CHAPTER 13

DUALITY, REAL STRUCTURES AND PERVERSE SHEAVES: AN OVERVIEW

Summary. The duality functor acts nicely on strictly holonomic $\widetilde{\mathcal{D}}_X$-modules. In this chapter, we are concerned with its action on sesquilinear pairings between $\mathcal{D}_X$-modules, in order to obtain a duality functor for strictly holonomic $\widetilde{\mathcal{D}}$-Triples. For that purpose, we visit the category of perverse sheaves via the de Rham functor and we make use of the Hermitian duality functor introduced by Kashiwara. Duality combined with Hermitian duality yields the notion of a real structure on an object of $\widetilde{\mathcal{D}}$-Triples. We compare it with the notion of real structure coming from that on a perverse sheaf via the Riemann-Hilbert correspondence.

13.1. Introduction

All functors considered in Chapter 12 on the category $\widetilde{\mathcal{D}}$-Triples are obtained by mixing functor on the category of $\widetilde{\mathcal{D}}_X$-modules and by analyzing the behaviour of their restriction at $z = 1$ with respect to sesquilinear pairings. The duality functor of Section 8.8.g is an exception. We define it on holonomic[1] $\mathcal{D}$-Triples, which consist of a pair of holonomic $\mathcal{D}_X$-modules and a sesquilinear pairing with values in $\mathfrak{Db}_X$ or $\mathfrak{C}_X$ between them, by means of a similar functor on the category of triples of perverse sheaves via the de Rham functor. Its definition has thus a topological origin, and not a purely analytic one. In order to understand this fact, let us compare with what happens with (regular) holonomic $\mathcal{D}_X$-modules.

For such a module $\mathcal{M}$, we directly define its dual $\boldsymbol{D}\mathcal{M}$ in $\mathcal{D}$-module theory (see Section 8.8.g). However, the notion of complex conjugate of $\mathcal{M}$ as a $\mathcal{D}_X$-module cannot just be defined by the naive conjugation functor, as the latter transforms a $\mathcal{D}_X$-module into a $\mathcal{D}_{\overline{X}}$-module. In order to obtain the conjugate $\mathcal{D}_X$-module $c(\mathcal{M})$ (that is characterized by the property that ${}^{\mathrm{p}}\mathrm{DR}_X(c(\mathcal{M})) = \overline{{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}}$), we first define the Hermitian dual $\mathcal{M}^*$ in the sense Kashiwara (see Section 13.3), and we realize conjugation as the composition, in any order, of duality and Hermitian duality.

[1] For the sake of simplicity, and because Hodge modules are regular holonomic, we in fact restrict to the case of regular holonomic $\mathcal{D}_X$-modules.

On the other hand, by definition, on the category of $\mathcal{D}$-Triples we have at our disposal the Hermitian duality functor, so that duality is the functor that needs a finer definition. Combining Hermitian duality with the putative duality functor, we may obtain a conjugation functor.

We provide in Section 13.2 a necessarily very short review on the theory of perverse sheaves on complex analytic manifolds and on the Riemann-Hilbert correspondence. A more precise presentation of the theory can be found in [**Dim04**] and [**MS22**], and complete references include [**BBDG82**] and [**KS90**].

13.2. Perverse sheaves and the de Rham functor

13.2.a. A short reminder on perverse sheaves. Let $\boldsymbol{k}$ be a field. A *$\boldsymbol{k}$-constructible sheaf* on X is a sheaf $\mathcal{F}$ for which there exists a locally finite stratification $(X_i)_{i\in I}$ by submanifolds whose closure is a closed analytic subset of X, such that $\mathcal{F}|_{X_i}$ is a locally constant sheaf of finite-dimensional $\boldsymbol{k}$-vector spaces. The bounded derived category $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\boldsymbol{k}_X)$ of complexes with $\boldsymbol{k}$-constructible cohomology is equipped with a natural *t-structure* $({}^{\mathrm{p}}\mathsf{D}^{\mathrm{b},\leqslant 0}_{\mathbb{C}\text{-c}}(\boldsymbol{k}_X), {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b},\geqslant 0}_{\mathbb{C}\text{-c}}(\boldsymbol{k}_X))$, whose heart is the abelian full subcategory $\mathsf{Perv}(\boldsymbol{k}_X)$ of $\boldsymbol{k}$-perverse sheaves on X. The Poincaré-Verdier duality functor $\boldsymbol{D}_{\mathrm{PV}}(\bullet) = \boldsymbol{R}\mathcal{H}om_{\boldsymbol{k}_X}(\bullet, \boldsymbol{k}_X[2n])$ on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\boldsymbol{k}_X)$ exchanges both terms of the t-structure and thus preserves $\mathsf{Perv}(\boldsymbol{k}_X)$.

To each bounded complex in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\boldsymbol{k}_X)$ is associated a *characteristic cycle*, which consists of the formal linear combination of homogeneous irreducible closed analytic Lagrangian subvarieties of the cotangent bundle T^*X with coefficients in $\mathbb{Z}$. For a perverse sheaf, the coefficients are non-negative. The characteristic cycle behaves additively with respect to distinguished triangles in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\boldsymbol{k}_X)$ and with respect to short exact sequences in $\mathsf{Perv}(\boldsymbol{k}_X)$. Furthermore, the characteristic cycle of a $\boldsymbol{k}$-perverse sheaf is zero if and only if the perverse sheaf is zero. We then deduce:

13.2.1. Lemma. *Let $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$ be an exact sequence of perverse sheaves. If $\mathcal{G}$ is isomorphic to $\mathcal{F}$, then $\mathcal{H} = 0$ and the morphism $\mathcal{G} \to \mathcal{F}$ is an isomorphism.* □

13.2.2. Pairings. Let $\mathcal{F}, \mathcal{G}$ be objects of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_X)$. We have a natural isomorphism (see [**KS90**, (2.6.8)])

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \boldsymbol{k}_X[2n]) \simeq \mathrm{Hom}(\mathcal{F}, \boldsymbol{D}_{\mathrm{PV}}\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{G}, \boldsymbol{D}_{\mathrm{PV}}\mathcal{F}).$$

Giving a pairing $\varphi : \mathcal{F} \otimes \mathcal{G} \to \boldsymbol{k}_X[2n]$ amounts thus to giving a morphism $\mathcal{F} \to \boldsymbol{D}_{\mathrm{PV}}\mathcal{G}$, or as well a morphism $\mathcal{G} \to \boldsymbol{D}_{\mathrm{PV}}\mathcal{F}$ (the latter being obtained from the former by duality, taking into account the biduality isomorphism $\boldsymbol{D}_{\mathrm{PV}}\boldsymbol{D}_{\mathrm{PV}} \simeq \mathrm{Id}$). We say that the pairing φ is *perfect* or *nondegenerate* if the corresponding morphism $\mathcal{F} \to \boldsymbol{D}_{\mathrm{PV}}\mathcal{G}$ (or as well $\mathcal{G} \to \boldsymbol{D}_{\mathrm{PV}}\mathcal{F}$) is an isomorphism.

13.2.3. Conjugation, Hermitian duality and real structure. Let $\overline{\mathbb{C}}$ denote the $\mathbb{C}$-vector space $\mathbb{R} + \mathrm{i}\,\mathbb{R}$ with action defined by $c \cdot (a + \mathrm{i}\, b) := \overline{c}(a + \mathrm{i}\, b)$. Given any $\mathbb{C}$-vector space V, we denote by $\overline{V}$ the conjugate vector space $\overline{\mathbb{C}} \otimes_{\mathbb{C}} V$. Given a sheaf $\mathcal{F}$ or a complex of $\mathbb{C}$-vector spaces on X, we similarly set $\overline{\mathcal{F}} := \overline{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{F}$. Then $\overline{\mathcal{F}} = \mathcal{F}$ as sheaves of $\mathbb{R}$-vector spaces. The *Hermitian dual* $\mathcal{F}^*$ of $\mathcal{F}$ is the dual conjugate of $\mathcal{F}$:

$$\mathcal{F}^* := \boldsymbol{D}_{\mathrm{PV}}(\overline{\mathcal{F}}).$$

A *real structure* on $\mathcal{F}$ is a $\mathbb{C}$-*linear isomorphism* $\kappa : \mathcal{F} \xrightarrow{\sim} \overline{\mathcal{F}}$ such that $\overline{\kappa} \circ \kappa = \mathrm{Id}$. If $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vector spaces or an object of $\mathsf{Perv}(\mathbb{C}_X)$, giving a real structure is equivalent to giving a subsheaf $\mathcal{F}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces, or a subobject in $\mathsf{Perv}(\mathbb{R}_X)$ such that the natural morphism $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{F}_{\mathbb{R}} \to \mathcal{F}$ is an isomorphism: indeed, one defines $\mathcal{F}_{\mathbb{R}}$ in the corresponding abelian category as $\mathrm{Ker}(\kappa - \mathrm{Id})$, where κ and Id are considered as $\mathbb{R}$-linear isomorphisms.

13.2.4. Triples of perverse sheaves. In order to mimic the construction for $\mathcal{D}_X$-modules and to make clear the Riemann-Hilbert correspondence in this context, we consider the abelian category $\mathsf{PervTriples}(\mathbb{C}_X)$ whose objects consist of triples ${}^{\mathrm{p}}\mathcal{T} = (\mathcal{F}', \mathcal{F}'', s)$, where s is a pairing $\mathcal{F}' \otimes_{\mathbb{C}} \overline{\mathcal{F}''} \to \mathbb{C}_X[2n]$, that we call a sesquilinear pairing between $\mathcal{F}'$ and $\mathcal{F}''$. Giving s is equivalent to giving one of the morphisms $s' : \mathcal{F}' \to \boldsymbol{D}_{\mathrm{PV}}(\overline{\mathcal{F}''})$ or $s'' : \mathcal{F}'' \to \boldsymbol{D}_{\mathrm{PV}}(\overline{\mathcal{F}'})$, related by $s'' = \boldsymbol{D}_{\mathrm{PV}}(\overline{s'})$ modulo the biduality isomorphism. Morphisms are defined as in Section 5.2.b. A triple is said to be *nondegenerate* if s', equivalently s'', is an isomorphism, and the full subcategory $\mathsf{PervTriples}^{\mathrm{nd}}(\mathbb{C}_X)$ of nondegenerate perverse triples is equivalent, via the functor

$$\mathcal{F} \longmapsto (\mathcal{F}, \boldsymbol{D}\overline{\mathcal{F}}, s),$$

where $s : \mathcal{F} \otimes \overline{\boldsymbol{D}\overline{\mathcal{F}}} \to \mathbb{C}_X[2n]$ is the tautological pairing $\mathcal{F} \otimes \boldsymbol{D}\mathcal{F} \to \mathbb{C}_X[2n]$.

The Hermitian dual $({}^{\mathrm{p}}\mathcal{T})^*$ of a triple ${}^{\mathrm{p}}\mathcal{T} = (\mathcal{F}', \mathcal{F}'', s)$ is the triple $(\mathcal{F}'', \mathcal{F}', s^*)$, where s^* satisfies $(s^*)' = s''$, equivalently $(s^*)'' = s'$, and the dual $\boldsymbol{D}_{\mathrm{PV}}({}^{\mathrm{p}}\mathcal{T})$ of a *nondegenerate* triple is defined by

$$\boldsymbol{D}_{\mathrm{PV}}({}^{\mathrm{p}}\mathcal{T}) = (\boldsymbol{D}_{\mathrm{PV}}\mathcal{F}', \boldsymbol{D}_{\mathrm{PV}}\mathcal{F}'', \boldsymbol{D}_{\mathrm{PV}}s),$$

with $(\boldsymbol{D}_{\mathrm{PV}}s)' = \boldsymbol{D}_{\mathrm{PV}}(\overline{s''})^{-1}$, equivalently $(\boldsymbol{D}_{\mathrm{PV}}s)'' = \boldsymbol{D}_{\mathrm{PV}}(\overline{s'})^{-1}$. Duality and Hermitian duality commute.

The *conjugation functor*[(2)] $c : {}^{\mathrm{p}}\mathcal{T} \to c({}^{\mathrm{p}}\mathcal{T})$ acting on nondegenerate perverse triples is the composition, in any order, of duality and Hermitian duality. The formula is

$$c(\mathcal{F}', \mathcal{F}'', s) = (\boldsymbol{D}\mathcal{F}'', \boldsymbol{D}\mathcal{F}', c(s)), \quad c(s)' = \boldsymbol{D}_{\mathrm{PV}}(\overline{s'})^{-1},\ c(s)'' = \boldsymbol{D}_{\mathrm{PV}}(\overline{s''})^{-1}.$$

A real structure on a nondegenerate triple ${}^{\mathrm{p}}\mathcal{T}$ is an isomorphism $\kappa : {}^{\mathrm{p}}\mathcal{T} \to c({}^{\mathrm{p}}\mathcal{T})$ such that $c(\kappa) \circ \kappa = \mathrm{Id}$.

13.2.5. Remark. It is easy to check that the equivalence $\mathsf{Perv}(\mathbb{C}_X) \simeq \mathsf{PervTriples}^{\mathrm{nd}}(\mathbb{C}_X)$ transforms the various functors and notions (duality, Hermitian duality, conjugation, real structure) on the source category to the corresponding ones on the target category.

(2) We avoid the notation $\overline{{}^{\mathrm{p}}\mathcal{T}}$ in order not to confuse with the naive conjugation.

13.2.b. The de Rham and the solution functor of a holonomic $\mathcal{D}_X$-module

Let $\mathcal{M}$ be a holonomic left $\mathcal{D}_X$-module. As recalled at the beginning of Section 8.8.h, the de Rham functor ${}^{\mathrm{p}}\mathrm{DR}$ attaches to a holonomic $\mathcal{D}_X$-module a bounded $\mathbb{C}$-constructible complex on X. That this complex satisfies the perversity condition follows from the local duality theorem that we review now.

The contravariant functor $\mathrm{Sol}(\mathcal{M}) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ also attaches to a holonomic $\mathcal{D}_X$-module a $\mathbb{C}$-constructible complex on X. In order to deal with perversity, we rather consider the shifted complex

$$ {}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[n]. $$

In a symmetric way, we recall that (see Remark 8.4.4(2))

$$ {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[n]. $$

The canonical morphism

$$ \begin{aligned} \eta_X : \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[n] \otimes_{\mathbb{C}_X} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[n] \\ \longrightarrow \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)[2n] \simeq \mathbb{C}_X[2n] \end{aligned} $$

can be regarded as a morphism

$$ \eta_X : {}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}) \longrightarrow \boldsymbol{D}_{\mathrm{PV}}\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}). $$

The *local duality theorem* already referred to in Section 8.8.h asserts that η_X is an isomorphism in the derived category $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ of bounded complexes with $\mathbb{C}$-constructible cohomology:

13.2.6. Theorem. *The morphisms*

$$ \begin{aligned} \eta_X : {}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}) &\longrightarrow \boldsymbol{D}_{\mathrm{PV}}\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) = \boldsymbol{R}\mathcal{H}om_{\mathbb{C}_X}({}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}), \mathbb{C}_X[2n]), \\ \boldsymbol{D}_{\mathrm{PV}}(\eta_X) : {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) &\longrightarrow \boldsymbol{D}_{\mathrm{PV}}\, {}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}) = \boldsymbol{R}\mathcal{H}om_{\mathbb{C}_X}({}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}), \mathbb{C}_X[2n]) \end{aligned} $$

are isomorphisms. □

This topological duality theorem is completed with an analytic one. There exists a canonical bi-functorial isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$, for $\mathcal{M}, \mathcal{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ (see [**Kas03**, (3.14)]):

$$ \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{M}) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\boldsymbol{D}\mathcal{M}, \boldsymbol{D}\mathcal{N}). $$

Since $\boldsymbol{D}\mathcal{O}_X \simeq \mathcal{O}_X$, we deduce a functorial isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ when $\mathcal{M}$ is holonomic:

$$ {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{Sol}(\boldsymbol{D}\mathcal{M}). \tag{13.2.7} $$

Together with Theorem 13.2.6 and biduality $\boldsymbol{D}\boldsymbol{D}\mathcal{M} \simeq \mathcal{M}$ for $\mathcal{M}$ holonomic, we obtain a functorial isomorphism for such an $\mathcal{M}$:

$$ {}^{\mathrm{p}}\mathrm{DR}\, \boldsymbol{D}\mathcal{M} \simeq {}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}) \simeq \boldsymbol{D}_{\mathrm{PV}}\, {}^{\mathrm{p}}\mathrm{DR}\, \mathcal{M}. \tag{13.2.8} $$

A well-known consequence of (13.2.8) is the perversity of ${}^{\mathrm{p}}\mathrm{DR}(\mathcal{M})$ and ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M})$ when $\mathcal{M}$ is holonomic. Indeed, by construction, ${}^{\mathrm{p}}\mathrm{DR}(\mathcal{M})$ is an object of ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b},\leqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, and (13.2.8) implies that so does $\boldsymbol{D}_{\mathrm{PV}}\, {}^{\mathrm{p}}\mathrm{DR}\, \mathcal{M}$, since $\boldsymbol{D}\mathcal{M}$ also belongs to $\mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$.

At this point we can also recall:

13.2.9. Lemma. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. We have the following equivalences:*

$$ {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) = 0 \iff {}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}) = 0 \iff \mathcal{M} = 0. \qquad \square $$

13.2.c. The Riemann-Hilbert correspondence. We have recalled the notion of regularity of a holonomic $\mathcal{D}_X$-module in Section 10.7.b. We denote by $\mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X)$ the abelian full subcategory of $\mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ whose objects are regular holonomic. The fundamental theorem for these objects is:

***13.2.10. Theorem* (Riemann-Hilbert correspondence).** *The functor DR_X induces an equivalence of categories*

$$ \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X) \xrightarrow{\sim} \mathsf{Perv}(\mathbb{C}_X). \qquad \square $$

As an application, we obtain a characterization of semi-simple regular holonomic $\mathcal{D}_X$-modules, by applying the criterion of semi-simplicity for $\mathbb{C}$-perverse sheaves of [**BBDG82**, Th. 4.3.1].

Let Z be a closed irreducible analytic subset of X and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module supported on Z. There exists a Zariski dense open subset Z^o of Z such that the restriction of $\mathcal{M}$ to Z^o is a vector bundle with flat connection $\mathcal{M}^o$.

13.2.11. Corollary. *Assume that $\mathcal{M}$ is regular holonomic and has no nonzero submodule or quotient module supported on a proper closed analytic subset of Z. Then $\mathcal{M}$ is semi-simple (as a holonomic $\mathcal{D}_X$-module) if and only if $\mathcal{M}^o$ is semi-simple (as a vector bundle with flat connection).* $\square$

13.3. The Hermitian duality functor for holonomic $\mathcal{D}_X$-modules

Let $\mathcal{M}$ be a holonomic (left, say) $\mathcal{D}_X$-module. We denote by $\overline{\mathcal{M}}$ its naive conjugate, which is a $\mathcal{D}_{\overline{X}}$-module (see Section 12.3). The sheaves $\mathcal{E}xt^i_{\mathcal{D}_{\overline{X}}}(\overline{\mathcal{M}}, \mathfrak{Db}_X)$ are naturally equipped with a $\mathcal{D}_X$-module structure by using that on $\mathfrak{Db}_X$, since the latter commutes with the $\mathcal{D}_{\overline{X}}$-module structure used for $\mathcal{E}xt^i$.

13.3.1. Theorem. *The $\mathcal{D}_{\overline{X}}$-modules $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X)$ vanish for $i \neq 0$ and the $\mathcal{D}_{\overline{X}}$-module $\mathcal{C}_X(\mathcal{M}) := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X)$ is holonomic.* $\square$

The proof of [**Kas86a**] only applies to *regular* holonomic $\mathcal{D}_X$-modules, which are in fact the only ones of interest for us (see Theorem 14.7.1), and the conclusion also provides a *regular* holonomic $\mathcal{D}_X$-module. However, the theorem holds true for any holonomic $\mathcal{D}_X$-module.

13.3.2. Definition. The *Hermitian dual* $\mathcal{M}^*$ is the holonomic $\mathcal{D}_X$-module $\overline{\mathcal{C}_X(\mathcal{M})} = \mathcal{C}_{\overline{X}}(\overline{\mathcal{M}})$.

13.3.3. Corollary. *For $\mathcal{M}$ holonomic, the natural $\mathcal{D}_X$-linear morphism*

$$\mathcal{M} \longrightarrow \mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X), \mathfrak{Db}_X) = \mathcal{C}_{\overline{X}} \circ \mathcal{C}_X(\mathcal{M}) = (\mathcal{M}^*)^* \tag{13.3.3 $*$}$$

is an isomorphism, and Hermitian duality induces an equivalence of categories $\mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}}$.

13.3.4. Lemma. *For $\mathcal{M}$ holonomic, there exists a functorial isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$:*

$${}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}(\mathcal{C}_X(\mathcal{M})) \simeq {}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{M});$$

We deduce from (13.2.8) a functorial isomorphism in $\mathsf{Perv}(\mathbb{C}_X)$ for $\mathcal{M}$ holonomic:

$${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}^*) \simeq \overline{{}^{\mathrm{p}}\mathrm{DR}_X(\boldsymbol{D}\mathcal{M})} \simeq \overline{\boldsymbol{D}_{\mathrm{PV}}\, {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M})} = {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M})^*. \tag{13.3.5}$$

***Proof of Lemma* 13.3.4.** Let $\mathcal{I}^\bullet$ be a $\mathcal{D}_{X,\overline{X}}$-resolution of $\mathfrak{Db}_X$ by $\mathcal{D}_X$-injective modules. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ we can write

$$\begin{aligned}\mathrm{DR}_{\overline{X}}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X)) &\simeq \mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathrm{Sp}(\mathcal{D}_{\overline{X}}), \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{I}^\bullet))\\ &\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathrm{Sp}(\mathcal{D}_{\overline{X}}), \mathcal{I}^\bullet))\end{aligned}$$

and each term of the double complex $\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathrm{Sp}(\mathcal{D}_{\overline{X}}), \mathcal{I}^\bullet)$ is still $\mathcal{D}_X$-injective, so that the last term reads

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{DR}_{\overline{X}}(\mathfrak{Db}_X)) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

according to the Dolbeault-Grothendieck lemma, that we write as

$$\mathcal{O}_X \xrightarrow{\sim} \mathrm{DR}_{\overline{X}}(\mathfrak{Db}_X),$$

and Remark 8.4.4(2). For $\mathcal{M}$ holonomic, we thus find the desired isomorphism. ☐

***Proof of Corollary* 13.3.3.** We first show that this morphism is injective. From Theorem 13.3.1 we deduce that $\mathcal{C}_X$ is an anti-exact functor $\mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_{\overline{X}})^{\mathrm{op}}$. Let m be a local section of $\mathcal{M}$ and let us assume that, for any $\varphi \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X)$, we have $\varphi(m)=0$. This implies that the injective morphism

$$\mathcal{C}_X(\mathcal{M}/\mathcal{D}_X m) \longrightarrow \mathcal{C}_X(\mathcal{M})$$

is also surjective. Therefore, $\mathcal{C}_X(\mathcal{D}_X m) = 0$. Let us denote by $\mathrm{Sol}_X(\mathcal{M})$ the complex $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Lemma 13.3.4 implies

$${}^{\mathrm{p}}\mathrm{Sol}_X(\mathcal{D}_X m) = {}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}\big(\mathcal{C}_X(\mathcal{D}_X m)\big) = 0.$$

Therefore, $\mathcal{D}_X m = 0$, according to Lemma 13.2.9.

Let us now show surjectivity. Let $\mathcal{Q}$ denote the quotient module $\mathcal{C}_{\overline{X}} \circ \mathcal{C}_X(\mathcal{M})/\mathcal{M}$. Then $\mathcal{Q}$ is also a holonomic $\mathcal{D}_X$-module. We thus obtain a short exact sequence of perverse sheaves

$$0 \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{C}_{\overline{X}} \circ \mathcal{C}_X(\mathcal{M})) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{Q}) \longrightarrow 0$$

and it is enough to show that ${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{Q}) = 0$, according to Lemma 13.2.9. But there exists an isomorphism ${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}) \simeq {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{C}_{\overline{X}} \circ \mathcal{C}_X(\mathcal{M}))$: indeed, one shows that

$$\begin{aligned}
{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{C}_{\overline{X}} \circ \mathcal{C}_X(\mathcal{M})) &\simeq \mathrm{Sol}_{\overline{X}}\big(\mathcal{C}_X(\mathcal{M})\big) \quad \text{by Lemma 13.3.4}\\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathbb{C}_X}\big({}^{\mathrm{p}}\mathrm{DR}_{\overline{X}} \circ \mathcal{C}_X(\mathcal{M}), \mathbb{C}_X\big) \quad \text{by Theorem 13.2.6}\\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathbb{C}_X}\big(\mathrm{Sol}_X(\mathcal{M}), \mathbb{C}_X\big) \quad \text{by Lemma 13.3.4}\\
&\simeq {}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}) \quad \text{by Theorem 13.2.6.}
\end{aligned}$$

The desired vanishing is thus a direct consequence of Lemma 13.2.1, which ends the proof of the corollary. $\square$

13.3.6. Proposition **(Behaviour with respect to (dual) localization)**

Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then the Hermitian dual of the natural morphism $\mathcal{M} \to \mathcal{M}(*g)$ is the natural morphism $\mathcal{M}^*(!g) \to \mathcal{M}^*$ and conversely. Furthermore, the property of being a minimal extension is preserved by Hermitian duality, that is, $(\mathcal{M}(!*g))^* = \mathcal{M}^*(!*g)$.

Let us first emphasize some properties of the localization and the dual localization of holonomic $\mathcal{D}_X$-modules. The localized $\mathcal{D}_X$-module $\mathcal{M}(*g)$ and the dual localized one $\mathcal{M}(!g)$ correspond to the restriction to $z = 1$ of those considered in Chapter 11. Furthermore, $\mathcal{M}(*g)$ is also the naive localization of $\mathcal{M}$. There is a natural morphism $\mathcal{M}(!g) \to \mathcal{M}(*g)$ whose kernel and cokernel are supported on the divisor (g). It is known (according to [**Kas78**]) that $\mathcal{M}(*g)$ is holonomic. In order to see that $\mathcal{M}(!g)$, as defined in Section 11.4.b, is also holonomic, one can argue as follows. Firstly, one reduces to the case of the dual localization along a smooth hypersurface $H \subset X$. By construction, $\mathcal{M}(!H)$ is $\mathbb{R}$-specializable along H and is characterized by the property that can_t is an isomorphism, while $\mathcal{M}(*H)$ is characterized by the property that var_t is an isomorphism. The behaviour of the V-filtration by duality (see Remark 10.7.16) shows that the natural morphism $\mathcal{M} \to \mathcal{M}(*H)$ is transformed by duality to the natural morphism $(\boldsymbol{D}\mathcal{M})(!H) \to \boldsymbol{D}\mathcal{M}$ and conversely. In particular, we have $\boldsymbol{D}(\mathcal{M}(*H)) \simeq (\boldsymbol{D}\mathcal{M})(!H)$, showing holonomicity of the latter.

Let also emphasize that, from the definitions in Chapter 11, the natural morphisms below are isomorphisms, for $\mathcal{M}$ holonomic:

$$\mathcal{M}(!g) \longrightarrow [\mathcal{M}(*g)](!g), \quad [\mathcal{M}(!g)](*g) \longrightarrow \mathcal{M}(*g).$$

Furthermore, if $\varphi : \mathcal{N} \to \mathcal{M}$ is a morphism between holonomic $\mathcal{D}_X$-modules which is an isomorphism when restricted to $X \smallsetminus g^{-1}(0)$, then φ induces isomorphisms

$$\varphi(*g) : \mathcal{N}(*g) \xrightarrow{\sim} \mathcal{M}(*g), \qquad \varphi(!g) : \mathcal{N}(!g) \xrightarrow{\sim} \mathcal{M}(!g).$$

Indeed, the kernel and cokernel of $\varphi(*)$ are localized along $g^{-1}(0)$ and supported on $g^{-1}(0)$, therefore vanish. For $\varphi(!)$, one argues by duality.

If there exists a morphism $\varphi : \mathcal{N} \to \mathcal{M}(*g)$ which induces an isomorphism after restricting to $X \smallsetminus g^{-1}(0)$, then it induces an isomorphism after localization, it is unique, and there exists a unique morphism $\psi : \mathcal{M}(!g) \to \mathcal{N}$ which induces an isomorphism after localization; furthermore, $\varphi \circ \psi : \mathcal{M}(!g) \to \mathcal{M}(*g)$ is the natural morphism.

A similar statement holds by exchanging the roles of φ and ψ. Let us check the first statement:

• since the kernel and cokernel of φ are supported on $g^{-1}(0)$, φ induces an isomorphism after localization;

• for the existence of ψ, as $\varphi(*g)$ is invertible, we obtain an invertible morphism $[\varphi(*g)^{-1}](!g) : \mathcal{M}(!g) \to \mathcal{N}(!g)$ and ψ is obtained by composing with the natural morphism $\mathcal{N}(!g) \to \mathcal{N}$;

• for the uniqueness of φ (that of ψ is obtained similarly), we assume that $\eta : \mathcal{N} \to \mathcal{M}(*g)$ induces zero after localization; then the following commutative diagram yields the conclusion:

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\eta"] \arrow[d, "\mathrm{loc}"'] & \mathcal{M}(*g) \arrow[d, equal] \\ \mathcal{N}(*g) \arrow[r, "0", "\sim"'] & \mathcal{M}(*g) \end{tikzcd}$$

Lastly, if there exists a morphism $\varphi : \mathcal{N}(*g) \to \mathcal{M}$ which is an isomorphism after localization, then $\mathcal{M} = \mathcal{N}(*g) \oplus \mathcal{P}$ with $\mathcal{P}$ supported on $g^{-1}(0)$: indeed, the morphism $\psi := \varphi(*g)^{-1} \circ \mathrm{loc} : \mathcal{M} \to \mathcal{N}(*g)$ satisfies $\psi \circ \varphi = \mathrm{Id}$.

***Proof of Proposition* 13.3.6.** Let us set $\mathcal{N} = \mathcal{M}(*g)^*$. We have trivially $\mathcal{N}(*g) \simeq \mathcal{M}^*(*g)$ because this holds away from $g^{-1}(0)$. By applying Hermitian duality to $\mathcal{N} \to \mathcal{N}(*g)$, we obtain a morphism $\mathcal{N}(*g)^* \to \mathcal{M}(*g)$ which is an isomorphism away from $g^{-1}(0)$. There exists thus a unique morphism $\mathcal{M}(!g) \to \mathcal{N}(*g)^*$ which is an isomorphism away from $g^{-1}(0)$, which yields a morphism $\mathcal{N}(*g) \to \mathcal{M}(!g)^*$ which is an isomorphism away from $g^{-1}(0)$. It follows that $\mathcal{M}(!g)^* \simeq \mathcal{N}(*g) \oplus \mathcal{P}$ with $\mathcal{P}$ supported on $g^{-1}(0)$. Applying Hermitian duality once more, we find that $\mathcal{P}^*$ is a quotient of $\mathcal{M}(!g)$ supported on $g^{-1}(0)$, hence is zero. Therefore, $\mathcal{M}(!g)^* \simeq \mathcal{N}(*g) \simeq \mathcal{M}^*(*g)$. The isomorphism $\mathcal{M}(*g)^* \simeq \mathcal{M}^*(!g)$ is obtained similarly. □

We now restrict to the category of regular holonomic $\mathcal{D}_X$-modules in order to make full use of the Riemann-Hilbert correspondence 13.2.10.

13.3.7. Proposition. *There exists a canonical isomorphism of functors* $\mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X) \to \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_{\overline{X}})$*:*

$$\mathcal{C}_X \circ \boldsymbol{D}_X \simeq \boldsymbol{D}_{\overline{X}} \circ \mathcal{C}_X.$$

Proof. By definition, we have $\mathrm{DR}_{\overline{X}}(\overline{\mathcal{M}}) = \overline{\mathrm{DR}_X(\mathcal{M})}$ in $\mathsf{Perv}(\mathbb{C}_X)$. By the Riemann-Hilbert correspondence (for regular holonomic $\mathcal{D}_{\overline{X}}$-modules), it suffices thus to show that both functors ${}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}(\mathcal{C}_X \circ \boldsymbol{D}_X(\bullet))$ and ${}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}(\boldsymbol{D}_{\overline{X}} \circ \mathcal{C}_X(\bullet))$ are canonically isomorphic to the same functor $\mathrm{DR}_X(\bullet)$.

On the one hand, we have, by (13.3.5),

$${}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}(\mathcal{C}_X \circ \boldsymbol{D}_X(\bullet)) \simeq {}^{\mathrm{p}}\mathrm{DR}_X(\boldsymbol{D}_X \circ \boldsymbol{D}_X(\bullet)) \simeq \mathrm{Id}(\bullet).$$

On the other hand, we have, by (13.3.5) and (13.2.8),

$$\begin{aligned}{}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}(\boldsymbol{D}_{\overline{X}} \circ \mathcal{C}_X(\bullet)) &\simeq \boldsymbol{D}_{\mathrm{PV}}({}^{\mathrm{p}}\mathrm{DR}_{\overline{X}}\, \mathcal{C}_X(\bullet)) \\ &\simeq \boldsymbol{D}_{\mathrm{PV}}({}^{\mathrm{p}}\mathrm{DR}_X\, \boldsymbol{D}_X(\bullet)) \simeq {}^{\mathrm{p}}\mathrm{DR}_X(\boldsymbol{D}_X \boldsymbol{D}_X(\bullet)) \simeq \mathrm{Id}(\bullet). \qquad \square\end{aligned}$$

We deduce an isomorphism of functors $\mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X) \to \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X)$:

$$(\boldsymbol{D}_X(\bullet))^* \simeq \boldsymbol{D}_X((\bullet)^*). \tag{13.3.8}$$

***13.3.9. Definition* (Conjugation and real structure).** The conjugation functor

$$c : \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X) \longrightarrow \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X)$$

is the composition

$$c(\mathcal{M}) = \boldsymbol{D}(\mathcal{M}^*) \simeq (\boldsymbol{D}\mathcal{M})^*.$$

A real structure on $\mathcal{M} \in \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X)$ is an isomorphism $\kappa : \mathcal{M} \xrightarrow{\sim} c(\mathcal{M})$ such that $c(\kappa) \circ \kappa = \mathrm{Id}$.

13.3.10. Proposition. *The Riemann-Hilbert correspondence* $\mathrm{DR}_X : \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X) \to \mathsf{Perv}(\mathbb{C}_X)$ *is compatible with duality, Hermitian duality, conjugation, and transforms real structures to real structures.*

Proof. Compatibility with duality has been seen in (13.2.8), with Hermitian duality in (13.3.5), and the remaining assertions follow. $\square$

13.4. Duality for nondegenerate regular holonomic triples

13.4.a. Nondegenerate sesquilinear pairings. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing between holonomic $\mathcal{D}_X$-modules. It induces two $\mathcal{D}_X$-linear morphisms

$$\mathfrak{s}' : \mathcal{M}' \longrightarrow (\mathcal{M}'')^*, \quad \mathfrak{s}'' : \mathcal{M}'' \longrightarrow (\mathcal{M}')^*. \tag{13.4.1}$$

***13.4.2. Corollary* (of Corollary 13.3.3).** *The morphism $\mathfrak{s}'$ is an isomorphism if and only if $\mathfrak{s}''$ is so.*

Proof. Let us assume that $\mathfrak{s}'$ is an isomorphism. Then we have a commutative diagram

$$\begin{tikzcd} & ((\mathcal{M}'')^*)^* \arrow[d, "(\mathfrak{s}')^*"] \\ \mathcal{M}'' \arrow[ur, "(13.3.3\,*)"] \arrow[r, "\mathfrak{s}''"'] & (\mathcal{M}')^* \end{tikzcd}$$

Therefore, $\mathfrak{s}''$ is also an isomorphism. The converse is proved similarly. $\square$

13.4.3. Definition. Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', \mathfrak{s})$ be an object of $\mathcal{D}$-$\mathsf{Triples}(X)$. We say that $\mathcal{T}$ is *holonomic* if both $\mathcal{M}'$ and $\mathcal{M}''$ are holonomic. We then say that $\mathcal{T}$ is *nondegenerate* if one of the morphisms $\mathfrak{s}'$ or $\mathfrak{s}''$ is an isomorphism (hence so is the other one).

The full subcategory $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}(X)$ whose objects are the nondegenerate holonomic objects of $\mathcal{D}\text{-}\mathsf{Triples}(X)$ is abelian. The functors defined for $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ in Section 12.7 obviously descend, by restriction to $z=1$, to similar functors on $\mathcal{D}\text{-}\mathsf{Triples}(X)$, and we use similar notations. For the sake of simplicity, we implicitly assume that the holonomic $\mathcal{D}_X$-modules we consider are $\mathbb{R}$-specializable along any function (in general they are specializable, but the eigenvalues of monodromies may not belong to S^1). When considering pushforward, we also implicitly assume that each component $\mathcal{M}',\mathcal{M}''$ of a holonomic triple admits (globally) a coherent F-filtration.(3)

***13.4.4. Corollary* (of Proposition 13.3.6).** *Let $g:X\to\mathbb{C}$ be a holomorphic function, set $D=(g)$, and let $\mathcal{T}=(\mathcal{M}',\mathcal{M}'',\mathfrak{s})$ be a holonomic object of $\mathcal{D}\text{-}\mathsf{Triples}(X)$. If $\mathcal{T}|_{X\smallsetminus D}$ is nondegenerate, then so are $\mathcal{T}(*D)$, $\mathcal{T}(!D)$ and $\mathcal{T}(!*D)$.*

***Proof*.** Recall that localization, dual localization and middle extension of objects of $\mathcal{D}\text{-}\mathsf{Triples}(X)$ is defined in Section 12.7.25. The sesquilinear pairing $\mathfrak{s}^{(*D)}$ yields the morphisms

$$\mathfrak{s}^{(*D)\prime}:\mathcal{M}'(*D)\longrightarrow\mathcal{M}''(!D)^*,\quad \mathfrak{s}^{(*D)\prime\prime}:\mathcal{M}''(!D)\longrightarrow\mathcal{M}'(*D)^*,$$

which read

$$\mathfrak{s}^{(*D)\prime}:\mathcal{M}'(*D)\longrightarrow(\mathcal{M}'')^*(*D),\quad \mathfrak{s}^{(*D)\prime\prime}:\mathcal{M}''(!D)\longrightarrow(\mathcal{M}')^*(!D).$$

In other words, $\mathfrak{s}^{(*D)\prime}=\mathfrak{s}'(*D)$ and $\mathfrak{s}^{(*D)\prime\prime}=\mathfrak{s}''(!D)$. The assumption that $\mathfrak{s}'|_{X\smallsetminus D},\mathfrak{s}''|_{X\smallsetminus D}$ are isomorphisms implies that so are $\mathfrak{s}^{(*D)\prime}$ and $\mathfrak{s}^{(*D)\prime\prime}$.

We argue similarly for $\mathcal{T}(!D)$. For the case of $\mathcal{T}(!*D)$, we interpret the commutative diagram (12.5.39 $*$) as the commutative diagram

$$\begin{tikzcd}
\mathcal{M}'(!D) \arrow[r, "\mathfrak{s}^{(!D)\prime}"] \arrow[d, "\mathrm{loc}'\circ\mathrm{dloc}'"'] & \mathcal{M}''^*(!D) \arrow[d, "\mathrm{loc}''^*\circ\mathrm{dloc}''^*"] \\
\mathcal{M}'(*D) \arrow[r, "\mathfrak{s}^{(*D)\prime}"] & \mathcal{M}''^*(*D)
\end{tikzcd}$$
□

***13.4.5. Proposition* (Kashiwara's equivalence).** *Let $\iota:Z\hookrightarrow X$ be the inclusion of a smooth submanifold and let $\mathcal{T}$ be a holonomic object of $\mathcal{D}\text{-}\mathsf{Triples}(Z)$. Then $\mathcal{T}$ belongs to $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}(Z)$ if and only if ${}_{\mathrm{T}}\iota_*\mathcal{T}$ belongs to $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}(X)$.*

***Proof*.** Since the question is local, we can assume, by induction on $\dim X$, that Z is a smooth hypersurface that we denote by H. We first check that there is a natural isomorphism of functors from $\mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_H)$ to $\mathsf{Mod}_{\mathrm{hol}}(\mathcal{D}_{\overline{X}})$:

$$\mathcal{C}_X\otimes{}_{\mathrm{D}}\iota_*\simeq{}_{\mathrm{D}}\overline{\iota}_*\circ\mathcal{C}_H. \tag{13.4.6}$$

(3)This follows from a theorem of Malgrange **[Mal04]**, but for our purpose, we only consider those holonomic $\mathcal{D}_X$-modules which come from a holonomic $\widetilde{\mathcal{D}}_X$-module by restriction to $z=1$, and we do not need to use this theorem.

The integration morphism for a closed inclusion is an isomorphism, so we have an isomorphism

$$\mathfrak{C}_H \otimes_{\mathcal{D}_{H,\overline{H}}} \mathcal{D}_{(H,\overline{H})\to(X,\overline{X})} \xrightarrow{\sim} \mathfrak{C}_X .$$

We note that

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}({}_{\mathrm{D}}\iota_*\mathcal{M}, \mathfrak{C}_X(*H)) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X(*H)}(({}_{\mathrm{D}}\iota_*\mathcal{M})(*H), \mathfrak{C}_X(*H)) = 0,$$

and thus (see Section 12.2.c), we have a natural isomorphism

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}({}_{\mathrm{D}}\iota_*\mathcal{M}, \mathfrak{C}_{X,H}) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}({}_{\mathrm{D}}\iota_*\mathcal{M}, \mathfrak{C}_X).$$

We also note that, by adjunction and Kashiwara's equivalence, we have

$$\begin{aligned}\boldsymbol{R}\iota_*\,\boldsymbol{R}\mathcal{H}om_{\iota^{-1}\mathcal{D}_X}(\mathcal{M}\otimes_{\mathcal{D}_H}\mathrm{Sp}_{H\to X}, \mathfrak{C}_H\otimes_{\mathcal{D}_H}\mathrm{Sp}_{H\to X}) &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}({}_{\mathrm{D}}\iota_*\mathcal{M}, {}_{\mathrm{D}}\iota_*\,\mathfrak{C}_H)\\ &\simeq \boldsymbol{R}\iota_*\,\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_H}(\mathcal{M}, \mathfrak{C}_H).\end{aligned}$$

It follows that, for $\mathcal{M}$ holonomic on H,

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X}({}_{\mathrm{D}}\iota_*\mathcal{M}, \mathfrak{C}_X) &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}({}_{\mathrm{D}}\iota_*\mathcal{M}, \mathfrak{C}_{X,H})\\ &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\boldsymbol{R}\iota_*(\mathcal{M}\otimes_{\mathcal{D}_H}\mathrm{Sp}_{H\to X}), \boldsymbol{R}\iota_*(\mathfrak{C}_H\otimes_{\mathcal{D}_{H,\overline{H}}}\mathrm{Sp}_{H,\overline{H}\to X,\overline{X}}))\\ &\simeq \boldsymbol{R}\iota_*\,\boldsymbol{R}\mathcal{H}om_{\iota^{-1}\mathcal{D}_X}(\mathcal{M}\otimes_{\mathcal{D}_H}\mathrm{Sp}_{H\to X}, \mathfrak{C}_H\otimes_{\mathcal{D}_{H,\overline{H}}}\mathrm{Sp}_{H,\overline{H}\to X,\overline{X}})\\ &\simeq \boldsymbol{R}\mathcal{H}om_{\iota^{-1}\mathcal{D}_X}(\mathcal{M}\otimes_{\mathcal{D}_H}\mathrm{Sp}_{H\to X}, \mathfrak{C}_H\otimes_{\mathcal{D}_H}\mathrm{Sp}_{H\to X})\otimes_{\mathcal{D}_{\overline{H}}}\boldsymbol{R}\iota_*\,\mathrm{Sp}_{\overline{H}\to\overline{X}}\\ &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_H}(\mathcal{M}, \mathfrak{C}_H)\otimes_{\mathcal{D}_{\overline{H}}}\boldsymbol{R}\iota_*\,\mathrm{Sp}_{\overline{H}\to\overline{X}}\\ &\simeq {}_{\overline{\mathrm{D}}}\iota_*\mathcal{C}_H(\mathcal{M}),\end{aligned}$$

hence (13.4.6) holds true. We can read it as ${}_{\mathrm{D}}\iota_*(\mathcal{M})^* \xrightarrow{\sim} ({}_{\mathrm{D}}\iota_*\mathcal{M})^*$.

Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', \mathfrak{s})$ be an object of $\mathcal{D}\text{-}\mathsf{Triples}(X)$. The composition

$${}_{\mathrm{D}}\iota\mathcal{M}' \xrightarrow{{}_{\mathrm{D}}\iota_*(\mathfrak{s}')} {}_{\mathrm{D}}\iota_*(\mathcal{M}'')^* \xrightarrow{\sim} ({}_{\mathrm{D}}\iota_*\mathcal{M}'')^*$$

defines a new sesquilinear pairing $\mathfrak{s}_1$ between ${}_{\mathrm{D}}\iota\mathcal{M}'$ and ${}_{\mathrm{D}}\iota\mathcal{M}''$ which restricts to $\mathfrak{s}$ on H. Kashiwara's equivalence (Section 12.7.29) implies that $\mathfrak{s}_1 = {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}$. Then $\mathfrak{s}$ is nondegenerate iff $\mathfrak{s}'$ is an isomorphism iff ${}_{\mathrm{D}}\iota_*(\mathfrak{s}')$ is an isomorphism iff $\mathfrak{s}_1$ is nondegenerate iff ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}$ is nondegenerate. $\square$

13.4.b. Duality and real structure on $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$. The dual object of a holonomic triple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', \mathfrak{s})$ of $\mathcal{D}\text{-}\mathsf{Triples}(X)$ should be an object

$$\boldsymbol{D}\mathcal{T} := (\boldsymbol{D}\mathcal{M}', \boldsymbol{D}\mathcal{M}'', \boldsymbol{D}\mathfrak{s}),$$

provided we are able to define the sesquilinear pairing $\boldsymbol{D}\mathfrak{s}$. For the sake of simplicity, we restrict to the category of $\mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules. In such a case, we can make use of the *Riemann-Hilbert correspondence*:

13.4.7. Definition. Let $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', \mathfrak{s})$ be an object of $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$ (i.e., $\mathcal{M}', \mathcal{M}''$ are regular holonomic). The sesquilinear pairing $\boldsymbol{D}\mathfrak{s} : \boldsymbol{D}\mathcal{M}' \otimes \overline{\boldsymbol{D}\mathcal{M}''} \to \mathfrak{Db}_X$ (or $\mathfrak{C}_X$) is defined by the data of $(\boldsymbol{D}\mathfrak{s})', (\boldsymbol{D}\mathfrak{s})''$, which are the composition

$$(\boldsymbol{D}\mathfrak{s})' : \boldsymbol{D}\mathcal{M}' \xrightarrow{\boldsymbol{D}(\mathfrak{s}')^{-1}} \boldsymbol{D}_X((\mathcal{M}'')^*) \xrightarrow{\sim} (\boldsymbol{D}_X\mathcal{M}'')^*$$

and

$$(\boldsymbol{D}\mathfrak{s})'' : \boldsymbol{D}\mathcal{M}'' \xrightarrow{\boldsymbol{D}(\mathfrak{s}'')^{-1}} \boldsymbol{D}_X((\mathcal{M}')^*) \xrightarrow{\sim} (\boldsymbol{D}_X\mathcal{M}')^*.$$

***13.4.8. Remark* (Linear pairings).** We took advantage of the $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$ structure of $\mathfrak{Db}_X$ (or $\mathfrak{C}_X$) to define the notion of a *sesquilinear* pairing between a $\mathcal{D}_X$-module and the naive conjugate of another one. We proceed similarly in order to define the notion of a *linear* pairing between two $\mathcal{D}_X$-modules. For that purpose, we use both left and right $\mathcal{D}_X$-module structures of $\mathcal{D}_X$. For the sake of simplicity, we consider a left holonomic $\mathcal{D}_X$-module $\mathcal{M}'^{\text{left}}$ and a right one $\mathcal{M}''^{\text{right}}$. A linear pairing between them is a $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X$-linear morphism in the derived category $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X)$:

$$\mathsf{q} : \mathcal{M}'^{\text{left}} \otimes_{\mathbb{C}} \mathcal{M}''^{\text{right}} \longrightarrow \mathcal{D}_X[n],$$

that we regard as a morphism of left $\mathcal{D}_X$-modules

$$\mathsf{q}' : \mathcal{M}'^{\text{left}} \longrightarrow \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}''^{\text{right}}, \mathcal{D}_X)[n] \simeq \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}''^{\text{right}}, \mathcal{D}_X),$$

or as a morphism of right $\mathcal{D}_X$-modules

$$\mathsf{q}'' : \mathcal{M}''^{\text{right}} \longrightarrow \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}'^{\text{left}}, \mathcal{D}_X)[n] \simeq \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}'^{\text{left}}, \mathcal{D}_X).$$

If we compose with side-changing, we regard q' and q'' as morphisms

$$\mathsf{q}' : \mathcal{M}' \longrightarrow \boldsymbol{D}\mathcal{M}'', \quad \mathsf{q}'' : \mathcal{M}'' \longrightarrow \boldsymbol{D}\mathcal{M}'$$

which are related, up to the biduality isomorphism, by $\boldsymbol{D}(\mathsf{q}') = \mathsf{q}''$.

13.4.9. Conjugation and real structure of an object of $\mathcal{D}$-$\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$. The conjugation functor $c : \mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X) \to \mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$, $\mathcal{T} \mapsto c(\mathcal{T})$ is composed of the duality functor $\boldsymbol{D}$ of Definition 13.4.7 and the Hermitian duality functor $\mathcal{T} \mapsto \mathcal{T}^*$, with $\mathcal{T}^* = (\mathcal{M}'', \mathcal{M}', \mathfrak{s}^*)$ (see Definition 12.7.4).

13.4.10. Lemma. *Duality and Hermitian duality commute on* $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$.

It follows that the conjugation functor c satisfies $c \circ c \simeq \mathrm{Id}$.

Proof. This amounts to proving the equality $(\boldsymbol{D}\mathfrak{s})^* = \boldsymbol{D}(\mathfrak{s}^*)$. We note that $\mathfrak{s}^*$ satisfies, by definition, $(\mathfrak{s}^*)' = \mathfrak{s}''$ and $(\mathfrak{s}^*)'' = \mathfrak{s}'$. The assertion follows. ☐

A *real structure* on an object $\mathcal{T}$ of $\mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$ is an isomorphism

$$\kappa : \mathcal{T} \longrightarrow c(\mathcal{T})$$

satisfying

$$c(\kappa) \circ \kappa = \mathrm{Id}\,.$$

We can extend the Riemann-Hilbert correspondence of Proposition 13.3.10 to triples.

13.4.11. Proposition. *We have a commutative diagram of equivalences of categories, compatible with duality, Hermitian duality and conjugation, which transforms real structures to real structures:*

$$\begin{tikzcd} \mathsf{Mod}_{\mathrm{holreg}}(\mathcal{D}_X) \arrow[r, "\sim"] \arrow[d, "{}^{\mathrm{p}}\mathrm{DR}_X"', "\wr"] & \mathcal{D}\text{-}\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X) \arrow[d, "\wr"] \\ \mathsf{Perv}(\mathbb{C}_X) \arrow[r, "\sim"] & \mathsf{PervTriples}^{\mathrm{nd}}(\mathbb{C}_X) \end{tikzcd}$$

13.4.c. Duality and real structure on $\widetilde{\mathcal{D}}$-$\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$. We now switch back to the filtered setting. For an object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ we denote by $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', \mathfrak{s})$ the associated object of $\mathcal{D}$-$\mathsf{Triples}(X)$. We denote by $\widetilde{\mathcal{D}}$-$\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$ the full subcategory of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ consisting of objects $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ such that

- $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are *strictly holonomic*,
- $\mathcal{T}$ is an object of $\mathcal{D}$-$\mathsf{Triples}^{\mathrm{nd}}_{\mathrm{reg}}(X)$.

The dual object $\boldsymbol{D}\widetilde{\mathcal{T}}$ is defined as $(\boldsymbol{D}\widetilde{\mathcal{M}}', \boldsymbol{D}\widetilde{\mathcal{M}}'', \boldsymbol{D}\mathfrak{s})$, where $\boldsymbol{D}\widetilde{\mathcal{M}}', \boldsymbol{D}\widetilde{\mathcal{M}}''$ are defined in Section 8.8.g, and $\boldsymbol{D}\mathfrak{s}$ is such that $(\boldsymbol{D}\mathcal{M}', \boldsymbol{D}\mathcal{M}'', \boldsymbol{D}\mathfrak{s}) = \boldsymbol{D}\mathcal{T}$. The conjugate object $c(\widetilde{\mathcal{T}})$ is defined as $\boldsymbol{D}(\widetilde{\mathcal{T}}^*) = (\boldsymbol{D}\widetilde{\mathcal{T}})^*$, and a real structure is an isomorphism $\kappa : \widetilde{\mathcal{T}} \to c(\widetilde{\mathcal{T}})$ such that $c(\kappa) \circ \kappa = \mathrm{Id}$.

13.5. Comments

The notion of Hermitian duality and conjugation for regular holonomic $\widetilde{\mathcal{D}}_X$-modules has been introduced by Kashiwara in [**Kas86a**], where Theorem 13.3.1 is proved (see also [**Bjö93**]). The condition of regularity can be relaxed, as proved by T. Mochizuki [**Moc11b**, Cor. 4.19] (see also [**Sab13**, Th. 12.20] and [**Sab00**, Th. II.3.1.2]), relying on a difficult structure theorem for meromorphic connections of Kedlaya [**Ked11**] and Mochizuki [**Moc11a**] (see also [**Sab00**]). The content of Sections 13.4.a and 13.4.b, together with compatibility between various functors, is treated with details in the holonomic (possibly non regular) case in [**Moc15**, Chap. 12].