Derived categories

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1 Motivation

In any homology theory, the homology groups take the form $H(X, \mathcal{A})$ where X is typically a space and $\mathcal{A}$ is typically a coefficient set. The main technique to compute these groups is to understand their behavior under the change of X and of $\mathcal{A}$. In this notation, $\mathcal{A}$ is call the *abelian* variable and X is called the *non-abelian* variable. We will be concerned with the dependence on the abelian variable in this lecture.
The first step will be to study the most general type of objects that can be used as the abelian variable, leading to the notion of abelian category.
Since most constructions in homology originate at the complex level, the second step will be to study complexes modulo quasi-isomorphisms, leading to the notion of derived category.
The third step will be to extend operations on the abelian variable to derived categories so that these have the best possible homological properties, leading to the notion of derived functor.

2 Categories

We start by recalling the first important notion for this lecture.

Definition 2.1. *A* category $\mathcal{C}$ *consists of the following data:*

(i) *a class* $\mathrm{Ob}(\mathcal{C})$*, whose elements are called* objects *of* $\mathcal{C}$*,*

(ii) *for all pairs* (X, Y) *of objects in* $\mathrm{Ob}(\mathcal{C})$*, a set* $\mathrm{Hom}(X, Y)$*, whose elements are called* morphisms *from* X *to* Y *and denoted by* $f : X \to Y$*,*

(iii) *for any triple* (X, Y, Z) *of objects in* $\mathrm{Ob}(\mathcal{C})$*, a map* $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$*, called the* composition *map and denoted* $(f, g) \mapsto g \circ f$*.*

These data satisfy:

(i) the composition of morphisms is associative,

(ii) for any $X \in \mathrm{Ob}(\mathcal{C})$*, there exists the* identity *morphism* $\mathrm{id}_X \in \mathrm{Hom}(X, X)$*; it is uniquely determined by the conditions* $f \circ \mathrm{id}_X = f$ *for any* $f \in \mathrm{Hom}(X, Y)$ *and* $\mathrm{id}_X \circ g = g$ *for any* $g \in \mathrm{Hom}(Y, X)$*.*

A morphism $f \in \mathrm{Hom}(X, Y)$ is called an *isomorphism* if there exists $g \in \mathrm{Hom}(Y, X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Examples 2.2. The following categories will be useful in this working group.

1. The category $\mathfrak{Sets}$ of sets, consisting of all sets, all maps between them and their natural composition law.

2. The category $\mathfrak{Top}$ of topological spaces, with continuous maps between them.

3. The category $\mathfrak{Diff}$ of smooth manifolds, with smooth maps between them.

4. The category $\mathfrak{Ab}$ of abelian groups, with group homomorphisms between them.

5. The category $\mathbf{k}$-$\mathfrak{Mod}$ of (left) modules over a fixed ring $\mathbf{k}$, with module maps between them.

6. The category $\mathfrak{Sh}(M)$ of sheaves on a manifold M, consisting of all sheaves of abelian groups on M, all sheaf morphisms between them and their natural composition law. This category will be studied in more details during the next lecture.

Definition 2.3. *Let* $\mathcal{C}$ *be a category. The* opposite *category* $\mathcal{C}^\circ$ *is defined by* $\mathrm{Ob}(\mathcal{C}^\circ) = \mathrm{Ob}(\mathcal{C})$ *and* $\mathrm{Hom}_{\mathcal{C}^\circ}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ *for all* $X, Y \in \mathrm{Ob}(\mathcal{C}^\circ)$*, with the obvious composition maps.*

We now recall the second important notion for this lecture.

Definition 2.4. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two categories. A* functor F *from* $\mathcal{C}$ *to* $\mathcal{C}'$ *consists of the following data:*

(i) a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$*,*

(ii) for all pairs (X, Y) *of objects in* $\mathrm{Ob}(\mathcal{C})$*, a map* $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$*.*

These data satisfy the condition $F(f \circ g) = F(f) \circ F(g)$ whenever the composition $f \circ g$ is defined in $\mathcal{C}$. In particular, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

Examples 2.5. The following functors will be useful in this working group.

1. The forgetful functors from $\mathfrak{Top}$, $\mathfrak{Diff}$, $\mathfrak{Ab}$, ... to $\mathfrak{Sets}$ associating to each object the underlying set, or from $\mathfrak{Diff}$ to $\mathfrak{Top}$ associating to each smooth manifold the underlying topological space.

2. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$. The functor $\mathrm{Hom}(X, \cdot)$ from $\mathcal{C}$ to $\mathfrak{Sets}$ is defined by $Y \mapsto \mathrm{Hom}(X, Y)$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and $f \mapsto f \circ \cdot$ for all $f \in \mathrm{Hom}(Y, Z)$.

3. The functor $\mathrm{Hom}(\cdot, X)$ is defined analogously. It can also be seen as the functor $\mathrm{Hom}(X, \cdot)$ from $\mathcal{C}^\circ$ to $\mathfrak{Sets}$.

4. Let M be a $\mathbf{k}$-bimodule. The functor $\otimes_{\mathbf{k}} M$ from $\mathbf{k}$-$\mathfrak{Mod}$ to itself is defined by $N \mapsto N \otimes_{\mathbf{k}} M$.

5. With M as above, the functors $\mathrm{Hom}_{\mathbf{k}}(M, \cdot)$ and $\mathrm{Hom}_{\mathbf{k}}(\cdot, M)$ from $\mathbf{k}$-$\mathfrak{Mod}$ to itself is defined by $N \mapsto \mathrm{Hom}_{\mathbf{k}}(M, N)$ and $N \mapsto \mathrm{Hom}_{\mathbf{k}}(N, M)$ respectively.

6. Let M be a smooth manifold and $U \subset M$ an open subset. The functor $\Gamma(U, \cdot)$ from $\mathfrak{Sh}(M)$ to $\mathfrak{Ab}$ is defined by $\mathcal{F} \mapsto \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

There is also a notion of morphism between functors.

Definition 2.6. *Let F_1 and F_2 be two functors from $\mathcal{C}$ to $\mathcal{C}'$. A* morphism *or* natural transformation *θ from F_1 to F_2 consists of the following data:*

$$\text{for any } X \in \mathrm{Ob}(\mathcal{C}), \text{ a morphism } \theta(X) \in \mathrm{Hom}_{\mathcal{C}'}(F_1(X), F_2(X)).$$

These data satisfy the condition:

$$\text{for any } f \in \mathrm{Hom}(X, Y), \text{ the diagram below is commutative.}$$

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\theta(X)} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow[\theta(Y)]{} & F_2(Y) \end{array}$$

Note that the collection of all functors between two categories $\mathcal{C}$ to $\mathcal{C}'$, with morphisms between them and their natural compositions, defines a category $\mathfrak{Fun}(\mathcal{C}, \mathcal{C}')$.
In particular, the notion of isomorphism of functors is defined.

Definition 2.7. *A functor $F : \mathcal{C} \to \mathfrak{Sets}$ is* representable *if there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that F is isomorphic to the functor* $\mathrm{Hom}(X, \cdot)$.

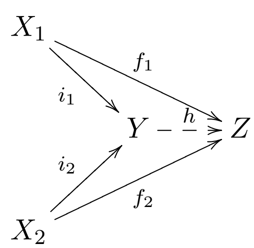
3 Additive categories

Definition 3.1. *A category $\mathcal{C}$ is* additive *if:*

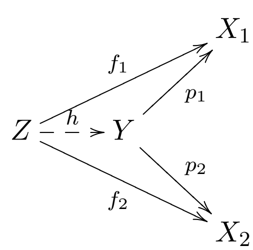
(i) for all pairs (X, Y) of objects in $\mathrm{Ob}(\mathcal{C})$, the set $\mathrm{Hom}(X, Y)$ has the structure of an additive (i.e. abelian) group, and the composition law is bilinear,

(ii) there exists an object $0 \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}(0, 0) = 0$,

(iii) for any $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, there exists $Y \in \mathrm{Ob}(\mathcal{C})$ and $i_1 \in \mathrm{Hom}(X_1, Y)$, $i_2 \in \mathrm{Hom}(X_2, Y)$ such that, for any $f_1 \in \mathrm{Hom}(X_1, Z)$ and $f_2 \in \mathrm{Hom}(X_2, Z)$, there exists $h \in \mathrm{Hom}(Y, Z)$ making the diagram below commutative,



(iv) for any $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, there exists $Y \in \mathrm{Ob}(\mathcal{C})$ and $p_1 \in \mathrm{Hom}(Y, X_1)$, $p_2 \in \mathrm{Hom}(Y, X_2)$ such that, for any $f_1 \in \mathrm{Hom}(Z, X_1)$ and $f_2 \in \mathrm{Hom}(Z, X_2)$, there exists $h \in \mathrm{Hom}(Z, Y)$ making the diagram below commutative.



Remark 3.2. The last two conditions can be better understood using some reformulations.

1. Under conditions (i) and (ii), the conditions (iii) and (iv) are equivalent, and the objects Y provided by these conditions are isomorphic.

2. Condition (iii) is equivalent to the property that the functor from $\mathcal{C}$ to $\mathfrak{Sets}$
$$Z \mapsto \mathrm{Hom}(X_1, Z) \times \mathrm{Hom}(X_2, Z)$$
is representable. The representing object Y in condition (iii) is called the *direct sum* of X_1 and X_2, and the maps i_1 and i_2 are *inclusion* maps. The pair (i_1, i_2) corresponds to id_Y via the isomorphism
$$\mathrm{Hom}(X_1, Y) \times \mathrm{Hom}(X_2, Y) \simeq \mathrm{Hom}(Y, Y).$$

3. Condition (iv) is equivalent to the property that the functor from $\mathcal{C}^\circ$ to $\mathfrak{Sets}$
$$Z \mapsto \mathrm{Hom}(Z, X_1) \times \mathrm{Hom}(Z, X_2)$$
is representable. The representing object Y in condition (iv) is called the *direct product* of X_1 and X_2, and the maps p_1 and p_2 are *projection* maps. The pair (p_1, p_2) corresponds to id_Y via the isomorphism
$$\mathrm{Hom}(Y, X_1) \times \mathrm{Hom}(Y, X_2) \simeq \mathrm{Hom}(Y, Y).$$

4. Conditions (iii) and (iv) can therefore be summarized by saying that direct sums and products exist and coincide.

Examples 3.3. Let us revisit our favorite categories.

1. The category $\mathfrak{Sets}$, as well as the categories $\mathfrak{Top}$ and $\mathfrak{Diff}$, are not additive categories. This is not surprising, since the latter typically play the role of the non-abelian variable.

2. The category $\mathfrak{Ab}$ is an additive category. Note that it is essential that the groups are abelian.

3. The category $\mathbf{k}$-$\mathfrak{Mod}$ is additive, even if the ring $\mathbf{k}$ is not commutatative and the category consists of left modules only.

4. The category $\mathfrak{Sh}(M)$ of sheaves of abelian groups on M is an additive category. This will be explained in the next lecture.

Definition 3.4. *A functor F from $\mathcal{C}$ to $\mathcal{C}'$, two additive categories, is said to be* additive *if the maps $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$ are group morphisms.*

In this lecture, the main feature of additive categories is that one can define the notion of (cochain) complex in such a category.

Definition 3.5. *A (cochain)* complex *X in an additive category $\mathcal{C}$ consists of the data $\{X^n, d^n\}_{n\in\mathbb{Z}}$ such that*

$$X^n \in \mathrm{Ob}(\mathcal{C}), \qquad d^n \in \mathrm{Hom}(X^n, X^{n+1}) \quad \textit{and} \quad d^{n+1} \circ d^n = 0,$$

for all $n \in \mathbb{Z}$.
A morphism *between complexes X and Y in $\mathcal{C}$ consists of the data $\{f^n\}_{n\in\mathbb{Z}}$ such that*

$$f^n \in \mathrm{Hom}(X^n, Y^n), \quad \textit{and} \quad d^n_Y \circ f^n = f^n \circ d^n_X,$$

for all $n \in \mathbb{Z}$.

The collection of all complexes in an additive category $\mathcal{C}$, with their morphisms and their natural composition, defines a category $\mathbf{C}(\mathcal{C})$. This is also an additive category.

We say that a complex X is bounded (resp. bounded above, resp. bounded below) if $X^n = 0$ for $|n|$ (resp. n, resp. $-n$) large enough. We denote by $\mathbf{C}^b(\mathcal{C})$ (resp. $\mathbf{C}^+(\mathcal{C})$, resp. $\mathbf{C}^-(\mathcal{C})$) the full subcategories of $\mathbf{C}(\mathcal{C})$ consisting of bounded (resp. bounded above, resp. bounded below) complexes.

The translation functor T^n is defined on all these categories (to themselves). The complex $T^n(X) = X[n]$ is defined by $T^n(X)^i = X^{n+i}$ and $d^i_{X[n]} = (-1)^n d^i_X$.

Definition 3.6. *Two morphisms $f, g : X \to Y$ in $\mathbf{C}(\mathcal{C})$ are* homotopic *if there exist morphisms $s^n : X^n \to Y^{n-1}$ in $\mathcal{C}$ such that*

$$f^n - g^n = s^{n+1} \circ d^n_X + d^{n+1}_Y \circ s^n,$$

for all $n \in \mathbb{Z}$.

Let $\mathbf{K}(\mathcal{C})$ be the category consisting of all complexes in $\mathcal{C}$, with their morphisms modulo homotopy and their induced composition law. There are similar definitions for $\mathbf{K}^b(\mathcal{C})$, $\mathbf{K}^+(\mathcal{C})$ and $\mathbf{K}^-(\mathcal{C})$. These are also additive categories.

Note, however, that the homology of a complex in an additive category cannot always be defined. This motivates the introduction of an even more particular class of categories.

4 Abelian categories

The notion of kernel is essential for the definition of homology. It can be generalized to morphisms of a category via the representation of a functor.

Definition 4.1. *Let* $f \in \mathrm{Hom}(X, Y)$*. Consider the functor* $\ker f$ *from* $\mathcal{C}^\circ$ *to* $\mathfrak{Sets}$ *defined by* $(\ker f)(Z) = \{g \in \mathrm{Hom}(Z, X) : f \circ g = 0\}$*. The kernel of* f*, denoted by* $\mathrm{Ker} f$*, is the object of* $\mathcal{C}$ *representing this functor* $\ker f$*, if it exists.*

If it exists, the kernel of f comes naturally with a morphism $k \in \mathrm{Hom}(\mathrm{Ker} f, X)$ satisfying the following universal property: if $g \in (\ker f)(Z)$, there is a unique morphism in $\mathrm{Hom}(Z, \mathrm{Ker} f)$ making the diagram below commutative:

$$\begin{array}{ccccc} Z & \xrightarrow{g} & X & \xrightarrow{f} & Y \\ & \searrow & \uparrow k & & \\ & & \mathrm{Ker} f & & \end{array}$$

Similarly, after dualizing twice, we obtain the definition of cokernel.

Definition 4.2. *Let* $f \in \mathrm{Hom}(X, Y)$*. Consider the functor* $\mathrm{coker} f$ *from* $\mathcal{C}$ *to* $\mathfrak{Sets}$ *defined by* $(\mathrm{coker} f)(Z) = \{g \in \mathrm{Hom}(Y, Z) : g \circ f = 0\}$*. The cokernel of* f*, denoted by* $\mathrm{Coker} f$*, is the object of* $\mathcal{C}$ *representing this functor* $\mathrm{coker} f$*, if it exists.*

If it exists, the cokernel of f comes naturally with a morphism $c \in \mathrm{Hom}(Y, \mathrm{Coker} f)$ satisfying the following universal property: if $g \in (\mathrm{coker} f)(Z)$, there is a unique morphism in $\mathrm{Hom}(\mathrm{Coker} f, Z)$ making the diagram below commutative:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & & c \downarrow & \nearrow & \\ & & \mathrm{Coker} f & & \end{array}$$

Remark 4.3. Note that the functor $Z \mapsto \{g \in \mathrm{Hom}(Z, Y)\}/f \circ \mathrm{Hom}(Z, X)$ does not lead to the correct definition of cokernel. In $\mathfrak{Ab}$, if $f : \mathbb{Z} \to \mathbb{Z}$ is multiplication by n and $Z = \mathbb{Z}_n$, then $Z \mapsto 0$ while $\mathrm{Hom}(Z, \mathrm{Coker} f = \mathbb{Z}_n) \neq 0$.

Definition 4.4. *An additive category* $\mathcal{C}$ *is* abelian *if any morphism* $f \in \mathrm{Hom}(X, Y)$ *admits a canonical decomposition*

$$\mathrm{Ker} f \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} \mathrm{Coker} f$$

where $j \circ i = f$ *and* $I = \text{Coker } k = \text{Ker } c$.

Remark 4.5. This is equivalent to the following two conditions:

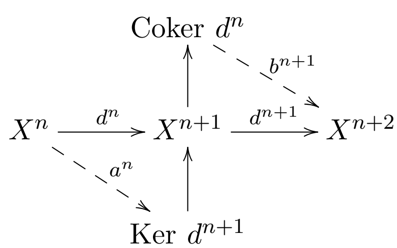
(i) for any $f \in \text{Hom}(X, Y)$, Ker$f$ and Coker$f$ exist,

(ii) the canonical morphism Coker $k \to$ Ker c is an isomorphism.

Examples 4.6. The additive categories $\mathfrak{Ab}$, $\mathbf{k}$-$\mathfrak{Mod}$ and $\mathfrak{Sh}(M)$ from the previous section are all abelian. These (as well as local systems of coefficients) are the ones playing the role of the abelian variable in classical homology theories.

We are now in position to define the homology of a complex in an abelian category.

Definition 4.7. *Let* $\mathcal{C}$ *be an abelian category and* $X \in \text{Ob}(\mathbf{C}(\mathcal{C}))$. *The homology of* X *is the object* $H(X) \in \text{Ob}(\mathbf{C}(\mathcal{C}))$ *the complex with trivial differentials defined by* $H^n(X) = \text{Coker}(a^n) = \text{Ker}(b^{n+1})$ *and the commutative diagram*



Note in addition that this abstract definition of homology induces a functor H from $\mathbf{C}(\mathcal{C})$ to itself.

Definition 4.8. *A morphism* $f \in \text{Hom}_{\mathbf{C}(\mathcal{C})}(X, Y)$ *is a* quasi-isomorphism *if* $H(f)$ *is an isomorphism.*

If $f, g \in \text{Hom}_{\mathbf{C}(\mathcal{C})}(X, Y)$ are homotopic, then $H(f) = H(g)$. In particular, the functor H naturally induces a functor, still denoted by H, from $\mathbf{K}(\mathcal{C})$ to itself.

It is then routine (but a useful exercise) to check that the definition of exact sequence naturally extends to this context and that a short exact sequence of complexes induces a long exact sequence in homology.

5 Derived categories

To go one step further in the systematic study of complexes (but having their homology in mind as the real object of interest), we would like to consider quasi-isomorphisms as isomorphisms of a new category. In other words, we would like to define a new category by formally inverting some class of morphisms, via some kind of localization procedure.

The following example shows that there exist quasi-isomorphisms that are not homotopic to an isomorphism (and hence the above localization procedure is indeed necessary).

Examples 5.1. Consider the two complexes in $\mathcal{C}(\mathbb{Z}\text{-}\mathfrak{Mod})$ and the morphism between them given in the commutative diagram

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{mod } 2} & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

The complex morphism is a quasi-isomorphism, but is not homotopic to an isomorphism, since there is no nontrivial morphism from the bottom complex to the top complex.

Definition 5.2. *The* derived category $\mathbf{D}(\mathcal{C})$ *of an abelian category* $\mathcal{C}$ *is unique category such that there exists a functor* Q *from* $\mathbf{K}(\mathcal{C})$ *to* $\mathbf{D}(\mathcal{C})$ *satisfying:*

(i) $Q(f)$ *is an isomorphism for any quasi-isomorphism* f*,*

(ii) any functor F *from* $\mathbf{K}(\mathcal{C})$ *to some category* $\mathcal{D}$ *transforming quasi-isomorphisms into isomorphisms uniquely factorizes through* Q *(i.e. there exists a functor* G *from* $\mathbf{D}(\mathcal{C})$ *to* $\mathcal{D}$ *such that* $F = G \circ Q$*).*

There are analogous definitions for $\mathbf{D}^b(\mathcal{C})$, $\mathbf{D}^+(\mathcal{C})$ and $\mathbf{D}^-(\mathcal{C})$ starting from $\mathbf{K}^b(\mathcal{C})$, $\mathbf{K}^+(\mathcal{C})$ and $\mathbf{K}^-(\mathcal{C})$ respectively.

Here is an explicit construction of the derived category $\mathbf{D}(\mathcal{C})$. We denote by S the class of quasi-isomorphisms in $\mathbf{K}(\mathcal{C})$.

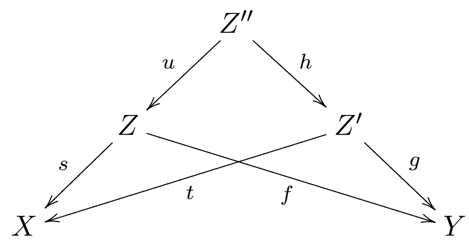
Definition 5.3. *The* derived category $\mathbf{D}(\mathcal{C})$ *of an abelian category* $\mathcal{C}$ *is defined by*

$$\mathrm{Ob}(\mathbf{D}(\mathcal{C})) = \mathrm{Ob}(\mathbf{K}(\mathcal{C}))$$

and

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{C})}(X, Y) = \{(s, f) \mid s \in \mathrm{Hom}_{\mathcal{C}}(Z, X) \cap S, f \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)\}/\sim$$

where $(s, f) \sim (t, g)$ *iff there exists a commutative diagram*



with $u, h \in S$.

The functor Q from the first definition is then defined by $Q(X) = X$ for all $X \in \mathrm{Ob}(\mathbf{K}(\mathcal{C}))$ and $Q(f) = [\mathrm{id}_X, f]$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

In order to check that this definition gives indeed rise to a category, it is sufficient for S to satisfy the following definition.

Definition 5.4. *A family* S *of morphisms in a category* $\mathcal{C}'$ *is a* localizing system *if:*

(i) for any $X \in \mathrm{Ob}(\mathcal{C}')$, $\mathrm{id}_X \in S$,

(ii) for any $f, g \in S$, $f \circ g \in S$ *whenever it is defined,*

(iii) for any $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Z, Y) \cap S$, *there exists a commutative diagram*

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \Big\downarrow h & & \Big\downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

with $h \in S$, *and similarly with all arrows reversed,*

(iv) for any $f, g \in \mathrm{Hom}(X, Y)$, *there exists* $t \in S$ *such that* $t \circ f = t \circ g$ *iff there exists* $s \in S$ *such that* $f \circ s = g \circ s$.

Proposition 5.5. *The class* S *of quasi-isomorphisms in* $\mathbf{K}(\mathcal{C})$ *is a localizing system.*

The proof uses the notion of mapping cone. Given $f \in \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(X, Y)$, the mapping cone $M(f) \in \mathrm{Ob}(\mathbf{C}(\mathcal{C}))$ is defined by $M(f) = X[1] \oplus Y$ and

$$d_{M(f)} = \begin{pmatrix} d_{X[1]} & 0 \\ f & d_Y \end{pmatrix}.$$

Then we have a sequence of morphisms $X \to Y \to M(f) \to X[1]$ in $\mathbf{C}(\mathcal{C})$ transformed by the functor H into the long exact sequence in homology of this mapping cone.

A remarkable property of mapping cone sequences is that they can be considered, modulo an isomorphism in $\mathbf{K}(\mathcal{C})$, as a mapping cone sequence with respect to any of the three morphisms in them.

Proof. Properties (i) and (ii) are obvious.

The commutative diagram in (iii) is the first square in the diagram

$$\begin{array}{ccccccc}
M(j)[-1] & \xrightarrow{h} & X & \xrightarrow{j} & M(g) & \longrightarrow & M(j) \\
\downarrow & & \downarrow{\scriptstyle f} & & \| & & \downarrow \\
Z & \xrightarrow{g} & Y & \longrightarrow & M(g) & \longrightarrow & Z[1]
\end{array}$$

Since $g \in S$, $M(g)$ is acyclic and hence $h \in S$ as well. Note that this diagram commutes only up to homotopy, i.e. commutes in $\mathbf{K}(\mathcal{C})$ but not in $\mathbf{C}(\mathcal{C})$.

To prove (iv), we can assume $g = 0$ since $\mathcal{C}$ is additive. If $t \in \operatorname{Hom}(Y, Z) \cap S$ satisfies $t \circ f = 0$ in $\mathbf{K}(\mathcal{C})$, we have a homotopy $h : X \to Z[-1]$ from $t \circ f$ to 0. Then we define $k = f \oplus h : X \to M(t)[-1]$ and s is obtained from the commutative diagram

$$\begin{array}{ccccc}
M(t)[-1] & \longrightarrow & Y & \xrightarrow{t} & Z \\
\| & & \uparrow{\scriptstyle f} & & \\
M(t)[-1] & \xleftarrow[k]{} & X & \xleftarrow[s]{} & M(k)[-1]
\end{array}$$

Then $f \circ s = 0$ because $k \circ s = 0$ and the diagram commutes. Since $t \in S$, $M(t)$ is acyclic and hence $s \in S$ as well. $\square$

These properties of S also suffice to verify that $\mathbf{D}(\mathcal{C})$ is an additive category.

Examples 5.6. If $\mathcal{C} = \mathfrak{Sh}(M)$ the categories of sheaves over M, then $\mathbf{D}(\mathcal{C})$ is called the derived category of M and is denoted by $\mathbf{D}(M)$. It will be the most important (derived) category in the next lectures.

6 Derived functors

Let F be an additive functor from the abelian category $\mathcal{C}$ to the abelian category $\mathcal{D}$. Acting on complexes in $\mathcal{C}$ componentwise, F induces an additive functor $\mathbf{C}(F)$ from $\mathbf{C}(\mathcal{C})$ to $\mathbf{C}(\mathcal{D})$. This functor preserves the notion of homotopy, so that it induces in turn an additive functor $\mathbf{K}(F)$ from $\mathbf{K}(\mathcal{C})$ to $\mathbf{K}(\mathcal{D})$. But in general this functor does not transform a quasi-isomorphism into a quasi-isomorphism, so that it does not induce a functor on the derived categories.

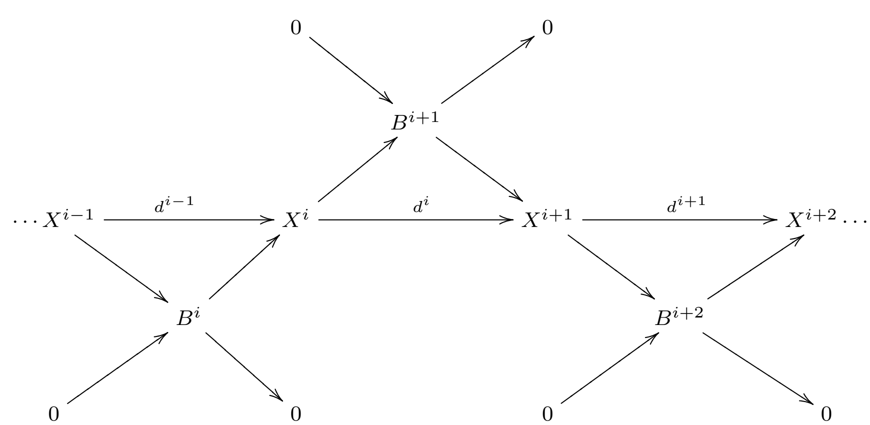
Definition 6.1. *An additive functor F is* exact *if it transforms a short exact sequence into a short exact sequence.*
It is left exact *if it transforms a short exact sequence into a sequence that is exact on the left side and in the middle.*
It is right exact *if it transforms a short exact sequence into a sequence that is exact in the middle and on the right side.*

Examples 6.2. The functors $\mathrm{Hom}_{\mathbf{k}}(\cdot, M)$, $\mathrm{Hom}_{\mathbf{k}}(M, \cdot)$ and $\Gamma(U, \cdot)$ are left exact. The functor $\otimes_{\mathbf{k}} M$ is right exact.

Proposition 6.3. *If the additive functor F from $\mathcal{C}$ to $\mathcal{D}$ is exact, then $\mathbf{K}(F)$ transforms a quasi-isomorphism into a quasi-isomorphism, and hence induces a functor $\mathbf{D}(F)$ from $\mathbf{D}(\mathcal{C})$ to $\mathbf{D}(\mathcal{D})$.*

Proof. Since a morphism of complexes is a quasi-isomorphism iff its mapping cone is acyclic, it is sufficient to show that F transforms an acyclic complex X into an acyclic complex $F(X)$.

Writing $B^i = \mathrm{Ker}\ d^i = \mathrm{Im}\ d^{i-1}$, we have the commutative diagram

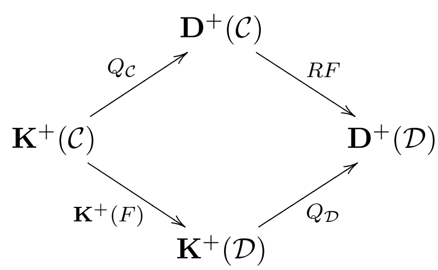


The exact functor F transforms this diagram into a diagram with the same properties, so that $F(X)$ is acyclic as well. □

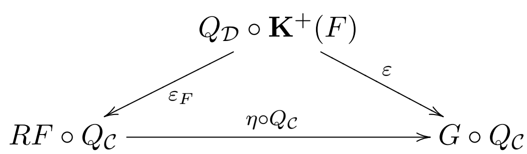
Note that the functor $\mathbf{D}(F)$ maps the sequence of morphisms $X \to Y \to M(f) \to X[1]$ into a sequence morphisms that is isomorphic to a mapping cone sequence. A functor between derived categories with this property is called exact (derived categories are additive but not abelian, so the usual definition cannot be used).

Since this componentwise construction does not work in general for non-exact functors, we need another definition for these.

Definition 6.4. *The* derived functor *of an additive left exact functor F from $\mathcal{C}$ to $\mathcal{D}$ is a pair consisting of an exact functor RF from $\mathbf{D}^+(\mathcal{C})$ to $\mathbf{D}^+(\mathcal{D})$ and a morphism of functors ε_F from $Q_\mathcal{D} \circ \mathbf{K}^+(F)$ to $RF \circ Q_\mathcal{C}$*



such that for any exact functor G from $\mathbf{D}^+(\mathcal{C})$ to $\mathbf{D}^+(\mathcal{D})$ and any morphism of functors ε from $Q_\mathcal{D} \circ \mathbf{K}^+(F)$ to $G \circ Q_\mathcal{C}$ there exists a unique morphism of functors η from RF to G making the diagram



commutative.

There is a similar definition for the derived functor LF of an additive right exact functor, defined from $\mathbf{D}^-(\mathcal{C})$ to $\mathbf{D}^-(\mathcal{D})$, obtained by reversing the arrows of the morphisms of functors.

The derived functor is constructed in the following way:

1. select a suitable subclass of objects of $\mathbf{K}(\mathcal{C})$,
2. if F acts nicely on this subclass, it induces componentwise a functor on the corresponding localization,
3. if the chosen subclass is large enough, its localization will be equivalent to the derived category, and hence the derived functor will be completely determined.