Quantization

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1 Introduction

We want to prove the quantization theorem of Guillermou-Kashiwara-Schapira's paper. Let M be a smooth manifold and I an open interval of $\mathbb{R}$ containing the origin. A *homogeneous Hamiltonian isotopy* is a smooth map $\Phi : \dot{T}^*M \times I \to \dot{T}^*M$ such that

- $\Phi(t, \cdot)$ is a homogeneous symplectic isomorphism for each $t \in I$;
- $\Phi(0, \cdot) = id_{\dot{T}^*M}$.

A homogeneous symplectic isomorphism on $\dot{T}^*M$ is a map $\psi : \dot{T}^*M \to \dot{T}^*M$ invariant under the $\mathbb{R}_{>0}$ action. We have the following classic theorem :

Theorem 1.1. *Φ is a homogeneous Hamiltonian isotopy if and only if it is the Hamiltonian flow of a homogeneous hamiltonian on $\dot{T}^*M$.*

To a homogeneous hamiltonian isotopy, we can associate a unique conic lagrangian submanifold of $\dot{T}^*M \times \dot{T}^*M \times T^*I$:

$$\Lambda_\Phi = \Big\{ (\Phi(t,\xi), -\xi, (t, -H_\Phi(t, \Phi(t,\xi)))) \, ; \xi \in \dot{T}^*M, t \in I \Big\} .$$

We shall sometimes write $\bar{\Lambda}$ for $\Lambda_\Phi \cup T^*_{M\times M\times I} M \times M \times I$. It is easy to check that $\bar{\Lambda}$ is closed in $T^*(M \times M \times I)$. The main result we want to prove is :

Theorem 1.2. *Let Φ be a homogeneous Hamiltonian isotopy. Then there exists a unique sheaf $\mathscr{K} \in \mathrm{D}^{\mathrm{lb}}(M \times M \times I)$ up to unique isomorphism such that*

(i) $\mathrm{SS}(\mathscr{K}) \subset \bar{\Lambda}$,

(ii) $\mathscr{K}|_{t=0} \simeq k_\Delta$.

Such a $\mathscr{K}$ is called a quantization of Φ.

1.1 An explicit quantization

This example will be important for the proof of theorem 1.2. Take M a smooth manifold and choose a metric on M. Then we define the homogeneous geodesic flow Φ_t as the hamiltonian flow of

$$(x,\xi) \mapsto |\xi|.$$

We shall take x only on a compact set $A \subset M$.

Lemma 1.3. *There exists $\epsilon > 0$ such that a local quantization of Φ on $A \times M \times [0,\epsilon]$ is given by k_F where F is the closed subset*

$$F := \{(y,x,t) \in M \times A \times I; d(x,y) \leq t\}.$$

Proof. We take ϵ smaller that the cut locus and the injectivity radius of every point in A. We see that $F = \bar{U}$ where U is the open set $\{(x,y,t) \in A \times M \times I; d(x,y) < t\}$ so we know that :

$$\mathrm{SS}(k_F) = T^*_{\partial U,\mathrm{int}} M \cup F$$

We will now compute $T\partial U$: take $(y,x,t) \in \partial U$. Name v the unique unitary vector of T_xM directing the geodesic linking x to y and $\tilde{v}$ its image by parallel transport. We will find an explicit basis of $T_{y,x,t}\partial U$ which is a vector space of dimension $2n$. By Gauss Lemma, we know that

$$\text{If } \langle \tilde{w}; \tilde{v} \rangle = 0, \text{ then } (\tilde{w},0,0) \in T_{y,x,t}\partial U.$$

This is just a traduction of the fact that geodesics are orthogonals to spheres. By symmetry, we have also

$$\text{If } \langle w; v \rangle = 0, \text{ then } (0,w,0) \in T_{y,x,t}\partial U.$$

And we can see explicitly that

$$(\tilde{v},0,1) \in T_{y,x,t}\partial U \text{ and that } (0,v,-1) \in T_{y,x,t}\partial U.$$

Indeed $(\exp_x((t+\theta)v), x, t+\theta) \in \partial U$ and $(y, \exp_x(\theta v), t-\theta) \in \partial U$ for small θ. So we have found a basis of $T_{y,x,t}\partial U$ and we can check that

$$T_{y,x,t}\partial U = \{(\tilde{w}, w, \tau); \langle \tilde{w}; \tilde{v} \rangle - \langle w; v \rangle - \tau = 0\}.$$

And we know that

$$((y, \langle \tilde{v}; \cdot \rangle), (x, -\langle v; \cdot \rangle), (t,-1)) \in \Lambda_\Phi.$$

So we have proved that

$$\mathrm{SS}(k_F) \subset \Lambda_\Phi \cup T^*_{M\times M\times I} M \times M \times I. \qquad \square$$

2 Technical tools

2.1 Kernels and microsupport

A kernel $\mathscr{K}$ is an element of $\mathrm{D^b}(X \times Y)$. We will say that $\mathscr{K}$ is a good kernel if the application $SS(K) \to T^*X$ is proper. Note that if we see an element $F \in \mathrm{D^b}(X)$ as a kernel of $\mathrm{D^b}(X \times \{pt\})$, it is automatically a good kernel because the application is an inclusion map and is therefore proper. Recall that if $K_1 \in \mathrm{D^b}(X \times Y)$ and $K_2 \in \mathrm{D^b}(Y \times Z)$, we defined :

$$K_1 \circ K_2 := Rq_{XZ!}(q_{XY}^{-1} K_1 \otimes q_{YZ}^{-1} K_2) \in \mathrm{D^b}(Y \times Z).$$

We have the following theorem :

Theorem 2.1. *If K_1 and K_2 are good kernels then $K_1 \circ K_2$ is a good kernel and*

$$SS(K_1 \circ K_2) \subset SS(K_1) \circ SS(K_2)$$

Proof. Recall that if $\Lambda_1 \subset T^*(X \times Y)$, $\Lambda_2 \subset T^*(Y \times Z)$ then we defined

$$\Lambda_1 \circ \Lambda_2 := \left\{ (\xi, \theta) \in T^*(X \times Z) \text{ such that } \exists \eta \in T^*Y \begin{cases} (\xi, -\eta) \in \Lambda_1 \\ (\eta, \theta) \in \Lambda_2 \end{cases} \right\}.$$

We will use without proof the following formulas :

$$\mathrm{SS}(F \otimes G) \subset \mathrm{SS}(F) \hat{+} \mathrm{SS}(G).$$

If $q : M \times N \to M$ is the projection and $F \in \mathrm{D^b}(M \times N)$ such that q is proper on $\mathrm{supp}(F)$ then :

$$\mathrm{SS}(Rq_! F) \subset \{(x, \xi) \in T^*M;\ \text{there exists } y \in N \text{ such that } (x, \xi, y, 0) \in \mathrm{SS}(F)\}.$$

$$\mathrm{SS}(q_{XY}^{-1} K_1) = \{(\xi, \eta, 0) \in T^*(X \times Y \times Z); (\xi, \eta) \in \mathrm{SS}(K_1)\}.$$

The hypothesis that K_1 and K_2 are good kernels ensure us that

$$\mathrm{SS}(q_{XY}^{-1} K_1) \hat{+} \mathrm{SS}(q_{YZ}^{-1} K_2) = \mathrm{SS}(q_{XY}^{-1} K_1) + \mathrm{SS}(q_{YZ}^{-1} K_2).$$

Putting all this together (we need maybe an additional condition), we get that

$$\begin{aligned} \mathrm{SS}(K_1 \circ K_2) &\subset \{(\xi, \theta); (\xi, 0, \theta) \in \mathrm{SS}(q_{XY}^{-1} K_1) + \mathrm{SS}(q_{YZ}^{-1} K_2)\} \\ &\subset \mathrm{SS}(K_1) \circ \mathrm{SS}(K_2). \end{aligned}$$ ∎

We'll also need a relative version of the kernel composition : if we have a kernel $K_1 \in \mathrm{D^b}(X \times Y \times I)$ and $K_2 \in \mathrm{D^b}(Y \times Z \times I)$ where I is a manifold, we define

$$K_1 \circ|_I K_2 := Rq_{XZI}(q_{XYI}^{-1} K_1 \otimes q_{YZI}^{-1} K_2)$$

and we define for $\Lambda_1 \subset T^*(X \times Y \times I)$ and $\Lambda_2 \subset T^*(Y \times Z \times I)$

$$\Lambda_1 \circ|_I \Lambda_2 := \left\{ (\xi, \theta, \tau) \in T^*(X \times Z \times I); \exists (\eta, \tau_1, \tau_2) \in T^*(Y \times I \times I); \begin{cases} (\xi, -\eta, \tau_1) \in \Lambda_1 \\ (\eta, \theta, \tau_2) \in \Lambda_2 \\ \tau_1 + \tau_2 = \tau \end{cases} \right\}$$

2.2 Dual sheaf

Take $\mathscr{K}$ a kernel of $\mathrm{D^b}(X \times Y)$. Note Δ the diagonal in $X \times X$. Then we can check that

$$k_\Delta \circ \mathscr{K} \simeq \mathscr{K}.$$

Indeed we have

$$\begin{aligned} k_\Delta \circ \mathscr{K} &= Rq_{13!}(q_{12}^{-1} k_\Delta \otimes q_{23}^{-1} \mathscr{K}) \\ &= Rq_{13!}(k_{\Delta \times Y} \otimes q_{23}^{-1} \mathscr{K}) \\ &= Rq_{13!} \left(q_{23}^{-1} F\right)_{\Delta \times Y}. \end{aligned}$$

On $\Delta \times Y$, $q_{13} = q_{23}$ so we have

$$\begin{aligned} k_\Delta \circ \mathscr{K} &= Rq_{13!}(k_{\Delta \times Y} \otimes q_{13}^{-1} \mathscr{K}) \\ &\simeq Rq_{13!} k_{\Delta \times Y} \otimes \mathscr{K} \\ &\simeq Rq_{13!} R\delta_! k_{M \times N} \otimes \mathscr{K} \simeq k_{M \times N} \otimes \mathscr{K} \\ &\simeq \mathscr{K}. \end{aligned}$$

When we have a kernel $\mathscr{K} \in \mathrm{D^b}(X \times Y)$, we would like to define a kernel $\mathscr{K}^{-1} \in \mathrm{D^b}(Y \times X)$ such that

$$\mathscr{K}^{-1} \circ \mathscr{K} \simeq k_\Delta.$$

This won't be always possible. We can still define a dual sheaf by :

$$\mathscr{K}^{-1} = v^{-1} R\mathscr{H}om(\mathscr{K}, q_Y^! k_Y) \in \mathrm{D^b}(Y \times X)$$

where $v : Y \times X \to X \times Y$ is the swap.
This definition makes sense because there is a natural morphism $\mathscr{K}^{-1} \circ \mathscr{K} \to k_{\Delta_Y}$. Indeed let q_{ij} be the $(i,j)-$th projection from $Y \times X \times Y$ and $\delta : Y \to Y \times Y$ the diagonal embedding. Then we have

$$\delta^{-1}(\mathscr{K}^{-1} \circ \mathscr{K}) = \delta^{-1} Rq_{13!}(q_{12}^{-1} \mathscr{K}^{-1} \otimes q_{23}^{-1} \mathscr{K})$$

But we have the following cartesian square

$$\begin{tikzcd}
X \times Y \arrow[r, "\bar{\delta}"] \arrow[d, "q_Y"'] \arrow[dr, phantom, "\square"] & Y \times X \times Y \arrow[d, "q_{13}"] \\
Y \arrow[r, "\delta"'] & Y \times Y
\end{tikzcd}$$

so we can apply the proper base change formula :

$$\begin{aligned}
\delta^{-1}(\mathscr{K}^{-1} \circ \mathscr{K}) &\simeq Rq_{2!}\bar{\delta}^{-1}(q_{12}^{-1}\mathscr{K}^{-1} \otimes q_{23}^{-1}\mathscr{K}) \\
&\simeq Rq_{2!}(\bar{\delta}^{-1}q_{12}^{-1}\mathscr{K}^{-1} \otimes \bar{\delta}^{-1}q_{23}^{-1}\mathscr{K}) \\
&\simeq Rq_{2!}(\mathscr{K} \otimes R\mathscr{H}om(\mathscr{K}, q_Y^! k_Y) \\
&\to Rq_{2!}(q_2^! k_Y) \\
&\to k_Y
\end{aligned}$$

We have found a natural morphism in $\mathrm{Hom}(\delta^{-1}\mathscr{K}^{-1} \circ \mathscr{K}, k_Y) \simeq \mathrm{Hom}(\mathscr{K}^{-1} \circ \mathscr{K}, \delta_* k_Y)$ so we have found a morphism

$$\mathscr{K}^{-1} \circ \mathscr{K} \to k_{\Delta_Y}.$$

With good conditions on $\mathscr{K}$, we could ensure that this morphism is an isomorphism but this is not the subject of this talk.

One another important remark is that we have the following expected formula :

$$k_\Delta^{-1} \simeq k_\Delta.$$

Indeed

$$\begin{aligned}
k_\Delta^{-1} &= v^{-1}R\mathscr{H}om(k_\Delta, q_2^! k_M) \\
&\simeq v^{-1}R\Gamma_\Delta(q_2^! k_M) \\
&\simeq v^{-1}\delta_!\delta^{-1}R\Gamma_\Delta(q_2^! k_M) && (1) \\
&\simeq v^{-1}\delta_!\delta^! q_2^! k_M && (2) \\
&\simeq v^{-1}\delta_! k_M \\
&\simeq v^{-1}k_\Delta \\
&\simeq k_\Delta
\end{aligned}$$

To understand equalities (1) and (2), you must remember that if $j : Z \hookrightarrow X$ is the embedding of a locally closed subset then we have an equivalence of category between $\mathrm{Sh}(Z)$ and the sheaves on X with support included in Z. Two inverse functors are given by $j_!$ and j^{-1}. As the support of $R\Gamma_\Delta(F)$ is included in Δ, we have indeed $\delta_! \delta^{-1} R\Gamma_\Delta F \simeq R\Gamma_\Delta F$. (2) is just the definition : $\delta^! = \delta^{-1} R\Gamma_\Delta$ proved in my last talk.

Before going into the proof of the main theorem, we can summarize the last two sections by highlighting what will be useful in the course of the proofs about our two tools :

- If $K_1 \in \mathrm{D}(X \times Y)$ and $K_2 \in \mathrm{D}(Y \times Z)$ then we defined $K_1 \circ K_2 \in \mathrm{D}(Y \times Z)$ and we have a formula for the microsupport : $\mathrm{SS}(K_1 \circ K_2) \subset \mathrm{SS}(K_1) \circ \mathrm{SS}(K_2)$. The relative version $\circ|_I$ with the analogous formula for the microsupport will be also very useful.
- $k_{\Delta_X} \circ K_1 \simeq K_1$
- If $K \in \mathrm{D}(X \times Y)$ then we defined $K^{-1} \in \mathrm{D}(Y \times X)$. We will see later the following formula for the micro support : $\mathrm{SS}(K^{-1}) \subset -v(\mathrm{SS}(K))$.
- $k_\Delta^{-1} \simeq k_\Delta$.

3 Uniqueness of the quantization

In this section, we prove

Proposition 3.1. *Take Φ a homogeneous hamiltonian isotopy and assume we have found $\mathscr{K} \in \mathrm{D}^{\mathrm{lb}}(M \times M \times I)$ with I an interval containing 0 a quantization of Φ. Then we have*

(i) $\mathscr{K}$ is a good kernel;

(ii) $\mathscr{K}_t \circ \mathscr{K}_t^{-1} \simeq \mathscr{K}_t^{-1} \circ \mathscr{K}_t \simeq k_\Delta$ for all $t \in I$;

(iii) such a $\mathscr{K}$ quantizing Φ is unique up to a unique isomorphism

where $\mathscr{K}_{t_0} := \mathscr{K}|_{t=t_0} \simeq K \circ k_{t=t_0} \in \mathrm{D}^{\mathrm{lb}}(M \times M)$.

Proof. (i) is clear : the projection $\mathrm{SS}(\mathscr{K}) \to T^*(M \times I)$ is the composition

$$\mathrm{SS}(\mathscr{K}) \hookrightarrow \Lambda \cup T^*_{M \times M \times I} M \times M \times I \to T^*(M \times I)$$

and the first one is an inclusion so is proper. The second one is clearly proper when you look at the definition of Λ so $\mathscr{K}$ is a good kernel. As the composition of two good kernels $\mathscr{K}_t$ is also a good kernel and so is $\mathscr{K}_t^{-1}$.

Let's prove (ii). Consider $F = \mathscr{K} \circ|_I \mathscr{K}^{-1}$. We will prove that

(a) $F_t \simeq F_0$ for all $t \in I$

(b) $F_t \simeq \mathscr{K}_t \circ \mathscr{K}_t^{-1}$

and this will clearly imply (ii).
To prove (a), we have to compute the micro support of F. As all kernels are good kernels, we have :

$$\mathrm{SS}(F) \subset \mathrm{SS}(\mathscr{K}) \circ|_I \mathrm{SS}(\mathscr{K}^{-1})$$

where $\mathscr{K}^{-1} = (v \times Id)^{-1} R\mathscr{H}om(\mathscr{K}, \omega_M \boxtimes k_M \boxtimes k_I)$ and where v is the swap. This implies that we have

$$\mathrm{SS}(\mathscr{K}^{-1}) = v(-\mathrm{SS}(K)).$$

$$\mathrm{SS}(F) \subset v(-\Lambda) \circ|_I \Lambda \cup T^*_{M\times M\times I} M \times M \times I.$$

Recall that

$$\Lambda = \{(\Phi_t(\xi), -\xi, (t, \tau(t,\xi)); t \in I, \xi \in \dot{T}^*M\}.$$

We have

$$v(-\Lambda) = \{(\xi', -\Phi_t(\xi'), (t', -\tau(t', \xi'))); t \in I, \xi' \in \dot{T}^*M\}$$

and

$$v(-\Lambda) \circ|_I \Lambda = \{(\Phi_t(\xi), -\Phi_t(\xi), (t, 0)); t \in I, \xi \in \dot{T}^*M\}.$$

Finally we have :

$$\mathrm{SS}(F) \subset T^*(M \times M) \times T^*_I I.$$

This implies that we have (see lemma 3.2 below):

$$F \simeq p^{-1} Rp_* F$$

where p is the projection $M \times M \times I \to M \times M$. So :

$$F_t \simeq i_t^{-1} p^{-1} Rp_* F \simeq Rp_* F \simeq i_0^{-1} p^{-1} Rp_* F \simeq F_0.$$

and this concludes the proof of (a). Now we want to simplify

$$i_t^{-1} F \simeq i_t^{-1} Rq_{13I!} \left(q_{12I}^{-1} K \otimes q_{23I}^{-1} K^{-1}\right)$$

We have the following cartesian square :

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\tilde{i_t}"] \arrow[d, "q_{13}"'] \arrow[dr, phantom, "\square"] & M \times M \times M \times I \arrow[d, "q_{13I}"] \\
M \times M \arrow[r, "i_t"'] & M \times M \times I
\end{tikzcd}$$

so we have :

$$\begin{aligned} i_t^{-1}F &\simeq Rq_{13!}\tilde{i_t}^{-1}\left(q_{12I}^{-1}K\otimes q_{23I}^{-1}K^{-1}\right) \\ &\simeq Rq_{13!}\left(\tilde{i_t}^{-1}q_{12I}^{-1}K\otimes \tilde{i_t}^{-1}q_{23I}^{-1}K^{-1}\right) \\ &\simeq (i_t^{-1}K)\circ(i_t^{-1}K^{-1}). \end{aligned}$$

It is now enough to prove that

$$i_t^{-1}K^{-1}\simeq K_t^{-1}. \tag{1}$$

This is a direct consequence of exercise VI.4 of [2][1] to conclude the proof of (ii). We just note that to prove (1), we use the fact that

$$\mathrm{SS}(K)\cap T^*_{M\times M}(M\times M)\times T^*I\subset T^*_{M\times M\times I}(M\times M\times I).$$

For the proof of (iii), we must precise now what F is : we know that $F\simeq p^{-1}Rp_*F$. This implies that $Rp_*F\simeq F_0\simeq k_\Delta$. This gives us that $F\simeq p^{-1}k_\Delta$:

$$F\simeq k_{\Delta\times I}$$

(iii) is now just an easy consequence of what we have said until now : take K_1 and K_2 two quantification of Φ. Take $L=K_1^{-1}\circ|_I K_2$. The proof of (ii) still applies : we have

$$\mathrm{SS}(L)\subset T^*(M\times M)\times T^*_I I,$$

$$L_0\simeq k_\Delta.$$

This implies that $L\simeq k_{\Delta\times I}$ and we can also prove with the same methods that $K_1\circ|_I K_1^{-1}\simeq k_{\Delta\times I}$ so we have

$$K_2\simeq\left(K_1\circ|_I K_1^{-1}\right)\circ|_I K_2\simeq K_1\circ|_I L\simeq K_1.$$

The uniqueness of the isomorphism comes from the uniqueness of the isomorphisms $K_1\circ|_I K_1^{-1}\simeq k_{\Delta\times I}$ and $L\simeq k_{\Delta\times I}$. □

During the proof we have used the following lemma which we will now prove.

Lemma 3.2. *Let I be a contractible manifold and let $p: M\times I\to M$ be the projection. If $F\in \mathrm{D^b}(M\times I)$ satisfies $\mathrm{SS}(F)\subset T^*M\times T^*_I I$, then $F\simeq p^{-1}Rp_*F$.*

Proof. Proposition 2.7.8 □

[1] To prove this would take too long. [1] gives a proof of this exercise using 3.1.13, 6.4.3(ii) and 6.4.4 (iii) of [2]

An important corollary of this uniqueness theorem is the following :

Lemma 3.3. *Let A be a relatively compact open subset of M. There exists an open subset U of $M \times A$ containing the diagonal such that if we take $Z = \bar{U}$ and $L = k_Z \in \mathrm{D^b}(M \times A)$,*

$$L^{-1} \circ L \simeq k_{\Delta_U}$$

Proof. Take the explicit quantization we found for the homogeneous geodesic flow and take $U = \{(x, y) \in A \times M; d(x, y) < \epsilon\}$. Then the uniqueness theorem ensures us that

$$k_{\bar{U}}^{-1} \circ k_{\bar{U}} \simeq k_{\Delta_U}.$$

□

4 Local existence in the compact case

We will use without proof the following lemma

Lemma 4.1. *Let M be a manifold and I an open interval of $\mathbb{R}$ containing 0. Take $(\Lambda_t)_{t\in I}$ a smooth family of lagrangian submanifolds of $\dot{T}^*M$. Write*

$$\Lambda = \cup_{t\in I}\Lambda_t \times \{t\}$$

the conic lagrangian of $\dot{T}^*M \times T^*I$ associated to this deformation. Suppose there exists U_0 an open subset with smooth boundary such that $\Lambda_0 = T^*_{int,\partial U_0}M$. Suppose also that outside a compact subset A, there is no deformation :

$$\Lambda \cap (T^*(M \setminus C) \times T^*I) = \Lambda_0 \times I.$$

Then there exists $\epsilon > 0$ with $\pm\epsilon \in I$ and an open subset $V \subset M\times]-\epsilon; \epsilon[$ with smooth boundary such that

(i) $\Lambda = \mathrm{SS}(k_{\bar{V}}) \cap \dot{T}^*M$*;*

(ii) $\Lambda_t = \mathrm{SS}(k_{\bar{V}_t}) \cap \dot{T}^*M$ *for any* $t \in]-\epsilon; \epsilon[$*;*

(iii) $V_0 = U_0$.

We will deduce of this lemma the following theorem

Theorem 4.2. *Let Φ be a homogeneous hamiltonian isotopy with compact support (outside of $\dot{T}^*V$ where V is a relatively compact open subset of M, Φ_t is equal to the identity of $\dot{T}^*M$ for all t). Then there exists $\epsilon > 0$ and $K \in \mathrm{D^b}(M \times M\times]-\epsilon; \epsilon[)$ a quantization of Φ on $]-\epsilon; \epsilon[$.*

Proof. Take $V \subset A \subset U$ where A is compact and U is a relatively compact open set and take $Z \subset M \times U$ as in lemma 3.3 and define $L := k_Z \in \mathrm{D^b}(M \times U)$. We have $\mathrm{SS}(k_Z) = Z \cup \Gamma_Z$ where $\Gamma_Z \subset \dot{T}^*(M \times U)$. Take $\Lambda_{\Phi,U}$ the restriction of Λ_Φ on $\dot{T}^*(U \times U \times I)$. Define

$$\widetilde{\Lambda} := \Gamma_Z \circ \Lambda_{\Phi,U} \subset \dot{T}^*M \times \dot{T}^*U \times T^*I,$$

$$\widetilde{\Lambda}_t = \Gamma_Z \circ \Lambda_{\Phi,U,t} \subset \dot{T}^*M \times \dot{T}^*U$$

We can now check that $\widetilde{\Lambda}_t$ is a deformation of $\Gamma_Z = T^*_{int,\partial IntZ}M$ and outside of the compact $A \times A \subset M \times U$, there is no deformation. So we get $\epsilon > 0$ and $\widetilde{L} \in \mathrm{D^b}(M \times U \times]-\epsilon;\epsilon[)$ a quantization of this deformation. We have :

$$\mathrm{SS}(\widetilde{L}) = \Gamma_Z \circ \Lambda_{\Phi,U} \text{ and } \widetilde{L}_0 \simeq k_Z$$

Set now

$$K := L^{-1} \circ|_I \widetilde{L} \in \mathrm{D^b}(U \times U \times]-\epsilon;\epsilon[).$$

We can compute

$$\mathrm{SS}(K) \subset \mathrm{SS}(L^{-1}) \circ \Gamma_Z \circ \Lambda_{\Phi,U} \subset T^*_{\Delta_U}(U \times U) \circ \Lambda_{\Phi,U} \subset \Lambda_{\Phi,U}.$$

and

$$K_0 = (L^{-1}) \circ \widetilde{L}_0 \simeq k_\Delta.$$

So K is a quantization of Φ on $U \times U \times]-\epsilon;\epsilon[$. The uniqueness theorem ensures us that outside of $A \times A$, K is equal to k_Δ :

$$K|_{(U\times U)\setminus(A\times A)\times]-\epsilon;\epsilon[} \simeq k_{\Delta_M \times]-\epsilon;\epsilon[}|_{(U\times U)\setminus(A\times A)\times]-\epsilon;\epsilon[}$$

so we can extend K to $\widetilde{K} \in \mathrm{D^b}(M \times M \times]-\epsilon;\epsilon[)$ (see lemma 5.1 below) which will be a quantization of Φ on $M \times M \times]-\epsilon;\epsilon[$. □

5 Gluing sheaves

Lemma 5.1. *Let U_1 and U_2 be two open subsets of M and set $U_{12} := U_1 \cap U_2$. Let $F_i \in \mathrm{D}(U_i)$ for $i = 1, 2$ and assume we have an isomorphism $\phi_{21} : F_1|_{U_{12}} \simeq F_2|_{U_{12}}$. Then there exists $F \in \mathrm{D}(U_1 \cup U_2)$ and isomorphisms $\phi_i : F|_{U_i} \simeq F_i$ such that $\phi_{21} = \phi_2|_{U_{12}} \circ \phi_1|_{U_{12}}^{-1}$. Moreover such a triple (F, ϕ_1, ϕ_2) is unique up to a (non-unique) isomorphism.*

Proof. For a more formal proof of this lemma, you can see [1]. There is a natural way of extending $F_i \in \mathrm{D}(U_i)$ as an element $\widetilde{F_i} \in \mathrm{D}(U_1 \cup U_2)$ which will have same stalks as F_i on U_i and will have trivial stalks outside of U_i. Formally $\widetilde{F_i} = j_{i!}F_i$ where $j_i : U_i \to U_1 \cup U_2$ is the injection. We know that

$$\left(\widetilde{F_1}\right)_{U_{12}} \simeq \left(\widetilde{F_2}\right)_{U_{12}}$$

thanks to the isomorphism ϕ_{12}. For all $Z \subset X$ locally closed, we have a natural morphism $k_Z \to k_X$ and this gives a natural morphism

$$F_Z \to F.$$

So we can define natural morphisms

$$\left(\widetilde{F_1}\right)_{U_{12}} \to \widetilde{F_1} \text{ and } \left(\widetilde{F_1}\right)_{U_{12}} \to \widetilde{F_2}$$

Consider now

$$\left(\widetilde{F_1}\right)_{U_{12}} \longrightarrow \widetilde{F_1} \oplus \widetilde{F_2}$$

and define F as the mapping cone of this morphism. We have

$$\left(\widetilde{F_1}\right)_{U_{12}} \longrightarrow \widetilde{F_1} \oplus \widetilde{F_2} \longrightarrow F \longrightarrow +1.$$

If we look at this exact sequence on $U_1 \setminus U_{12}$, we see

$$0 \longrightarrow F_1 \oplus 0 \longrightarrow F \longrightarrow 0.$$

On U_{12} we see that F is the mapping cone of

$$F_1 \longrightarrow F_1 \oplus F_1$$

so F is isomorphic to $F_1 \simeq F_2$ on U_{12}. This is exactly what we want. The uniqueness of F comes from the fact if F is a solution of our gluing problem then $F_{U_1} \simeq \widetilde{F_1}$, $F_{U_2} \simeq \widetilde{F_2}$ and $F_{U_{12}} \simeq \left(\widetilde{F_1}\right)_{U_{12}}$ and of the following distinguished triangle

$$F_{U_{12}} \to F_{U_1} \oplus F_{U_2} \to F \to +1.$$

The non uniqueness of the isomorphism comes from the non uniqueness of the isomorphism between mapping cones (see [2] lemma 1.4.2). □

Lemma 5.2. *Let $j_n : U_n \hookrightarrow M$ be an increasing sequence of open embeddings of M with $\cup_n U_n = M$. We consider a sequence $\{F_n\}_n$ with $F_n \in \mathrm{D}^{\mathrm{lb}}(U_n)$ together with isomorphisms $u_{n+1,n} : F_n \simeq F_{n+1}|_{U_n}$. Then there exists $F \in \mathrm{D}^{\mathrm{lb}}(M)$ and isomorphisms $u_n : F|_{U_n} \simeq F_n$ such that $u_{n+1,n} = u_{n+1} \circ u_n^{-1}$ for all n. Moreover such a family $(F, \{u_n\})$ is unique up to a (non-unique) isomorphism.*

Proof. See [1] for a more formal proof. The idea is essentially the same as the previous lemma : take $\widetilde{F_n} \in \mathrm{D^{lb}}(M)$ an extension of $F_n \in \mathrm{D^{lb}}(U_n)$. Then, as $\left(\widetilde{F_{n+1}}\right)_{U_n} \simeq \widetilde{F_n}$, we have natural morphisms

$$\widetilde{F_n} \to \widetilde{F_{n+1}}$$

and we can take F as the direct limit of the diagram $\{\widetilde{F_n}\}$. $\square$

6 Quantization theorem

Theorem 6.1. *Let $\Phi : \dot{T}^*M \times I \to \dot{T}^*M$ be a homogeneous hamiltonian isotopy with compact support with I an interval of $\mathbb{R}$. Then there exists $K \in \mathrm{D^b}(M \times M \times I)$ a quantization of Φ.*

Proof. Consider the set of pairs (J, K_J) where J in an open interval contained in I and containing 0 and K_J is quantization of $\{\phi_t\}_{t \in J}$. This set is ordered by

$$(J, K_J) \leq (J', K_{J'}) \iff \begin{cases} J \subset J' \\ K_{J'}|_{M \times M \times J} \simeq K_J. \end{cases}$$

The lemma 5.2 ensures us that every totally ordered subset of this set has an upper bound. Take now (J, K_J) a maximal element of this set and we'll show that $J = I$.

Write $J =]t_0, t_1[$ and suppose $t_1 \in I$. Then consider the homogeneous hamiltonian isotopy $\{\phi_t \circ \phi_{t_1}^{-1}\}_{t \in I}$. The local existence in the compact case ensures us that there exists $t_0 < t_3 < t_1 < t_4$ and $L \in \mathrm{D^{lb}}(M \times M \times]t_3; t_4[)$ a quantization of this isotopy for small time. Choose t_2 with $t_3 < t_2 < t_1$ and set

$$F = K|_{]t_3;t_1[} \circ K_{t_2}^{-1},$$

$$F' = L|_{]t_3;t_1[} \circ L_{t_2}^{-1}.$$

Then it is easy to verify that F and F' are quantization of the isotopy $\{\phi_t \circ \phi_{t_2}^{-1}\}_{t \in]t_3;t_1[}$. By uniqueness of the quantization, F and F' are isomorphic and

$$K|_{]t_3;t_1[} \simeq L|_{]t_3;t_1[} \circ L_{t_2}^{-1} \circ K_{t_2}$$

using lemma 5.1, we can find $\widetilde{K}$ such that

$$\widetilde{K}\Big|_{]t_0;t_1[} \simeq K \text{ and } \widetilde{K}\Big|_{]t_3;t_4[} \simeq L \circ L_{t_2}^{-1} \circ K_{t_2}.$$

It is easy to verify that $\widetilde{K}$ is a quantization of ϕ on $]t_0; t_4[$. So this is absurd because J is supposed maximal : $J = I$. $\square$

We are now ready to prove the main theorem :

Theorem 6.2. *Let $\Phi : \dot{T}^*M \times I \to \dot{T}^*M$ be a homogeneous hamiltonian isotopy. There exists a quantization $K \in \mathrm{D}^{\mathrm{lb}}(M \times M \times I)$ of Φ.*

Proof. Take increasing sequence of relatively compact $U_n \times J_n$ of $M \times I$ where U_n is an open set of M and J_n an interval containing 0. We write

$$\dot{\pi} : \dot{T}^*M \to M.$$

As Φ is homogeneous, we know that $\dot{\pi}(\Phi(J_n, \dot{T}^*U_n))$ is relatively compact. Take $g_n : M \to \mathbb{R}$ a function with compact support such that $g_n = 1$ on $\dot{\pi}(\Phi(J_n, \dot{T}^*U_n))$. If we name H the homogeneous hamiltonian associated to Φ then define Φ_n as the homogeneous hamiltonian isotopy associated to

$$(x, \xi, t) \mapsto g_n(x)H(x, \xi, t).$$

Φ_n has compact support and verifies

$$\Phi_n|_{\dot{T}^*U_n \times J_n} = \Phi|_{\dot{T}^*U_n \times J_n}.$$

Take now $L_n \in \mathrm{D}^{\mathrm{lb}}(M \times M \times I)$ a quantization of Φ_n as given by theorem 6.1 and take $K_n = L|_{M \times U_n \times J_n}$. The uniqueness theorem ensures us that $K_{n+1}|_{M \times U_n \times J_n} \simeq K_n$. We can apply lemma 5.2 and we get $K \in \mathrm{D}^{\mathrm{lb}}(M \times M \times I)$. As $(K_n)_0 \simeq k_\Delta$ for all n, we know that

$$K_0 \simeq k_\Delta.$$

Moreover

$$\begin{aligned} \mathrm{SS}(K) \cap \dot{T}^*(M \times U_n \times J_n) &= \mathrm{SS}(K_n) \cap \dot{T}^*(M \times U_n \times J_n) \\ &\subset \Lambda_{\Phi_n} \cap \dot{T}^*(M \times U_n \times J_n) \\ &\subset \Lambda_{\Phi} \cap \dot{T}^*(M \times U_n \times J_n). \end{aligned}$$

We have proved that K is a quantization of Φ. □

7 Some useful reminders

7.1 Definitions

Let Y, X be topological spaces and $f : Y \to X$ a continuous map. k is a field. O will be an open subset of X and F a closed subset of X. W will be either an open or a closed subset of X[2]. $\mathscr{F}$, $\mathscr{H}$ and $\mathscr{L}$ will be sheaves on X and $\mathscr{G}$ a sheaf on Y. The set of sections of $\mathscr{F}$ over an open set U will be noted either $\Gamma(U; \mathscr{F})$ or $\mathscr{F}(U)$.

[2]all the definitions can be adapted for W locally closed

- $k_O(U) := \{s : O \cap U \to k$ locally constant with closed support in $O\}$.
- $k_F(U) := \{s : F \cap U \to k$ locally constant$\}$.
- The functor of global sections is $\Gamma(X; \cdot) : \mathrm{Sh}(X) \to \mathrm{Mod}$.
- $f_*\mathscr{G}(U) := \mathscr{G}(f^{-1}(U))$.
- $f^{-1}\mathscr{F}$ is the sheafification of the presheaf $U \to \lim_{V \supset f(U)} \mathscr{F}(V)$.
- If Z is a subset of X and if $j : Z \hookrightarrow X$ is the inclusion map, $\Gamma(Z; F) := \Gamma(Z; j^{-1}\mathscr{F})$. Be careful that $\Gamma(Z; \mathscr{F})$ is not $\lim_{V \supset Z} \mathscr{F}(V)$ because of the sheafification. This will be true if Z is closed and X is paracompact.
- $\mathscr{F} \otimes \mathscr{H}$ is the sheafification of the presheaf $U \to \mathscr{F}(U) \otimes \mathscr{H}(U)$.
- $\mathscr{H}om(\mathscr{F}, \mathscr{H})(U) := \{$the module of morphisms of sheaves from $\mathscr{F}|_U$ to $\mathscr{H}|_U\}$
- The sheaf of sections of $\mathscr{F}$ with closed support in W is $\mathscr{F}_W := k_W \otimes \mathscr{F}$. We can prove that $\mathscr{F}_X = \mathscr{F}$ and that $\mathscr{F}_F(U) = \Gamma(F \cap U; \mathscr{F})$.
- The sheaf of sections of F supported by W is $\Gamma_W\mathscr{F} := \mathscr{H}om(k_W, \mathscr{F})$. We can prove that $\Gamma_X\mathscr{F} = \mathscr{F}$ and that $\Gamma_O\mathscr{F}(U) = \mathscr{F}(O \cap U)$.
- $f_!\mathscr{G}(U) := \{s \in \mathscr{G}(f^{-1}(U)), f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U$ is proper$\}$.
- We won't define $f^!$, we just need to know that it is defined on $\mathrm{D^b}(X)$ and that it is a right adjoint for $Rf_!$.

7.2 On the stalk level

- $(k_W)_x = \begin{cases} k \text{ if } x \in W \\ 0 \text{ if } x \notin W \end{cases}$
- $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$.
- $(\mathscr{F} \otimes \mathscr{H})_x = \mathscr{F}_x \otimes \mathscr{H}_x$.
- $(F_W)_x = \begin{cases} F_x \text{ if } x \in W \\ 0 \text{ if } x \notin W \end{cases}$
- There are no easy formulas for the stalks of $f_*\mathscr{G}$, $\mathscr{H}om(\mathscr{F}, \mathscr{H})$, $\Gamma_W\mathscr{F}$ and $f_!\mathscr{F}$.

7.3 Some formulas

We will write the formulas on $\mathrm{Sh}(X)$ but they can be derived without any problems.

- $\mathscr{H}om(\mathscr{F}, f_*\mathscr{G}) \simeq f_* \mathscr{H}om(f^{-1}\mathscr{F}, \mathscr{G})$ as sheaves on X.
- $\mathscr{H}om(\mathscr{F}, \mathscr{H}om(\mathscr{H}, \mathscr{L})) \simeq \mathscr{H}om(\mathscr{F} \otimes \mathscr{H}, \mathscr{L})$.
- $\Gamma(U; \cdot) \circ \Gamma_O \simeq \Gamma(U \cap O; \cdot)$.
- $f^{-1}(\mathscr{F} \otimes \mathscr{H}) \simeq f^{-1}\mathscr{F} \otimes f^{-1}\mathscr{H}$.
- $f_!\mathscr{G} \otimes \mathscr{F} \simeq f_!(\mathscr{G} \otimes f^{-1}\mathscr{F})$
- If you have a cartesian square then $g^{-1} \circ Rf_! \simeq Rf'_! \circ \bar{g}^{-1}$
- On the derived category, we have

$$R\mathscr{H}om(Rf_!\mathscr{G}, \mathscr{F}) \simeq Rf_* \circ R\mathscr{H}om(\mathscr{G}, f^!\mathscr{G}).$$

As we have no idea of what $f^!$ is, this is the only way we have to deal with $f^!$. For example, an important trick is the following :

$$\begin{aligned} R\Gamma(U; f^!\mathscr{F}) &\simeq R\Gamma(X; \cdot) \circ R\Gamma_U f^!\mathscr{F} \\ &\simeq R\Gamma(X; \cdot) \circ R\mathscr{H}om(k_U, f^!\mathscr{F}) \\ &\simeq R\Gamma(Y; \cdot) \circ Rf_* \circ R\mathscr{H}om(k_U, f^!\mathscr{F}) \\ &\simeq R\Gamma(Y; \cdot) \circ R\mathscr{H}om(Rf_!k_U, \mathscr{F}). \\ &\simeq R\mathrm{Hom}(Rf_!k_U, \mathscr{F}) \end{aligned}$$

This remark is the main idea behind the construction of $f^!$.

References

[1] Guillermou S., Kashiwara M., Schapira P.: *Sheaf quantization of Hamiltonian isotopies and applications to non-displaceability problems*

[2] Kashiwara M., Schapira P.:*Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften 292, Springer, 1990