^1}(M; a, b)$ and $FH^*_{S^1}(M)$. The same proof as in proposition 1.3 shows that $FH^*_{S^1}(M, -\delta, \delta) \simeq H^*_{S^1}(M, \partial M) \simeq H^*(M, \partial M) \otimes H^*(\mathbb{C}P^\infty)$, the S^1 action on M being trivial. And as in proposition 1.4 we have an exact sequence

$$\to FH^*_{S^1}(M;\delta,+\infty)\to FH^*_{S^1}(M)\to H^*(M,\partial M)\otimes H^*(\mathbb{C}P^\infty)$$
$$\to FH^{*+1}_{S^1}(M,\delta,+\infty)\to$$

5.1 Product structures, spectral sequences and localization

In this section we shall work with rational coefficients.

Corresponding to the $H^*(\Lambda M)$ module structure on $FH^*(M;a,b)$, there is an $H^*_{S^1}(\Lambda M)$ module structure on $FH^*_{S^1}(M;a,b)$, defined as in [V8] by integrating (equivariant) forms, that we may identify as forms on $\Lambda M\times S^{2N-1}/S^1$, on the trajectory spaces.

Now according to [V10] there is a spectral sequence with $E_2^{p,q}=H^p(\mathbb{C}P^\infty)\otimes FH^q(M;a,b)$ converging to $E_\infty^{p,q}$ such that $\sum_{p+q=n}E_\infty^{p,q}$ is a graded ring associated to $FH^*_{S^1}(M;a,b)$ (We refer to [Spanier] for the definition of spectral sequences).

Note that the $H^*(\mathbb{C}P^\infty)$ module structure is induced by the $H^*_{S^1}(\Lambda M)$ module structure, through the map $H^*(\mathbb{C}P^\infty)\to H^*_{S^1}(\Lambda M)$ induced by the constant map $\Lambda M\to pt$.

For M closed (and as above $[\omega]\pi_2(M)=0$) we have $FH^*_{S^1}(M)=H^*(M)\otimes H^*(\mathbb{C}P^\infty)$ the graded module associated to some filtration of $FH^n(M;a,b)$. Note that there is a natural $H^*(\mathbb{C}P^\infty)$ module structure on $FH^*_{S^1}(M;a,b)$ that corresponds to cup product with the first factor of $E_2^{*,*}$.

Now set $H^*(\mathbb{C}P^\infty)=\mathbb{Q}[u]$ where u is a degree two element. For an $\mathbb{Q}[u]$ module A, we denote by $A_{(u)}$ the localization of A at the multiplicative subset $\{1,u,u^2,\ldots\}$. Then the localization of $\mathbb{Q}[u]$ is $\mathbb{Q}[u,u^{-1}]$ (and we have $A_{(u)}=\mathbb{Q}[u,u^{-1}]\otimes_{\mathbb{Q}[u]}A$). Now for **finite values** of a,b we have

Proposition 5.1. $\{FH^*_{S^1}(H;a,b)\}_{(u)}\simeq H^*(M^b,M^a)\otimes\mathbb{Q}[u,u^{-1}]$ *where* $M^b=\{x\in M|H(x)\leq b\}$.

Proof. (sketch) This follows because after localization the inclusion of the fixed point set induces an isomorphism. One can see that at the cochain level, the fixed points are the cochains generated by the critical points on the set of constant loops, the cohomology of which is given by $H^*(M^b,M^a)\otimes\mathbb{Q}[u]$, it localizes to $H^*(M^b,M^a)\otimes\mathbb{Q}[u,u^{-1}]$. □

Finally for $M = DT^*N$, we have $FH^*_{S^1}(M) \simeq H^*_{S^1}(\Lambda M)$ as an $H^*_{S^1}(\Lambda M)$ module (see Part II).

Theorem 5.2. Main theorem, equivariant case.
Under assumption (A) the following is a commutative diagram

$$\begin{array}{ccc} FH^*_{S^1}(W) & \xrightarrow{(F_j)^!} & FH^*_{S^1}(M) \\ \downarrow & & \downarrow \\ H^*(W,\partial W)\otimes\mathbb{Q}[u] & \xrightarrow{j^!\otimes 1} & H^*(M,\partial M)\otimes\mathbb{Q}[u] \end{array}$$

Proof

It is in all respects isomorphic to the nonequivariant case (Theorem 3.1), since it was based on manipulations of critical levels, and the critical levels in the equivariant and nonequivariant case obviously coincide.

5.2 The equivariant version of the algebraic Weinstein conjecture

First consider the case $[\omega]\pi_2(M) = 0$. We shall define a new version of algebraic Weinstein conjecture, using Floer's equivariant cohomology, and such that the property is easier to "inherit".

let us consider the localization isomorphism

$$\{FH^*_{S^1}(M;a,b)\}_{(u)} \xrightarrow{\sim} H^*(M^b,M^a)\otimes\mathbb{Q}[u,u^{-1}].$$

Note that this is the localization of the "natural map" from $FH^*_{S^1}(M;a,b)$ to $H^*(M^b,M^a)\otimes\mathbb{Q}[u]$, and is an isomorphism for a,b finite.

Practically this means that given $\alpha \in H^*(M^b,M^a), \alpha u^k$ is in the image of $FH^*_{S^1}(M;a,b)$ for k large enough, and if ξ,η in $FH^*_{S^1}(M;a,b)$ have the same image in $H^*(M^b,M^a)\otimes\mathbb{R}[u]$, then, $u^\ell\xi = u^\ell\eta$ for ℓ large enough.

Of course, k (and ℓ) usually depend on a,b.

Now as a goes to-∞ and b to $+\infty$, $(M^b,M^a) \simeq (M,\partial M)$, and we get maps

$$FH^*_{S^1}(M;a,b) \to H^*(M,\partial M)\otimes\mathbb{Q}[u]$$

which are isomorphisms after localization. However, since k as defined above may go to infinity, the map

$$FH^*_{S^1}(M) \to H^*(M,\partial M)\otimes\mathbb{Q}[u]$$

need not be surjective after localization.

DEFINITION We shall say that $(M,\partial M)$ satisfies the strong equivariant algebraic Weinstein conjecture (EWC) if the class μ_{2n} is not in the image of $\{FH^*_{S^1}(M)\}_{(u)}$. In other words, if for all k, $\mu_{2n}u^k$ is not in the image of $FH^*_{S^1}(M)$ in $H^*(M,\partial M)\otimes\mathbb{Q}[u]$

EXAMPLES

1. $M=D^{2n}$, then $FH^*_{S^1}(D^{2n})=0$, hence it satisfies the EWC.
2. M is closed with $[\omega]\pi_2(M)=0$. Then $FH^*_{S^1}(M)\simeq H^*(M)\otimes\mathbb{Q}[u]$, so that it never satisfies the EWC.
3. $M=DT^*N$, where N is a closed manifold closed manifold. Then according to Part II, $FH^*_{S^1}(M,\partial M)\simeq H^*_{S^1}(\Lambda N)$, and $\{H^*_{S^1}(\Lambda N)\}_{(u)}$ only depends on $\pi_1(N)\otimes\mathbb{Q}$, according to a result of [Go][1].

Remark. As pointed out by V.L. Ginzburg ([Gi 2]), this last example implies existence of periodic trajectories for any Hamiltonian system with a magnetic field derived from a potential α (identified to a 1-form on N) in T^*N for N as above, near any energy level. This corresponds to finding periodic orbits of Y_H where Y_H is the Hamiltonian of H for the symplectic form $dp\wedge dq+q^*(d\alpha)$, and this is the same as X_K, the standard Hamiltonian of K where $K(q,p)=H(q,p-\alpha)$.

For example if $\pi_1(N)\otimes\mathbb{Q}$ is trivial, then $\{H^*_{S^1}(\Lambda N)\}_{(u)}\simeq H^0(N)\otimes\mathbb{Q}[u,u^{-1}]$ and so M satisfies the EWC.

The main advantage of the EWC, is explained by the following theorem:

Theorem 5.3. *Let $(M,\partial M)$ satisfy the EWC, and W be a codimension zero restricted contact type submanifold.*

Then $(W,\partial W)$ satisfies the EWC.

Proof. There is a commutative diagram

$$\begin{array}{ccc}
FH^*_{S^1}(W,\partial W)) & \xrightarrow{(F_j)^!} & FH^*_{S^1}(M,\partial M) \\
\downarrow & & \downarrow \\
H^*(W,\partial W)\otimes\mathbb{Q}[u] & \xrightarrow[j^!]{} & H^*(M,\partial M)\otimes\mathbb{Q}[u]
\end{array}$$

[1] where $\pi_1(N)\otimes\mathbb{Q}$ is the $\mathbb{Q}$ unipotent completion of $\pi_1(N)$

The map $j^!$ sends the fundamental class μ_W of $H^*(W, \partial W)$ to the fundamental class μ_M of $H^*(M, \partial M)$.

If for all $k, \mu_M u^k$ is not in the image of $FH^*_{S^1}(M, \partial M)$, then $\mu_W u^k$ cannot be in the image of $FH^*_{S^1}(W, \partial W)$, and thus W satisfies the EWC. □

In particular, we get

Corollary 5.4. *If μ_{2n} is in the image of $\{H_{S^1}(\Lambda L)\}_{(u)}$, in particular if L is a $K(\pi, 1)$, and N is simply connected, there is no exact embedding of L into T^*N.*

It is sometimes hoped that the following would be true:

Conjecture. Let W be a Stein manifold. Then W satisfies the EWC.

We had originally conjectured that $FH^*_{S^1}(W) = H^*_{S^1}(\Lambda W)$. Paul Seidel pointed out that plumbing two copies of T^*S^1 is a counterexample to this statement!

Conjecture. For which manifolds does the boundary ∂W determines whether W is AWC or EWC

For example it was proved by Renyi Ma, that any symplectic manifold with boundary S^{2n-1} (with canonical contact structure) satisfies Weinstein's conjecture ([Ma2]). It is reasonable to expect (and should be not too hard to prove) that such a manifold is AWC.

5.3 Capacity in AWC and EWC manifolds

Let M be a AWC manifold. One can the define a capacity as follows.

Let us consider the map $FH^*(M; -\infty, c) \to FH^*(M; -\infty, \delta)$. For δ positive, small enough, the right hand side is equal to $H^*(M; \partial M)$ (note that there are no characteristics of negative action, so $FH^*(M; -\infty, \delta) = FH^*(M; -\delta, \delta)$), and for c large enough the above map is zero (because M is an AWC manifold).

Definition. The capacity of M, $c(M)$ is defined as the smallest value of c such that $FH^{2n}(M; -\infty, c) \to FH^{2n}(M; -\infty, \delta) = H^{2n}(M, \partial M)$ vanishes.

It is not hard to check that c is a capacity in the sense of [E-H1]. It follows from the above that the capacity is invariant by symplectomorphism, is monotone, and $c(B^2(r) \times \mathbb{R}^{2n-2}) = c(B^{2n}(r)) = \pi r^2$.

In the equivariant case, we may define a sequence of capacities γ_k, by setting

$$\gamma_k(W) = \inf\{c \mid \text{The image of} \quad FH^{2n}(M; -\infty, c) \quad \text{in} \quad u^k H^{2n}(M, \partial M) \quad \text{vanishes}\}$$

We leave it to the reader to show that the $\gamma_k(W)$ make an increasing sequence of symplectic invariants, such that $lim_{k\to\infty}\gamma_k(W) = \infty$ (thus generalizing the construction in [E-H 2]).

6 The existence of holomorphic curves and applications to real forms and complex analysis

In this section we consider symplectic manifolds with a given admissible almost complex structure. Note that in the previous sections we were allowed to change the almost complex structure. It will be useful to notice that application of the results of the previous section usually requires some genericity of the almost complex structure. However, we shall only used facts which hold for a closed subset of the set of admissible structures. For example under certain assumptions, we prove existence of pseudo-holomorphic curves. If we have existence of such a curve for a generic structure, and a bound on the area, we get existence for any admissible almost complex structure, by Gromov's compactness argument.

Another result we will use when we apply our results to Lagrangians is the following

Lemma 6.1. *Let L^n be a totally real submanifold L in the almost complex manifold (M^{2n}, J) of dimension n. Then L has a basis of pseudoconvex neighbourhoods in M.*

Proof. Indeed let us consider L as the zero section in T^*L. This is possible, since the normal bundle of L in M is isomorphic to the normal bundle of L in T^*L. Moreover we may assume that $JH_q = V_q$ where H_q is the tangent subspace to the zero section, and V the vertical subbundle.

Now let $| \quad |_g$ be a metric on L; and $\psi(q,p) = |p|_g$ the induced norm on V_q. We have that $Jd\psi - pdq = \alpha$ near the zero section, where pdq is the Liouville form on T^*L, and α a 1-form vanishing to second order on L. Thus $dJd\psi - \omega = d\alpha$ and $d\alpha$ vanishes to first order on L. As a result, $dJd\psi$ is positive on J-complex lines for p small enough, since this is the case for ω. As a consequence $\psi^{-1}(c)$ is J-pseudoconvex for c small enough. We thus get our basis of pseudoconvex neighbourhoods. $\square$

Our first result shows that knowing that condition (A) is not fullfilled has interesting consequences.

Theorem 6.2. *Assume condition (A) is violated for any sufficiently small neighbourhood of a Lagrangian submanifold. Then, there is a holomorphic surface S such that $\partial S \subset L$.*

Remember that condition (A) says that for a Hamiltonian H in a certain family $K_{\lambda,\mu}^{\epsilon,\alpha}$, associated to the neighbourhood W of L we have:

$$(A)\begin{cases} \text{Any solution} & u:\mathbb{R}\times S^1\to M \text{ of} \\ \overline{\partial}u=\nabla H(u) & \lim\limits_{t\to\pm\infty} u(t,\theta)=x_\pm(\theta) \\ \text{where} \quad x_\pm(\theta) \quad \text{solves} & \dot{x}_\pm(\theta)=X_H(x_\pm(\theta)) \\ \text{has its image in W} & \end{cases}$$

Here X_H is the Hamiltonian vector field of H.

We use the following lemma:

Lemma 6.3. *Let S_n be a sequence of non-trivial holomorphic curves with bounded area, and boundary in W_α, where W_α is a fundamental family of neighbourhoods of a totally real submanifold L. Then S_n converges as a current to S, a non-trivial holomorphic curve with boundary in L.*

Proof. The proof is left to the reader, using the result of Bishop-Gromov [Gr1] (i.e. Bishop for the integrable case, and Gromov for the non-integrable case, even though Gromov's convergence is stronger than Bishop's) on the local convergence of the S_n and an obvious almost complex extension of Chirka ([Chi]) or Sibony (see [Si]) to control the boundary behaviour.

Let B be a ball in the complement of L. We must prove that $S_n \cap B$ converges as a current. But we have a sequence of holomorphic curves in B and this sequence has bounded area. Thus according to Gromov it will converge to a holomorphic curve in B. Now we get as limiting current a holomorphic (normal and integral) current T in $M - L$. Since L is totally real, the quoted paper by Sibony allows us to extend T to a holomorphic (normal and integral) current on M with boundary in L. According to Chang ([Chang]) and Chirka, such a current is given by a holomorphic curve with boundary in L.

The non-triviality of the limit follows from tha fact that the area of the S_n outside W is bounded from below (again by Lemma 4.2), hence the lower bound holds for the limit.

□

Proof of Theorem 6.2. Suppose assumption (A) is violated. So for each value of the parameters α, ε we get a Floer trajectory with endpoints in W_ε exiting from W_α. Let S_n be the part of the trajectory outside of W_α, for $\alpha = 1/n$.

Then S_n satisfies the assumptions of the Lemma, the bound on the area being given by the following computation.

First, let $\Gamma = \Gamma_+ \cup \Gamma_-$ where Γ_- is the interior of $u^{-1}(\partial W \times \{\alpha\})$.

Let x, y be the two periodic solutions. Then $A_H(y) - A_H(x) = \int_\Gamma u^*\omega - (H(y) - H(x)) \simeq \int_\Gamma u^*\omega$

$$= \int_{\Gamma_-} u^*(d(t\lambda_0)) + \int_{\Gamma_+} u^*\omega$$

$$= \int_{\partial\Gamma_-} u^*(t\lambda_0) + \int_{\Gamma_+} u^*\omega$$

Now $\partial\Gamma_-$ has two parts, one in $u^{-1}(\partial W \times \{\varepsilon\})$ (the two periodic orbits) and one in $u^{-1}(\partial W \times \{\alpha\})$, that we denote by $\partial\Gamma_-^\alpha$ and $\partial\Gamma_-^\varepsilon$. Thus

$$\begin{aligned}
\int_{\partial\Gamma_-} u^*(t\lambda_0) &= \int_{\partial\Gamma_-^\alpha} u^*(t\lambda_0) - \int_{\partial\Gamma_-^\varepsilon} u^*(t\lambda_0) \\
&= \alpha \int_{\partial\Gamma_-^\alpha} u^*(\lambda_0) - \varepsilon \int_{\partial\Gamma_-^\varepsilon} u^*(\lambda_0) \\
&= \alpha \int_{\Gamma_-} u^*(d\lambda_0) + (\alpha - \varepsilon) \int_{\partial\Gamma_-^\varepsilon} u^*(\lambda_0)
\end{aligned}$$

but on Γ_-, $d\lambda_0$ is positive.

Indeed

$$u^*(d\lambda_0) = d\lambda_0\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial \theta}\right) = d\lambda_0\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s} + J\nabla H(u)\right)$$

Now $J\nabla H(u) = X_H \in \ker(d\lambda_0)$ while $d\lambda_0(\xi, J\xi) \geq 0$ since if π denotes the projection of TW on $\ker(\lambda_0)$ parallel to the space spanned by $\frac{\partial}{\partial t}$ and X, then π satisfies $d\lambda_0(\xi, J\xi) = \omega(\pi(\xi), J\pi(\xi))$

As a result, $A_H(y) - A_H(x) \geq \int_{\Gamma_+} u^*\omega + 0(\alpha)$.

Since H is constant on $u(\Gamma_+)$, u is J-holomorphic on Γ_+.

□

Remark. Note that our proof is not really complete. Indeed assumption (A) is only assumed for "generic" Hamiltonians, and we need to use a Floer trajectory for such a generic Hamiltonian to a trajectory for a Hamiltonian

H constant outside a neighbourhood of the Lagrangian. This is non trivial, and we refer to the PhD dissertation of David Hermann (see [Her1][Her2]) where this has been worked out with full details.

Remark. One could wonder if the boundary of the above curve may be empty. Of course the question only makes sense if W contains a closed curve. However, even in this case the curve we obtain is non-empty. Indeed, since $[\omega]\pi_2(W) = 0$, condition (A) is not violated for trajectories connecting two constant solutions inside U. Thus the trajectory connects two closed geodesics (or a constant and a closed geodesic).

We may thus prove:

Theorem 6.4. *The curve S in theorem 6.2 has a non-empty boundary.*

Proof. Note that the question may in fact be reformulated as follows. Let S be the holomorphic curve of the Proposition 6.2, does the area of the curve $S \cap W_\alpha$ go to zero as ε goes to zero. In this case S is a closed curve (i.e. $\partial S = 0$). And under some genericity assumption, this sufficient condition is also necessary. We shall prove that this does not happen.

Indeed we have

$$\int_{\Gamma_-} u^*\omega = \int_{\Gamma_-} u^*(d(t\lambda_0)) = \int_{\partial\Gamma_-} u^*(t\lambda_0) = \alpha \int_{\partial\Gamma_-^\alpha} u^*(\lambda_0) - \varepsilon \int_{\partial\Gamma_-^\varepsilon} u^*(\lambda_0)$$

where $\partial\Gamma_-^\alpha$ (resp. $\partial\Gamma_-^\varepsilon$) is the component of $\partial\Gamma_-$ in $t = \alpha$ (resp. $t = \varepsilon$).

The last sum is equal to $\alpha l(\partial\Gamma_-^\alpha) - \varepsilon l(\partial\Gamma_-^\varepsilon)$ where $l(\gamma)$ is the integral of λ_0 on γ, and is bounded by the length (but is equal to the length for a closed characteristic). Thus if the length of $\partial\Gamma_-^\alpha$ goes to zero, the above sum goes to zero faster than α.

Now we also have:

$$\alpha \int_{\partial\Gamma_-^\alpha} u^*(\lambda_0) - \varepsilon \int_{\partial\Gamma_-^\varepsilon} u^*(\lambda_0) = \alpha\left(\int_{\partial\Gamma_-^\alpha} u^*(\lambda_0) - \int_{\partial\Gamma_-^\varepsilon} u^*(\lambda_0)\right) + (\alpha - \varepsilon) \int_{\partial\Gamma_-^\varepsilon} u^*(\lambda_0)$$

$$= \alpha \int_{\Gamma_-^\alpha} u^*(d\lambda_0) + (\alpha - \varepsilon) l(\partial\Gamma_-^\varepsilon) \geq (\alpha - \varepsilon) l(\partial\Gamma_-^\varepsilon)$$

since $u^*(d\lambda_0)$. But $l(\partial\Gamma_-^\varepsilon)$ is bounded from below (the shortest length of a closed geodesic).

We thus get a lower bound of the order of α, a contradiction.

$\square$

Remark. Of course, one could imagine a curve, S, such that $S \cap L$ is one dimensional. Then the length of $S \cap \partial W_\alpha$ may be bounded from below by a positive constant. This is a non generic situation. It does not contradict our statement, since the complement of a non-discrete set in a closed Riemann surface is the image of a surface with (nonempty) boundary. Note however, that if L is real analytic (in particular if it is the real part of the complex manifold), this does not happen. Indeed $S \cap L$ can have no accumulation point in this case (unless it is an analytic curve (see [Mi]).

One can apply the above theorem to all the cases of the previous section where we proved that condition (A) is violated.

Proposition 6.5. *Let L be a $K(\pi, 1)$ and N be simply connected. Then if L is Lagrangian in T^*N, there exists a holomorphic curve S (of unknown genus) such that $\partial S \subset L$.*

Proposition 6.6. *Let W be an SAWC manifold. Then any closed Lagrangian submanifold bounds a holomorphic curve. As a consequence, if W has a closed real form, it contains a closed holomorphic curve.*

We shall here say that a submanifold is *holomorphically convex*, if $L = \hat{L}$ where $\hat{L} = \{x \mid \forall f \in \mathcal{H}(W) |f(x)| \leq \sup_{z \in L} |f(z)|\}$. Here $\mathcal{H}(W)$ is the set of globally defined holomorphic functions on W.

This is equivalent to saying that there is a plurisubharmonic function ψ on W such that $\psi \geq 0$ and $L = \{z \mid \psi(z) = 0\}$. One should be careful, as this is usually called (for domains in $\mathbb{C}^n$) *polynomial convexity.* Note that this last definition is also valid for a non-integrable almost complex structure (while the previous one does not make sense). Also, since Duval and Sibony proved that for a totally real submanifold, *meromorphic convexity* is equivalent to the existence of an exact positive symplectic form $\omega = \partial\bar{\partial}\phi$ (ϕ is then plurisubharmonic) making L Lagrangian, we may take this as a definition in the almost complex case.

In other words, meromorphically convex means that for some positive exact symplectic form L is Lagrangian, while holomorphically convex implies that L is exact.

With these generalized definitions, holomorphically convex still implies meromorphically convex and the following theorems then also hold in the *almost complex* setting.

Proposition 6.7. *Let L be totally real in an AWC manifold. Then L cannot be holomorphically convex (relative to W, in other words, (W, U) is not a Runge pair for sufficiently small neighbourhoods U of L). Similarly, if W is EWC, and L carries a metric with negative curvature, then L cannot be holomorphically convex. In particular, in both cases, L bounds a holomorphic surface.*

Proof. Indeed, if L is holomorphically convex, then it is meromorphically convex. Then, according to [D-S1], L is Lagrangian for some Kähler form ω_1 coinciding with the standard symplectic form outside a compact set. Then according to Proposition 1.6, $FH^*(W)$ depends only on the value of the symplectic form outside a compact set, thus our theory applies to (W, ω_1), and L bounds a holomorphic disc in W, hence cannot be holomorphically convex (it is enough to notice that L cannot be exact, according to 4.3). This type of argument had been used in [D-S2] for their proof of Proposition 4.1. □

Remark. This was proved for $\mathbb{C}^n$ by Alexander ([Al]).

Corollary 6.8. *Let L be the real form of a Stein EWC manifold M. Then L does not carry a hyperbolic metric.*

In a forthcoming paper [V12], we shall extend such results to the projective case, and we also give sufficient conditions for a Stein manifold to be *EWC*.

The same type of argument may be used to remove an assumption in Theorem 1.5 from [D-S2]:

Theorem 6.9. *Let L be a totally real submanifold of dimension n in an exact symplectic manifold W^{2n}. If either*

-W is AWC or

- W is EAWC and L is hyperbolic

then for any sufficiently small neighbourhood U of L, there is a Riemann surface in the closure of $W - U$ with boundary in U.

Proof. (sketch) Let us consider L in W, and ϕ be the restriction of the Liouville form to L. Then L has a neighbourhood symplectomorphic to a neighbourhood of $L_\phi = \{(q, p) \in T^*L \mid p = \phi(q)\}$, the graph of the one form ϕ. Now let us take the Hamiltonians $K_{\lambda,\mu}^{\varepsilon,\alpha}$, and shift them in the p direction, so that they vanish on L_ϕ, i.e. consider $K_{\lambda,\mu}^{\varepsilon,\alpha}(q, p - \phi(q))$.

Then, as in the case of a Lagrange submanifold (i.e. $\phi = 0$), we have that for λ going to ∞, $FH^*(K_\lambda^{\varepsilon,\alpha})$ converges to $H^*(\Lambda L)$. Note that the difference here, is that we cannot let ε, α go to zero, for fixed μ, since $K_\lambda^{\varepsilon,\alpha}$ may have closed characteristics with arbitrarily negative action (if this is not the case, we can go to the limit, and get a Riemann surface bounded by L as in the Lagrangian case).

It is easy to show that the functional A_K has the same critical levels as E_K, where $E_K(q) = \int_{S^1}[K^*(\dot{q})dt + q^*\phi]$ where K^* is the Legendre dual of K. But for $\phi \neq 0$ this function will have in general a negative minimum, so we cannot get estimates for the area independent from U as before.

As in the Lagrangian case, one proves that assumption (A) is violated, hence there is a Floer trajectory outside a neighbourhood of L. Then outside the support U of $\nabla K_\lambda^{\varepsilon,\alpha}$ this yields a Riemann surface with boundary in U. □

Remarks. 1. For example if W is Stein and subcritical, it is AWC. The fact that it has no closed real part follows from the fact that the real part is holomorphically convex in W, and thus according to Andreotti and Narasimhan ([A-N]), $H_n(L) \to H_n(W)$ is injective. In our case $H_n(W) = 0$, hence L cannot be compact.

As pointed out to me by V. Kharlamov, this also follows from the fact that using Kalinin's spectral sequence either the generator of $H_n(L)$ has nonzero image in $H_n(W)$ or there is a nonzero class in $H_{n+k}(W)$ with $k > 0$, hence the real form is never homologous to zero in a Stein manifold W.

However to be AWC is more general than to be Stein and subcritical.

2. One may also estimate the area of S, which makes the Proposition more efficient. For this one has to follow the numbers involved.

The area of S is easily seen to be less than $c(W)$. Thus if W has a closed real form, it will contain a curve of area less than $2c(W)$. Moreover this curve has a nonempty real part.

Remark. Note that a priori, as U gets smaller, the bounds on the area of the surface goes to infinity, so we cannot conclude that L itself contains the boundary of a Riemann surface. However by normalization, we see that L will contain the boundary of a positive current.

One may use this to prove the following result:

Define $w(L)$ to be $w(L) = \sup_J \inf_{S|\partial S\subset L} \int_S \omega$, where S is a holomorphic surface. Note that our notation extends in some sense the notion of width defined by Gromov. Note also that there is some ambiguity, since ∂S could be empty. This can happen if W contains a non-trivial holomorphic curve.

We have that for any Lagrange submanifold L in W, $0 < w(L)$. The number $w(L)$ is invariant by symplectic diffeomorphism of W (but could be equal to $+\infty$, which means that there exist a J such that L does not bound any J-holomorphic curve).

Theorem 6.10. *If W is an AWC manifold, $w(L) < c(W) < \infty$. Similarly, if L is hyperbolic and W is EWC, then $w(L) \leq \gamma_1(W)$*

Proof. The proof is left to the reader.

□

Remarks. 1. Note that by doing a "small" Polterovich surgery on the manifold ([LS, Pol]), we may see that the same is also true for an immersed Lagrangian (considered as a subset of W). Similarly, we get a bounding holomorphic curve, except that it now bounds the immersed Lagrangian, and may thus go through double points.

2. Note that Chekanov proved in [Che] the following result:

   Theorem. Let P be any tame symplectic manifold. Then the symplectic displacement energy of L is greater than $\sigma(P, L) = \sup_J \inf_{D|\partial D\subset L} \int_S \omega$. Here D is a holomorphic disc.

   In fact Chekanov even proved an estimate of the number of intersection points of L and $g(L)$ by the sum of Betti numbers of L.

   Chekanov's result, like Gromov's theorem, implies that L bounds a holomorphic disc whenever there is a Hamiltonian isotopy moving L away from itself. The class of such manifolds contains $P \times \mathbb{C}$, but it is unclear what are other manifolds in this class. Often this result is useful when arguing by contradiction, to prove that there can not be exact submanifolds of a certain type, since it is easier to find conformal maps moving a submanifold to infinity, and conformal maps restrict to Hamiltonian maps on exact submanifolds.

For example there is no exact submanifold in $T^*N - 0_N$ since the radial map would move this set at infinity, hence it should bound a holomorphic disc, a contradiction since L is exact.

7 Appendix: Checking assumption (A) for neighbourhoods of *weakly exact* Lagrangians

Proposition 7.1. *Let L be a Lagrange submanifold in (M,ω). Then if $[\omega]\pi_2(M,L) = 0$, assumption (A) is satisfied for W a tubular neighbourhood of L and J any almost complex structure making ∂W pseudoconvex; provided x or y is null homotopic in W.*

Proof. Identify (possibly a subset of)W with $\{(q,p) \in T^*L//p/ \leq 1\} = W_1$. Then for given J making W pseudoconvex, any closed holomorphic curve in $W - W_\alpha(\alpha \leq \frac{1}{2})$ has area bounded from below.

Now for $H = K^\varepsilon_{\lambda,W}$, and x, y be non trivial 1-periodic solutions, we claim that

$$\Delta A_H(u) = \int \widehat{u}^*\omega + H(x) - H(y)$$

where $\overline{x}, \overline{y}$ are the projections of x and y on L, and $\widehat{u}$ is obtained from u by adding the rings

$$\begin{array}{rcl} S^1 \times [0,1] & \to & T^*L \\ (\theta, s) & \to & (q(\theta), sp(\theta)) \end{array}$$

where $x(\theta)$ (resp. $y(\theta)$) is $(\overline{x}(\theta), p(\theta))$ (resp. $(\overline{y}(\theta), p(\theta))$). (see Fig.7)

Note that $\overline{x}, \overline{y}$ are closed geodesics, and $\widehat{u}$ is holomorphic, and exits from W, hence its area is bounded from below. on the other hand, the term $(H(x) - H(y))$ goes to zero with ε.

Now assume $\overline{x}$ is contractible in L. Then, we may add to the cylinder $\widehat{u}(\mathbb{S}^1 \times \mathbb{R})$ a disc in L, so that we get a disc (the union of the disc in L and the cylinder) with boundary $\overline{y}(\in L)$. Moreover the integral of ω over this disc is obviously positive, hence we proved that $[\omega]\pi_2(M,L) \neq 0$, a contradiction. $\square$

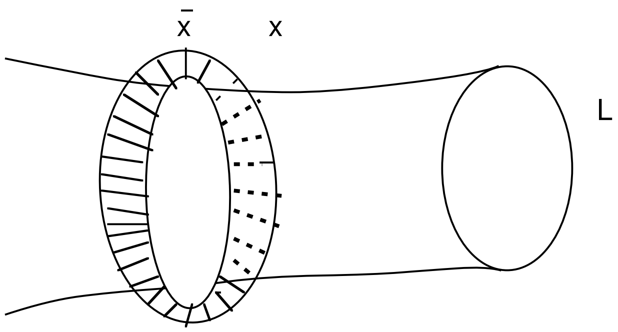

Figure 9: x, its projection $\overline{x}$ and the ring $\widehat{u} - u$

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