

Estimates of Characteristic numbers of real algebraic varieties

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Abstract

We give some explicit bounds for the number of cobordism classes of real algebraic manifolds of real degree less than d , and for the size of the sum of Betti numbers with $\mathbb{Z}/2$ coefficients for the real form of complex manifolds of complex degree less than d .

Introduction

In complex as well as in real algebraic geometry, it is useful to understand what are the simplest manifolds, and try to measure the complexity of algebraic manifolds. *A priori* there are many ways to define what “simple” manifolds should be and for no obvious reason should they all agree. Just to mention a few examples, among the most frequently used (see [Kol01]), there are low degree manifolds, rational manifolds, Fano and uniruled manifolds. These manifolds have sometimes bounded Betti numbers, some belong to finitely many deformation classes (low degree and Fano).

Once we decide on a given notion of complexity for complex manifolds, we may try to understand the properties of their real part. Are the real part of “simple” complex manifolds “simpler” ?

The situation is far from being understood. Among the oldest results in real algebraic geometry, we find the problem of estimating the Betti numbers of a real algebraic manifold.

Besides Harnack’s result for the maximal number of connected components of a real algebraic curve, dating back to 1876, and later work, for example by Severi and Commessatti for surfaces, the literature we are aware of, may be divided in two classes.

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In the first approach, the n -dimensional manifold $M_{\mathbb{R}}^n$ is the set of real points of a complex manifold, $M_{\mathbb{C}}^n$. The inequality between the Betti numbers with $\mathbb{Z}/2$ coefficients

$$(1) \quad \sum_{j=0}^n b_j(M_{\mathbb{R}}^n) \leq \sum_{j=0}^{2n} b_j(M_{\mathbb{C}}^n)$$

is then given by the so-called Smith's theory. The first occurrence of this theory can be found in [Smi38] and [RS38] explicitly for the number of connected components of a real algebraic set, but the same ideas can be applied to estimate higher Betti numbers, as hinted in the paper (see particularly [RS38] page 620, and [Smi38], page 509). The explicit inequality ((1)) can first be found in [Flo52] (theorem 4.4 page 146) and the modern presentation, using spectral sequences for the $\mathbb{Z}/2$ -equivariant cohomology, is due to [Bor60] (page 55, § 4.1). These results imply that whenever there are bounds on the Betti numbers with $\mathbb{Z}/2$ coefficients of the set of complex points, there are bounds on the set of real points.

Another approach defines $M_{\mathbb{R}}^n$ by equations $F_1 = 0, \dots, F_k = 0$ of degree d_1, \dots, d_k , in $\mathbb{R}P^{n+k}$. The Oleinik inequalities for hypersurfaces, extended to the general case by Thom and Milnor (see [PO49], [Ole51], [Tho65] and [Mil64]), are all based on applying Morse theory, usually to $F = \sum_{j=1}^k F_j^2$. Thus their estimate will depend on $\delta = \max d_k$ and yield a bound of the order of δ^{n+k-1} .

Definition 0.1. The **real degree** of a smooth manifold M is the minimal degree of a projective embedding of M in a real projective space. By definition, that is also the minimal degree of a projective embedding of all algebraic manifolds $M_{\mathbb{C}}$ such that $M_{\mathbb{R}} = M$ in the complex projective space, which are equivariant for the anti-holomorphic involution of $M_{\mathbb{C}}$ (i.e. such that $\text{Fix}(\tau) = M$) and the canonical anti-holomorphic involution of the projective space. It is also equal to the smallest degree of a very ample line bundle L on $M_{\mathbb{C}}$ such that $\tau^*(L) = \bar{L}$.

Now applying the above results, and considering the fact that a general algebraic manifold of **real** degree d can be realized as a real algebraic manifold in $\mathbb{R}P^{2n+1}$, defined by equations of degree less than d , we get an estimate of the type¹

$$\sum_{j=0}^n b_j(M_{\mathbb{R}}^n) \leq Cd^{2n+1}$$

¹Alternatively, if one wants to use the **complex** degree, one could use the Oleinik-Thom-Milnor inequality for the complex case, and then apply Smith theory, but the estimates would then be in d^{4n+2}

We will show in the second part of the paper, that bounds on the **complex** degree of the complex manifold are sufficient to yield bounds on the Betti numbers with $\mathbb{Z}/2$ coefficients, and these bounds are explicit and of the order of Cd^{n+1} .

It is also known according to Eliashberg and the second author (cf. [Kha02, EGH00], and [Kol99] for the 3 dimensional case), that complex uniruled manifolds cannot have a real form with negative sectional curvature and this means that negatively curved manifolds are "complicated": in particular they can not be represented by equations of low degree.

Besides this, little is known about the following conjectures, where the word *simple* has no precise meaning²

Conjecture 1. Let $M(\mathbb{C})$ be a smooth projective manifold. Then the set of diffeomorphism types of its real forms is finite

Conjecture 2. *Simple* complex manifolds have *simple* real forms.

To prove this even in the more accessible simply connected case, one would need bounds on the Betti numbers, which follows from the Smith-Thom inequalities, the multiplicative structure, and the Pontriagin classes. Our goal here is to prove that for manifolds of low degree, the Pontriagin numbers are explicitly bounded by the degree of a real embedding, and thus the number of cobordism classes. If the complex cotangent bundle is nef, this bound depends only on the degree of a complex embedding: thus the Pontriagin numbers are also a measure of the complexity at least in this case.

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1 Main theorem

In this paper we shall work with complex projective manifolds $M_{\mathbb{C}}$, with an anti-holomorphic involution τ . The set of fixed points is denoted by $M_{\mathbb{R}}$. Another point of view is to look at a smooth projective real variety X defining

$$M_{\mathbb{C}} = X(\mathbb{C}) \text{ and } M_{\mathbb{R}} = X(\mathbb{R}).$$

We'll be mainly interested in the case where $M_{\mathbb{R}}$ is further assumed to be orientable: we'll say simply in this case that M (or $M_{\mathbb{R}}$) is an orientable real algebraic manifold.

²However we refer to [BS64, DIK00a, DIK00b, DIK04] for some examples.

We'll use without any further comment these two points of view (with this notation). Of course for a given $M_{\mathbb{C}}$ there may be many non equivalent $M_{\mathbb{R}}$: in fact as pointed out in the introduction, it is not even known that there are finitely diffeomorphism, homeomorphism or homotopy types of $M_{\mathbb{R}}$. Vice-versa, it is not known, given $M_{\mathbb{R}}$, what are the different complex manifolds $M_{\mathbb{C}}$ having $M_{\mathbb{R}}$ as real form. But here even the formulation of the problem is unclear, since we may obviously blow up $M_{\mathbb{C}}$ along subvarieties invariant by τ but not intersecting $M_{\mathbb{R}}$, without changing $M_{\mathbb{R}}$. Even in the two dimensional case, where we may ask how many minimal $M_{\mathbb{C}}$ yield the given real form³: the Klein bottle is the real form of all the odd Hirzebruch surfaces, $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2k+1))$ ([Sil89]).

Now given a real algebraic manifold, we shall assume it is endowed with a *real* very ample line bundle H , where *real* means that $\tau^*(H) = \overline{H}$. Then we denote by $\deg(M) = c_1(H)^n$. Note that if H is a (non real) very ample line bundle over $M_{\mathbb{C}}$, then $L = H \otimes \tau^*(\overline{H})$ is a real very ample line bundle, but there is no general bound of its degree depending only on the degree of H .

Our main results are as follows

Theorem 1.1. *Let $M_{\mathbb{R}}$ be an orientable real algebraic manifold of degree d and dimension n . Then the Pontriagin numbers satisfy the following inequalities*

$$|p_I(M_{\mathbb{R}})| \leq 2^{n^2+3n} d(d+n-2)^n$$

In particular the number of cobordism classes of manifolds representable as real algebraic manifolds of a given real degree can be explicitly bounded.

Note that the existence of such a bound can be proved a priori, since one can prove that real manifolds of a given (real) degree fall into finitely many diffeomorphism classes, but we believe making these bounds explicit is useful. The only result we know of, related to the above, is due to Thom, who proves in [Tho51] that if V is a compact real affine complete intersection (i.e. given by k polynomial equations in \mathbb{R}^{n+k}) then it is an (unoriented) boundary, and thus all Stiefel-Whitney numbers are zero.

The next step in our program should then be to show that such a bound also holds for d the complex degree. This would then imply that the real part of a complex manifold of given degree falls in finitely

³If one wants to have surfaces which are not deformation equivalent one could consider $\mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}$ and the product of two elliptic curves with the standard antiholomorphic involution.

many cobordism classes, a first step towards the proof of conjecture 1. For the moment, there are only few classes for which this can be proved. Note that, as pointed out by Kharlamov, using Torelli's theorem for K3 surfaces, it is easy to construct surfaces with complex degree bounded and the real degree can be arbitrarily large.

We also give some estimate on the size of the homology for the real part of a complex manifold. Note that contrary to the previous theorem, the degree used here is the complex degree, that is the minimal degree of any algebraic embedding, not necessarily real.

Let $b(X)$ represent the sum of the Betti numbers with $\mathbb{Z}/2$ coefficients.

Theorem 1.2. *Let X be a real subvariety of the projective space. Assume that $X_{\mathbb{C}}$ is smooth, connected of complex degree d and dimension n . Then, one has $b(X(\mathbb{R})) \leq b(X(\mathbb{C})) \leq 2^{n^2+2}d^{n+1}$.*

The first inequality is just the Smith-Thom inequality. As for the second one, according to Milnor ([Mil64]), a complex submanifold in $\mathbb{C}P^m$ given by equations of degree less than δ has $b(X_{\mathbb{C}}) \leq C_n \delta^{2m+2}$. Since if $X(\mathbb{C})$ has degree d , it is given by polynomial equations of degree at most d in $\mathbb{C}P^{2n+1}$ (see [Mum69]), Milnor's inequality yields $b(X_{\mathbb{C}}) \leq C_n d^{4n+4}$. In the real case, for a submanifold in $\mathbb{R}P^m$, Milnor gives $b(X_{\mathbb{R}}) \leq C_n \delta^{m+1}$ and the same argument yields $b(X_{\mathbb{R}}) \leq C_n d^{2n+2}$. However we cannot say that we improve on Milnor's inequality, since there seems to be no clear lower bound for the maximal degree of the equations defining X in $\mathbb{C}P^{2n+1}$ as a function of the degree of X .

2 Some lemmata on real and complex cycles

Since we shall deal with non-orientable situations, we shall use homology and cohomology with local systems of coefficients⁴, \mathcal{L} . Unless otherwise stated, all our manifolds are closed and connected. We denote by $H^*(X, \mathcal{L})$ and $H_*(X, \mathcal{L})$ such groups, by $H_c^*(X, \mathcal{L})$ its compact supported version. We denote by o_X the orientation bundle of X .

Moreover we have cup and cap product map

⁴We shall only use the case of local coefficients with fibre \mathbb{Z} , and this implies that we always have $\mathcal{L} \otimes \mathcal{L} = \mathbb{Z}$. In particular we identify \mathcal{L} to \mathcal{L}^* . Of course with $\mathbb{Z}/2\mathbb{Z}$ coefficients, \mathcal{L} is trivial and much of what follows is much simpler.

$$\begin{aligned}\cup &: H^k(M, \mathcal{L}) \otimes H^l(M, \mathcal{L}') \longrightarrow H^{k+l}(M, \mathcal{L} \otimes \mathcal{L}') \\ \cap &: H^k(M, \mathcal{L}) \otimes H_l(M, \mathcal{L}') \longrightarrow H_{l-k}(M, \mathcal{L} \otimes \mathcal{L}')\end{aligned}$$

The first map is a non-degenerate pairing for $k+l = n$, $\mathcal{L} \otimes \mathcal{L}' = \mathcal{O}_M$.

Also a map $f : V \rightarrow M$ induces maps

$$\begin{aligned}f_* &: H_*(V, f^*(\mathcal{L})) \longrightarrow H_*(M, \mathcal{L}) \\ f^* &: H^*(V, \mathcal{L}) \longrightarrow H^*(M, f^*(\mathcal{L}))\end{aligned}$$

Moreover we have a Poincaré duality map:

$$PD : H_k(M, \mathcal{L}) \longrightarrow H^{n-k}(M, \mathcal{L} \otimes \mathcal{O}_M)$$

This implies in particular that $H_n(M, \mathcal{O}_M) = H^0(M, \mathbb{Z})$ is canonically isomorphic to \mathbb{Z} .

Finally to a k -dimensional vector bundle, $\pi : E \rightarrow X$, we associate the relative orientation bundle $\mathcal{O}_{E/X}$, as the set of orientations of the fiber, namely $\mathcal{O}_E \otimes \pi^*(\mathcal{O}_X^{-1})$. The Thom isomorphism theorem is a map

$$T : H^*(X, \mathcal{L}) \longrightarrow H_c^{*+k}(E, \mathcal{L} \otimes \mathcal{O}_{E/X})$$

and this is given by $T(u) = \pi^*(u) \cup T_E$ where $T_E \in H_c^k(E, \mathcal{O}_{E/X})$. The class T_E is uniquely defined as the preimage of 1 by the canonical isomorphism $H_c^k(E, \mathcal{O}_{E/X}) \rightarrow \mathbb{Z}$. Naturality implies that for every x in X , T_E is uniquely determined by the property that it restricts on $H_c^k(E_x, \mathcal{O}_{E_x})$ to the canonical generator.

From this it easily follows⁵, using the canonical isomorphism $\pi^* \mathcal{O}_{E/X} \otimes \pi'^*(\mathcal{O}_{E'/X}) \simeq \mathcal{O}_{E \oplus E'/X}$, that $T_{E \oplus E'} = T_E \cup T_{E'}$, since we just have to check this over a point.

Finally note that Poincaré duality on E is an isomorphism

$$H_{n-d}(E, \mathcal{L}) \rightarrow H_c^{k+d}(E, \mathcal{L} \otimes \mathcal{O}_M)$$

but since M is a retract of E , and

$$H_{n-d}(E, \mathcal{L}) \simeq H_{n-d}(M, \mathcal{L}) \simeq H^d(M, \mathcal{L} \otimes \mathcal{O}_M)$$

the second isomorphism being again Poincaré duality, we get an isomorphism

$$H^d(M, \mathcal{L} \otimes \mathcal{O}_M) \rightarrow H_c^{k+d}(E, \mathcal{L} \otimes \mathcal{O}_M)$$

⁵Note that if π, π' are the projections of $E \oplus E'$ on E, E' , then $\pi^*(T_E)$ has compact support only in the E direction, and $(\pi')^*(T_{E'})$ in the E' direction, but then $\pi^*(T_E) \cup (\pi')^*(T_{E'})$ has compact support, and we denote this class by $T_E \cup T_{E'}$.

and so the Thom isomorphism is induced by Poincaré duality.

Now if \mathcal{L} is a local system and we have isomorphism $\varepsilon : H_n(M, \mathcal{L}) \rightarrow \mathbb{Z}$ that we call an \mathcal{L} pseudo orientation, and a map $j : V \rightarrow M$ and $j^*(\mathcal{L}) \otimes_{o_V} \simeq \mathbb{Z}$ (or equivalently $j^*(\mathcal{L}) \simeq o_V$), then we say that V is \mathcal{L} -coorientable. If moreover one of these two isomorphism is given, we say that V is \mathcal{L} -cooriented. In the case $\mathcal{L} = o_M$ we just say that V is coorientable (resp. cooriented).

Now if V is k -dimensional, connected and \mathcal{L} -cooriented, we may associate to V a class $[V] \in H_k(M, \mathcal{L})$ as follows. By Poincaré duality we have an isomorphism

$$H_k(V, j^*(\mathcal{L})) \rightarrow H^0(V, j^*(\mathcal{L}) \otimes_{o_V} \simeq \mathbb{Z}$$

and the last isomorphism is canonically given by the \mathcal{L} -coorientation of V . In the non connected case, we just do the same for each connected component.

Then we have $j_* : H_k(V, j^*(\mathcal{L})) \rightarrow H_k(M, \mathcal{L})$ and we still denote by $[V]$ the class $j_*([V]) \in H_k(M, \mathcal{L})$.

The submanifold V also has a fundamental class μ_V in $H^{n-k}(M, \mathcal{L} \otimes_{o_M})$ defined as the extension of the Thom class of the normal bundle of V in M . It is the image of $[V] \in H_k(M, \mathcal{L})$ by Poincaré duality.

fundamental class $[V] \in \mathbb{Z}$).

Note that if $\beta \in H^k(M, \mathcal{L}^*)$ then $\langle \beta, [V] \rangle \in H_0(M, \mathbb{Z}) \simeq \mathbb{Z}$ is given by $\mu_V \cup \beta \in H^n(M, \mathcal{L} \otimes \mathcal{L}^* \otimes_{o_M}) \simeq H^n(M, o_M) \simeq \mathbb{Z}$.

Lemma 2.1. *Let V, W be \mathcal{L} -cooriented and \mathcal{L}' -cooriented submanifolds of M . Assume V, W intersect transversally, and set $Z = V \cap W$. Then Z is $\mathcal{L} \otimes \mathcal{L}' \otimes_{o_M}$ cooriented and if $\mu_V \in H^k(M, \mathcal{L} \otimes_{o_M})$ and $\mu_W \in H^l(M, \mathcal{L}' \otimes_{o_M})$ then $\mu_V \cup \mu_W = \mu_Z \in H^{k+l}(M, \mathcal{L} \otimes \mathcal{L}')$*

Proof. The class μ_V is related to the Thom class as follows: if V is an \mathcal{L} cooriented submanifold of M , $N(V, M)$ its normal bundle, and $T_{(V, M)} \in H_c^{n-k}(N(V, M), \otimes_{o_{N(V, M)}})$ its Thom class, then the extension of T_V to $H^{n-k}(M, \otimes_{o_M})$ is precisely μ_V .

Moreover by naturality of the Thom class, if $i : Z \rightarrow V$, E is a bundle over V , and we denote by \tilde{i} the natural map $i^*(E) \rightarrow E$, that $T_{i^*(E)} = (\tilde{i})^*(T_E)$.

Now let $j_V : V \cap W \rightarrow V, j_W : V \cap W \rightarrow W$ be the inclusions, since $j_V^*(N(V, M)) = N(V \cap W, W), j_V^*(N(W, M)) = N(V \cap W, V)$, and of course by general position

$$N(V \cap W, M) = N(V \cap W, W) \oplus N(V \cap W, V)$$

that is

$$N(V \cap W, M) = j_W^* N(W, M) \oplus j_V^*(N(V, M))$$

and this induces an isomorphism $o_{N(V \cap W, M)/V \cap W} \simeq j_V^* o_{N(V, M)/V} \otimes j_W^* o_{N(W, M)/W}$ and we get we get that $T_{N(V \cap W, M)} = T_{N(W, M)} \cup T_{N(V, M)}$, that is

$$T_{V \cap W} = T_V \cup T_W$$

Thus if V, W are \mathcal{L} and \mathcal{L}' -coorientable submanifolds, in general position, we have

$$T_{V \cap W} = T_V \cup T_W$$

this implies

$$\mu_{V \cap W} = \mu_V \cup \mu_W$$

□

In particular if V is oriented (and therefore \mathbb{Z} -cooriented) and W cooriented, we have that $\mu_V \in H^*(M, o_M)$, $\mu_W \in H^*(M, \mathbb{Z})$ then Z is oriented and dual to $\mu_V \cup \mu_W \in H^*(M, o_M)$.

Remark 2.2. This extends to the case where V, W are singular submanifolds with codimension 2 singular locus. Indeed in this case the fundamental class is also well defined, and provided V, W are in general position, $V \cap W$ is also a singular manifold with codimension 2 singular locus and the above result also applies. (see [BH64] for more details).

Let's now consider a smooth real projective manifold M of dimension n .

Let V, W be two real submanifolds of M , so that we have $V_{\mathbb{C}}, W_{\mathbb{C}} \subset M_{\mathbb{C}}$ and $V_{\mathbb{R}}, W_{\mathbb{R}} \subset M_{\mathbb{R}}$. Note that $V_{\mathbb{C}}, W_{\mathbb{C}}, M_{\mathbb{C}}$ are always orientable

Let us denote by “ \cdot ” the algebraic intersection of cycles, and assume $V_{\mathbb{C}}, W_{\mathbb{C}}$ are in general position and have complementary dimension. For example if $V_{\mathbb{C}}, W_{\mathbb{C}}$ are zero sets of sections of very ample bundles, we can always move V, W so that this assumption is satisfied.

Lemma 2.3. *Assume $V_{\mathbb{R}}$ is \mathcal{L} -coorientable, $W_{\mathbb{R}}$ is \mathcal{L}^* -orientable, and $V_{\mathbb{R}}, W_{\mathbb{R}}$ intersect transversally. Then*

$$|V_{\mathbb{R}} \cdot W_{\mathbb{R}}| \leq V_{\mathbb{C}} \cdot W_{\mathbb{C}}$$

Proof. The assumption is only needed so that the number $V_{\mathbb{R}} \cdot W_{\mathbb{R}}$ is well defined. Indeed assume the submanifolds are in general position. Then $|V_{\mathbb{R}} \cdot W_{\mathbb{R}}|$ counts with sign the number of intersection points of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. But these are contained in the set of intersection points of $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ while $V_{\mathbb{C}} \cdot W_{\mathbb{C}}$ counts all intersection points with positive sign. The inequality is now obvious. □

Remark 2.4. The same proof shows that if $V_{\mathbb{R}}^j$ are \mathcal{L}_j -coorientable, with

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \dots \otimes \mathcal{L}_k \otimes (o_M)^k = \mathbb{Z}$$

then have

$$|V_{\mathbb{R}}^1 \cdot V_{\mathbb{R}}^2 \dots V_{\mathbb{R}}^k| \leq V_{\mathbb{C}}^1 \cdot V_{\mathbb{C}}^2 \dots V_{\mathbb{C}}^k$$

Let f be a "real" holomorphic map from a real manifold V to M , that is $f : V_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$ commutes with the anti-holomorphic involution, hence sends $V_{\mathbb{R}}$ to $M_{\mathbb{R}}$. Then f sends $[V_{\mathbb{C}}]$ to a class $(f_{\mathbb{C}})_*([V_{\mathbb{C}}])$, and $[V_{\mathbb{R}}]$ to $(f_{\mathbb{R}})_*([V_{\mathbb{R}}])$. Consider a similar "real" holomorphic map g from W to M . We claim that the above result also implies that

Proposition 2.5. *Assume $V_{\mathbb{R}}$ is $f_{\mathbb{R}}^*(\mathcal{L})$ -coorientable, $W_{\mathbb{R}}$ is $f_{\mathbb{R}}^*(\mathcal{L}^*)$ -orientable. If $f(V_{\mathbb{C}}) \cap g(W_{\mathbb{C}})$ is finite, we have*

$$|(f_{\mathbb{R}})_*([V_{\mathbb{R}}]) \cdot (g_{\mathbb{R}})_*([W_{\mathbb{R}}])| \leq (f_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cdot (g_{\mathbb{C}})_*([W_{\mathbb{C}}])$$

Proof. With the same argument as above, we just have to prove that the multiplicity of a complex intersection is always larger than the multiplicity of the real intersection. But this is a local statement, and we can then of course move locally V and W to put them in general position, in which case the statement is obvious. \square

Remark 2.6. If $V \cap W$ contains a component of positive dimension, the above inequality may fail, since $V_{\mathbb{C}} \cdot W_{\mathbb{C}}$ can be negative. One could however hope for an inequality with $V_{\mathbb{C}} \cdot W_{\mathbb{C}}$ replaced by $|V_{\mathbb{C}} \cdot W_{\mathbb{C}}|$. Note that the main example, is given by a "real" holomorphic bundle E over $V_{\mathbb{C}}$ and $W = V$. Then $V \cdot W$ denoted by $\chi(E)$ is the number of zeros of a generic smooth section of E counted with sign. In the complex case this coincides with $c_n(E)$ ($n = \dim(E)$). In both real and complex case, if $E = TV$ it coincides with

$$\sum_{j=0}^{\dim(V)} (-1)^j \dim H^j(V, \mathbb{Z}).$$

Then $V_{\mathbb{C}} \cap W_{\mathbb{C}} = \chi(E_{\mathbb{C}})$, and $V_{\mathbb{R}} \cap W_{\mathbb{R}} = \chi(E_{\mathbb{R}})$. Thus we could expect

$$|\chi(E_{\mathbb{R}})| \leq |\chi(E_{\mathbb{C}})|$$

However there is the following counterexample, suggested by Slava Kharlamov:

let $M_{\mathbb{C}}$ be a four dimensional complex manifold such that $M_{\mathbb{R}}$ is orientable, and $|\chi(M_{\mathbb{R}})|$ is arbitrarily large.

Let P be the blow up of M along the curve C . The formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ follows from the Mayer-Vietoris exact sequence and implies that

$$\chi(P_{\mathbb{C}}) = \chi(M_{\mathbb{C}}) + (\nu - 1)\chi(C_{\mathbb{C}})$$

where $\nu = \text{codim}(C) = 3$, and

$$\chi(P_{\mathbb{R}}) = \chi(M_{\mathbb{R}}) + 2\chi(C_{\mathbb{R}})$$

Since $C_{\mathbb{R}}$ is a union of circles, $\chi(C_{\mathbb{R}}) = 0$ and thus $\chi(P_{\mathbb{R}}) = \chi(M_{\mathbb{R}})$, while by a suitable choice of C , we may arrange that $\chi(P_{\mathbb{C}})$ is small. Indeed, take $M_{\mathbb{C}}$ to be the product of a surface with real part of arbitrarily large Euler characteristic, and the surface

$$S = \{(x, y, z, t) \in \mathbb{C}P^3 \mid x^2 + y^2 + z^2 + t^2 = 0\}$$

Then $S_{\mathbb{R}}$ is a sphere, $\chi(S_{\mathbb{R}}) = 2$ and $S_{\mathbb{C}}$ contains a curve of genus 4 by taking the intersection of $S_{\mathbb{C}}$ with a generic hypersurface of degree 3 yields a curve of genus 4, thus $\chi(C) = -6$. Since we may move C , after blowing-up we may find a new curve of genus 4 in the blown-up manifold, and thus we may repeat the blow up, eventually get a manifold P such that $|\chi(P_{\mathbb{C}})|$ is between 0 and 5, while $|\chi(P_{\mathbb{R}})| = |\chi(M_{\mathbb{R}})|$ is large.

Thus we see that $|\chi(P_{\mathbb{R}})| = |\chi(TP_{\mathbb{R}})|$ can be arbitrarily large, depending on the choice of the first surface, while $|\chi(P_{\mathbb{C}})| = |\chi(TP_{\mathbb{C}})|$ is between 0 and 5, and $TP_{\mathbb{R}}$ and $TP_{\mathbb{C}}$ are the counter-examples we were looking for.

Lemma 2.7. *Let X, Y be respectively \mathcal{L} -cooriented and \mathcal{L}' -cooriented submanifolds of M , and Z be the clean intersection of X and Y , that is $T_Z X \cap T_Z Y = T_Z$. Then if μ_X, μ_Y are the cohomology classes associated to X, Y in $H^*(M, \mathcal{L} \otimes o_M)$ and $H^*(M, \mathcal{L}' \otimes o_M)$ we have $\mu_X \cup \mu_Y = \mu_N \in H^*(M, \mathcal{L} \otimes \mathcal{L}')$ where N is the zero set of a generic section of νZ , where $\nu Z = T_Z M / (T_Z X + T_Z Y)$*

Proof. Let us take local coordinates in a neighborhood of Z , associated to the decomposition $T_Z M = T_Z \oplus \nu_X \oplus \nu_Y \oplus \nu Z$, where ν_X (resp. ν_Y) is the normal bundle of Z in X (resp Y). Then X is parameterized by the total space of $\nu_X: (z, \xi_X) \rightarrow (z, \xi_X, 0, 0)$, Y by the total space of $\nu_Y: (z, \xi_Y) \rightarrow (z, 0, \xi_Y, 0)$. A generic perturbation \tilde{X} of X will be given by $(z, \xi_X) \rightarrow (z, \xi_X, 0, \varepsilon(z, \xi_X))$, where $\varepsilon(z, \xi_X)$ vanishes for $|\xi_X| \geq \delta$. Then the intersection $\tilde{X} \cap Y$ is given by the equation $\varepsilon(z, 0) = 0$, where ε is a section of νZ . This intersection is transverse provided $\frac{\partial \varepsilon}{\partial z}(z, 0)$ has maximal rank at points where $\varepsilon(z, 0) = 0$, and $\tilde{X} \cap Y$ is equal to the zero set of the section ε . We let the reader check that the coorientations match. \square

In the sequel, given a submanifold V of M , we denote by $V \cdot V$ the intersection of V with the image of V by a small generic isotopy. If V is \mathcal{L} -coorientable, then $V \cdot V$ is $\mathcal{L}^{\otimes 2} \otimes o_M$ -coorientable. The class μ_V is in $H^*(M, \mathcal{L} \otimes o_M)$, and $V \cdot V$ is Poincaré dual to $\mu_V^2 \in H^*(M, \mathcal{L}^{\otimes 2})$.

We now denote by c_V, c_W the codimension of $V_{\mathbb{R}}, W_{\mathbb{R}}$ in $M_{\mathbb{R}}$, so that these are also the complex codimensions of $V_{\mathbb{C}}, W_{\mathbb{C}}$ in $M_{\mathbb{C}}$. Let $\gamma_V \in H^{2c_V}(M_{\mathbb{C}}, \mathbb{Z})$ be the class Poincaré dual to $V_{\mathbb{C}}$. Assume V is coorientable and $\rho_V \in H^{c_V}(M_{\mathbb{R}}, o_V)$ be the class Poincaré dual to $V_{\mathbb{R}}$. Then $[V_{\mathbb{R}}] \cdot [V_{\mathbb{R}}] \in H^{2c_V}(M_{\mathbb{R}}, \mathbb{Z})$ is defined as $(T_{\nu V_{\mathbb{R}}} \cap [V_{\mathbb{R}}])$ is dual to $\rho_V^2 \in H^{2c_V}(M_{\mathbb{R}}, \mathbb{Z})$.

Let i be the inclusion of $M_{\mathbb{R}}$ into $M_{\mathbb{C}}$, we want to compute $i^*(\gamma_V)$ in terms of ρ_V .

Proposition 2.8. *We have*

$$i^*(\gamma_V) = \rho_V^2$$

Proof. Applying lemma 2.7 to $X = V_{\mathbb{C}}$ and $Y = M_{\mathbb{R}}$, with $Z = V_{\mathbb{R}}$, we have that if $\nu(V_{\mathbb{R}})$ is the normal bundle of $V_{\mathbb{R}}$ in $M_{\mathbb{R}}$, it is easy to see that $J\nu(V_{\mathbb{R}})$ is normal to the space $T(V_{\mathbb{C}}) + T(M_{\mathbb{R}})$ in $TM_{\mathbb{C}}$, and thus, a perturbation of $V_{\mathbb{R}}$ will intersect $V_{\mathbb{C}}$ along the zeros of a section of the bundle $J\nu(V_{\mathbb{R}})$ over $V_{\mathbb{R}}$. Since this bundle is isomorphic to $\nu(V_{\mathbb{R}})$, this is the same as the zero set of a section of the normal bundle, and this coincides with $V_{\mathbb{R}} \cdot V_{\mathbb{R}}$. \square

Proposition 2.9. *If $V_{\mathbb{R}}$ is \mathcal{L} -coorientable, we have*

$$(f_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cap M_{\mathbb{R}} = (f_{\mathbb{R}})_*([V_{\mathbb{R}}]) \cdot (f_{\mathbb{R}})_*([V_{\mathbb{R}}])$$

Remark 2.10. Note that this is still true for $V_{\mathbb{C}}$ a variety with singularities of codimension at least 2, so that the same holds for $V_{\mathbb{R}}$. This does not hold if the singularities have codimension 1: in this case we cannot even guarantee that $V_{\mathbb{R}}$ is a cycle.

Remark 2.11. Given the map $f : X \rightarrow Y$ the normal bundle is the quotient $f^*(TY)/TX$

Proof. If f is an embedding, this follows from the previous proposition. Consider now the case where $V \subset E$ and $f = \pi|_V$ where $\pi : E \rightarrow B$ is a real holomorphic projection (so that $\pi^{-1}(B_{\mathbb{R}}) = E_{\mathbb{R}}$). Let $Z \subset B_{\mathbb{R}}$ be a cocycle and consider

$$\begin{aligned} (\pi_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cdot B_{\mathbb{R}} \cdot Z &= V_{\mathbb{C}} \cdot (\pi_{\mathbb{C}})^{-1}(B_{\mathbb{R}} \cap Z) = (V_{\mathbb{C}} \cdot E_{\mathbb{R}}) \cap \pi^{-1}(Z) = \\ &= (V_{\mathbb{R}} \cdot V_{\mathbb{R}}) \cap \pi^{-1}(Z) = \pi_*(V_{\mathbb{R}}) \cdot \pi_*(V_{\mathbb{R}}) \cap Z \end{aligned}$$

Since this holds for any cycle, we have that

$$(\pi_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cdot B_{\mathbb{R}} = \pi_*(V_{\mathbb{R}}) \cdot \pi_*(V_{\mathbb{R}})$$

Now the general case follows from the fact that a general real holomorphic map f is the composition of an embedding $\tilde{f} : V \rightarrow V \times M$ given by $x \rightarrow (x, f(x))$ and the restriction of the projection $V \times M \rightarrow M$. \square

Remark 2.12. After proving the above result, we realized that it can be traced back to [AH93], (cf. Theorem A, (b), page 311) where this was proved mod 2, and is stated there as the identification

$$\bar{H}_{\mathbb{C}-alg}^{2k}(M_{\mathbb{R}}) = \{\alpha^2 \mid \alpha \in H_{alg}^k(M_{\mathbb{R}})\}$$

where $\bar{H}_{\mathbb{C}-alg}^{2k}(M_{\mathbb{R}})$ is the set of pull-backs in $M_{\mathbb{R}}$ of the Poincaré dual classes in $M_{\mathbb{C}}$ of real algebraic submanifolds ⁶, and $H_{alg}^k(M_{\mathbb{R}})$ is the set of classes Poincaré dual to a real algebraic set. The proof in [AH93] is more algebraic, but proves also that any "square of an algebraic class" is induced by a complex class mod 2. However we really need the integral coefficient case in order to get our estimates on the Pontriagin classes (mod 2 estimates would be useless here).

One should always be careful that $[V_{\mathbb{R}}] \cdot [V_{\mathbb{R}}]$ is only a square if $V_{\mathbb{R}}$ is orientable, as we see from the following example.

Example 2.13. 1) Let us consider the inclusion $\mathbb{R}P^n \rightarrow \mathbb{C}P^n$, and the pull-back of the generator $u \in H^2(\mathbb{C}P^n)$ to $H^2(\mathbb{R}P^n)$. Remember that $H^q(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ for q even different from 0, n and is equal to \mathbb{Z} for $q = 0$ and for $q = n$ if n is even. The pull-back of u is then equal to the generator of $H^2(\mathbb{R}P^n)$. This generator can be identified with the class $a^2 \in H^2(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$, where a is the generator of $H^1(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$. Even though this class is a square in $H^2(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$, it isn't a square in $H^2(\mathbb{R}P^n, \mathbb{Z})$

2) Let Q be the manifold of degree 2^{n-1} ,

$$Q_{\mathbb{C}} = \{(z_0, \dots, z_{2n-1}) \mid z_0^2 - z_{2j}^2 - z_{2j+1}^2 = 0, j = 1..n-1\}$$

in $\mathbb{C}P^{2n}$. Then $Q_{\mathbb{R}} = T^{n-1}$. The pull back of the hyperplane class, dual to $\mathbb{C}P^{2n-1}$ is given by the square of $T^{n-1} \cap \mathbb{R}P^{2n-1} = T^{n-2} \cup T^{n-2}$

For $I = (a_1, \dots, a_q)$, let

$$S_I(k) = \{L \mid \dim(L \cap k^{n-k+i+a_i}) \geq i\} \subset G_q(k^n)$$

This is the Schubert cycle associated to the multi-index I . If $|I| = \sum_{l=1}^q a_l$ the cycle has dimension $|I|$. If 1_r denotes a sequence of r ones,

⁶This should not be confused with $H_{\mathbb{C}-alg}^{2k}(M_{\mathbb{R}})$ obtained by pulling back the classes dual to any complex submanifold of $M_{\mathbb{C}}$.

$S_{1_r}(\mathbb{C})$ is Poincaré dual to the r -th Chern class. More generally it is important to notice that the S_I are submanifolds with codimension 2 singularities (cf.[Pon47]), so that according to remark 2.2 the previous results apply.

Remark 2.14. The cycle $\tilde{S}_{1_r}(\mathbb{R})$, lift of $S_{1_r}(\mathbb{R})$ to the Grassmannian of oriented k -subspaces, $\tilde{G}_k(\mathbb{R}^n)$, is orientable in $\tilde{G}_k(\mathbb{R}^n)$ provided r, k are both even ([Pon47]). Note that $\tilde{G}_k(\mathbb{R}^n)$ being simply connected is orientable, and orientability and coorientability coincide in this case.

Note also that the $\tilde{S}_I(\mathbb{R})$ are invariant by the involution τ of $\tilde{G}_k(\mathbb{R}^n)$ that sends a space to the same one with opposite orientation. If this involution is orientation preserving, then $G_k(\mathbb{R}^n)$ will be orientable, and this happens precisely for k even. However, the involution changes the orientation of $S_I(\mathbb{R})$, except when I corresponds to the Euler class. So in the sequel, squares are never real squares, and in particular, one should not conclude that our Pontriagin classes are squares (see example 3.13) !

Now let S be a positive linear combination of Schubert cycles, and c_S be the Poincaré dual class of $S(\mathbb{C})$, o_S be either the trivial bundle or the unique non-trivial local coefficients . We denote by j the inclusion $j : G_k(\mathbb{R}) \rightarrow G_k(\mathbb{C})$. Then $p_S = j^*(c_S)$. Note that if $c_S = \sum_{\alpha \in \{1,n\}^r} a_\alpha c_1^{\alpha_1} \cdots c_r^{\alpha_r}$ (the a_α need not be all non-negative) then ${}^7 p_S = \sum_{\alpha \in \{1,n\}^r} a_\alpha p_2^{\alpha_1} \cdots p_{2r}^{\alpha_r}$. We denote by o the orientation bundle of $G_k(\mathbb{R}^n)$.

Corollary 2.15. *Let $S(\mathbb{C})$ be a positive linear combination of Schubert cycles in $G_q(\mathbb{C}^n)$, and $S(\mathbb{R})$ the analogous cycle in $G_q(\mathbb{R}^n)$. Consider the inclusion $i : G_q(\mathbb{R}^n) \rightarrow G_q(\mathbb{C}^n)$ and $\gamma_S \in H^{2\text{codim}(S)}(G_q(\mathbb{C}^n), \mathbb{Z})$ and $\rho_S \in H^{\text{codim}(S)}(G_q(\mathbb{R}^n), o)$ be the cohomology classes Poincaré dual to $S(\mathbb{C})$ and $S(\mathbb{R})$. Then*

$$j^*(\gamma_S) = \rho_S^2$$

In other words, the cohomology class Poincaré dual to $S(\mathbb{C})$ in $G_q(\mathbb{C}^n)$ induces on $G_q(\mathbb{R}^n)$ a cohomology class Poincaré dual of $S(\mathbb{R}) \cdot S(\mathbb{R})$.

Proof. Follows immediately from prop 2.8. Note that we assume here that $S(\mathbb{R})$ is coorientable, and according to Pontriagin, this is the case exactly when $S(\mathbb{R})$ represents a non-zero Pontriagin class. Had we taken $\mathbb{Z}/2\mathbb{Z}$ coefficients, we could also take the pull-backs of the odd Chern classes c_{2k+1} . □

⁷remember that only the p_{4k} are non-zero

Let now $E \rightarrow M$ be a "real" holomorphic bundle, that is, there is a real bundle $E_{\mathbb{R}}$ over $M_{\mathbb{R}}$ such that

$$E|_{M_{\mathbb{R}}} = (E_{\mathbb{R}} \otimes \mathbb{C})$$

and the anti-holomorphic involution τ lifts to $E_{\mathbb{C}}$ in such a way $E_{\mathbb{R}}$ is the set of fixed points. In a more algebraic setting, E comes from an algebraic locally free sheaf on real algebraic variety X . Assume now that $E_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$ is generated by sections, that is there are sections s_1, \dots, s_m such that for each point $x \in M_{\mathbb{C}}$, $s_1(z), \dots, s_m(z)$ generate E_z . A holomorphic section is said to be real, if $s(\tau(z)) = \bar{s}(z)$. Clearly, if E is generated by sections, it is generated by real holomorphic sections. Indeed, the sections $\Re(s_j) = \frac{s_j(z) + \bar{s}_j(\tau(z))}{2}$, $\Im(s_1) = \frac{s_1(z) - \bar{s}_1(\tau(z))}{2i}$ are real holomorphic and generate $E_{\mathbb{C}}$.

Now the real holomorphic sections induce a real holomorphic map $\tau_{\mathbb{C}} : M_{\mathbb{C}} \rightarrow G_k(\mathbb{C}^m)$, such that its restriction $\tau_{\mathbb{R}}$ to $M_{\mathbb{R}}$ has its image in $G_k(\mathbb{R}^m)$. Moreover, $(\tau_{\mathbb{C}})^*(U_{\mathbb{C}}) = E_{\mathbb{C}}$, $(\tau_{\mathbb{R}})^*(U_{\mathbb{R}}) = E_{\mathbb{R}}$.

This last remark together with lemma 2.3 implies

Theorem 2.16. *Let $E_{\mathbb{C}}$ be a bundle generated by its sections over $M_{\mathbb{C}}$ and assume $E_{\mathbb{C}}$ to be real. Let $J = (j_1, \dots, j_q)$ be such that $4 \sum_{t=1}^q j_t = \dim M$ then we have the inequality*

$$\left| \left\langle \prod_{t=1}^q p_{4j_t}(E_{\mathbb{R}}), [M_{\mathbb{R}}] \right\rangle \right| \leq \left\langle \prod_{t=1}^q c_{2j_t}^2(E_{\mathbb{C}}), [M_{\mathbb{C}}] \right\rangle$$

that we write for short

$$|\langle p_J(E_{\mathbb{R}}), [M_{\mathbb{R}}] \rangle| \leq \langle c_J(E_{\mathbb{C}})^2, [M_{\mathbb{C}}] \rangle$$

Proof. With i denoting the inclusion $i : M_{\mathbb{R}} \rightarrow M_{\mathbb{C}}$ we have $\tau_{\mathbb{C}} \circ i = j \circ \tau_{\mathbb{R}}$. Then if S is a cycle as in the previous corollary we have

$$\begin{aligned} \langle p_S(E_{\mathbb{R}}), M_{\mathbb{R}} \rangle &= \langle \tau_{\mathbb{R}}^* j^*(PD(S(\mathbb{C}))), M_{\mathbb{R}} \rangle = \\ \langle \tau_{\mathbb{R}}^*(PD(S(\mathbb{R})) \cdot PD(S(\mathbb{R}))), M_{\mathbb{R}} \rangle &= \langle \tau_{\mathbb{R}}^*(PD(S(\mathbb{R})) \cdot \tau_{\mathbb{R}}^*(PD(S(\mathbb{R}))), M_{\mathbb{R}} \rangle \end{aligned}$$

According to proposition 2.5, we have the inequality

$$\begin{aligned} |\langle \tau_{\mathbb{R}}^*(PD(S(\mathbb{R})) \cdot \tau_{\mathbb{R}}^*(PD(S(\mathbb{R}))), M_{\mathbb{R}} \rangle| &\leq \\ \langle \tau_{\mathbb{C}}^*(PD(S(\mathbb{C})) \cdot \tau_{\mathbb{C}}^*(PD(S(\mathbb{C}))), M_{\mathbb{C}} \rangle &= \langle c_S^2, M_{\mathbb{C}} \rangle \end{aligned}$$

□

Remarks 2.17. (a). Of course the same statement holds for S representing an effective ample cycle and as a consequence of Giambelli's formula, this is the case for positive combination of Schur polynomials ⁸

$$P = \det(c_{a_i - i + j}) \text{ where } r \geq a_1 \geq a_2 \geq \dots \geq a_r \geq 0$$

For example we get that for a bundle generated by its sections over a surface,

$$|\langle p_1(E_{\mathbb{R}})^2 - p_2(E_{\mathbb{R}}), M_{\mathbb{R}} \rangle| \leq \langle c_1(E_{\mathbb{C}})^2 - c_2(E_{\mathbb{C}}), M_{\mathbb{C}} \rangle$$

(b). Of course the same holds, with obvious changes, if $E_{\mathbb{C}}^*$ is generated by sections, since then $c_j(E_{\mathbb{C}}^*) = (-1)^j c_j(E_{\mathbb{C}})$, and $p_j(E_{\mathbb{R}}^*) = p_j(E_{\mathbb{R}})$

Corollary 2.18 ([Kha04]). *Let $M_{\mathbb{R}}^4$ be real algebraic manifold, and assume $T^*M_{\mathbb{C}}^4$ is generated by sections. If $\sigma(M_{\mathbb{R}}^4)$ is the signature of the 4-dimensional manifold $M_{\mathbb{R}}^4$, then we have*

$$|3\sigma(M_{\mathbb{R}}^4)| \leq c_2^2(M_{\mathbb{C}})$$

Proof. This is obvious once we notice that

$$\sigma(M_{\mathbb{R}}) = \frac{1}{3} \langle p_1(M_{\mathbb{R}}^4), M_{\mathbb{R}}^4 \rangle$$

□

A similar proof yields a result for Pontriagin classes, not only Pontriagin numbers as follows:

Proposition 2.19. *With the assumptions of the previous theorem, let $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ be cohomology classes Poincaré dual to $Z_{\mathbb{R}}$ and $Z_{\mathbb{C}}$, where $Z_{\mathbb{C}}$ is a singular cycle in $M_{\mathbb{C}}$ with codimension two singularities. Then we have*

$$|\langle p_J(E_{\mathbb{R}}) \cup \mu_{\mathbb{R}}, [M_{\mathbb{R}}] \rangle| \leq \langle c_J(E_{\mathbb{C}})^2 \cup \mu_{\mathbb{C}}, [M_{\mathbb{C}}] \rangle$$

Corollary 2.20. *Let E be a bundle of rank r , Λ be a real positive line bundle, and p be such that $E \otimes \Lambda^p$ is globally generated. Then we have as a consequence of the above theorem*

$$c_k(E \otimes \Lambda^p) = \sum_{j=0}^k p^{k-j} c_1(\Lambda)^{k-j} \cdot c_j(E)$$

⁸And it follows from [F 83] that there are no other such classes. Note that [F 83] is also a basic ingredient of [DPS94].

Taking p even, we have that $\Lambda_{|M_{\mathbb{R}}}$ is trivial, hence $p_{4r}(E_{\mathbb{R}} \otimes \Lambda_{\mathbb{R}}) = p_{4r}(E_{\mathbb{R}})$. Thus denoting $\Gamma_r(E; t) = \sum_{j=0}^r t^{r-j} \cdot c_j(E)$, we have

$$|\langle p_J(E_{\mathbb{R}}), [M_{\mathbb{R}}] \rangle| \leq \langle \Gamma_J^2(E_{\mathbb{C}}; pc_1(\Lambda)), [M_{\mathbb{C}}] \rangle$$

3 Bounds

Let X be a $n \geq 1$ -dimensional connected smooth subvariety of $\mathbb{C}P^m$ of degree d . We denote by \mathcal{L}_X the line bundle $\omega_X \otimes \mathcal{O}(n+2)$ where $\omega_X = \bigwedge^n T^*X$ is the canonical bundle. Recall that \mathcal{L}_X is ample⁹. We denote by L_X, K_X and h the (first) Chern classes of \mathcal{L}_X, ω_X and $\mathcal{O}(1)$ respectively.

Proposition 3.1. *One has the inequalities $1 \leq h^{n-i}L_X^i \leq d^{i+1}$ for $i = 0, \dots, n$.*

Proof. The left hand side inequality follows from the ampleness of both $\mathcal{O}(1)$ and \mathcal{L}_X .

Let us prove the other inequality by induction on n . For $i = 0$, this inequality reduces to the equality $h^n = d = \deg(X)$.

Assume that $n > 1$. Let H be a smooth (connected) hyperplane section of X . It's an $n-1$ -dimensional subvariety of $\mathbb{C}P^{m-1}$ of degree d . By adjunction, one has

$$\omega_H = \omega_X \otimes \mathcal{O}(1)|_H$$

and therefore

$$(3.1) \quad (\mathcal{L}_X)|_H = \mathcal{L}_H.$$

First, one has the equalities (projection formula)

$$h^{n-i}L_X^i = h^{n-i-1}L_X^i|_H = h^{n-1-i}L_H^i$$

for all $i < n$.

Because both $\mathcal{O}(1)$ and \mathcal{L}_X are ample, one has $hL_X^{n-1} > 0$ and therefore there exists some rational $\alpha \in \mathbb{Q}$ such that

$$(L_X + \alpha h)L_X^{n-1} = 0.$$

In other words, $L_X + \alpha h$ is a primitive class in $H^2(X, \mathbb{Q})$. By the Hodge index theorem, one gets

$$(L_X + \alpha h)^2 L_X^{n-2} \leq 0.$$

⁹One proves by induction that $K_X + (n+1)H$ is globally generated by sections by using Kodaira vanishing theorem. More precise results can be found in [SV87].

Thus the real quadratic form

$$t \mapsto (L_X + th)^2 L_X^{n-2}$$

has non negative discriminant (because it is negative for $t = 0$ and positive for large t). This gives the inequality

$$0 < L_X^n \leq \frac{(hL_X^{n-1})^2}{(h^2L_X^{n-2})}$$

Applying the above inequality to a smooth codimension k -plane section, we get

$$0 < h^k L_X^{n-k} \leq \frac{(h^{k+1} L_X^{n-k-1})^2}{(h^{k+2} L_X^{n-k-2})}$$

Set now $l_k = \log(h^{n-k} L_X^k)$. Since $h^{n-k} L_X^k \geq 1$ we have $l_k \geq 0$ and the previous inequality implies $l_k + l_{k-2} \leq 2l_{k-1}$, or else $l_k - l_{k-1}$ is decreasing.

We claim that $l_1 - l_0 \leq \log(d)$. Indeed, first one has certainly $l_0 = \log(h^n) = \log(d)$. Second, observe that l_1 is obtained for curves as follows. One could use Castelnuovo's inequalities, but let us just give these elementary bounds. Let C be a generic intersection of X and $(n-1)$ hyperplane. It is a genus g curve of degree d . By 3.1, one has

$$l_1 = \log(\deg(\mathcal{L}_C)) = \log(2g - 2 + 3d)$$

One just has to bound g in terms of d . A generic projection defines a birational morphism onto a plane degree d curve C . One deduces the inequality

$$1 - g = \chi(\mathcal{O}_X) \geq \chi(\mathcal{O}_C) = -d(d-3)/2$$

and therefore

$$l_1 \leq \log(d^2)$$

giving the inequality in this case.

Thus $l_k \leq (k+1)\log(d)$ and this concludes our proof. \square

Remark 3.2. The referee suggested an alternative proof reducing to the hypersurface case by projection.

4 Chern classes

Using the splitting principle, we get the formula

$$c_i(E \otimes L) = \sum_j \binom{n-j}{i-j} c_1(L)^{i-j} c_j(E)$$

for E a (complex) vector bundle and L a line bundle on any variety. Applying this identity to $E = \Omega(2)$ and $L = \mathcal{O}(-2)$

$$(4.1) \quad c_i(\Omega) = \sum_j (-2)^{i-j} \binom{n-j}{i-j} h^{i-j} c_j(\Omega(2)).$$

Because $\Omega_{CP^m}(2)$ is globally generated, so is its quotient $\Omega_X(2)$ which is therefore nef. By [DPS94], corollary 2.6, every Chern number is controlled by K_X and h , or, what's the same by L_X and h . More precisely for every multi index $I = (i_1, \dots, i_r)$ of $r \leq n$ integers in $[1, \dots, n]$, one has

$$0 \leq c_I(\Omega(2)) h^{n-|I|} \leq c_1(\Omega_X(2))^{|I|} h^{n-|I|} = (L_X + (n-2)h)^{|I|} h^{n-|I|}.$$

where $c_I = c_{i_1} \cdots c_{i_r}$. Let's assume $n > 1$ (the case $n = 1$ is left to the reader!). Using 3.1, we get the estimate

$$(4.2) \quad 0 \leq c_I(\Omega(2)) h^{n-|I|} \leq d(d+n-2)^{|I|}.$$

To bound $c_I(\Omega) h^{n-|I|}$, let's denote the multi-index $(n, \dots, n) \in \mathbb{Z}^r$ by \underline{n} . Using 4.1, one gets

$$\begin{aligned} |c_I(\Omega) h^{n-|I|}| &\leq \sum_J 2^{|I|-|J|} \binom{\underline{n}-J}{I-J} h^{n-|J|} c_J(\Omega(2)) \\ &\leq \sum_J 2^{|I|-|J|} \binom{\underline{n}-J}{I-J} d(d+n-2)^{|J|} \text{ by (4.2)} \\ &= d(d+n-2)^{|I|} \sum_J \binom{\underline{n}-I+I-J}{\underline{n}-I} 2^{|I|-|J|} (d+n-2)^{|J|-|I|} \end{aligned}$$

Applying the identity

$$\sum_i \binom{i+m}{m} t^i = \frac{1}{(1-t)^{m+1}} \text{ with } t^{-1} = (d+n-2)/2 \text{ and } m = n-i_\alpha,$$

we obtain that the last sum is bounded by

$$\prod_\alpha \frac{1}{(1-t)^{n-i_\alpha+1}}$$

which in turn is less or equal than 2^{n^2} because $0 \leq t \leq 1/2$ provided $d+n \geq 6$. This yields the estimate (the cases where $d+n \leq 6$ can be dealt with by inspection)

$$(4.3) \quad |c_I(\Omega) h^{n-|I|}| \leq 2^{n^2} d(d+n-2)^{|I|}$$

where $I = (i_1, \dots, i_r)$.

5 Proof of the main theorem 1.1

Assume that X is a real smooth subvariety of degree d of some projective space $\mathbb{R}P^m$. Recall that this means that X is defined by polynomials with real coefficients and that the complex corresponding variety is smooth.

The (twisted) cotangent bundle $\Omega_X(2)$ is therefore a quotient of the twisted cotangent bundle $\Omega_P(2)$. Because the latter is globally generated (straightforward computation), so is $\Omega_X(2)$. By 2.20 and 4.3, we get easily the claimed inequality

$$|p_I(X(\mathbb{R}))| \leq 2^{n^2+3n}d(d+n-2)^n.$$

Once this is proved, the finiteness of the number of cobordism classes follows from the fact that the cobordism ring as the work of [Wal60], completing the results of Thom, Rokhlin and Milnor, is determined by the Pontriagin numbers.

Remark 5.1. According to remark (a), there are also inequalities for the Schur's polynomials. If

$$P(c_k) = \det(c_{a_i-i+j}) \text{ o } r \geq a_1 \geq a_2 \geq \dots \geq a_r \geq 0$$

we have at least that for any Schur's polynomial P there is a universal function $F_P(d, n)$ such that

$$|\langle P(p_{2k}), X(\mathbb{R}) \rangle| \leq F_P(d, n)$$

6 Betti numbers

Let's explain how to obtain bounds for the Betti numbers of X using 4.3. It is important to note that the results of this section only depend on the complex degree of $M_{\mathbb{C}}$, that is lowest degree of an ample (not necessarily real) line bundle. In fact this section is about bounding Betti numbers of complex projective manifolds, as functions of their degree, and the application to real manifolds is a consequence of the Smith-Thom inequality.

If $n = 1$, we have

$$h^{1,0} = g(X) \leq \frac{(d-1)(d-2)}{2}$$

which gives

$$b(X) \leq 2 + (d-1)(d-2).$$

Let's assume $n > 1$. Let H be a smooth hyperplane section as above.

By the Lefschetz Hyperplane theorem, we have the relations

$$b_i(X) = b_i(H) \text{ if } i < n - 1 \text{ and } b_{n-1}(X) \leq b_{n-1}(H).$$

By Poincaré duality, we have

$$b_i(X) = b_{2n-i}(X).$$

Therefore, one gets

$$b_i(X) \leq b_{i-2}(H) \text{ if } i > n.$$

It remains to control the middle term, $b_n(X)$. But the holomorphic Gauss-Bonnet formula (due to Chern) says

$$\chi(X) = c_n(TX).$$

Notice that this formula follows easily from the Riemann-Roch theorem and from the well known formula

$$c_n(TX) = \text{td}(TX) \sum (-1)^p \text{ch}(\bigwedge^p(\Omega_X))$$

([FL85], proposition 5.3). We get therefore¹⁰

$$\begin{aligned} b_n(X) &\leq |\chi(X)| + \left| \sum_{i \neq n} (-1)^i b_i(X) \right| \\ &\leq |\chi(X)| + \left| \sum_{i \neq n, n-1, n+1} (-1)^i b_i(X) \right| + b_{n-1}(X) + b_{n+1}(X) \\ &= |\chi(X)| + |\chi(H)| + b_{n-1}(X) + b_{n+1}(X) \\ &\leq |c_n(TX)| + |c_{n-1}(TH)| + 2b_{n-1}(H) \\ &\leq 2(2^{n^2} d^{n+1} + b_{n-1}(H)) \text{ thanks to (4.3)} \end{aligned}$$

If $b(X) = \sum b_i(X)$ is the total Betti number, one gets

$$b(X) \leq 4b(H) + 2 \cdot 2^{n^2} d^{n+1}$$

and finally

$$b(X) \leq 4^n(d^2 + 1) + 2 \sum_{k=2}^n 4^{n-k} \cdot 2^{k^2} d^{k+1} \leq 2 \sum_{k=0}^n 4^{n-k} \cdot 2^{k^2} d^{k+1}$$

¹⁰Notice that the Euler characteristic is independent from the coefficient field

To get a bound without summation, one can for instance bound $4^{n-k} \cdot 2^{k^2}$ by $2^{n+2+(k+1)(n-2)}$. The last sum is bounded by

$$2^{n+3} \sum_{k=0}^n (2^{n-2}d)^{k+1} = 2^{n+3} \frac{(2^{n-2}d)^{n+2} - 2^{n-2}d}{2^{n-2}d - 1}$$

and, at last,

$$(6.1) \quad b(X) \leq 2^{n^2+2}d^{n+1}$$

(of course it's easy to get a better bound, but with the same growth in d at fixed n).

Using the Smith-Thom's inequality, one gets

Proposition 6.1. *Let X be a real subvariety of the projective space. Assume that $X_{\mathbb{C}}$ is smooth, connected of degree d and dimension n . Then, one has $b(X(\mathbb{R})) \leq 2^{n^2+2}d^{n+1}$.*

Notice that this bound does not depend on the real structure. Also the bound on mod 2 Betti numbers of course implies the same on rational Betti numbers.

Remark 6.2. It should be interesting to have some control on the fundamental group of the (non connected) real part. For instance, given a finite covering of $X(\mathbb{R})$, can one lift this covering to a finite ramified covering of $X_{\mathbb{C}}$ ramified over an hypersurface of controlled degree (without real point of course) ?

7 Appendix: Some simple finiteness results

Let X be a n dimensional complex algebraic variety. We denote by \bar{X} the conjugate variety obtained by the base change by the conjugation $\mathbb{C} \rightarrow \mathbb{C}$. Explicitly, if X is (locally) defined by polynomial equations $\sum a_{i,j}x^i = 0$, the variety \bar{X} is defined by the conjugate equations $\sum \bar{a}_{i,j}x^i = 0$. Of course, one has $\bar{\bar{X}} = X$. To give a real structure on X remains to give a complex morphism $t : X \rightarrow \bar{X}$ such that $\bar{t} \circ t = \text{Id}_X$ (we'll say simply that t is a skew involution of X). The real points of this real complex structure we'll be denoted by $X_t(\mathbb{R})$ (the set of fixe points). If t_1, t_2 are two skew involutions, the composite $t_1 \circ \bar{t}_2$ belongs to $\text{Aut}(X)$ We get therefore the following well known result

Proposition 7.1. *A complex variety of general type has a finite number of real structures.*

Proof. Observe that $\text{Aut}(X)$ is finite in this case ([Kob59]). □

7.1 Final remarks

The set of morphisms $\text{Hom}(X, \bar{X})$ is parameterized by the complex points of a countable union of quasiprojective varieties, which is a locally closed subvariety of the Hilbert scheme of $X \times \bar{X}$ (look at the graph), and therefore the set of skew morphisms has the same property. Recall (Chow) that the subvariety of the Hilbert scheme of a smooth projective (polarized) variety parameterizing the reduced subvarieties of bounded degree is quasiprojective. If L is an ample line bundle on X , it defines a conjugate line bundle \bar{L} on \bar{X} and therefore $X \times \bar{X}$ is polarized by $L \boxtimes \bar{L}$. The degree of the graph of a morphism $t \in \text{Hom}(X, \bar{X})$ is $(c_1(L) + t^*c_1(\bar{L}))^n$. In particular, if $t^*\bar{L}$ and L are numerically equivalent, this degree is bounded by $2^n \deg_L(X)$. This is certainly the case if for instance $H^{1,1}(X) = \mathbb{C}$. Another good situation is when X is Fano. Indeed, one can take $L = \omega_X^{-1}$ (observe that $\bar{\omega}_X = \omega_{\bar{X}}$).

Corollary 7.2. *The real structures of a given complex Fano variety have a finite number of deformation classes.*

Remark 7.3. If the automorphism group of X is linear algebraic, its action on the Picard group factors through the finite group of the connected components. One deduces (averaging a given ample bundle) that the number of deformation classes is finite in this case. A more general result using [BS64] can be found in [DIK04] (Section D.1.10).

In particular, the number of diffeomorphism types of $X_t(\mathbb{R})$ when t runs over all real structures is then finite. This is for instance the case for toric varieties (due to Demazure). This last observation was first made in [Del04]. The same averaging process can more generally be achieved if for instance the closure of the ample cone is (rational) polyhedral (which is also the case for toric varieties), giving a more elementary proof of the former finiteness result (use the fact that the 1-dimensional edges are permuted by the automorphisms and that an automorphism fixing the edges is the identity because it is an integral dilatation on each edge on one hand and, on the other hand, it is volume preserving, its determinant being equal to ± 1).

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