CHAPTER 6

VARIATIONS OF HODGE STRUCTURE ON CURVES
PART 1: METRIC PROPERTIES NEAR PUNCTURES

Summary. We consider polarizable variations of $\mathbb{C}$-Hodge structure on a punctured smooth projective curve. This is the first occurrence of polarizable variations of $\mathbb{C}$-Hodge structure with singularities. It is essential to understand their local behaviour in the neighbourhood of a singular point. In this part of Chapter 6, we state the main results and, as an application, we prove the semi-simplicity theorem analogue to that proved in Chapter 4.

6.1. Introduction

A Hodge structure, as explained in Section 2.5, can be considered as a Hodge structure on a vector bundle supported by a point, that is, a vector space. The case where the underlying space is a complex manifold is called a *variation of Hodge structure*. It has been explained in Section 4.1 from a local point of view. The global properties have been considered in Section 4.2.

The question we address in this chapter is the definition and properties of Hodge structures on a vector bundle on a punctured complex projective curve (punctured compact Riemann surface) in the neighbourhood of the punctures (also called the *singularities* of the variation). The notion of a polarized variation of Hodge structure on a non-compact curve is analytic in nature, and a control near the punctures is needed in order to obtain interesting global results. Let us emphasize that, nevertheless, the approach is local, and we will mainly restrict the study to a local setting, where the base manifold is a disc Δ centered at the origin in $\mathbb{C}$ of radius 1 for convenience (or simply the germ of Δ at the origin), and we will denote by t its coordinate.

This chapter is divided in three parts, due to the length of the arguments. In this part, we state the fundamental properties of the variation near a puncture. We first focus on metric properties without paying much attention to the Hodge filtration itself. Our interest lies in the relations between two possible extensions of the holomorphic bundle with connection and Hermitian metric underlying a variation of $\mathbb{C}$-Hodge structure from the punctured Riemann surface X^* to the compact one X. We then explain how to extend the Hodge filtration at the punctures and provide the main statement for the limiting Hodge-Lefschetz structure. As an application of the metric properties, we prove the semi-simplicity theorem analogue of Theorem 4.3.3.

6.2. Variations of Hodge structure on a punctured disc

We consider the behaviour of a variation of $\mathbb{C}$-Hodge structure near a singular point. From now on, we will work on a disc Δ of radius 1 with coordinate t, as indicated in the introduction of this chapter and we will denote by Δ^* the punctured disc $\Delta \smallsetminus \{0\}$. Assume that H is a variation of Hodge structure on Δ^* (Definitions 4.1.4, 4.1.5 and 5.4.3). Our goal is to define a suitable restriction of these data to the origin. As for the case of a point in Δ^*, the underlying vector space of the restricted object should have a dimension equal to the rank of the bundle on Δ^*.

6.2.a. Reminder on holomorphic vector bundles with connection. We recall in this section the equivalence between the category of holomorphic vector bundle with connection $(\mathcal{V}, \nabla)$ on Δ^* and the category of finite dimensional vector spaces equipped with an automorphism. We shall first construct a functor from the first one to the second one.

If we are given a holomorphic vector bundle with connection $(\mathcal{V}, \nabla)$ on Δ^*, there exists a canonical meromorphic extension, called the *Deligne meromorphic extension*, of the bundle $\mathcal{V}$ to a meromorphic bundle $\mathcal{V}_*$ (that is, a free sheaf of $\mathcal{O}_\Delta[1/t]$-modules) equipped with a connection ∇. It consists of all local sections of $j_*\mathcal{V}$ (where $j : \Delta^* \hookrightarrow \Delta$ is the inclusion) whose coefficients in some (or any) basis of multivalued ∇-horizontal sections have moderate growth in any sector with bounded arguments. Equivalently, it is characterized by the property that the coefficients of any multivalued horizontal section expressed in some basis of $\mathcal{V}_*$ are multivalued functions on Δ^* with moderate growth in any sector with bounded arguments.

Similarly, there exists a canonical free $\mathcal{O}_\Delta$-submodule $\mathcal{V}^0_*$ of $\mathcal{V}_*$, called the *Deligne canonical lattice*, consisting of *all* local sections of $j_*\mathcal{V}$ whose coefficients in any basis of horizontal sections on any bounded sector are holomorphic functions on this sector *with at most logarithmic growth*. On this bundle $\mathcal{V}^0_*$, the connection ∇ has a pole of order 1. The residue $\mathcal{R}$ of the connection on $\mathcal{V}^0_*$ is an endomorphism of the vector space $\mathcal{V}^0_*/t\mathcal{V}^0_*$. The real parts of its eigenvalues belong to $[0, 1)$. The latter two properties also characterize $\mathcal{V}^0_*$ among all lattices of $\mathcal{V}_*$ (i.e., free $\mathcal{O}_\Delta$-submodules of $\mathcal{V}_*$ which generate $\mathcal{V}_*$ as a $\mathcal{O}_\Delta[t^{-1}]$-module).

The existence of a free $\mathcal{O}_\Delta$-submodule $\mathcal{V}^0_*$ of $\mathcal{V}_*$ such that $\mathcal{O}_\Delta[t^{-1}] \otimes \mathcal{V}^0_* = \mathcal{V}_*$ and on which ∇ has a pole of order 1 is by definition the condition ensuring that $(\mathcal{V}_*, \nabla)$ has a *regular singularity* at the origin of Δ.

A classical result (see e.g. [**Mal91**, (2.6) p. 24]) asserts that $\mathcal{V}^0_*$ has an $\mathcal{O}_\Delta$-basis with respect to which the matrix of ∇ is constant. More precisely, any $\mathbb{C}$-basis of $\mathcal{V}^0_*/t\mathcal{V}^0_*$ can be lifted to an $\mathcal{O}_\Delta$-basis of $\mathcal{V}^0_*$, and the matrix of ∇ is then equal to the matrix of the residue $\mathcal{R}$ in the given basis of $\mathcal{V}^0_*/t\mathcal{V}^0_*$. These results can be reformulated as follows.

6.2.1. Theorem. *The construction $(\mathcal{V}, \nabla) \mapsto (\mathcal{V}_*, \nabla)$ induces an equivalence between the category of vector bundles with connection on the punctured disc Δ^* and that of free $\mathcal{O}_\Delta[1/t]$-modules with a connection ∇ having a regular singularity at the origin.* □

Of course, an inverse functor is the restriction of $(\mathcal{V}_*, \nabla)$ to Δ^*. Notice also that this result implies that any morphism $\varphi : (\mathcal{V}_1, \nabla) \to (\mathcal{V}_2, \nabla)$ can be extended in a unique way as a morphism $(\mathcal{V}_{1*}, \nabla) \to (\mathcal{V}_{2*}, \nabla)$. The proof is obtained by interpreting φ as a horizontal section of $\mathcal{H}om_{\mathcal{O}_{\Delta^*}}(\mathcal{V}_1, \mathcal{V}_2)$ and by using the property that, for a connection with regular singularity (as ∇ on $\mathcal{H}om_{\mathcal{O}_\Delta[1/t]}(\mathcal{V}_{1*}, \mathcal{V}_{2*})$), any horizontal section on Δ^* extends in a unique way as a ∇-horizontal section on Δ (see Exercise 6.1(4)).

We can then more generally consider a whole family of Deligne canonical lattices: for every $\beta \in \mathbb{R}$, we denote by $\mathcal{V}_*^\beta$ the lattice defined by the property that the eigenvalues of the residue of the connection have their real part in $[\beta, \beta+1)$. If we set $\mathcal{V}_*^{>\beta} = \bigcup_{\beta'>\beta} \mathcal{V}_*^{\beta'}$, then $\mathcal{V}_*^{>\beta}$ is the Deligne canonical lattice for which the eigenvalues of the residue of the connection have real part in $(\beta, \beta+1]$. We use the notation

$$\mathrm{gr}^\beta \mathcal{V}_* := \mathcal{V}_*^\beta / \mathcal{V}_*^{>\beta}. \tag{6.2.2}$$

See Exercise 6.2 for the properties of the canonical lattices.

If we denote by $\mathcal{V}_*^{>-1}$ the lattice on which $\operatorname{Res}\nabla$ has eigenvalues with real part in $(-1, 0]$, and if $\beta \in (-1, 0]$, then $\mathrm{gr}^\beta \mathcal{V}_*$ is identified with the generalized eigenspace of $\operatorname{Res}\nabla$ on $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ corresponding to the eigenvalues $\beta + \mathrm{i}\,\beta''$ ($\beta'' \in \mathbb{R}$) with real part β. We set $\mathrm{N} = -(\operatorname{Res}\nabla)^{\mathrm{nilp}}$ (nilpotent part). This is the endomorphism induced by $\bigoplus_{\beta''}[-(t\partial_t - \beta - \mathrm{i}\,\beta'')]$ on $\mathrm{gr}^\beta \mathcal{V}_*$. [This choice is suggested by the property that the unipotent part of the monodromy operator on the locally constant sheaf $\mathcal{V}^\nabla := \operatorname{Ker}\nabla$ can be identified with $\exp 2\pi \mathrm{i}\,\mathrm{N}$.]

6.2.3. Remark **(Behaviour with respect to operations).** Let $(\mathcal{V}, \nabla)$ be a holomorphic bundle with connection on Δ^*.

(1) Let $\mathcal{V}_1$ be a holomorphic subbundle of $\mathcal{V}$ which is preserved by the connection. Then, by construction, the Deligne canonical lattice $\mathcal{V}_{1,*}^0$ of $(\mathcal{V}_1, \nabla)$ is nothing but $j_*\mathcal{V}_1 \cap \mathcal{V}_*^0$, and similarly, for any β, $\mathcal{V}_{1,*}^\beta = j_*\mathcal{V}_1 \cap \mathcal{V}_*^\beta$.

(2) Let $(\mathcal{V}^\vee, \nabla)$ be the dual bundle with the dual connection. Using that the residue of the connection on $(\mathcal{V}_*^\beta)^\vee$ is minus the transposed of that on $\mathcal{V}_*^\beta$, one deduces that

$$(\mathcal{V}_*^\vee)^\beta \simeq (\mathcal{V}_*^{>-\beta-1})^\vee.$$

As a consequence, the natural pairing

$$(\mathcal{V}_*^\vee)^\beta \otimes \mathcal{V}_*^{-\beta-1} \longrightarrow \mathcal{O}_\Delta[1/t]$$

induces, by composing with the residue at $t=0$, a perfect pairing

$$\mathrm{gr}^\beta \mathcal{V}_*^\vee \otimes \mathrm{gr}^{-\beta-1}\mathcal{V}_* \longrightarrow \mathbb{C}.$$

Equivalently, after multiplication by t, the natural pairing

$$\langle \bullet, \bullet \rangle : (\mathcal{V}_*^\vee)^\beta \otimes \mathcal{V}_*^{-\beta} \longrightarrow \mathcal{O}_\Delta$$

induces, by composing with restriction at $t=0$, a perfect pairing

$$\mathrm{gr}^\beta \mathcal{V}_*^\vee \otimes \mathrm{gr}^{-\beta}\mathcal{V}_* \longrightarrow \mathbb{C}.$$

In particular, for any section v of $\mathcal{V}_*^{-\beta}$ whose class in $\mathrm{gr}^{-\beta}\mathcal{V}_*$ is nonzero, there exists a section $v^\vee$ of $(\mathcal{V}_*^\vee)^\beta$ (whose class in $\mathrm{gr}^\beta \mathcal{V}_*^\vee$ is nonzero) such that $\langle v^\vee, v \rangle = 1$.

(3) Let $\det(\mathcal{V},\nabla)$ be the determinant bundle (maximal exterior power) with connection. Given a frame $\boldsymbol{e}$ of $\mathcal{V}$, the matrix of the connection on $\det\mathcal{V}$ in the frame $e_1\wedge\cdots\wedge e_r$ is the trace of that of ∇ on $\mathcal{V}$ in the frame $\boldsymbol{e}$. Let $\gamma\geqslant 0$ be the sum of the real parts of the eigenvalues of the residue at the origin of ∇ on $\mathcal{V}^0_*$. We thus find

$$(\det\mathcal{V})^\gamma_* = \det(\mathcal{V}^0_*)\quad\text{and}\quad \dim\operatorname{gr}^\gamma(\det\mathcal{V})_* = 1.$$

6.2.4. Theorem. *The correspondence*

$$(\mathcal{V}_*,\nabla)\longmapsto(\mathcal{H}^o,\mathrm{T}) = \bigoplus_{\beta\in(-1,0]}(\operatorname{gr}^\beta\mathcal{V}_*, \mathrm{e}^{-2\pi\mathsf{i}\,\beta}\,\mathrm{T}_\beta\cdot\mathrm{e}^{2\pi\mathsf{i}\,\mathrm{N}}),$$

with T_β semi-simple with positive eigenvalues, is an equivalence between the category of free $\mathcal{O}_\Delta[1/t]$-modules with a connection ∇ having a regular singularity at the origin and the category of finite dimensional vector spaces with an automorphism.

Here is a quasi-inverse functor. Given $(\mathcal{H}^o,\mathrm{T})$, we group the generalized eigenspaces corresponding to the eigenvalues μ of T which share the same value $\lambda=\mu/|\mu|$, and denote this space $\mathcal{H}^o_\lambda$. On such a subspace, the action of T reads $\lambda\mathrm{T}_\lambda\mathrm{e}^{2\pi\mathsf{i}\,\mathrm{N}}$ with N nilpotent and T_λ semi-simple with positive eigenvalue commuting with N. We thus obtain a decomposition $(\mathcal{H}^o,\mathrm{T}) = \bigoplus_{|\lambda|=1}(\mathcal{H}^o_\lambda,\lambda\mathrm{T}_\lambda\mathrm{e}^{2\pi\mathsf{i}\,\mathrm{N}})$. Furthermore, we write each λ as $\exp-2\pi\mathsf{i}\,\beta$ with $\beta\in(-1,0]$. We then associate to $(\mathcal{H}^o_\lambda,\lambda\mathrm{T}_\lambda\mathrm{e}^{2\pi\mathsf{i}\,\mathrm{N}})$ the free $\mathcal{O}_\Delta[1/t]$-module $\mathcal{H}^o_\lambda\otimes_{\mathbb{C}}\mathcal{O}_\Delta[1/t]$ with connection $\nabla=\mathrm{Id}\otimes\mathrm{d}+(\beta\,\mathrm{Id}+\frac{\mathsf{i}}{2\pi}\log\mathrm{T}_\lambda-\mathrm{N})\mathrm{d}t/t$.

The canonical decomposition of the right-hand side of the correspondence of Theorem 6.2.4 corresponds to a canonical decomposition of the left-hand side:

6.2.5. Corollary. *There exists a canonical decomposition*

$$(\mathcal{V}_*,\nabla)\simeq\bigoplus_{\beta\in(-1,0]}(\mathcal{V}_{*\beta},\nabla)\tag{6.2.5 $*$}$$

for which $\mathcal{V}_{*\beta}$ has a frame $\boldsymbol{v}_\beta$ in which ∇ has matrix $(\beta\,\mathrm{Id}+\frac{\mathsf{i}}{2\pi}\,\mathsf{D}_\beta-\mathsf{N})\mathrm{d}t/t$ with N nilpotent and D_β diagonal with positive eigenvalues.

Proof. We denote by $\mathcal{V}_\beta$ the subsheaf of $\mathcal{V}$ consisting of $\mathcal{O}_{\Delta^*}$-linear combinations of local sections of $\mathcal{V}$ annihilated by some power of $t\partial_t-(\beta+\mathsf{i}\,b'')$ for all possible $b''\in\mathbb{R}$. We then have a canonical decomposition

$$(\mathcal{V},\nabla)\simeq\bigoplus_{\beta\in(-1,0]}(\mathcal{V}_\beta,\nabla).\tag{6.2.5 $**$}$$

The correspondence of Theorem 6.2.1 induces a canonical decomposition $(\mathcal{V}_*,\nabla)\simeq\bigoplus_{\beta\in(-1,0]}(\mathcal{V}_{\beta*},\nabla)$ and we set $\mathcal{V}_{*\beta}=\mathcal{V}_{\beta*}$. □

It follows from this decomposition that the space of multi-valued horizontal sections of $(\mathcal{V},\nabla)$ on Δ^* decomposes correspondingly with respect to the eigenvalues λ of the monodromy, which take the form $\lambda=\exp(-2\pi\mathsf{i}(\beta+\mathsf{i}\,b''))$ for any $(\beta+\mathsf{i}\,b'')$ occurring in (6.2.5 $*$). In particular, the absolute value of the eigenvalues of the monodromy are all equal to one if and only if $\mathsf{D}_\beta=0$ for any β.

6.2.b. Reminder on Hermitian bundles on the punctured disc. Let $\mathcal{V}$ be a holomorphic vector bundle on Δ^* and let h be a Hermitian metric on the associated C^∞-bundle $\mathcal{H} := \mathcal{C}^\infty_{\Delta^*} \otimes_{\mathcal{O}_{\Delta^*}} \mathcal{V}$. We denote by $\mathcal{V}_{\mathrm{mod}}$ the subsheaf of $j_*\mathcal{V}$ consisting of local sections whose h-norms have *moderate growth* in the neighbourhood of the origin, i.e., bounded by some (negative) power of $|t|$. This is an $\mathcal{O}_\Delta[1/t]$-module, which coincides with $\mathcal{V}$ when restricted to Δ^*.

The *parabolic filtration* $\mathcal{V}^\bullet_{\mathrm{mod}}$ is the decreasing filtration, indexed by $\mathbb{R}$, defined as follows. For any $\beta \in \mathbb{R}$, we define $\mathcal{V}^\beta_{\mathrm{mod}}$ as consisting of local sections v of $j_*\mathcal{V}$ such that, for any $\varepsilon > 0$, there exists $C_\varepsilon(v) > 0$ such that $\|v\|_{\mathrm{h}} \leqslant C_\varepsilon(v)|t|^{\beta-\varepsilon}$. For $\beta' > \beta$, we have $\mathcal{V}^{\beta'}_{\mathrm{mod}} \subset \mathcal{V}^\beta_{\mathrm{mod}}$ and we set $\mathcal{V}^{>\beta}_{\mathrm{mod}} = \bigcup_{\beta'>\beta} \mathcal{V}^{\beta'}_{\mathrm{mod}}$.

Clearly, each $\mathcal{V}^\beta_{\mathrm{mod}}$ is an $\mathcal{O}_\Delta$-submodule of $\mathcal{V}_{\mathrm{mod}}$, which coincides with $\mathcal{V}$ when restricted to Δ^*, and we have

$$\mathcal{V}_{\mathrm{mod}} = \bigcup_\beta \mathcal{V}^\beta_{\mathrm{mod}}, \qquad \text{and} \qquad \forall\, k \in \mathbb{Z}, \quad t^k\mathcal{V}^\bullet_{\mathrm{mod}} = \mathcal{V}^{\bullet+k}_{\mathrm{mod}}.$$

A *jump* (or, more correctly, jumping index) of the parabolic filtration is a real number β such that the quotient $\mathrm{gr}^\beta(\mathcal{V}_{\mathrm{mod}}) := \mathcal{V}^\beta_{\mathrm{mod}}/\mathcal{V}^{>\beta}_{\mathrm{mod}}$ is nonzero. Clearly, if β is a jump, then $\beta + k$ is a jump for every $k \in \mathbb{Z}$. We denote by $J(\beta)$ the set of jumping indices which belong to $[\beta, \beta + 1)$. We have $J(\beta + k) = J(\beta) + k$ for every $k \in \mathbb{Z}$.

6.2.6. Definition. We say that the metric is *moderate* if each $\mathcal{V}^\beta_{\mathrm{mod}}$ ($\beta \in (-1, 0]$) is $\mathcal{O}_\Delta$-locally free.

If the metric is moderate, $\mathcal{V}^\beta_{\mathrm{mod}}$ is $\mathcal{O}_\Delta$-locally free for any $\beta \in \mathbb{R}$ and $\mathcal{V}_{\mathrm{mod}} = \mathcal{O}_\Delta[1/t] \otimes_{\mathcal{O}_\Delta} \mathcal{V}^\beta_{\mathrm{mod}}$ (any β) is $\mathcal{O}_\Delta[1/t]$-locally free. Furthermore, the induced decreasing filtration $\mathcal{V}^\bullet_{\mathrm{mod}}(\mathcal{V}^\beta_{\mathrm{mod}}/\mathcal{V}^{\beta+1}_{\mathrm{mod}})$ is finite, so that $J(\beta)$ is finite. It follows that $\mathcal{V}^{>\beta}_{\mathrm{mod}} = \mathcal{V}^{\beta'}_{\mathrm{mod}}$ for some $\beta' > \beta$. We also have

$$\mathcal{V}^\beta_{\mathrm{mod}}/t\mathcal{V}^\beta_{\mathrm{mod}} = \bigoplus_{\beta' \in J(\beta)} \mathrm{gr}^{\beta'}\mathcal{V}_{\mathrm{mod}}.$$

***6.2.7. Remark* (Behaviour with respect to operations).** Let $(\mathcal{V}, \mathrm{h})$ be a holomorphic bundle with a *moderate* Hermitian metric.

(1) Let $\mathcal{V}_1$ be a holomorphic subbundle of $\mathcal{V}$ and let h_1 be the Hermitian metric induced by h on $\mathcal{V}_1$. Then, by construction, $\mathcal{V}_{1,\mathrm{mod}} = j_*\mathcal{V}_1 \cap \mathcal{V}_{\mathrm{mod}}$ and, for any β, $\mathcal{V}^\beta_{1,\mathrm{mod}} = j_*\mathcal{V}_1 \cap \mathcal{V}^\beta_{\mathrm{mod}}$. However, *we cannot claim that* $(\mathcal{V}_1, \mathrm{h}_1)$ *is moderate,* i.e., that $\mathcal{V}^\beta_{1,\mathrm{mod}}$ is locally free for any β (see Exercise 6.3).

(2) Let $\boldsymbol{v}$ be a frame of $\mathcal{V}^0_{\mathrm{mod}}$ lifting a basis of $\mathcal{V}^0_{\mathrm{mod}}/t\mathcal{V}^0_{\mathrm{mod}}$ adapted to the filtration induced by $\mathcal{V}^\bullet_{\mathrm{mod}}$. The diagonal entries of the matrix A of h in this frame have thus a controlled behaviour. The determinant bundle $\det \mathcal{V}$ is naturally equipped with a metric, and using this frame, one finds that it is moderate. Furthermore, setting $\gamma = \sum_{\beta \in J(0)} \beta$, one has $(\det \mathcal{V})^\gamma_{\mathrm{mod}} = \det \mathcal{V}^0_{\mathrm{mod}}$.

(3) We do not have much information on the other entries of the matrix A. Similarly, we do not have much information on the matrix ${}^t\mathsf{A}^{-1}$ of the metric on the dual bundle $\mathcal{V}^\vee$ in the dual frame $\boldsymbol{v}^\vee$.

We will make use in Part 2 of a criterion of moderateness in terms of the curvature, which goes back to [**CG75**] and [**Sim88**], and that we will not prove here (see [**Sim88**, §10] and [**Sim90**, Prop. 3.1]). For a Hermitian holomorphic bundle $(\mathcal{V}, \mathrm{h})$, the curvature operator R_{h} of the Chern connection of the metric is a linear morphism $\mathcal{H} \to \mathcal{E}^2_{\Delta^*} \otimes \mathcal{H}$, where $\mathcal{H}$ is the C^∞ bundle associated with $\mathcal{V}$. By fixing a constant norm on the trivial bundle $\mathcal{E}^2_{\Delta^*}$ (e.g. $\mathrm{d}t \wedge \mathrm{d}\bar{t}$ has norm one), we can consider the norm of R_{h} considered as a section of $\mathcal{E}nd(\mathcal{V}) \otimes \mathcal{E}^2_{\Delta^*}$, that we denote by $\|R_{\mathrm{h}}\|_{\mathrm{h}}$.

***6.2.8. Notation* (for $\mathrm{L}(t)$).** We consider on Δ^* the function

$$\mathrm{L}(t) = -\log |t|^2 = -\log t\bar{t}.$$

The main properties we use are given as an exercise (see Exercise 6.5).

***6.2.9. Theorem* (Criterion of moderateness).** *Assume that the curvature R_{h} satisfies $\|R_{\mathrm{h}}\|_{\mathrm{h}} \leqslant C/|t|^2\mathrm{L}(t)^2$ for some constant $C > 0$. Then the Hermitian holomorphic bundle $(\mathcal{V}, \mathrm{h})$ is moderate.* □

6.3. Metric properties near a puncture

6.3.a. The Deligne and parabolic filtrations for a polarized variation of Hodge structure. Let us consider a polarized variation of $\mathbb{C}$-Hodge structure $(H, \mathcal{S})$ of weight w on the punctured disc Δ^* (see Definitions 4.1.4 and 4.1.5). We set $H = (\mathcal{H}, D, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$. We thus have a positive definite Hermitian metric h on $\mathcal{H}$. On the other hand, we set $\mathcal{V} = \operatorname{Ker} D''$, on which the filtration $F'^{\bullet}\mathcal{H}$ induces a filtration $F^{\bullet}\mathcal{V}$ by holomorphic sub-bundles. We aim at comparing the canonical filtration $\mathcal{V}^{\bullet}_*$ and the filtration $\mathcal{V}^{\bullet}_{\mathrm{mod}}$ relative to the Hodge metric h, and more precisely at showing that they coincide. In particular, this implies that the Hodge metric is moderate.

***6.3.1. Example* (The unitary case).** In the simple case where the connection is compatible with the Hermitian metric h, we claim that the metric is moderate.

The assumption corresponds to a variation of Hodge structure of pure type $(0, 0)$. Then the norm of any horizontal section of $\mathcal{V}$ is constant, hence bounded. The monodromy matrix being unitary, its eigenvalues have absolute value equal to 1, and the matrices T_β considered in Corollary 6.2.5 are the identity matrices, so that $\log \mathsf{T}_\beta = 0$.

The decomposition (6.2.5 $**$) is compatible with the metric, and we are reduced to proving the claim on each term. We can then assume for simplicity that $\beta = 0$ by multiplying by $|t|^{2\beta}$. It is then enough to identify $\mathcal{V}^0_*$ and $\mathcal{V}^0_{\mathrm{mod}}$.

Given any section v of $\mathcal{V}$, we express it on a unitary frame of multivalued horizontal sections, and v is a section of $\mathcal{V}^0_{\mathrm{mod}}$ if and only if its multivalued coefficients are bounded by $|t|^{-\varepsilon}$ in any bounded angular sector of sufficiently small radius. Similarly, by definition, a section of $\mathcal{V}$ is a section of $\mathcal{V}^0_*$ if and only if its mutlivalued coefficients have logarithmic growth, and equivalently satisfy the same growth condition as for $\mathcal{V}^0_{\mathrm{mod}}$, hence the claim.

The properties of the previous example hold true for any polarized variation of $\mathbb{C}$-Hodge structure: this is the main results in this part of Chapter 6.

6.3.2. Theorem. *Let* $(\mathcal{V}, \nabla, \mathrm{h})$ *be a Hermitian holomorphic bundle with connection underlying a polarized variation of* $\mathbb{C}$*-Hodge structure on* Δ^**. Then,*

(1) *the metric* h *on* $\mathcal{H}$ *is moderate and the parabolic filtration* $\mathcal{V}^\bullet_{\mathrm{mod}}$ *on* $\mathcal{V}_*$ *induced by the metric* h *is equal to the filtration* $\mathcal{V}^\bullet_*$*;*

(2) *furthermore, the eigenvalues of the monodromy have absolute value equal to* 1*.*

6.3.3. Remark. This result justifies the need of refining the filtration $\mathcal{V}^\bullet_*$ indexed by $\mathbb{Z}$ and its graded spaces with a filtration indexed by $\mathbb{R}$ and the corresponding graded spaces (6.2.2).

Theorem 6.3.2 characterizes sections of $\mathcal{V}^\beta_*$ in terms of growth of their norm with respect to real powers of $|t|$. In order to analyze the L^2 behaviour of the norm, we will need to refine this result by using a logarithmic scale.

6.3.4. Definition **(Lift of the monodromy filtration).** For each $\beta \in \mathbb{R}$, we denote by $\mathrm{M}_\bullet \mathrm{gr}^\beta \mathcal{V}_*$ the monodromy filtration relative to the nilpotent endomorphism N of $\mathrm{gr}^\beta \mathcal{V}_*$ (see Theorem 6.2.4). The *lift* $\mathrm{M}_\bullet \mathcal{V}^\beta_*$ of $\mathrm{M}_\bullet \mathrm{gr}^\beta \mathcal{V}_*$ is the pullback by the projection $\mathcal{V}^\beta_* \to \mathrm{gr}^\beta \mathcal{V}_*$ of $\mathrm{M}_\bullet \mathrm{gr}^\beta \mathcal{V}_*$. This is a locally free extension of $\mathcal{V}$ to Δ.

6.3.5. Theorem **(Finer norm estimates).** *A section of* $\mathcal{V}$ *on* Δ^* *extends as a section of* $\mathrm{M}_\ell \mathcal{V}^\beta_*$ *and not as a section of* $\mathrm{M}_{\ell-1} \mathcal{V}^\beta_*$ *(i.e., with non-zero image in* $\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^\beta \mathcal{V}_*$*) if and only if its* h*-norm has the same order of growth as* $|t|^\beta \mathrm{L}(t)^{\ell/2}$*.*

Theorems 6.3.2 and 6.3.5, while depending on the Hodge structure in their assumptions, do not involve Hodge properties in their conclusions. As a matter of fact, the statements hold for harmonic flat bundles (Definition 4.2.5) on the punctured disc whose Higgs field is nilpotent. We will prove them in that setting. We thus forget the Hodge filtration for a while and consider a vector bundle $(\mathcal{V}, \nabla)$ equipped with a *harmonic metric* h. We now assume that $(\mathcal{H}, \mathrm{h}, D)$ is a harmonic flat bundle on Δ^* and we consider the associated metric connection $D_\mathrm{h} = D'_\mathrm{h} + D''_\mathrm{h}$ and Higgs field $\theta = \theta' + \theta''$. We recall that the Hermitian holomorphic Higgs bundle $(\mathcal{E}, \mathrm{h}, \theta)$ is defined by $\mathcal{E} = \operatorname{Ker} D''_\mathrm{h}$ and θ is induced by θ' (see Definition 4.2.8). In other words, for a polarized variation of Hodge structure, we also pay attention to the graded bundle $\mathcal{E} = \mathrm{gr}_F \mathcal{V}$ equipped with its Higgs field induced by $\theta := \mathrm{gr}^{-1}_F \nabla$, as in (4.2.12). However, we forget the grading of this bundle and only remember that θ is nilpotent.

6.3.6. Definition **(Nilpotent harmonic bundle).** We say that the harmonic bundle is *nilpotent* if the coefficient of $\mathrm{d}t$ in θ' is a nilpotent endomorphism of $\mathcal{H}$.

6.3.7. Remarks.

(1) By Hermitian adjunction, the coefficient of $\mathrm{d}t$ in θ' is nilpotent if and only if the coefficient of $\mathrm{d}\bar{t}$ in θ'' is nilpotent.

(2) The harmonic bundle associated with a polarized variation of Hodge structure on Δ^* is nilpotent. Indeed, θ' has bidegree $(-1,1)$ with respect to the Hodge decomposition.

In this part, we give a proof of these theorems and we give some important consequences, in particular concerning semi-simplicity.

6.3.8. Remarks.

(1) In Section 6.2.a, when extending the vector bundle $\mathcal{V}$ with holomorphic connection ∇ from Δ^* to Δ, we have chosen Deligne's meromorphic extension, that is, we have chosen the (unique) meromorphic extension on which the extended connection is meromorphic and has *regular singularities.* Such a choice, while being canonical and, in some sense, as simple as possible, was not the only possible one. We could have chosen other kinds of meromorphic extensions, on which the extended meromorphic connection has irregular singularities. A posteriori, when considering variations of *polarized* Hodge structures, Theorem 6.3.2 strongly justify the previous choice.

(2) One may wonder why we have considered the filtration $\mathcal{V}_*^\bullet$ decreasing and the filtration $\mathrm{M}_\bullet\mathrm{gr}^\beta\mathcal{V}_*$ increasing. The answer is that this reflects the scale of growth of the family of functions $|t|^\beta\mathrm{L}(t)^{\ell/2}$ ($\beta\in(-1,0]$ and $\ell\in\mathbb{Z}$): the function $|t|^\beta\mathrm{L}(t)^{\ell/2}$ grows faster (or decreases slower) than $|t|^{\beta'}\mathrm{L}(t)^{\ell'/2}$ when $t\to 0$ if and only if either $\beta<\beta'$ or $\beta=\beta'$ and $\ell>\ell'$.

6.3.b. Sketch of the proof of Theorems 6.3.2 and 6.3.5 for nilpotent harmonic bundles. Let $(\mathcal{V},\nabla,\mathrm{h})$ be a nilpotent harmonic flat bundle.

Step 1. The first objective is to show that the eigenvalues of the monodromy have absolute value one (Theorem 6.3.2(2)). This point relies on the estimate of the h-norm of a multi-valued horizontal section of $\mathcal{V}$ which is an eigenvector for the monodromy operator. Due to Exercise 4.6, the h-norm of any multi-valued horizontal section v satisfies

$$\partial_t\|v\|_{\mathrm{h}}^2=-2\mathrm{h}(\theta_0'v,\overline{v}),\quad \partial_{\overline{t}}\|v\|_{\mathrm{h}}^2=-2\mathrm{h}(\theta_0''v,\overline{v}),$$

where we have set $\theta'=\theta_0'\mathrm{d}t$ and $\theta''=\theta_0''\mathrm{d}\overline{t}$. Making use of the norm of the Higgs field computed with the metric on the bundle of endomorphisms of $\mathcal{E}$, we deduce

$$\big|\partial_t\|v\|_{\mathrm{h}}\big|\leqslant 2\|v\|_{\mathrm{h}}\|\theta_0'\|_{\mathrm{h}}^{1/2},\quad \big|\partial_{\overline{t}}\|v\|_{\mathrm{h}}\big|\leqslant 2\|v\|_{\mathrm{h}}\|\theta_0''\|_{\mathrm{h}}^{1/2}=2\|v\|_{\mathrm{h}}\|\theta_0'\|_{\mathrm{h}}^{1/2},$$

where the latter equality follows from the fact that θ_0'' is the h-adjoint of θ_0'. The main tool for the proof is then an estimate for the norm of the Higgs field, proved in Section 6.3.d.

6.3.9. Theorem **(Simpson's estimate).** *If $(\mathcal{H},D,\mathrm{h})$ is a nilpotent harmonic bundle, the Higgs field $\theta'=\theta_0'\mathrm{d}t$ satisfies $\|\theta_0'\|_{\mathrm{h}}\leqslant C/|t|\mathrm{L}(t)$ on Δ^*, for some $C>0$.*

6.3.10. Remark. By choosing a volume form vol on Δ^*, giving rise to a norm on differential forms, one can consider the norm $\|\theta'\|_{\mathrm{h},\mathrm{vol}}$. In the Poincaré metric that we will consider in Section 6.12.c, the norm of $\mathrm{d}t/t$ and $\mathrm{d}\overline{t}/\overline{t}$ is $\mathrm{L}(t)$. The theorem thus asserts that the norm $\|\theta'\|_{\mathrm{h,vol}}$ (and that of $\|\theta''\|_{\mathrm{h,vol}}$ since θ'' is the h-adjoint of θ') is bounded.

This estimate leads to the following:

$$\bigl|t\partial_t \log\|v\|_{\mathrm{h}}\bigr| \leqslant C'/\mathrm{L}(t), \quad \bigl|\bar t\partial_{\bar t} \log\|v\|_{\mathrm{h}}\bigr| \leqslant C'/\mathrm{L}(t).$$

If v is an eigenvector of the monodromy operator T corresponding to the eigenvalue λ, then $\log\|\mathrm{T}v\|_{\mathrm{h}} - \log\|v\|_{\mathrm{h}} = \log|\lambda|$. Expressing this difference by an integral formula in the universal covering of Δ^* involving the above partial derivatives of $\log\|v\|_{\mathrm{h}}$ one finds

$$\bigl|\log|\lambda|\bigr| \leqslant C''/\mathrm{L}(t)$$

for a suitable constant $C'' > 0$. Since the right-hand side tends to zero when t tends to 0, this implies $\log|\lambda| = 0$, that is, $|\lambda| = 1$.

Step 2. The next step (Section 6.3.c) is, starting with the only data of $(\mathcal{V}, \nabla)$ without any other assumption, to construct a model harmonic metric, that we call the Deligne harmonic model, and to show that, if we moreover assume that the eigenvalues of the monodromy have absolute value equal to one, this model satisfies the conclusions of Theorems 6.3.2(1) and 6.3.5 (the conclusion of Theorem 6.3.2(2) being part of the assumption).

Step 3. If $(\mathcal{V}, \nabla)$ satisfying 6.3.2(2) is equipped with two comparable harmonic metrics, and if it satisfies the conclusions of Theorems 6.3.2(1) and 6.3.5 for one of both, it does so for the other one. These theorems are thus a consequence of the following.

6.3.11. Theorem. *Let $(\mathcal{V}, \nabla, \mathrm{h})$ be a nilpotent harmonic flat bundle. Then the metrics* h *and* $\mathrm{h}^{\mathrm{Del}}$ *are mutually bounded, that is, there exist constants $C_1, C_2 > 0$ such that, on Δ^*,*

$$C_1|\mathrm{h}^{\mathrm{Del}}(\bullet, \bar\bullet)| \leqslant |\mathrm{h}(\bullet, \bar\bullet)| \leqslant C_2|\mathrm{h}^{\mathrm{Del}}(\bullet, \bar\bullet)|.$$

Proof. The filtration $\mathcal{V}_*^\bullet$ is the parabolic filtration both for h and $\mathrm{h}^{\mathrm{Del}}$. The identity morphism $(\mathcal{H}, D, \mathrm{h}) \to (\mathcal{H}, D, \mathrm{h}^{\mathrm{Del}})$ or vice versa, regarded as a flat section of $\mathcal{H}om(\mathcal{H}, \mathcal{H})$ satisfies thus the metric assumption of Lemma 6.3.12 below. Recall that $\mathcal{H}om(\mathcal{H}, \mathcal{H})$, equipped with its natural metric and flat connection, is harmonic (Exercise 4.8). By Lemma 6.3.12 below, the identity morphism, in both directions, is bounded, which is equivalent to the mutual boundedness of h and $\mathrm{h}^{\mathrm{Del}}$. ☐

6.3.12. Lemma. *Let $(\mathcal{H}, D, \mathrm{h})$ be a flat bundle with metric on Δ^*. Assume that $(\mathcal{H}, D, \mathrm{h})$ is harmonic. Let $u \in \Gamma(\Delta^*, \mathcal{H})$ be a D-flat section of $\mathcal{H}$ such that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying $\|u(t)\|_{\mathrm{h}} \leqslant C_\varepsilon|t|^{-\varepsilon}$ on Δ^*. Then $\|u(t)\|_{\mathrm{h}}$ is bounded near the origin.*

A proof of this lemma is given in Section 6.3.d. This concludes the proof of Theorems 6.3.2 and 6.3.5 for nilpotent harmonic bundles. ☐

6.3.c. The Deligne harmonic model. We will now construct a model of such a vector bundle, starting from an $\mathfrak{sl}_2$-representation with a positive definite Hermitian form, and we will check whether Theorem 6.3.2 holds on such a model. We will link Property 6.3.2(2) with the nilpotency of the Higgs field of the model. This model only relies on the datum of the flat vector bundle $(\mathcal{V}, \nabla)$ on Δ^* and is built so that the Deligne canonical filtration $\mathcal{V}_*^\bullet$ is equal to the parabolic filtration of the harmonic metric. This is why we call it the Deligne harmonic model.

Let $\mathcal{H}^o$ be a $\mathbb{C}$-vector space of dimension d equipped with an $\mathfrak{sl}_2$-representation, hence of endomorphisms $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ (see Section 3.1.a). Since we do not deal with Hodge filtrations for the moment, we do not introduce a polarization S and only consider the resulting positive definite Hermitian form h^o. In order to prepare compatibility with the notion of polarization, we impose that

$$\mathrm{h}^o(\mathrm{X}\bullet, \overline{\bullet}) = \mathrm{h}^o(\bullet, \overline{\mathrm{Y}\bullet}), \quad \mathrm{h}^o(\mathrm{Y}\bullet, \overline{\bullet}) = \mathrm{h}^o(\bullet, \overline{\mathrm{X}\bullet}), \quad \mathrm{h}^o(\mathrm{H}\bullet, \overline{\bullet}) = \mathrm{h}^o(\bullet, \overline{\mathrm{H}\bullet}), \tag{6.3.13}$$

as suggested by Remark 3.2.8(2). Let us fix an h^o-orthonormal basis $\boldsymbol{v}^o = (v_1^o, \dots, v_d^o)$ consisting of eigenvectors for H. If we denote by A the matrix of the endomorphism A in a given basis, this identification leads to the notation

$$(\mathrm{X}v_1^o, \dots, \mathrm{X}v_d^o) = (v_1^o, \dots, v_d^o) \cdot \mathsf{X}.$$

Similarly, for a Hermitian form $\mathfrak{s}^o$ on $\mathcal{H}^o$, we also denote by s^o the matrix (s_{ij}^o) defined by $\mathsf{s}_{ij}^o = \mathfrak{s}^o(v_i^o, \overline{v_j^o})$.

6.3.14. Simple example. We suggest the reader to follow the next computations on the simple example where Y consists of a single Jordan block of size $\ell + 1$, so that $\dim \mathrm{P}_\ell = 1$ and $\mathrm{P}_{\ell'} = 0$ if $\ell' \neq 0$. Then each $\boldsymbol{v}_{\ell,j}^o$ consists of a single element $v_{\ell,j}$.

It will be convenient to assume that the basis $\boldsymbol{v}^o$ is obtained as follows: for each $\ell \geqslant 0$, let us fix an h^o-orthonormal basis $\boldsymbol{v}_{\ell,0}^o$ of the ℓ-th primitive part $\mathrm{P}_\ell \mathcal{H}^o \subset \operatorname{Ker} \mathrm{X}$ made with eigenvectors of H (with eigenvalue ℓ); for any $j \geqslant 0$, consider the basis of $\mathrm{Y}^j \mathrm{P}_\ell \mathcal{H}^o$

$$\boldsymbol{v}_{\ell,j}^o = \star_{\ell,j} \boldsymbol{v}_{\ell,0}^o \, \mathsf{Y}^j, \tag{6.3.15}$$

where $\star_{\ell,j}$ is some constant. In such a way, $\boldsymbol{v}_{\ell,j}^o$ is a basis of the Lefschetz component $\mathrm{Y}^j \mathrm{P}_\ell$, the basis $\boldsymbol{v}^o := (\boldsymbol{v}_{\ell,j}^o)_{\ell,j}$ is h^o-orthogonal, and one can (and will) choose $\star_{\ell,j}$ (with $\star_{\ell,0} = 1$) such that this basis is h^o-orthonormal. The formula of Exercise 3.1(2) shows that these constants are positive. Then the matrix H of H in this basis is diagonal with integral entries, while X (resp. Y) is block upper (resp. lower) triangular whose entries are positive or zero, X being the transpose of Y.

6.3.16. Definition **(The model bundle with connection).** Let $\mathcal{H} = \mathcal{C}_{\Delta^*}^\infty \otimes_{\mathbb{C}} \mathcal{H}^o$ be the trivial C^∞-bundle on Δ^* and let $\boldsymbol{v}$ be the basis $\boldsymbol{v} = 1 \otimes \boldsymbol{v}^o$. Let b be a complex number, that we write $b = b' + \mathrm{i}\, b''$ $(b', b'' \in \mathbb{R})$ and a real number $\beta \in (-1, 0]$. We endow $\mathcal{H}$ with the connection D such that

$$D'' \boldsymbol{v} = 0, \quad D' \boldsymbol{v} = \boldsymbol{v} \cdot (b \operatorname{Id} - \mathsf{Y}) \frac{\mathrm{d}t}{t}, \tag{6.3.16 $*$}$$

so that $\mathcal{V} := \operatorname{Ker} D''$ is the holomorphic trivial bundle $\mathcal{O}_{\Delta^*} \cdot \boldsymbol{v}$ and the connection ∇ on $\mathcal{V}$ induced by D' has matrix $(b\,\mathrm{Id} - \mathsf{Y})\mathrm{d}t/t$.

Let $\boldsymbol{\varepsilon}$ be the basis obtained from $\boldsymbol{v}$ by the change of basis of having inverse matrix

$$\mathsf{P}_\beta(t) = \mathrm{e}^{\mathsf{X}}|t|^\beta \mathrm{L}(t)^{\mathsf{H}/2} = |t|^\beta \mathrm{L}(t)^{\mathsf{H}/2}\mathrm{e}^{\mathsf{X}/\mathrm{L}(t)} \quad \text{(see Exercise 6.5)}, \tag{6.3.17}$$

that is,

$$\boldsymbol{v} = \boldsymbol{\varepsilon} \cdot \mathsf{P}_\beta(t). \tag{6.3.18}$$

The bases $\boldsymbol{\varepsilon}$ and $\boldsymbol{v}$ are decomposed as $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_{\ell,j})_{\ell,j}$ and $\boldsymbol{v} = (\boldsymbol{v}_{\ell,j})_{\ell,j}$, so that (6.3.18) reads

$$\boldsymbol{v}_{\ell,j} = |t|^\beta \sum_{k\geqslant 0} c_{\ell,j,k}\boldsymbol{\varepsilon}_{\ell,j+k}\mathrm{L}(t)^{\mathsf{H}/2} = |t|^\beta \mathrm{L}(t)^{\ell/2-j}\sum_{k\geqslant 0} c_{\ell,j,k}\mathrm{L}(t)^{-k}\boldsymbol{\varepsilon}_{\ell,j+k}, \tag{6.3.19}$$

for some nonnegative numbers $c_{\ell,j,k}$ with $c_{\ell,j,0} = 1$.

6.3.20. Definition **(The model metric on the model bundle with connection)**
We equip $\mathcal{H}$ with the Hermitian metric h such that $\boldsymbol{\varepsilon}$ is an orthonormal basis.

We now group the terms $\boldsymbol{v}_{\ell,j}, \boldsymbol{\varepsilon}_{\ell,j}$ corresponding to the same $w = \ell - 2j$ and we set

$$\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_w)_{w\in\mathbb{Z}}, \quad \boldsymbol{v} = (\boldsymbol{v}_w)_{w\in\mathbb{Z}} \quad \text{with } v \in \boldsymbol{v}_w \iff \|v\|_{\mathrm{h}} \underset{t\to 0}{\sim} |t|^\beta \mathrm{L}(t)^{w/2}. \tag{6.3.21}$$

Moreover, the basis $\boldsymbol{v}$ is *asymptotically* h*-orthogonal*, with logarithmic decay.

The metric h and the connection D on $\mathcal{H}$ enable us to define operators D'_{h}, D''_{h}, θ' and θ'' (see Lemma 4.2.2).

6.3.22. Proposition. *With the previous assumptions, the metric* h *on* $(\mathcal{H}, D)$ *is harmonic.*

Proof. Let us write

$$D'\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \mathsf{M}' \,\frac{\mathrm{d}t}{t}, \quad D''\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \mathsf{M}'' \,\frac{\mathrm{d}\bar t}{\bar t}.$$

Applying the base change formula for connections, we find

$$\mathsf{M}' = b\,\mathrm{Id} - \mathsf{P}_\beta\,\mathsf{Y}(\mathsf{P}_\beta)^{-1} + \mathsf{P}_\beta\, t\partial_t(\mathsf{P}_\beta)^{-1} \quad \text{and} \quad \mathsf{M}'' = \mathsf{P}_\beta\, \bar t\partial_{\bar t}(\mathsf{P}_\beta)^{-1}.$$

According to the identities of Exercise 6.5 we obtain

$$\begin{aligned} \mathsf{M}' &= \Big(b - \frac{\beta}{2}\Big)\,\mathrm{Id} - \frac{\mathsf{Y} + \mathsf{H}/2}{\mathrm{L}(t)} \\ \mathsf{M}'' &= -\frac{\beta}{2}\,\mathrm{Id} + \frac{\mathsf{H}/2 - \mathsf{X}}{\mathrm{L}(t)} \end{aligned} \tag{6.3.23}$$

$$\begin{aligned} \theta' &= \frac{1}{2}(\mathsf{M}' + \mathsf{M}''^*)\frac{\mathrm{d}t}{t} = \Big(\frac{(b-\beta)}{2}\,\mathrm{Id} - \frac{\mathsf{Y}}{\mathrm{L}(t)}\Big)\frac{\mathrm{d}t}{t} \\ \theta'' &= \frac{1}{2}(\mathsf{M}'^* + \mathsf{M}'')\frac{\mathrm{d}\bar t}{\bar t} = \Big(\frac{(\bar b-\beta)}{2}\,\mathrm{Id} - \frac{\mathsf{X}}{\mathrm{L}(t)}\Big)\frac{\mathrm{d}\bar t}{\bar t} \end{aligned} \tag{6.3.24}$$

and

$$D''_{\mathrm{h}}\boldsymbol{\varepsilon} = (D'' - \theta'')\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \Big(-\frac{\bar b}{2}\,\mathrm{Id} + \frac{\mathsf{H}/2}{\mathrm{L}(t)}\Big)\frac{\mathrm{d}\bar t}{\bar t}.$$

We need to prove that the matrix of θ' is holomorphic when expressed in a D''_{h}-holomorphic basis of $\mathcal{H}$. We note that, for any complex number c, the diagonal matrix

$$\mathsf{A}_c(t) = |t|^c \operatorname{L}(t)^{\mathsf{H}/2} \tag{6.3.25}$$

satisfies, according to Exercise 6.5,

$$t\partial_t \mathsf{A}_c(t) = \bar t\partial_{\bar t} \mathsf{A}_c(t) = \Big(\frac{c}{2}\operatorname{Id} - \frac{\mathsf{H}/2}{\operatorname{L}(t)}\Big).$$

Therefore, after the base change with matrix

$$\mathsf{A}_{\bar b}(t) := |t|^{\bar b} \operatorname{L}(t)^{\mathsf{H}/2}, \tag{6.3.26}$$

the basis $\boldsymbol{e} = \boldsymbol{\varepsilon}\cdot \mathsf{A}_{\bar b}(t)$ is D''_{h}-holomorphic: indeed, the coefficient of $\mathrm{d}\bar t/\bar t$ in the matrix of D''_{h} with respect to the basis $\boldsymbol{e}$ is

$$(\mathsf{A}_{\bar b})^{-1}\frac{\mathsf{H}/2 - \bar b \operatorname{Id}}{\operatorname{L}(t)}\mathsf{A}_{\bar b} + (\mathsf{A}_{\bar b})^{-1}\bar t\partial_{\bar t}\mathsf{A}_{\bar b} = 0,$$

hence the assertion. Let us notice that $\boldsymbol{e}$ decomposes as $(\boldsymbol{e}_w)_{w\in\mathbb{Z}}$ according to the decomposition $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_w)_{w\in\mathbb{Z}}$ analogous to (6.3.21), and each element of $\boldsymbol{e}_w$ has norm $|t|^{b'}\operatorname{L}(t)^{w/2}$. Moreover, $\boldsymbol{e}$ is h-orthogonal.

The coefficient of $\mathrm{d}t/t$ in the matrix of θ' in the basis $\boldsymbol{e}$ is therefore

$$(\mathsf{A}_{\bar b})^{-1}\Big(\frac{(b-\beta)}{2}\operatorname{Id} - \frac{\mathsf{Y}}{\operatorname{L}(t)}\Big)\mathsf{A}_{\bar b} = \frac{(b-\beta)}{2}\operatorname{Id} - \mathsf{Y}, \tag{6.3.27}$$

according to Exercise 6.5. □

Proof of Theorem* 6.3.2 *for the model. The norm of a holomorphic section of $\mathcal{V}$ is easily computed with its coefficients in the orthonormal basis $\boldsymbol{\varepsilon}$. Since the entries of the matrices P_β and $(\mathsf{P}_\beta)^{-1}$ defined by (6.3.17) have moderate growth, this norm is moderate if and only if the coefficients of the section in the holomorphic basis $\boldsymbol{v}$ have moderate growth on Δ^*, i.e., if and only if they are meromorphic functions. Therefore, $\mathcal{V}_*$ is determined by the moderate growth condition on the norm of holomorphic sections.

In order to determine the parabolic filtration, we need to compute the norm of the elements v of $\boldsymbol{v}$. This norm is computed by (6.3.21). The parabolic filtration $\mathcal{V}^\bullet_{\mathrm{mod}}$ is thus given by $\mathcal{V}^{\beta+k}_{\mathrm{mod}} = t^k\mathcal{O}_\Delta\cdot\boldsymbol{v}$ (the jumps occur only at $\beta+\mathbb{Z}$).

On the other hand, the filtration by Deligne canonical lattices $\mathcal{V}^\bullet_*$ is given by $\mathcal{V}^{b'+k}_* = t^k\mathcal{O}_\Delta\cdot\boldsymbol{v}$ (the jumps occur only at $b'+\mathbb{Z}$).

If $b'=\beta$, then both filtrations coincide, and (6.3.21) also shows that $\mathrm{M}_w\mathcal{V}^\beta_*$ is determined by the norm condition of Theorem 6.3.5.

We notice that the model is a *nilpotent* harmonic bundle if and only if $b=\beta$. The nilpotency condition thus implies both properties in Theorem 6.3.2 as well as the conclusion of Theorem 6.3.5. □

Let $(\mathcal{V},\nabla)$ be a holomorphic bundle with connection. Tensoring the decomposition of (6.2.5 $**$) with $\mathcal{C}^\infty_{\Delta^*}$ leads to a decomposition

$$(\mathcal{H}, D) \simeq \bigoplus_{\beta\in(-1,0]}(\mathcal{H}_\beta, D), \tag{6.3.28}$$

and $\boldsymbol{v}_\beta$ is a holomorphic frame of $(\mathcal{H}_\beta, D)$ on Δ^*. Besides, the C^∞ bundle $\mathcal{H}_\beta$ is in fact defined all over Δ, since the frame $\boldsymbol{v}_\beta$ is so, and its fiber $\mathcal{H}^o_\beta$ at the origin is isomorphic to $\mathrm{gr}^\beta \mathcal{V}_*$ via an identification of bases. For every $b'' \in \mathbb{R}$ such that $\exp(2\pi b'')$ is an eigenvalue of T_β in Theorem 6.2.4, we associate a model metric $\mathrm{h}^{\mathrm{Del}}_b$ as in Section 6.3.c with $b = \beta + \mathrm{i}\, b''$ and the nilpotent endomorphism N given by that theorem. Summing over all such possible $b'' \in \mathbb{R}$, we obtain a model metric $\mathrm{h}^{\mathrm{Del}}_\beta$ on each $(\mathcal{H}_\beta, D)$, and summing over all $\beta \in (-1, 0]$, we obtain a model metric $\mathrm{h}^{\mathrm{Del}}$ in such a way that the decomposition (6.3.28) is $\mathrm{h}^{\mathrm{Del}}$-orthogonal and that the restriction of $\mathrm{h}^{\mathrm{Del}}$ to each $\mathcal{H}_\beta$ is the model metric $\mathrm{h}^{\mathrm{Del}}_\beta$. Since $b' = \beta$, the parabolic filtration of $(\mathcal{V}, \mathrm{h}^{\mathrm{Del}})$ is the canonical filtration $\mathcal{V}_*$. If we consider the Higgs bundle $(\mathcal{E}^{\mathrm{Del}}, \mathrm{h}^{\mathrm{Del}}, \theta^{\mathrm{Del}})$, the parabolic filtration is such that the frame $(\boldsymbol{e}_\beta)_{\beta \in (-1,0]}$ forms an adapted basis of $\mathcal{E}^{\mathrm{Del}}$.

6.3.29. Definition. We call $(\mathcal{V}, \mathrm{h}^{\mathrm{Del}}, \nabla)$ the *Deligne harmonic model* for $(\mathcal{V}, \nabla)$.

6.3.30. Remark. For the Deligne harmonic model, the statement of Theorem 6.3.2(2) is equivalent to the property that it is nilpotent.

6.3.d. Proof of Theorem 6.3.9 and Lemma 6.3.12.

We continue assuming that $(\mathcal{V}, \nabla, \mathrm{h})$ is a harmonic flat bundle on Δ^*. Let us start with a corollary of Theorem 6.3.9 that will be used when proving semi-simplicity in Section 6.4.

***6.3.31. Corollary* (Curvature properties).** *The curvature $R_\mathcal{V}$ of $(\mathcal{V}, \mathrm{h})$ and the curvature $R_\mathcal{E}$ of $(\mathcal{E}, \mathrm{h})$ satisfy an inequality*

$$\|R\|_{\mathrm{h}} \leqslant C/|t|^2 \mathrm{L}(t)^2 \quad \textit{for some } C > 0, \tag{6.3.31 $*$}$$

in particular they are L^1_{loc} on Δ.

6.3.32. Remark. This corollary follows from Simpson's estimate (Theorem 6.3.9) and can be combined with the criterion of moderateness provided by Theorem 6.2.9 to yield moderateness of the metric h. However, we do not make use of this criterion here, as moderateness follows from the identification of the parabolic filtration $\mathcal{V}^\bullet_{\mathrm{mod}}$ with the Deligne canonical filtration $\mathcal{V}^\bullet_*$, as follows from Theorem 6.3.11 and the properties of the Deligne harmonic model.

Proof. Let us emphasize that $R_\mathcal{V}$ is the curvature of the Chern connection of h on $\mathcal{H}$ with the holomorphic structure D'', while $R_\mathcal{E}$ is that of h on $\mathcal{H}$ with the holomorphic structure D''_{h}. The formula of Exercise 4.4(5) and the identities following (4.2.6) give

$$R_\mathcal{V} = -2(\theta' \wedge \theta'' + \theta'' \wedge \theta') = 2R_\mathcal{E}.$$

Since θ'' has the same h-norm as θ', it follows that both $R_\mathcal{V}$ and $R_\mathcal{E}$ satisfy (6.3.31 $*$), hence are L^1_{loc} on Δ, according to Exercise 6.6. □

***Proof of Theorem* 6.3.9.** We start with a variant of Ahlfors Lemma, whose proof is given as an exercise (Exercise 6.8). We denote by Δ_t the Euclidean Laplacian on the disc, that is, $\Delta_t = 4\partial_t \partial_{\bar{t}}$.

6.3.33. Lemma. *Let f be a C^2 function with nonnegative real values on the unit punctured disc Δ^*. Let us assume that the following inequality holds:*

$$\Delta_t \log f(t) \geqslant 4f(t). \tag{6.3.33 $*$}$$

Then

$$f(t) \leqslant \frac{1}{|t|^2 \mathrm{L}(t)^2} \quad \text{on } \Delta^*. \tag{6.3.33 $**$}$$

We thus aim at proving that $f = c\|\theta'\|_{\mathrm{h}}^2$ (for some $c > 0$) satisfies the assumption of the lemma. Let us set $\theta' = \theta'_0 \mathrm{d}t$ and $\theta'' = \theta''_0 \mathrm{d}\bar{t}$. Regarding θ'_0 as generating a line subbundle of $\mathcal{E}nd(\mathcal{E})$ with induced metric h, so that $\|\theta'\|_{\mathrm{h}}^2 = 2\|\theta'_0\|_{\mathrm{h}}$, the inequality for the curvature of a subbundle (see **[GH78**, p. 79]) implies

$$\|\theta'_0\|_{\mathrm{h}}^2 \cdot \mathrm{d}''\mathrm{d}' \log \|\theta'_0\|_{\mathrm{h}}^2 \leqslant \mathrm{h}\big(\mathrm{ad}(R_{\mathcal{E}})(\theta'_0), \overline{\theta'_0}\big),$$

in the sense that the coefficients of $\mathrm{d}t \wedge \mathrm{d}\bar{t}$ satisfy the corresponding inequality. The above expression of $R_{\mathcal{E}}$ amounts to $R_{\mathcal{E}} = -[\theta'_0, \theta''_0]\mathrm{d}t \wedge \mathrm{d}\bar{t}$ and the previous inequality reads

$$-\partial_t \partial_{\bar{t}} \log \|\theta'_0\|_{\mathrm{h}}^2 \leqslant -\frac{\mathrm{h}([[\theta'_0, \theta''_0], \theta'_0], \overline{\theta'_0})}{\|\theta'_0\|_{\mathrm{h}}^2}.$$

We write

$$\begin{aligned} -\mathrm{h}([[\theta'_0, \theta''_0], \theta'_0], \theta'_0) &= \mathrm{h}(\mathrm{ad}(\theta'_0)([\theta'_0, \theta''_0]), \overline{\theta'_0}) \\ &= \mathrm{h}([\theta'_0, \theta''_0], \overline{\mathrm{ad}(\theta''_0)(\theta'_0)}) = -\big\|[\theta'_0, \theta''_0]\big\|_{\mathrm{h}}^2, \end{aligned}$$

and the previous inequality reads

$$\Delta_t \log \|\theta'_0\|_{\mathrm{h}}^2 \geqslant 4\, \frac{\big\|[\theta'_0, \theta''_0]\big\|_{\mathrm{h}}^2}{\|\theta'_0\|_{\mathrm{h}}^2}.$$

Here comes the assumption that θ'_0 is nilpotent. We claim that there exists a constant $c > 0$ only depending on the rank of $\mathcal{E}$ such that $\big\|[\theta'_0, \theta''_0]\big\|_{\mathrm{h}} \geqslant c\|\theta'_0\|_{\mathrm{h}}^2$. Indeed, because we look for a universal constant c, it is enough to solve the question independently on each fiber, and we are reduced to a question on vector spaces, which is treated in see Exercise 6.12. As a consequence,

$$\Delta_t \log \|\theta'_0\|_{\mathrm{h}}^2 \geqslant 4c^2 \|\theta'_0\|_{\mathrm{h}}^2.$$

We conclude the proof of Theorem 6.3.9 by applying Lemma 6.3.33 to $f(t) = c^2\|\theta'_0\|_{\mathrm{h}}^2$. □

***Proof of Lemma* 6.3.12.** This lemma is a direct consequence of the following lemma, together with Exercise 6.7.

6.3.34. Lemma. *For u as in Lemma* 6.3.12*, the function* $\log \|u\|_{\mathrm{h}}^2$ *is subharmonic in* Δ^**, that is, we have the inequality*

$$\Delta_t \log \|u\|_{\mathrm{h}}^2 \geqslant 0.$$

Proof. Let us start by computing $\Delta_t\|u\|^2_{\mathrm{h}}$. On the one hand, we have

$$(\Delta_t\|u\|^2_{\mathrm{h}})\,\mathrm{d}t\wedge\mathrm{d}\bar t = 4\mathrm{d}'\mathrm{d}''\|u\|^2_{\mathrm{h}}.$$

On the other hand, u satisfies $D'u=0$ and $D''u=0$, that is, $D'_{\mathrm{h}}u=-\theta'u$ and $D''_{\mathrm{h}}u=-\theta''u$ (recall the notation of Section 4.2.b). Moreover, since $D''_{\mathrm{h}}(\theta')=0$ (see (4.2.6)), we find

$$D''_{\mathrm{h}}\theta'u = -\theta' D''_{\mathrm{h}}u = \theta'\theta''u,$$

and similarly $D'_{\mathrm{h}}\theta''u=\theta''\theta'u$. We thus obtain, since θ'' is the h-adjoint of θ' (we use the convention of Remark 4.2.3),

$$\begin{aligned}\mathrm{d}'\mathrm{d}''\|u\|^2_{\mathrm{h}} &= -\mathrm{d}'\big[\mathrm{h}(\theta''u,\overline{u})+\mathrm{h}(u,\overline{\theta'u})\big] = -2\mathrm{d}'\mathrm{h}(u,\overline{\theta'u})\\ &= 2\big[\mathrm{h}(\theta'u,\overline{\theta'u})-\mathrm{h}(u,\overline{\theta'\theta''u})\big]\\ &= 2\big[\mathrm{h}(\theta'u,\overline{\theta'u})-\mathrm{h}(\theta''u,\overline{\theta''u})\big].\end{aligned}$$

Since $\|\mathrm{d}t\|=\|\mathrm{d}\bar t\|=2$ with the metric induced by the Euclidean volume form, we find

$$\mathrm{h}(\theta'u,\overline{\theta'u}) = \frac14\,\|\theta'u\|^2_{\mathrm{h}}\,\mathrm{d}t\wedge\mathrm{d}\bar t \quad\text{and}\quad \mathrm{h}(\theta''u,\overline{\theta''u}) = -\frac14\,\|\theta''u\|^2_{\mathrm{h}}\,\mathrm{d}t\wedge\mathrm{d}\bar t,$$

we finally obtain

$$\Delta_t\|u\|^2_{\mathrm{h}} = 2\big(\|\theta'u\|^2_{\mathrm{h}}+\|\theta''u\|^2_{\mathrm{h}}\big). \tag{6.3.35}$$

Now,

$$\Delta_t\log\|u\|^2_{\mathrm{h}} = \frac{\Delta_t\|u\|^2_{\mathrm{h}}}{\|u\|^2_{\mathrm{h}}} - 4\,\frac{\partial_t\|u\|^2_{\mathrm{h}}}{\|u\|^2_{\mathrm{h}}}\,\frac{\partial_{\bar t}\|u\|^2_{\mathrm{h}}}{\|u\|^2_{\mathrm{h}}},$$

and $\partial_t\|u\|^2_{\mathrm{h}}\cdot\partial_{\bar t}\|u\|^2_{\mathrm{h}}$ is the coefficient of $\mathrm{d}t\wedge\mathrm{d}\bar t$ in $\mathrm{d}'\mathrm{h}(u,\overline{u})\wedge\mathrm{d}''\mathrm{h}(u,\overline{u})$. The previous arguments give

$$\begin{aligned}\mathrm{d}'\mathrm{h}(u,\overline{u})\wedge\mathrm{d}''\mathrm{h}(u,\overline{u}) &= 4\mathrm{h}(\theta'u,\overline{u})\wedge\mathrm{h}(u,\overline{\theta'u})\\ &= 4\mathrm{h}(\theta'u,\overline{u})\wedge\overline{\mathrm{h}(\theta'u,\overline{u})} = \|\mathrm{h}(\theta'u,\overline{u})\|^2\,\mathrm{d}t\wedge\mathrm{d}\bar t.\end{aligned}$$

Therefore, noticing that $\|\mathrm{h}(\theta'u,\overline{u})\|=\|\mathrm{h}(\theta''u,\overline{u})\|$,

$$\begin{aligned}\partial_t\|u\|^2_{\mathrm{h}}\cdot\partial_{\bar t}\|u\|^2_{\mathrm{h}} = \|\mathrm{h}(\theta'u,\overline{u})\|^2 &= \frac12\,(\|\mathrm{h}(\theta'u,\overline{u})\|^2+\|\mathrm{h}(\theta''u,\overline{u})\|^2)\\ &\leqslant \frac12\,\|u\|^2_{\mathrm{h}}\cdot(\|\theta'u\|^2_{\mathrm{h}}+\|\theta''u\|^2_{\mathrm{h}}),\end{aligned}$$

and the desired inequality follows. □

6.3.36. Remarks.

(a) Together with the conclusion of Lemma 6.3.12, (6.3.35) also implies that $\Delta_t\|u\|^2_{\mathrm{h}}$ and $\|\theta'u\|^2_{\mathrm{h}}+\|\theta''u\|^2_{\mathrm{h}}$ are L^1_{loc} at the origin (see Exercise 6.9).

(b) On a Riemann surface X equipped with a Kähler metric, the Laplacian Δ satisfies $\Delta=2\Delta''=-2\,\mathrm{i}\,\Lambda\mathrm{d}'\mathrm{d}''$. In the setting of Lemma 6.3.12 with a punctured X^* instead of Δ^*, then an argument similar to that leading to (6.3.35) gives

$$\Delta\|u\|^2_{\mathrm{h}} = -4\big(\|\theta'u\|^2_{\mathrm{h}}+\|\theta''u\|^2_{\mathrm{h}}\big).$$

Moreover, (a) implies that the right-hand side—hence the left-hand side also—is L^1_{loc} on X.

6.4. Semi-simplicity

As an application of the metric properties of Section 6.3, we extend in this section the results of Section 4.3 to the case of a punctured projective curve. Let X be a smooth projective curve and let X^* be a Zariski dense open subset of X (i.e., the complement of a finite set of points).

6.4.1. Theorem. *Let* $(\mathcal{H}, \mathrm{h}, D)$ *be a* nilpotent harmonic bundle *on* X^* *and let* $\underline{\mathcal{H}}$ *be the associated local system* $\operatorname{Ker} D$. *Then the complex local system* $\underline{\mathcal{H}}$ *is semi-simple.*

***6.4.2. Corollary* (of Theorem 6.4.1 and Remark 6.3.7(2)).** *Let*

$$H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}, D, \mathrm{S})$$

be a polarized variation of $\mathbb{C}$*-Hodge structure of weight* w *on* X^* *(see Definition* 4.1.4*), and let* $\underline{\mathcal{H}} = \operatorname{Ker}\nabla$ *be the associated complex local system. Then* $\underline{\mathcal{H}}$ *is semi-simple.* □

6.4.3. Remark. After having proved the Hodge-Zucker theorem 6.11.1, we will complement this corollary in a way analogous to that of Theorem 4.3.13, by showing (Theorem 6.14.17) that each irreducible component of $\underline{\mathcal{H}}$ underlies an essentially unique polarized variation of Hodge structure and we will show how to recover the polarized variation of Hodge structure H from its irreducible components. In other words, the underlying local system of a simple object in the category of polarized variations of Hodge structure on X^* is irreducible.

Before starting the proof of the semi-simplicity theorem, we notice useful consequences of Property (1) of Theorem 6.3.2.

6.4.4. Proposition. *Assume that* $(\mathcal{V}, \mathrm{h}, \nabla)$ *satisfies* 6.3.2(1). *Then*

(1) *any flat holomorphic subbundle with induced metric and connection* $(\mathcal{V}_1, \mathrm{h}, \nabla)$ *also satisfies* 6.3.2(1);

(2) *the determinant* $\det(\mathcal{V}, \mathrm{h}, \nabla)$ *also satisfies* 6.3.2(1).

Proof.

(1) In view of Remark 6.2.7(1), the question reduces to the $\mathcal{O}_\Delta$-coherence of $j_*\mathcal{V}_1 \cap \mathcal{V}^\beta_{\mathrm{mod}}$. But the latter is equal to $j_*\mathcal{V}_1 \cap \mathcal{V}^\beta_*$, which $\mathcal{O}_\Delta$-locally free, being equal to $\mathcal{V}^\beta_{1,*}$ (see Remark 6.2.3(1)).

(2) This point follows from Remarks 6.2.3(3) and 6.2.7(2). □

Let $(\mathcal{V}, \nabla, \mathrm{h})$ be a Hermitian holomorphic bundle with connection on a punctured compact Riemann surface X^*. Let R_h denote the curvature of $(\mathcal{V}, \mathrm{h})$: it is a section of $\mathcal{E}^{1,1}_{X^*} \otimes \mathcal{E}nd(\mathcal{V})$. The determinant bundle $\det(\mathcal{V}, \mathrm{h}, \nabla)$ has curvature $\operatorname{tr}(R_\mathrm{h})$. Assume that the curvature $\operatorname{tr}(R_\mathrm{h})$ of $\det(\mathcal{V}, \mathrm{h})$ is L^1_{loc} on X, i.e., in local coordinates near a puncture, it can be written as $k\,\mathrm{d}t \wedge \mathrm{d}\bar{t}$ with k being L^1_{loc}. We then set

$$\deg^{\mathrm{an}}(\mathcal{V}, \mathrm{h}) = \frac{\mathrm{i}}{2\pi} \int_X \operatorname{tr}(R_\mathrm{h}).$$

***6.4.5. Proposition* (Vanishing of the analytic degree).** *Assume that* $(\mathcal{V}, \mathrm{h}, \nabla)$ *is a Hermitian holomorphic bundle with connection on a punctured compact Riemann surface* X^* *that satisfies* 6.3.2(1) *as well as its dual* $(\mathcal{V}^\vee, \mathrm{h}, \nabla)$. *Assume that the curvature* $\mathrm{tr}(R_{\mathrm{h}})$ *of* $\det(\mathcal{V}, \mathrm{h})$ *is* L^1_{loc} *on* X. *Then for any flat holomorphic subbundle* $(\mathcal{V}_1, \nabla)$ *of* $(\mathcal{V}, \nabla)$ *equipped with the induced Hermitian metric, the curvature of* $\det(\mathcal{V}_1, \mathrm{h})$ *is* L^1_{loc} *on* X *and we have*

$$\deg^{\mathrm{an}}(\mathcal{V}_1, \mathrm{h}) = 0.$$

6.4.6. Lemma. *If* $(\mathcal{V}, \mathrm{h}, \nabla)$ *and its dual* $(\mathcal{V}^\vee, \mathrm{h}, \nabla)$ *satisfy* 6.3.2(1) *on* Δ^*, *then the* h*-norm of any local section* v *of* $\mathcal{V}^\beta_*$ *whose image in* $\mathrm{gr}^\beta \mathcal{V}_*$ *is nonzero satisfies the inequalities*

$$(6.4.6\,*) \qquad \forall\, \varepsilon > 0, \quad |t|^{\beta+\varepsilon} \leqslant \|v\|_{\mathrm{h}} \leqslant |t|^{\beta-\varepsilon} \quad (|t| < R_\varepsilon).$$

Proof. The right inequality is by assumption. Let $v^\vee$ be a local section of $(\mathcal{V}^\vee_*)^{-\beta}$ such that $\langle v^\vee, v\rangle = 1$ (see Remark 6.2.3(1)). Then $\|v^\vee\|_{\mathrm{h}} \leqslant |t|^{-\beta-\varepsilon}$ for all $\varepsilon > 0$ and $|t|$ correspondingly small enough, by assumption. By computing in an orthonormal frame, Schwartz inequality implies $\|v^\vee\|_{\mathrm{h}} \|v\|_{\mathrm{h}} \geqslant |\langle v^\vee, v\rangle| = 1$. Therefore, $\|v\|_{\mathrm{h}} \geqslant |t|^{\beta+\varepsilon}$, hence the assertion. ☐

***Proof of Proposition* 6.4.5.** Let $x \in X$ be a puncture and let $\gamma_x \in [0, 1)$ be the unique jumping index of the canonical filtration of $\mathcal{L}_* := \det \mathcal{V}_{1,*}$. For a local frame v at x of the Deligne canonical extension $\mathcal{L}^0_*$ obtained from a frame adapted to the filtration of $\mathcal{V}_{1,*}$, we deduce that $\|v\|_{\mathrm{h}}$ satisfies the inequalities of the lemma with γ_x instead of β (this is justified by Proposition 6.4.4(2)).

Let us prove the statement on the curvature of $(\mathcal{V}_1, \mathrm{h})$. This is a local statement near each puncture, so that we assume that $X^* = \Delta^*$. Let $\boldsymbol{v}$ be a frame of $\mathcal{V}$ inducing a frame of $\mathcal{V}^0_*/t\mathcal{V}^0_*$ adapted to the filtration induced by $\mathcal{V}^\bullet_*$ and to $\mathcal{V}_{1,*}$, so that part of $\boldsymbol{v}$, denoted $\boldsymbol{v}_1$, is a frame of $\mathcal{V}^0_{1,*}/t\mathcal{V}^0_{1,*}$ adapted to the filtration induced by $\mathcal{V}^\bullet_{1,*}$. Then the curvature matrix of $(\mathcal{V}_1, \mathrm{h})$ in the frame $\boldsymbol{v}_1$ is smaller than that of $(\mathcal{V}, \mathrm{h})$ in the frame $\boldsymbol{v}$, in the sense of [**GH78**, p. 79]. Taking traces, the same property holds for $\det \mathcal{V}_1$ with its frame $v_1 = \det \boldsymbol{v}_1$ and $\det \mathcal{V}$ with its frame $v = \det \boldsymbol{v}$. This reads

$$-\partial_t \partial_{\bar{t}} \log \|v_1\|^2_{\mathrm{h}} \leqslant -\partial_t \partial_{\bar{t}} \log \|v\|^2_{\mathrm{h}},$$

that is, with the Laplacian $\Delta_t = 4\partial_t\partial_{\bar{t}}$,

$$\Delta_t \log \|v_1\|^2_{\mathrm{h}} \geqslant \Delta_t \log \|v\|^2_{\mathrm{h}}.$$

We have $\Delta_t \log \|v_1\|^2_{\mathrm{h}} = \Delta_t f_1$ on Δ^* with $f_1 = \log \|v_1\|^2_{\mathrm{h}} - \log |t|^{2\gamma_1}$, where γ_1 is the exponent of (6.4.6 $*$) for v_1, and similarly f and γ for v. By assumption, f is L^1_{loc} on Δ. On the other hand, (6.4.6 $*$) for v_1 implies that for all $\varepsilon > 0$, $|f_1(t)| \leqslant \varepsilon \mathrm{L}(t)$ on $\Delta^*_{R_\varepsilon}(x)$ for $R_\varepsilon > 0$ small enough. Therefore, $\lim_{t\to 0} |f_1(t)|/\mathrm{L}(t) = 0$. The assumptions in Exercise 6.10 are thus fulfilled and we can conclude that $\Delta_t f_1 = \Delta_t \log \|v_1\|^2_{\mathrm{h}}$ is L^1_{loc} on Δ, as wanted.

By the residue theorem we have $\deg(\mathcal{L}^0_*)^\vee = -\deg \mathcal{L}^0_* = \sum_x \gamma_x$. Let us fix an arbitrary C^∞ metric on $(\mathcal{L}^0_*)^\vee$. We thus obtain a metric, that we still denote by h, on the trivial bundle $\mathcal{O} = (\mathcal{L}_*^0)^\vee \otimes \mathcal{L}_*^0$, such that the norm of the unit section 1 satisfies (6.4.6 $*$) (up to constants). We aim at proving that $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h})$ is well-defined and is equal to $\sum_x \gamma_x$.

Let us consider a model metric h^o on $\mathcal{O}$, such that $\|1\|_{\mathrm{h}^o}$ is C^∞ on X^*, equal to h on the complement of discs centered at the punctures, and *equal to* $|t|^{\gamma_x}$ for some local coordinate t at each puncture x. The curvature of h^o is $\mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o}$, and is meaningful as a $(1,1)$-current on X. In the neighbourhood of a puncture, we have

$$\frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o} = \frac{\gamma_x\,\mathrm{i}}{\pi}\,\mathrm{d}''\mathrm{d}' \log |t| = -\gamma_x \delta_x,$$

so that, on X, we have $\frac{\mathrm{i}}{2\pi}\mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o} = \eta - \sum_x \gamma_x\delta_x$, where $\eta \in \mathcal{E}^{1,1}_{\mathrm{c}}(X^*)$. Furthermore, $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) = \int_X \eta$. We then find

$$0 = \deg \mathcal{O} = \frac{\mathrm{i}}{2\pi}\langle 1, \mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o}\rangle = \deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) - \sum_x \gamma_x,$$

and thus $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) = \sum_x \gamma_x$.

Let us set $f = \log \|1\|_{\mathrm{h}} - \log \|1\|_{\mathrm{h}^o}$. It is supported on the union of discs $\Delta^*_R(x)$, where x is a puncture. Then, as above, (6.4.6 $*$) implies that for each puncture x, $\lim_{t\to 0} |f(t)|/\mathrm{L}(t) = 0$. On the other hand, we have seen that $\mathrm{d}''\mathrm{d}' f_{|X^*}$ is L^1_{loc} on X, so that $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h})$ is well-defined. The assumptions of Exercise 6.9 are thus fulfilled and we conclude that the current $\mathrm{d}''\mathrm{d}' f$ is L^1_{loc} on X. Furthermore,

$$\frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' \log \|1\|_{\mathrm{h}} = \eta + \frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' f - \sum_x \gamma_x \delta_x$$

as currents on X, where the first two terms of the right-hand side are L^1_{loc} on X. Since

$$\int_{X^*} \mathrm{d}''\mathrm{d}' f = \int_X \mathrm{d}''\mathrm{d}' f = \langle 1, \mathrm{d}''\mathrm{d}' f\rangle = 0,$$

we find $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}) = \deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) = \sum_x \gamma_x$, as wanted. □

***Proof of the semi-simplicity theorem* 6.4.1.** We argue by induction on the rank of $\mathcal{H}$, the case of rank 1 being clear. We first emphasize that we can apply Proposition 6.4.5 to $(\mathcal{H}, \mathrm{h}, D)$: indeed, the dual Hermitian bundle with connection $(\mathcal{H}, \mathrm{h}, D)^\vee$ is also harmonic, so that Theorem 6.3.2(1) applies to both $(\mathcal{H}, \mathrm{h}, D)$ and its dual.

Let $(\mathcal{H}_1, D)$ be a flat subbundle of $(\mathcal{H}, D)$, that we equip with the Hermitian metric h_1 induced by h. Proving that $(\mathcal{H}_1, \mathrm{h}_1, D)$ is a direct summand amounts to proving that the h-orthogonal projection $\pi : \mathcal{H} \to \mathcal{H}_1$ is compatible with D. However, in order to apply Theorem 6.4.1 by induction on the rank, we also need to prove that $(\mathcal{H}_1, \mathrm{h}_1, D)$ is a nilpotent harmonic bundle. Considering π as a section of the nilpotent harmonic bundle $(\mathcal{E}nd\,\mathcal{H}, \mathrm{h}, D)$ (see Exercise 6.11), we are thus left with proving

(1) $D(\pi) = 0$,
(2) $\theta(\pi) = 0$.

We first claim that the second property is a consequence of the first one. By (1), π is a flat section of $\mathcal{E}nd\,\mathcal{V}$ which preserves the metric, hence the filtration $\mathcal{V}^\bullet_{\mathrm{mod}}$, and satisfies thus the hypotheses in Lemma 6.3.12. It follows that $\|\pi\|^2_{\mathrm{h}}$ is bounded.

Furthermore, according to Remark 6.3.36(b), the function $\|\theta'(\pi)\|_{\mathrm{h}}^2 + \|\theta''(\pi)\|_{\mathrm{h}}^2$ is L^1_{loc} on X with integral equal to zero, since X is compact and $\langle 1, \Delta\|\pi\|_{\mathrm{h}}^2\rangle = 0$. Therefore, $\|\theta'(\pi)\|_{\mathrm{h}}^2 + \|\theta''(\pi)\|_{\mathrm{h}}^2 = 0$, as claimed.

Let us prove (1), that is, $D(\pi) = 0$. We set $(\mathcal{V}_1, \nabla) = \operatorname{Ker} D''$. Let us denote by h_1 the metric on $\mathcal{V}_1$ to avoid confusion, and let R_{h_1} denote the corresponding curvature.

6.4.7. Lemma. *With the previous notation we have, denoting by $\|\bullet\|_{\mathrm{HS}}^2$ the Hilbert-Schmidt norm,*

$$\operatorname{tr} R_{\mathrm{h}_1} = \frac{\mathrm{i}}{2}\,\|D(\pi)\|_{\mathrm{HS}}^2 \operatorname{vol}.$$

By Corollary 6.3.31 and Proposition 6.4.5, we have $\deg^{\mathrm{an}}(\mathcal{V}_1, \mathrm{h}_1) = 0$. On the other hand, the above lemma yields

$$\deg^{\mathrm{an}}(\mathcal{V}_1, \mathrm{h}_1) = -\frac{1}{4\pi}\int_X \|D(\pi)\|_{\mathrm{HS}}^2 \operatorname{vol},$$

hence $D(\pi) = 0$, and this concludes the proof of the theorem. □

***Proof of Lemma* 6.4.7.** We will use the formulas in Exercises 4.4–4.11 to compute the curvature of $\det(\mathcal{V}_1, \mathrm{h}_1)$. For any Hermitian holomorphic bundle with flat connection $(\mathcal{H}, \mathrm{h}, D)$, since $\dim X^* = 1$, we have

- $D''_{\mathrm{h}}(\theta') + D'_{\mathrm{h}}(\theta'') = -(\theta' \wedge \theta'' + \theta'' \wedge \theta')$,
- $(D^{\mathrm{c}})^2 = D''_{\mathrm{h}}(\theta') + D'_{\mathrm{h}}(\theta'') - (\theta' \wedge \theta'' + \theta'' \wedge \theta') = -2(\theta' \wedge \theta'' + \theta'' \wedge \theta')$,
- $4\overline{\mathcal{D}}^2 = DD^{\mathrm{c}} + D^{\mathrm{c}}D - 2(\theta' \wedge \theta'' + \theta'' \wedge \theta')$,

and the formula of Exercise 4.4(5) becomes

$$R_{\mathrm{h}} = -\tfrac{1}{2}(DD^{\mathrm{c}} + D^{\mathrm{c}}D) - (\theta' \wedge \theta'' + \theta'' \wedge \theta').$$

Taking trace, we obtain, since the trace of $(\theta' \wedge \theta'' + \theta'' \wedge \theta')$ is zero,

$$\operatorname{tr} R_{\mathrm{h}} = -\tfrac{1}{2}\operatorname{tr}(DD^{\mathrm{c}} + D^{\mathrm{c}}D).$$

Then Exercise 4.11(4) implies

$$\operatorname{tr} R_{\mathrm{h}_1} = -\tfrac{1}{2}\operatorname{tr}(D(\pi)D^{\mathrm{c}}(\pi)) = \tfrac{1}{2}\operatorname{tr}(D^{\mathrm{c}}(\pi)D(\pi)),$$

and this yields

$$\operatorname{tr} R_{\mathrm{h}_1} = (\Lambda \operatorname{tr} R_{\mathrm{h}_1})\operatorname{vol} = \tfrac{1}{2}\Lambda\operatorname{tr}(D^{\mathrm{c}}(\pi)D(\pi))\operatorname{vol}.$$

Since Λ commutes with π and acts by 0 except on $(1,1)$-forms with values in $\mathcal{H}$, we can write, according to Exercise 4.10(4),

$$\Lambda D^{\mathrm{c}}(\pi)D(\pi) = [[\Lambda, D^{\mathrm{c}}], \pi]D(\pi) = -\mathrm{i}\,D^\star(\pi)D(\pi).$$

But π being obviously self-adjoint with respect to h, and recalling that $f^\star = f^*$ for a $\mathcal{C}^\infty_{X^*}$-linear morphism between Hermitian bundles, we deduce

$$D^\star(\pi) = [D^\star, \pi] = -[D,\pi]^\star = -[D,\pi]^* = -D(\pi)^*.$$

If $\|D(\pi)\|_{\mathrm{HS}}^2$ denotes the square of the Hilbert-Schmidt norm of the $\mathcal{C}^\infty_{X^*}$-linear morphism $D(\pi) : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$, i.e., $\|D(\pi)\|_{\mathrm{HS}}^2 = \operatorname{tr}(D(\pi)^*D(\pi))$, we finally obtain the desired formula. □

6.5. Exercises

***Exercise 6.1* (The structure of $(\mathcal{V}_*, \nabla)$).** Assume that $(\mathcal{V}_*, \nabla)$ has a regular singularity at the origin of Δ and no other singularity.

(1) Show that $(\mathcal{V}_*, \nabla)$ is a successive extension of rank 1 meromorphic connections. [*Hint*: Use a Jordan basis for $\mathcal{R}$ of $\mathcal{V}_*^0/t\mathcal{V}_*^0$.]

(2) Assume that $\mathcal{V}$ has rank 1. Let v_γ be an $\mathcal{O}_\Delta$-basis of $\mathcal{V}_*^0$ in which the matrix of $t\nabla_{\partial_t}$ is constant. Show that $t\nabla_{\partial_t} v_\gamma = \gamma v_\gamma$ with $\operatorname{Re}\gamma \in [0,1)$. Identify $\mathcal{V}^\nabla$ with the subsheaf of $\rho_*\mathcal{O}_{\widetilde{\Delta}^*}$ consisting of multiples of some (or any) branch of the multivalued function $t^{-\gamma}$, by sending $ct^{-\gamma}$ to $ct^{-\gamma}v_\gamma$.

(3) For $\operatorname{Re}\gamma \in [0,1)$ and $p \geqslant 0$, set $\mathcal{J}_{\gamma,p} = (\mathcal{O}_\Delta[1/t]^{p+1}, \nabla)$, where the matrix of ∇_{∂_t} in the canonical basis $\boldsymbol{v}_{\gamma,p} = (v_{\gamma,0}, \dots, v_{\gamma,p})$ is given by $t\nabla_{\partial_t} v_{\gamma,k} = \gamma v_{\gamma,k} - v_{\gamma,k-1}$ (so that $v_{\gamma,p}$ is a generating section with respect to $t\nabla_{\partial_t}$). Show that $(\mathcal{V}_*, \nabla)$ has a decomposition

$$(\mathcal{V}_*, \nabla) \simeq \bigoplus_{\gamma \in [0,1)} \Big[\bigoplus_p (\mathcal{J}_{\gamma,p}, \nabla)\Big]. \tag{6.5.1}$$

[*Hint*: Use a Jordan decomposition for $\mathcal{R}$.]

(4) Compute $\operatorname{Ker}\nabla$ on $\mathcal{V}_*$ in terms of this decomposition.

(5) Show that there is no nonzero morphism $\mathcal{J}_{\gamma_1,p} \to \mathcal{J}_{\gamma_2,q}$ if $\gamma_1 \neq \gamma_2 \in [0,1)$, and conclude that the decomposition indexed by γ above is unique.

Exercise 6.2. Show the following properties.

(1) $\mathcal{V}_*^{\beta+k} = t^k \mathcal{V}_*^\beta$ for every $k \in \mathbb{Z}$.

(2) $\operatorname{gr}^\beta \mathcal{V}_*$ can be identified with the generalized β-eigenspace of the residue of ∇ on $\mathcal{V}_*^{[\beta]}/t\mathcal{V}_*^{[\beta]}$.

(3) The map induced by ∇_{∂_t} sends $\operatorname{gr}^\beta \mathcal{V}_*$ to $\operatorname{gr}^{\beta-1}\mathcal{V}_*$ and, if $\beta \neq 0$, it is an isomorphism. [*Hint*: Use that the composition $t\nabla_{\partial_t} : \operatorname{gr}^\beta\mathcal{V}_* \to \operatorname{gr}^\beta\mathcal{V}_*$ is identified with the restriction of the residue of ∇ on $\mathcal{V}_*^{[\beta]}/t\mathcal{V}_*^{[\beta]}$ to its generalized β-eigenspace.]

(4) The map $\nabla_{\partial_t} : \mathcal{V}_*^\beta \to \mathcal{V}_*^{\beta-1}$ is onto (equivalently, $t\nabla_{\partial_t} : \mathcal{V}_*^\beta \to \mathcal{V}_*^\beta$ is onto) provided that $\beta > 0$. [*Hint*: Reduce to the case where $\mathcal{V}_*$ has rank 1 by using Exercise 6.1 and has a basis v_γ which satisfies $t\nabla_{\partial_t} v_\gamma = \gamma v_\gamma$ for some $\gamma \in [0,1)$, and show that $\mathcal{V}_*^{\gamma+k} = t^k\mathcal{O}_\Delta v_\gamma$ for $k \in \mathbb{Z}$.]

(5) With respect to a decomposition of $(\mathcal{V}_*, \nabla)$ as in Exercise 6.1(3), show that, for $\gamma \in [0,1)$, we have, for $k \in \mathbb{Z}$,

$$\mathcal{V}_*^{\gamma+k} = \bigoplus_{i,\,\gamma_i \geqslant \gamma} t^k \mathcal{O}_\Delta \cdot \boldsymbol{v}_{\gamma_i,p_i} \oplus \bigoplus_{i,\,\gamma_i < \gamma} t^{k+1}\mathcal{O}_\Delta \cdot \boldsymbol{v}_{\gamma_i,p_i}.$$

(6) The subsheaf $\sum_{j \geqslant 0} (\nabla_{\partial_t})^j \mathcal{V}_*^\beta$ of $\mathcal{V}_*$ is an $\mathcal{O}_\Delta$-module equipped with a connection ∇, and

- does not depend on $\beta > -1$, or on $\beta \leqslant -1$,
- in the latter case, it is equal to $\mathcal{V}_*$,

• in the former case, we call it the *middle extension* of $(\mathcal{V},\nabla)$ and denote it by $\mathcal{V}_{\mathrm{mid}}$; then $\nabla_{\partial_t}:\mathcal{V}_{\mathrm{mid}}\to\mathcal{V}_{\mathrm{mid}}$ is *onto* and has kernel equal to the sheaf $j_*(\mathcal{V}^\nabla)$.

***Exercise 6.3* (Local freeness and subbundles).** Let F be a rank two free bundle on Δ, with basis f_1,f_2. Let $E\subset j^*F$ be the subbundle on Δ^* with basis $e=\exp(1/t)f_1+f_2$. Show that $j_*E\cap F$ is not locally free. [*Hint*: Show that the germ $(j_*E\cap F)_0$ consists of sections $a(t)e$, with $a(t)$ holomorphic on some punctured neighbourhood of 0 in Δ, such that both $a(t)$ and $\exp(1/t)a(t)$ belong to $\mathbb{C}\{t\}$; conclude that $(j_*E\cap F)_0=0$.]

Exercise 6.4. Prove the result of Theorem 6.3.2 in the unitary case of Example 6.3.1.

Exercise 6.5. Show the following identities on Δ^* for the function $\mathrm{L}(t)=-\log|t|^2=-\log t\bar t$:

$$(6.5*)\qquad \begin{aligned}\mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{Y}\,\mathrm{L}(t)^{\mp\mathrm{H}/2}&=\mathrm{L}(t)^{\mp1}\,\mathrm{Y},\quad \mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{X}\,\mathrm{L}(t)^{\mp\mathrm{H}/2}=\mathrm{L}(t)^{\pm1}\,\mathrm{X}\\ \mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{e}^{\mathrm{Y}}\,\mathrm{L}(t)^{\mp\mathrm{H}/2}&=\mathrm{e}^{\mathrm{L}(t)^{\mp1}\,\mathrm{Y}},\quad \mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{e}^{\mathrm{X}}\,\mathrm{L}(t)^{\mp\mathrm{H}/2}=\mathrm{e}^{\mathrm{L}(t)^{\pm1}\,\mathrm{X}}\end{aligned}$$

[*Hint*: Use Exercise 3.1(1)] and

$$(6.5{*}{*})\qquad \begin{aligned}-\,t\partial_t\mathrm{L}(t)^k/k!=-\bar t\partial_{\bar t}\mathrm{L}(t)^k/k!&=\mathrm{L}(t)^{k-1}/(k-1)!\quad(k\geqslant0),\\ \mathrm{L}(t)^{\mathrm{H}/2}\,t\,\frac{\partial}{\partial t}\big(\mathrm{L}(t)^{-\mathrm{H}/2}\big)=\mathrm{L}(t)^{\mathrm{H}/2}\,\bar t\,\frac{\partial}{\partial\bar t}\big(\mathrm{L}(t)^{-\mathrm{H}/2}\big)&=\frac{\mathrm{H}/2}{\mathrm{L}(t)}.\end{aligned}$$

Exercise 6.6. Let $R\in(0,1)$, let $\beta\in\mathbb{R}$ and $\ell\in\mathbb{Z}$. Show that the integral

$$\int_0^R r^{2\beta}\mathrm{L}(r)^\ell\,\frac{\mathrm{d}r}{r}$$

is finite iff $\beta>0$ or $\beta=0$ and $\ell\leqslant-2$ (recall that $\mathrm{L}(r):=2|\log r|=-2\log r$). Conclude that the function $t\mapsto|t|^{-2}\mathrm{L}(t)^{-2}$ is L^1_{loc} near the origin. [*Hint*: Recall that the volume form in polar coordinates is a multiple of $r\mathrm{d}r\mathrm{d}\theta$.]

***Exercise 6.7* (Subharmonic functions).** Let $R\in(0,1)$ and let Δ^*_R be the punctured open disc of radius R. Let f be a continuous subharmonic function on Δ^*_R.

(1) Assume that $\limsup_{t\to0}f(t)/\mathrm{L}(t)\leqslant0$. Show that $f\leqslant\sup_{\partial\Delta_{R'}}f(t)$ on $\Delta^*_{R'}$. [*Hint*: Reduce first to the case where $\sup_{\partial\Delta_{R'}}f(t)=0$ by considering $f-\sup_{\partial\Delta_{R'}}f(t)$. Then, prove that, for any $\varepsilon>0$, $f(t)-\varepsilon\mathrm{L}(t)\leqslant0$ on $\Delta^*_{R'}$ by showing first that $\limsup_{t\to0}(f(t)-\varepsilon\mathrm{L}(t))\leqslant0$ and by applying the maximum principle on $\Delta^*_{R'}$ for subharmonic functions, i.e., if g is subharmonic on $\Delta^*_{R'}$ and if for any $t_o\in\{0\}\cup\partial\Delta_{R'}$ it satisfies $\limsup_{t\to t_o}g(t)\leqslant0$, then $g\leqslant0$ on $\Delta^*_{R'}$.]

(2) Assume that, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ satisfying $f(t)\leqslant\log C_\varepsilon+\varepsilon\mathrm{L}(t)$ on $\Delta^*_{R'}$. Prove that $f\leqslant\sup_{\partial\Delta_{R'}}f(t)$ on $\Delta^*_{R'}$. [*Hint*: Show that f satisfies the assumptions in (1).]

***Exercise 6.8* (Proof of Lemma 6.3.33).** In this exercise, Δ, resp. Δ^*, denotes the open disc, resp. the punctured open disc, of radius 1 with coordinate τ, resp. t, and $\rho:\Delta\to\Delta^*$ is the model of a universal covering defined by $\rho(\tau)=\exp(\mathrm{i}(1+\tau)/(1-\tau))$. The Poincaré metric on Δ, resp. Δ^*, has volume form $\mathrm{vol}_\Delta=(1-|\tau|^2)^{-2}|\mathrm{d}\tau\,\mathrm{d}\bar\tau|$, resp. $\mathrm{vol}_{\Delta^*} = (|t|\mathrm{L}(t))^{-2}|\mathrm{d}t\,\mathrm{d}\bar{t}|$. Furthermore, $\rho^*\mathrm{vol}_{\Delta^*} = \mathrm{vol}_\Delta$. Let $\Delta_\tau = 4\partial_\tau\partial_{\bar\tau}$, resp. $\Delta_t = 4\partial_t\partial_{\bar t}$, be the corresponding Laplacians. Let $f : \Delta^* \to \mathbb{R}_+$ be a C^2 function satisfying the assumptions of Lemma 6.3.33. We first transfer the assumption on Δ^* to an assumption on Δ.

(1) For $R \in (0,1]$, set $\mathrm{v}_R(\tau) = R^2/(R-|\tau|^2)^2$ on the open disc Δ_R. Show that $\Delta_\tau \log \mathrm{v}_R = 4\mathrm{v}_R$ and $\mathrm{v}_{1|\Delta_R} \leqslant \mathrm{v}_R$.

(2) Express the equality $\rho^*\mathrm{vol}_{\Delta^*} = \mathrm{vol}_\Delta$ as $(|\rho(\tau)||\log\rho(\tau)|)^{-2} = \mathrm{v}_1 \cdot |\rho'(\tau)|^{-2}$.

(3) Set $g(\tau) = f\circ\rho(\tau)\cdot|\rho'(\tau)|^2$. Prove that the nonnegative real function g satisfies $\Delta_\tau \log g(\tau) \geqslant 4g(\tau)$. [*Hint*: Show that for a C^2 function $h(t)$, $\Delta_\tau(h\circ\rho) = (\Delta_t h)\circ\rho\cdot|\rho'(\tau)|^2$.]

(4) Let $U(R) \subset \Delta_R$ be the open set where $g(\tau) > \mathrm{v}_R(\tau)$. Show that $\log(g/\mathrm{v}_R)$ is subharmonic on $U(R)$. [*Hint*: Use that $U(R) \subset U(1)$.]

(5) Show that $\partial U(R) \cap \partial\Delta_R = \varnothing$. Deduce that $U(R) = \varnothing$. [*Hint*: Use that $\log(g/\mathrm{v}_R) = 0$ on $\partial U(R)$ and the maximum principle.]

(6) Conclude that $g \leqslant \mathrm{v}_R$ on Δ_R and the proof of Lemma 6.3.33. [*Hint*: Pass to the limit $R \to 1$.]

Exercise 6.9. Let $R \in (0,1)$ and Δ^*_R be as in Exercise 6.7. Let f be a C^2 function on Δ^*_R such that $\lim_{t\to 0}|f(t)|/\mathrm{L}(t) = 0$ (in particular, f is L^1_{loc} on Δ_R). We consider the Laplace operator $\Delta_t = 4\partial_t\partial_{\bar t}$. Then $\Delta_t f$ is a distribution on Δ_R and $\Delta_t(f_{|\Delta^*_R})$ is a continuous function on Δ^*_R. The aim of this exercise is to prove that *if $\eta := \Delta_t(f_{|\Delta^*_R})$ is L^1_{loc} on Δ_R, then $\Delta_t f = \eta$ as distributions on Δ_R, i.e., $\Delta_t f$ does not have components supported at the origin.*

(1) Let $\psi : \mathbb{R}_+ \to [0,1]$ be decreasing a C^∞ function such that $\psi(r) = 1$ for $r \in [0,1/2]$ and $\psi \equiv 0$ for $r \geqslant 1$. For any $N > 0$, set

$$\psi_N(r) = N\psi(r\mathrm{e}^N) + \mathrm{L}(r)(1-\psi(r\mathrm{e}^N)).$$

Show the following properties of the C^∞ function ψ_N on $(0,1)$:

(a) $0 \leqslant \psi_N(r) \leqslant \min(N+\log 2, \mathrm{L}(r))$ and $\psi_N(r) \equiv N$ if $\mathrm{L}(r) \geqslant N + \log 2$,

(b) $\psi_N(t) \to \mathrm{L}(r)$ and $\psi_N(r)/N \to 0$ pointwise when $N \to \infty$,

(c) setting $\Delta_t\psi_N(r) = \partial_r^2\psi_N(r)$ and $\partial_t\psi_N(r) = \partial_{\bar t}\psi_N(r) = \frac{1}{2}\partial_r\psi_N(r)$, show that the functions $\Delta_t\psi_N, \partial_t\psi_N, \partial_{\bar t}\psi_N$ are supported on the set

$$\{r \mid \mathrm{L}(r) \leqslant N + \log 2\} \subset \{r \mid \mathrm{L}(r) \leqslant 2N\},$$

and

$$\int_{\Delta_R} |\partial_t\psi_N(r)|\ \mathrm{vol},\quad \int_{\Delta_R} |\partial_{\bar t}\psi_N(r)|\ \mathrm{vol},\quad \int_{\Delta_R} |\psi_N(r)|\ \mathrm{vol}$$

are bounded by a constant independent of N.

(2) Let $\chi \in C^\infty_{\mathrm{c}}(\Delta_R)$ be a test function. Show that

$$\int_{\Delta^*_R} f\Delta_t[(1-\psi_N/N)\chi]\ \mathrm{vol} \xrightarrow[N\to\infty]{} \int_{\Delta^*_R} f\Delta_t\chi\ \mathrm{vol}$$

by showing first

$$\int_{\Delta_R^*} f(1-\psi_N/N)\Delta_t\chi \text{ vol} \xrightarrow[N\to\infty]{} \int_{\Delta_R^*} f\Delta_t\chi \text{ vol}.$$

[*Hint*: Use that $|f|\,|\Delta_t\psi_N|/N \leqslant 2(|f|/\mathrm{L}(r))|\Delta_t\psi_N|$ and similarly with $\partial_r\psi_N$.]

(3) Using that $(1-\psi_N/N)\chi$ is a test function on Δ_R^*, show that

$$\int_{\Delta_R^*} f\Delta_t[(1-\psi_N/N)\chi] \text{ vol} = \int_{\Delta_R^*} \eta(t)[(1-\psi_N/N)\chi] \text{ vol} \xrightarrow[N\to\infty]{} \int_{\Delta_R^*} \eta(t)\chi(t) \text{ vol},$$

and conclude.

Exercise 6.10. Same setting as in Exercise 6.9. Prove that *if there exists $\eta \in L^1_{\mathrm{loc}}(\Delta_R)$ such that $\Delta_t(f_{|\Delta_R^*}) \geqslant \eta_{|\Delta_R^*}$, then the distribution $\Delta_t f$ on Δ_R is in fact L^1_{loc}, i.e., $\Delta_t(f_{|\Delta_R^*})$ is L^1_{loc} on Δ_R and coincide with $\Delta_t f$ as distributions.*

(1) Prove that $\Delta_t f \geqslant \eta$ as distributions on Δ_R, i.e., for any nonnegative test function χ on Δ_R,

$$\langle \Delta_t f, \chi\rangle \geqslant \int_{\Delta_R^*} \eta\cdot\chi \text{ vol}.$$

[*Hint*: Keep 6.9(1) and (2) as they are with a nonnegative χ, and in (3) replace the equality with an inequality.]

(2) Deduce that the distribution $\Delta_t f - \eta$, hence also $\Delta_t f$, is the sum of a L^1_{loc} function on Δ_R and a multiple of the Dirac mass at the origin. [*Hint*: Use [**Hör03**, Th. 2.1.7] and the theorem of Radon-Nikodym.]

(3) Apply Exercise 6.9 to conclude.

Exercise 6.11. Let $(\mathcal{H}_1, D_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, D_2, \mathrm{h}_2)$ be nilpotent harmonic bundles. Show that $(\mathcal{H}_1\otimes\mathcal{H}_2, D, \mathrm{h})$ and $\mathcal{H}om(\mathcal{H}_1,\mathcal{H}_2), D, \mathrm{h})$ are also nilpotent. [*Hint*: Use Exercise 4.8.]

Exercise 6.12. Let E be a finite dimensional $\mathbb{C}$-vector space with a Hermitian metric h and a nilpotent endomorphism θ'_0.

(1) Show that there exists a h-orthonormal basis $\boldsymbol{\varepsilon}$ of E in which the matrix A of θ'_0 is strictly upper triangular.

(2) Let θ''_0 be the h-adjoint of θ'_0, with matrix ${}^t\overline{A}$. Show that there exists a positive constant c depending only on $\dim E$ such that $\|[A, {}^t\overline{A}]\| \geqslant c\|A\|^2$. [*Hint*: By homogeneity and compactness of the sphere $\|A\|=1$, it is enough to show that the function $A\mapsto \|[A,{}^t\overline{A}]\|$ (A strictly upper triangular) does not vanish on the sphere; use then that the only normal and nilpotent matrix is the zero matrix.]

6.6. Comments

The fundamental work of Griffiths on the period mapping attached to a polarized variation of Hodge structure (see [**Gri70b, Del71c**] and the references therein) leads to the analysis of degenerations of such variations, which was achieved in the fundamental article of Schmid [**Sch73**] (see also [**GS75**] and the references therein). Theorems 6.3.2(1) and 6.3.5, together with Theorems 6.7.3 and 6.8.7 are due to Schmid in loc. cit., and Theorem 6.3.2(2) is due to Borel (see [**Sch73**, Lem. 4.5]).

While Schmid's theory focuses on variations having unipotent local monodromies, it is well-known that the results can be extended to the case of quasi-unipotent local monodromies. The more general case treated here of local monodromies whose eigenvalues have absolute value equal to 1 is known to be a consequence of the methods of Schmid (see [**Del87**, §1.11]).

The idea of focusing on the harmonic aspect of the theory is due to Simpson [**Sim88, Sim90**]. A similar approach is considered in [**S-Sch22**], with a more precise estimate on constants involved, that can prove useful in higher dimensions. The proof of the semi-simplicity theorem 6.4.1 given here, in the framework of nilpotent harmonic bundles, is due to Simpson. The idea of considering the analytic degree $\deg^{\mathrm{an}}$ is instrumental in his proof of stability of general harmonic flat bundles.

CHAPTER 6

VARIATIONS OF HODGE STRUCTURE ON CURVES
PART 2: LIMITING HODGE PROPERTIES

Summary. We keep the local setting of Part 1. We state the fundamental theorems of Schmid concerning the limiting behavior of the Hodge filtration and give an idea of the proof, together with the example of the Deligne harmonic model.

6.7. The holomorphic Hodge filtration

We keep the setting of Section 6.3 and we assume (as justified by Theorem 6.3.2(2)) that the eigenvalues of the monodromy have absolute value equal to 1. We wish to extend the filtration $F^\bullet\mathcal{V}$ as a filtration $F^\bullet\mathcal{V}_*$ by sub-bundles satisfying the Griffiths transversality property with respect to the meromorphic connection ∇. A first natural choice would be to set

$$F^p\mathcal{V}_* := j_*F^p\mathcal{H}\cap\mathcal{V}_*,$$

where $j:\Delta^*\hookrightarrow\Delta$ denotes the inclusion. This choice can lead to a non-coherent $\mathcal{O}_\Delta$-module: for example, if $p\ll 0$, we have $F^p\mathcal{V}=\mathcal{V}$ and we would get $F^p\mathcal{V}_*=\mathcal{V}_*$, which is not $\mathcal{O}_\Delta$-coherent. Since we have at our disposal the locally free $\mathcal{O}_\Delta$-modules $\mathcal{V}_*^\beta$ for any $\beta\in\mathbb{R}$, it may be more clever to consider, for any such β,

$$F^p\mathcal{V}_*^\beta := j_*F^p\mathcal{H}\cap\mathcal{V}_*^\beta, \tag{6.7.1}$$

where the intersection is taken in $j_*\mathcal{V}$. The main question to address is whether these sheaves are $\mathcal{O}_\Delta$-coherent. If so, being torsion free, they would be $\mathcal{O}_\Delta$-locally free. Furthermore, we may wonder whether the filtration $F^\bullet\mathcal{V}_*^\beta$ of $\mathcal{V}_*^\beta$ which clearly satisfies $F^p\mathcal{V}_*^\beta=0$ for $p\gg 0$ and $F^p\mathcal{V}_*^\beta=\mathcal{V}_*^\beta$ for $p\ll 0$) is a filtration by *sub-bundles*, i.e., whether the quotients $F^p\mathcal{V}_*^\beta/F^{p+1}\mathcal{V}_*^\beta$ are locally free for any $p\in\mathbb{Z}$.

According to Theorem 6.3.2, we can interpret sections of $F^p\mathcal{V}_*^\beta$ on Δ as being the sections of $F^p\mathcal{V}$ on Δ^* whose h-norm on any punctured closed sub-disc $(\overline{\Delta'})^*$ ($\overline{\Delta'}\subset\Delta$) is bounded by $C_\varepsilon|t|^{\beta-\varepsilon}$ for any $\varepsilon>0$ and some $C_\varepsilon>0$. Let us already notice:

6.7.2. Lemma.

(1) *For $k \geqslant 0$ and any $\beta \in \mathbb{R}$, we have*
$$F^p\mathcal{V}_*^{\beta+k} = t^k F^p\mathcal{V}_*^\beta.$$

(2) *The following properties are equivalent:*

  (a) *there exists $\beta \in \mathbb{R}$ such that, for any $p \in \mathbb{Z}$, $F^p\mathcal{V}_*^\beta$ is $\mathcal{O}_X$-coherent,*

  (b) *for any $\beta \in \mathbb{R}$, the filtration $F^\bullet\mathcal{V}_*^\beta$ of $\mathcal{V}_*^\beta$ is a filtration by sub-bundles.*

Proof. The first point is clear since $\mathcal{V}_*^{\beta+k} = t^k\mathcal{V}_*^\beta$, as well as the implication (2b) ⇒ (2a). Let us show (2a) $\implies$ (2b). Let β be such that $F^p\mathcal{V}_*^\beta$ is $\mathcal{O}_X$-coherent for any p and let γ in $\mathbb{R}$. By the first point, any $F^p\mathcal{V}_*^{\beta+k}$ is $\mathcal{O}_X$-coherent, so we can assume that $\gamma \leqslant \beta$. Then $F^p\mathcal{V}_*^\gamma = (F^p\mathcal{V}_*^\beta) \cap \mathcal{V}_*^\gamma$ and, since both terms in the right-hand side are coherent, so is their intersection.[1]

Moreover, by the coherence property and the first point, $\dim(\mathrm{gr}_F^p\mathcal{V}_*^\gamma/\mathrm{gr}_F^p\mathcal{V}_*^{\gamma+1}) \geqslant \mathrm{rk}\,\mathrm{gr}_F^p\mathcal{V}$ for each p. Since the sum over p of both sides are equal (as $\mathcal{V}_*^\gamma$ is locally free), they are equal for each p, hence $\mathrm{gr}_F^p\mathcal{V}_*^\gamma$ is locally free. □

6.7.3. Theorem. *For any $\beta \in \mathbb{R}$, the filtration $F^p\mathcal{V}_*^\beta$ is a filtration of $\mathcal{V}_*^\beta$ by sub-bundles.*

Proof. According to Lemma 6.7.2, it is enough to prove that, for any p and any β, the $\mathcal{O}_X$-module $F^p\mathcal{V}_*^\beta$ is coherent. Let us fix p. Since we already know that $\mathcal{V}_*^\beta = \mathcal{V}_{\mathrm{mod}}^\beta$ (Theorem 6.3.2), it is enough to show that the Hermitian holomorphic bundle $(F^p\mathcal{V}, \mathrm{h})$, where h is the metric induced by h on $\mathcal{V}$, is *moderate*. As noticed in Remark 6.2.7(1), some care has to be taken. Exercise 4.4(7) together with Simpson's estimate (Theorem 6.3.9) show that $(F^p\mathcal{V}, \mathrm{h})$ satisfies the criterion of Theorem 6.2.9 for each $p \in \mathbb{Z}$. Therefore, $(F^p\mathcal{V}, \mathrm{h})$ is moderate. □

6.8. The limiting Hodge-Lefschetz structure

We will now describe the limiting Hodge-Lefschetz structure attached to a polarized variation of $\mathbb{C}$-Hodge structure (H, S) of weight w on Δ^*.

6.8.1. Convention. We use the simplified setting as in Proposition 5.2.16 and we now write (H, S) as $((\mathcal{V}, \nabla, F^\bullet\mathcal{V}), \mathcal{S})$ (see Definition 5.4.3, and 5.4.1 for $\mathcal{S}$).

For every $\beta \in (-1, 0]$, we define the object $\mathrm{gr}^\beta H$ as follows. We set
$$\mathrm{gr}^\beta(\mathcal{V}, \nabla, F^\bullet\mathcal{V}) = (\mathrm{gr}^\beta\mathcal{V}_*, F^\bullet\mathrm{gr}^\beta\mathcal{V}_*),$$
which is equipped with the nilpotent endomorphism N induced by the action of $-(t\partial_t - \beta)$:
$$(\mathrm{gr}^\beta\mathcal{V}_*, F^\bullet\mathrm{gr}^\beta\mathcal{V}_*) \xrightarrow{\;\mathrm{N}\;} (\mathrm{gr}^\beta\mathcal{V}_*, F[-1]^\bullet\mathrm{gr}^\beta\mathcal{V}_*).$$
It remains to define the sesquilinear pairing $\mathrm{gr}^\beta\mathcal{S}$

[1] Indeed, the sum $(F^p\mathcal{V}_*^\beta) + \mathcal{V}_*^\gamma$ is clearly locally of finite type in $\mathcal{V}_*$, hence coherent. Then one deduces the desired coherence from the isomorphism $\big[(F^p\mathcal{V}_*^\beta) + \mathcal{V}_*^\gamma\big]/\mathcal{V}_*^\gamma \simeq (F^p\mathcal{V}_*^\beta)/(F^p\mathcal{V}_*^\beta) \cap \mathcal{V}_*^\gamma$.

6.8.a. Behaviour of sesquilinear pairings. We will make explicit the behaviour of sesquilinear pairings (see Definition 4.1.2) with respect to the functor $(\mathcal{V},\nabla)\mapsto(\mathcal{H}^o,\mathrm{T})$ of Theorem 6.2.4. We assume in this section that the eigenvalues of the residue of ∇ are real, that is, each matrix D_β occurring in Corollary 6.2.5 is equal to zero. This is justified by Theorem 6.3.2(2).

We keep the notation of Exercise 6.1(3), but we choose the indices in $(-1,0]$ instead of $[0,1)$. Let $\beta',\beta''\in(-1,0]$ and let $\mathfrak{s}:\mathcal{J}_{\beta',p|\Delta^*}\otimes\overline{\mathcal{J}_{\beta'',q|\Delta^*}}\to\mathcal{C}^\infty_{\Delta^*}$ be a sesquilinear pairing as in Definition 5.4.1. We denote by $\boldsymbol{v}'_{\beta',p}$ (resp. $\boldsymbol{v}''_{\beta'',q}$) the basis considered in Exercise 6.1. Recall Notation 6.2.8. The compatibility of $\mathfrak{s}$ with the connection enables us to simplify its expression.

6.8.2. Lemma. *For $i=0,\dots,p$ and $j=0,\dots,q$, there exist complex numbers $c_k(i,j)$ such that*

$$\mathfrak{s}(v'_{\beta',i},\overline{v''_{\beta'',j}})=\begin{cases}0 & \text{if } \beta'\neq\beta'',\\ |t|^{2\beta}\sum_{k=0}^{\min(i,j)}c_k(i,j)\,\mathrm{L}(t)^k/k! & \text{if } \beta'=\beta''=:\beta.\end{cases}\tag{6.8.2 *}$$

Proof. Let us first assume that $i=j=0$. If we restrict on an open sector centered at the origin on which $t^{\beta'}$ and $t^{\beta''}$ are univalued holomorphic functions, then $\mathfrak{s}(t^{-\beta'}v'_{\beta',0},\overline{t^{-\beta''}v''_{\beta'',0}})$ is constant since it is annihilated by ∂_t and $\partial_{\bar t}$. Therefore, $\mathfrak{s}(v'_{\beta',0},\overline{v''_{\beta'',0}})=c\bar t^{\beta''}t^{\beta'}$ on such a sector. But $\mathfrak{s}(v'_{\beta',0},\overline{v''_{\beta'',0}})$ is a C^∞ function on the whole Δ^*, hence $\beta'-\beta''\in\mathbb{Z}$ unless $\mathfrak{s}(v'_{\beta',0},\overline{v''_{\beta'',0}})=0$. Since we assume $\beta',\beta''\in(-1,0]$, we obtain the assertion in this case.

In general, we argue similarly by using that, if $\eta\in C^\infty(\Delta^*)$ satisfies $(t\partial_t)^{i+1}\eta=(\bar t\partial_{\bar t})^{j+1}\eta=0$, then $\eta=\sum_{k=0}^{\min(i,j)}c_k\mathrm{L}(t)^k/k!$. $\square$

We conclude that any sesquilinear pairing $\mathfrak{s}:\mathcal{J}_{\beta',p|\Delta^*}\otimes\overline{\mathcal{J}_{\beta'',q|\Delta^*}}\to\mathcal{C}^\infty_{\Delta^*}$ is zero if $\beta'\neq\beta''$, and we are reduced to considering sesquilinear pairings

$$\mathfrak{s}:\mathcal{J}_{\beta,p|\Delta^*}\otimes\overline{\mathcal{J}_{\beta,q|\Delta^*}}\longrightarrow\mathcal{C}^\infty_{\Delta^*}.$$

Let us notice that, due to the explicit expression of $\mathfrak{s}$, we have

$$\mathfrak{s}(v',\overline{t\partial_t v''})=\mathfrak{s}(t\partial_t v',\overline{v''}).$$

We still denote by $\boldsymbol{v}'_{\beta,p}$ (resp. $\boldsymbol{v}''_{\beta,q}$) the basis induced on $\mathrm{gr}^\beta\mathcal{J}'_{\beta,p}=\mathcal{O}_\Delta\boldsymbol{v}'_{\beta,p}/t\mathcal{O}_\Delta\boldsymbol{v}'_{\beta,p}$ (resp. $\mathrm{gr}^\beta\mathcal{J}''_{\beta,q}$). We define $\mathrm{gr}^\beta\mathfrak{s}$ by the formula

$$(\mathrm{gr}^\beta\mathfrak{s})(v'_{\beta,i},\overline{v''_{\beta,j}})=c_0(i,j).\tag{6.8.3}$$

We conclude from the previous remark that $(\mathrm{gr}^\beta\mathfrak{s})(v',\overline{\mathrm{N}v''})=(\mathrm{gr}^\beta\mathfrak{s})(\mathrm{N}v',\overline{v''})$ (with N induced by $-(t\partial_t-\beta)$), that is, N is self-adjoint with respect to $\mathrm{gr}^\beta\mathfrak{s}$.

We can now define the pairing $\mathrm{gr}^\beta\mathfrak{s}:\mathrm{gr}^\beta\mathcal{V}_*\otimes_{\mathbb{C}}\overline{\mathrm{gr}^\beta\mathcal{V}_*}\to\mathbb{C}$ by using the decomposition (6.5.1) for $(\mathcal{V}_*,\nabla)$ and by applying (6.8.3) to each pair of terms corresponding to the same $\beta\in(-1,0]$. This can also be obtained by a residue formula, without explicitly referring to such a decomposition and showing also the independence with respect to it (see Exercise 6.13). We can regard $\mathrm{gr}^\beta\mathfrak{s}$ as a morphism of Lefschetz pairs

$$\mathrm{gr}^\beta\mathfrak{s}:(\mathrm{gr}^\beta\mathcal{V}_*,2\pi\mathsf{i}\,\mathrm{N})\longrightarrow(\mathrm{gr}^\beta\mathcal{V}_*,2\pi\mathsf{i}\,\mathrm{N})^*,\tag{6.8.4}$$

as $2\pi\mathsf{i}\,\mathrm{N}$ is skew-adjoint with respect to $\mathfrak{s}$.

We note that the coefficients $c_0(i,j)$ (for i,j varying) determine all the coefficients $c_k(i,j)$ $(0\leqslant k\leqslant \min(i,j))$. Indeed, if $i\geqslant 1$ we find, by compatibility of $\mathfrak{s}$ with ∇,

$$\begin{aligned}|t|^{2\beta}\sum_{k=1}^{\min(i,j)} c_k(i,j)\,\frac{\mathrm{L}(t)^{k-1}}{(k-1)!} &= -(t\partial_t-\beta)\mathfrak{s}(v'_{\beta,i},\overline{v''_{\beta,j}})\\ &= \mathfrak{s}(v'_{\beta,i-1},\overline{v''_{\beta,j}}) = |t|^{2\beta}\sum_{k=0}^{\min(i-1,j)} c_k(i-1,j)\,\frac{\mathrm{L}(t)^k}{k!},\end{aligned}$$

hence $c_k(i,j)=c_{k-1}(i-1,j)$ for $k\geqslant 1$. In such a way one reconstructs $\mathfrak{s}$ from the sesquilinear pairings $\mathrm{gr}^\beta\mathfrak{s}$ by means of (6.8.2 $*$).

6.8.5. Lemma. *The pairing* $\mathrm{gr}^\beta\mathfrak{s}$ *induces a pairing* $\mathrm{gr}^{\mathrm{M}}_{\bullet}\mathrm{gr}^\beta\mathcal{V}_*\otimes_{\mathbb{C}}\overline{\mathrm{gr}^{\mathrm{M}}_{-\bullet}\mathrm{gr}^\beta\mathcal{V}_*}\to\mathbb{C}$, *which is non-degenerate if and only if* $\mathfrak{s}$ *is non-degenerate.*

Proof. Being a morphism of Lefschetz pairs, $\mathrm{gr}^\beta\mathfrak{s}$ is therefore compatible with the monodromy filtrations (see Section 3.3.a). For the second assertion, we can assume that only terms $\mathcal{J}_{\beta,p}$ (with the same $\beta\in(-1,0]$) occur in the decomposition (6.5.1). Note that $\mathrm{gr}^{\mathrm{M}}\mathrm{gr}^\beta\mathfrak{s}$ is an isomorphism if and only if $\mathrm{gr}^\beta\mathfrak{s}$ is so (Exercise 3.8). In order to conclude, we can now interpret Lemma 6.8.2 as giving an asymptotic expansion of $\mathfrak{s}$ when $|t|\to 0$, and (6.8.3) as taking its dominant part. We then clearly obtain that $\mathfrak{s}$ is non-degenerate near the origin if and only if $\mathrm{gr}^\beta\mathfrak{s}$ is non-degenerate. The equivalence with non-degeneracy on the whole disk follows then from Remark 5.4.2. ∎

***6.8.6. Example* (A symbolic identity).** Let $\eta\in C^\infty_{\mathrm{c}}(\Delta)$ be any test function. Arguing as in Exercise 6.13(1), one shows that the function

$$F(s)=\int_\Delta |t|^{2s-2}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t$$

is holomorphic on the half-space $\mathrm{Re}\,s>0$ and extends as a meromorphic function on the s-plane with a simple pole at $s=0$. An integration by parts gives

$$F(s)=\frac{1}{s^2}\int_\Delta |t|^{2s}\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t=\int_\Delta\frac{|t|^{2s}-1}{s^2}\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t, \tag{6.8.6 $*$}$$

[for the first equality, apply Stokes formula first to $\mathrm{d}(|t|^{2s}\eta(t)\mathrm{d}\bar t/\bar t)$ and then to $\mathrm{d}(|t|^{2s}\partial_t\eta(t)\mathrm{d}t)$; for the second one, apply Stokes formula to obtain the vanishing of $\int_\Delta\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t$] and expanding with respect to s (taking into account that $|t|^{2s}=\mathrm{e}^{-s\mathrm{L}(t)}$) gives the residue:

$$\mathrm{Res}_{s=0}\,F(s)=-\int_\Delta \mathrm{L}(t)\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t,$$

and the regular part $F_{\mathrm{reg}}(s):=F(s)-\frac{1}{s}\mathrm{Res}_{s=0}\,F(s)$ of $F(s)$ writes

$$F_{\mathrm{reg}}(s)=\int_\Delta\frac{|t|^{2s}-1}{s}\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t.$$

Note that, by Exercise 6.13(1) and the residue interpretation, if χ is a cut-off function, we have

$$\int_\Delta \mathrm{L}(t)\,\partial_t\partial_{\bar t}\chi(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t=2\pi\mathrm{i}\,.$$

Let N be a nilpotent element of some $\mathbb{C}$-algebra. We identify $|t|^{-2\mathrm{N}}$ with $e^{Lt\mathrm{N}}$, which is a polynomial in $\mathrm{L}(t)$ with coefficients in this algebra. We are interested in rewriting the symbolic expression

$$F_{\mathrm{N}}(s) := \int_\Delta |t|^{2s-2-2\mathrm{N}}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t = \sum_{n=0}^\infty\Big(\int_\Delta \frac{\mathrm{L}(t)^n}{n!}\,|t|^{2s-2}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t\Big)\mathrm{N}^n$$

(which is in fact a finite sum) in a way that lets us analyze how it behaves near $s=0$. Formula (6.8.6 $*$) becomes the symbolic identity

$$(6.8.6\,{*}{*})\qquad \begin{aligned} F_{\mathrm{N}}(s) &= \int_\Delta \frac{|t|^{2s-2\mathrm{N}}}{(\mathrm{N}-s)^2}\,\partial_t\partial_{\bar t}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t\\ &= \int_\Delta \frac{|t|^{2s-2\mathrm{N}}-1}{(\mathrm{N}-s)^2}\,\partial_t\partial_{\bar t}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t. \end{aligned}$$

It should be understood as an identity between two families of holomorphic functions – namely the coefficients at N^p on both sides – on the half-space $\operatorname{Re} s>0$. Indeed, writing for $\operatorname{Re} s$ large enough, we can write

$$\begin{aligned} &\mathrm{d}(|t|^{2s-2\mathrm{N}}\eta\mathrm{d}\bar t/\bar t) = (s-\mathrm{N})|t|^{2s-2\mathrm{N}}\eta\mathrm{d}t\wedge\mathrm{d}\bar t + |t|^{2s-2\mathrm{N}}\partial_t\eta\mathrm{d}t\wedge\mathrm{d}\bar t/\bar t,\\ &\mathrm{d}(|t|^{2s-2\mathrm{N}}\partial_t\eta\mathrm{d}t) = -|t|^{2s-2\mathrm{N}}\partial_{\bar t}\partial_t\eta\mathrm{d}t\wedge\mathrm{d}\bar t - (s-\mathrm{N})|t|^{2s-2\mathrm{N}}\partial_t\eta\mathrm{d}t\wedge\mathrm{d}\bar t/\bar t. \end{aligned}$$

Since integration on Δ of the left-hand terms yields zero as the forms have compact support, we obtain the desired equality for $\operatorname{Re} s\gg0$, and it holds as an equality of meromorphic functions by unique analytic continuation. The (matrix) function $F_{\mathrm{N}}(s)$ has a pole of higher order at $s=0$, with residue equal to that of $F(s)\,\mathrm{Id}$ however, and the regular part of $F_{\mathrm{N}}(s)$ writes

$$F_{\mathrm{N},\mathrm{reg}}(s) = \int_\Delta \frac{|t|^{2s-2\mathrm{N}}-1}{(\mathrm{N}-s)}\,\partial_t\partial_{\bar t}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t.$$

In particular, evaluating at $s=0$ we find

$$F_{\mathrm{N},\mathrm{reg}}(0) = \int_\Delta \frac{|t|^{-2\mathrm{N}}-1}{\mathrm{N}}\,\partial_t\partial_{\bar t}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t = \sum_{p\geqslant1}\mathrm{N}^{p-1}\int_\Delta \frac{\mathrm{L}(t)^p}{p!}\,\partial_t\partial_{\bar t}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t.$$

6.8.b. The limiting Hodge-Lefschetz structure. We continue with Convention 6.8.1. In order to obtain a Hodge-Lefschetz structure, we use the sesquilinear pairing $\mathrm{gr}^\beta\mathcal{S}:\mathrm{gr}^\beta\mathcal{V}_*\otimes_{\mathbb{C}}\overline{\mathrm{gr}^\beta\mathcal{V}_*}\to\mathbb{C}$ defined by (6.8.3) (see also Exercise 6.13).

6.8.7. Theorem. *Let (H,S) be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on Δ^*. Then for every $\beta\in(-1,0]$, the data*

$$(\mathrm{gr}^\beta H,\mathrm{N},\mathrm{gr}^\beta\mathrm{S})$$

form a polarized Hodge-Lefschetz structure with central weight w (Definitions 3.4.3 *and* 3.4.14*).*

We will not give a proof of this theorem and refer to **[S-Sch22]** for a proof of it, by means of the analysis of the period mapping. We will content ourselves with illustrating it on the model of Section 6.3.c (from which we keep the notation), that we enrich with a Hodge filtration. So we start from a polarized $\mathfrak{sl}_2$-Hodge structure $(H^o, \mathrm{N}, \mathrm{S}^o)$ with central weight $w \in \mathbb{Z}$, so that S^o and h^o are related by (see Definition 3.2.7(2))

$$\mathrm{h}^o(u^o, \overline{v^o}) = \mathrm{S}^o(\mathrm{w}u^o, \overline{\mathrm{C}^o_{\mathrm{D}} v^o}).$$

Recall that, since X, Y are of type $(-1,-1)$, they anti-commute with $\mathrm{C}^o_{\mathrm{D}}$, while H commutes with $\mathrm{C}^o_{\mathrm{D}}$. Let us now examine the commutation of w with $\mathrm{C}^o_{\mathrm{D}}$. Let us consider the modified Weil operator $\mathrm{C}^{\mathrm{abs}}_{\mathrm{D}}$ on H^o obtained by removing in C_{D} the dependence in ℓ but keeping the dependence in p, that is, by setting

$$\mathrm{C}^{\mathrm{abs}}_{\mathrm{D}} = (-1)^{w-p} = (-1)^{\ell}\mathrm{C}^o_{\mathrm{D}} \quad \text{on } (H^o_\ell)^{p, w+\ell-p} \tag{6.8.8}$$

for any $\ell \in \mathbb{Z}$. Since w sends $(H^o_\ell)^{p-\ell, w-p}$ to $(H^o_{-\ell})^{p, w+\ell-p}$, we have

$$\mathrm{C}^o_{\mathrm{D}}\, \mathrm{w} = \mathrm{w}\, \mathrm{C}^{\mathrm{abs}}_{\mathrm{D}}.$$

We can then express the metric $\mathrm{h}(x, \overline{y}) := \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}^o_{\mathrm{D}} y})$ as (see Exercise 3.1(6))

$$\mathrm{h}(x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}\mathrm{C}^o_{\mathrm{D}} y}) = \mathrm{S}(x, \overline{\mathrm{C}^{\mathrm{abs}}_{\mathrm{D}} \mathrm{w} y}).$$

Let $\mathsf{C}^{\mathrm{abs}}_{\mathrm{H}}$ and w be the matrices of $\mathrm{C}^{\mathrm{abs}}_{\mathrm{D}}$ and w in the orthonormal basis $\boldsymbol{v}^o$. Then the matrix of S^o in this basis is

$$\mathsf{S}^o := \mathsf{C}^{\mathrm{abs}}_{\mathrm{H}} \cdot \mathsf{w},$$

since $\mathsf{C}^{\mathrm{abs}}_{\mathrm{H}}$ is real (its entries are ± 1 or 0), as well as the matrix w.

We consider the C^∞ bundle $\mathcal{H}$ on Δ^* with flat connection D as in Definition 6.3.16, that we equip with the metric h and orthonormal frame $\boldsymbol{\varepsilon}$ as in Definition 6.3.20. It has a holomorphic frame $\boldsymbol{v} = 1 \otimes \boldsymbol{v}^o$, which is now further decomposed as $(\boldsymbol{v}^p)_{p\in\mathbb{Z}}$, as well as the basis $\boldsymbol{\varepsilon}$ defined by (6.3.18).

Since, for each $p \in \mathbb{Z}$, Y sends F^pH^o to $F^{p-1}H^o$ and X sends it to $F^{p+1}H^o$, while H preserves F^pH^o, (6.3.19) now reads

$$\boldsymbol{v}^p_{\ell,j} = |t|^\beta \mathrm{L}(t)^{\ell/2-j}\Big[\boldsymbol{\varepsilon}^p_{\ell,j} + \textstyle\sum_{k\geqslant 1} c_{\ell,j,k} \mathrm{L}(t)^{-k} \boldsymbol{\varepsilon}^{p+k}_{\ell,j+k}\Big] \tag{6.8.9}$$

We denote by $\mathcal{H}^{p,w-p}$ the C^∞ bundle with basis $\boldsymbol{\varepsilon}^p$ on Δ^*, giving rise to a decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$, and by $F^p\mathcal{H}$ the C^∞ sub-bundle of $\mathcal{H}$ generated by the sub-basis $(\boldsymbol{\varepsilon}^{p'})_{p'\geqslant p}$, equivalently (due to (6.8.9)), the sub-basis $(\boldsymbol{v}^{p'})_{p'\geqslant p}$. This is clearly a holomorphic sub-bundle, either because $D''\boldsymbol{v}^p = 0$, or because the matrix M'' does not decrease p. Then $F^p\mathcal{V} := \operatorname{Ker} D''_{\mathcal{V}|F^p\mathcal{H}}$ is the $\mathcal{O}_{\Delta^*}$-submodule of $\mathcal{V} = \mathcal{O}_{\Delta^*}\cdot \boldsymbol{v}$ generated by the elements of $\boldsymbol{v}^{p'}_{\ell,j}$ for $p' \geqslant p$ and ℓ, j arbitrary. Since $\mathcal{V}^\beta_* = \mathcal{O}_\Delta \cdot \boldsymbol{v}$, we have $F^p\mathcal{V}^\beta_* = \mathcal{O}_\Delta \cdot \boldsymbol{v}^{\geqslant p}$, and the $\mathcal{O}_\Delta$-coherence is clear. Moreover, by construction, Griffiths transversality holds for $F^\bullet\mathcal{V}^\beta_*$ and the filtration induced on $\mathrm{gr}^\beta\mathcal{V}_* = \mathcal{H}^o$ is equal to $F^p\mathcal{H}^o$.

Let us now analyze the polarization. We define the sesquilinear form $\mathcal{S}$ on $\mathcal{H}$ by the expected rule

$$\mathcal{S}(\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{(\mathrm{C}_{\mathrm{D}})^{-1}\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet}),$$

where h is the metric for which $\boldsymbol{\varepsilon}$ is an orthonormal basis and C_{D} is relative to the decomposition $\mathcal{H} = \bigoplus \mathcal{H}^{p,w-p}$. By definition, when restricted to any point x of Δ^*, the sesquilinear form $\mathcal{S}$ is a polarization of the Hodge structure H_x. Furthermore, the matrix of $\mathrm{C_D}$ in the h-orthonormal frame $\boldsymbol{\varepsilon}$ is equal to $\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}$, so the matrix of $\mathcal{S}$ in the frame $\boldsymbol{\varepsilon}$ is $\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}$.

6.8.10. Lemma. *The sesquilinear form $\mathcal{S}$ is a polarization of the variation of Hodge structure H on Δ^* which satisfies* $\mathrm{gr}^\beta\mathcal{S} = \mathrm{S}^o$.

We thus find that $(\mathrm{gr}^\beta H, \mathrm{N}, \mathrm{gr}^\beta\mathcal{S})$, being identified with $(H^o, \mathrm{N}, \mathrm{S}^o)$, is a polarized Hodge-Lefschetz structure with central weight w.

Proof. In order to prove that $\mathcal{S}$ is D-horizontal, let us first compute the matrix S of $\mathcal{S}$ in the holomorphic basis $\boldsymbol{v}$. According to (6.3.18) and Exercise 6.5, we find

$$\begin{aligned}\mathsf{S} = {}^t\mathsf{P}_\beta\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \overline{\mathsf{P}}_\beta &= |t|^{2\beta}\mathrm{L}(t)^{\mathsf{H}/2}\,\mathrm{e}^{\mathsf{Y}}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathrm{e}^{\mathsf{X}}\,\mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,|t|^{2\beta}\mathrm{L}(t)^{\mathsf{H}/2}\,\mathrm{e}^{-\mathsf{Y}}\mathrm{e}^{\mathsf{X}}\,\mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,|t|^{2\beta}\mathrm{L}(t)^{\mathsf{H}/2}\,\mathrm{e}^{-\mathsf{X}}\,\mathsf{w}\,\mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,|t|^{2\beta}\mathrm{e}^{-\mathrm{L}(t)\,\mathsf{X}}\,\mathrm{L}(t)^{\mathsf{H}/2}\,\mathsf{w}\,\mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,|t|^{2\beta}\mathrm{e}^{-\mathrm{L}(t)\,\mathsf{X}}\,\mathsf{w}\\ &= |t|^{2\beta}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathsf{w} = \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathsf{w}\;|t|^{2\beta}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}.\end{aligned}$$

Recall (see (6.3.16 ∗)) that $\boldsymbol{v}\cdot t^{-\beta\,\mathrm{Id}+\mathsf{Y}}$ is a horizontal basis of the connection, and the matrix of $\mathcal{S}$ in this basis is, since the transpose of Y is X and both are real,

$$t^{-\beta\,\mathrm{Id}+\mathsf{X}}|t|^{2\beta}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathsf{w}\,\bar{t}^{-\beta\,\mathrm{Id}+\mathsf{Y}} = t^{\mathsf{X}}\,\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\bar{t}^{\mathsf{Y}} = t^{\mathsf{X}}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}\,\bar{t}^{\mathsf{Y}}.$$

Horizontality of $\mathcal{S}$ follows thus from the identities:

$$t\partial_t(t^{\mathsf{X}}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}) = 0 \quad\text{and}\quad \bar{t}\partial_{\bar{t}}(\mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}\,\bar{t}^{\mathsf{Y}}) = 0.$$

In order to prove the second part of the lemma, let us show that the matrix of $\mathrm{gr}^\beta\mathcal{S}$ is equal to S^o. By Exercise 6.13 (items (2) and (1)) the matrix of $\mathrm{gr}^\beta\mathcal{S}$ in the basis $\boldsymbol{v}^o$ is given by

$$\begin{aligned}\mathrm{Res}_{s=-\beta-1}\int_{\mathbb{C}}|t|^{2s}\,\mathsf{S}\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar{t}\qquad&\\ = \mathrm{Res}_{s+\beta=-1}\int_{\mathbb{C}}&|t|^{2(s+\beta)}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathsf{w}\,\mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar{t}\\ = \mathrm{Res}_{s+\beta=-1}\int_{\mathbb{C}}&|t|^{2(s+\beta)}\,\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathsf{w}\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar{t}\\ = \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\,\mathsf{w} = \mathsf{S}^o.&\end{aligned}$$

□

6.9. Exercises

Exercise 6.13 **(A residue formula for** $\mathrm{gr}^\beta\mathfrak{s}$**).** Let $\chi(t)$ be a C^∞ function with compact support on Δ which is $\equiv 1$ near $t=0$ (that we simply call a *cut-off function near* $t=0$). Assume that $\chi(t)$ only depends on $|t|$ (e.g. $\chi(t) = \widetilde{\chi}(|t|^2)$ where $\widetilde{\chi}$ is C^∞).

(1) Show that the function

$$s \longmapsto (s+1) \int_{\mathbb{C}} |t|^{2s} \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t}$$

is holomorphic for $\operatorname{Re} s > -1$ and extends as an entire function. Show that

$$\operatorname{Res}_{s=-1} \int_{\mathbb{C}} |t|^{2s} \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t} = 1.$$

[*Hint*: By expressing the integrand with respect to the real variables x, y with $t = x + \mathrm{i}\, y$, check the sign of the left-hand side; then compute with polar coordinates up to sign.]

(2) By differentiating k times for $\operatorname{Re} s > -1$, show that

$$\int_{\mathbb{C}} |t|^{2s} \, \frac{\mathrm{L}(t)^k}{k!} \, \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t} = \frac{(-1)^k}{(s+1)^{k+1}} + F_k(s),$$

where $F_k(s)$ is holomorphic for $\operatorname{Re} s > -1$ and extends as an entire function. Conclude that, for $k \geqslant 1$,

$$\operatorname{Res}_{s=-1} \int_{\mathbb{C}} |t|^{2s} \, \frac{\mathrm{L}(t)^k}{k!} \, \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t} = 0.$$

(3) Let $\mathfrak{s} : \mathcal{V}' \otimes \overline{\mathcal{V}}'' \to \mathcal{C}^\infty_{\Delta^*}$ be a sesquilinear pairing. For $\beta \in (-1, 0]$ and sections v' of $\mathcal{V}'^{\beta}_*$ and v'' of $\mathcal{V}''^{\beta}_*$, with respective classes $[v']$ and $[v'']$ in $\operatorname{gr}^\beta \mathcal{V}'_*$ and $\operatorname{gr}^\beta \mathcal{V}''_*$, show the formula

$$(\operatorname{gr}^\beta \mathfrak{s})([v'], \overline{[v'']}) = \operatorname{Res}_{s=-\beta-1} \int_{\mathbb{C}} |t|^{2s} \mathfrak{s}(v', \overline{v''}) \, \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

[*Hint*: Argue (in a simpler way) as in Proposition 12.5.4.]

6.10. Comments

The idea of defining the limiting Hodge filtration by a formula like (6.7.1) goes back to [**Sai84**], where M. Saito was inspired by the work of Steenbrink and Varchenko. This idea was further developed in his subsequent works, abutting to [**Sai88**]. The approach followed for the proof of Theorem 6.7.3 is that of Simpson [**Sim88, Sim90**], which was then extended in higher dimension by T. Mochizuki [**Moc11a**, Chap. 21] and revisited more recently by Deng [**Den22**]. In [**S-Sch22**], the results are obtained by means of the analysis of the period mapping and its convergence properties, more in the spirit of the fundamental work of Schmid [**Sch73**].

CHAPTER 6

VARIATIONS OF HODGE STRUCTURE ON CURVES
PART 3: THE HODGE-ZUCKER THEOREM

Summary. This part provides a proof of the Hodge-Zucker theorem 6.11.1. The notion of middle extension of a local system appears as the topological analogue of the L^2 extension of a Hermitian bundle with flat connection, and the main results consist in the algebraic computation of the L^2 de Rham and Dolbeault complexes.

6.11. Introduction

Our aim in this part is to present the proof of the *Hodge-Zucker theorem* 6.11.1 on a punctured compact Riemann surface, which is a Hodge theorem "with singularities". We mix the setting of Sections 4.2.c and 4.2.e, that is, we consider a polarized variation of Hodge structure (H, S) of weight w on a punctured compact Riemann surface $X^* \xhookrightarrow{j} X$.

***6.11.1. Theorem* (Hodge-Zucker).** *In such a case, the cohomology $H^k(X, j_*\underline{\mathcal{H}})$ carries a natural polarized Hodge structure of weight $w+k$ ($k = 0, 1, 2$).*

The way of using L^2 cohomology is exactly the same as in Section 4.2.e, provided that we replace D' and D'' with $\mathcal{D}'$ and $\mathcal{D}''$. Then we are left with the corresponding L^2 Poincaré and Dolbeault lemmas.

In any case, it is important to extend in some way the variation to the projective curve in order to apply algebraic techniques. What kind of an object should we expect on the projective curve? On the one hand, the theorems of Schmid enable us to extend each step of the Hodge filtration as an algebraic bundle over the curve. On the other hand, Zucker selects the interesting extension among all possible extensions in order to obtain the Hodge-Zucker theorem. This is the *middle extension* $(\mathcal{V}_{\mathrm{mid}}, \nabla)$ of the polarized variation of Hodge structure. This selection is suggested by the L^2 approach to the Hodge theorem. As in the previous parts of this chapter, we mainly work in a neighbourhood Δ of a puncture.

6.12. The holomorphic de Rham complexes

6.12.a. The meromorphic de Rham complex. Let $(\mathcal{V},\nabla)$ be any holomorphic bundle with connection on Δ^*. Recall that the holomorphic de Rham complex $\mathrm{DR}(\mathcal{V},\nabla)$ is the complex

$$0 \longrightarrow \mathcal{V} \xrightarrow{\ \nabla\ } \Omega^1_{\Delta^*}\otimes\mathcal{V} \longrightarrow 0,$$

whose cohomology is nonzero only in degree zero, with $H^0\,\mathrm{DR}(\mathcal{V},\nabla)=\mathcal{H}^\nabla := \operatorname{Ker}\nabla$.

Assume now that $(\mathcal{V}_*,\nabla)$ is a meromorphic bundle with connection on Δ, having a regular singularity at the origin and set $\mathcal{V}=\mathcal{V}_{*|\Delta^*}$. Let us consider the meromorphic de Rham complex $\mathrm{DR}(\mathcal{V}_*,\nabla)$, defined as the complex

$$0 \longrightarrow \mathcal{V}_* \xrightarrow{\ \nabla\ } \Omega^1_{\Delta}\otimes\mathcal{V}_* \longrightarrow 0.$$

Its restriction to Δ^* coincides with $\mathrm{DR}(\mathcal{V},\nabla)$, hence has nonzero cohomology in degree zero only. In other words, $H^1\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a skyscraper sheaf supported at the origin, and $H^0\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is some sheaf extension (across the origin) of the locally constant sheaf $\mathcal{V}^\nabla := \operatorname{Ker}\nabla$. We will determine these sheaves.

One can filter the de Rham complex, so that each term of the filtration is a complex whose terms are free $\mathcal{O}_\Delta$-modules of finite rank: for every β, we set

$$V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)=\{0 \longrightarrow \mathcal{V}_*^\beta \xrightarrow{\ \nabla\ } \Omega^1_\Delta\otimes\mathcal{V}_*^{\beta-1} \longrightarrow 0\}. \tag{6.12.1}$$

Since the action of t is invertible on $\mathcal{V}_*$, the latter complex is quasi-isomorphic to the complex

$$V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)=\{0 \longrightarrow \mathcal{V}_*^\beta \xrightarrow{\ t\nabla\ } \Omega^1_\Delta\otimes\mathcal{V}_*^{\beta} \longrightarrow 0\}.$$

***6.12.2. Lemma* (The de Rham complex of the canonical meromorphic extension)**

The inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)\hookrightarrow\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a quasi-isomorphism provided $\beta\leqslant 0$. Moreover, the germs at the origin of these complexes can be computed as the complex of finite dimensional vector spaces

$$0 \longrightarrow \mathrm{gr}^0\mathcal{V}_* \xrightarrow{\ t\partial_t\ } \mathrm{gr}^0\mathcal{V}_* \longrightarrow 0.$$

As a consequence, the natural morphism (in the derived category)

$$\mathrm{DR}(\mathcal{V}_*,\nabla) \longrightarrow \boldsymbol{R}j_*j^{-1}\,\mathrm{DR}(\mathcal{V}_*,\nabla)=\boldsymbol{R}j_*\,\mathrm{DR}(\mathcal{V},\nabla) \xleftarrow{\ \sim\ } \boldsymbol{R}j_*\mathcal{V}^\nabla$$

is an isomorphism.

Proof. For the first statement, we notice that it is enough to check that for every $\beta\leqslant 0$ and any $\gamma<\beta$, the inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla)\hookrightarrow V^\gamma\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a quasi-isomorphism. This amounts to showing that the quotient complex

$$0 \longrightarrow \mathcal{V}_*^\gamma/\mathcal{V}_*^\beta \xrightarrow{\ \partial_t\ } \mathcal{V}_*^{\gamma-1}/\mathcal{V}_*^{\beta-1} \longrightarrow 0$$

is quasi-isomorphic to zero for such pairs (β,γ), and an easy inductive argument reduces to proving that, for every $\gamma<0$, the complex

$$0 \longrightarrow \mathrm{gr}^\gamma\mathcal{V}_* \xrightarrow{\ t\partial_t\ } \mathrm{gr}^\gamma\mathcal{V}_* \longrightarrow 0$$

is quasi-isomorphic to zero. The result is now easy since $t\partial_t-\gamma$ is nilpotent on $\mathrm{gr}^\gamma\mathcal{V}_*$.

For the second statement, we are reduced to proving that the germ at the origin of the complex

$$0 \longrightarrow \mathcal{V}_*^{>0} \xrightarrow{t\partial_t} \mathcal{V}_*^{>0} \longrightarrow 0$$

is quasi-isomorphic to zero.[(2)]

Arguing as in Exercise 6.1, one can assume that $\mathcal{V}_*$ has rank 1, and has a basis v_γ ($\gamma \in [0,1)$) such that $t\nabla_{\partial_t} v_\gamma = \gamma \cdot v_\gamma$.

(1) If $\gamma \neq 0$, then $\mathcal{V}_*^{>0} = \mathcal{V}_*^0 = \mathcal{O}_\Delta v_\gamma$ and, setting $\mathcal{O} = \mathcal{O}_{\Delta,0}$, the result follows from the property that $(t\partial_t + \gamma) : \mathcal{O} \to \mathcal{O}$ is an isomorphism (easily checked on series expansions).

(2) If $\gamma = 0$, then $\mathcal{V}_*^{>0} = t\mathcal{V}_*^0 = t\mathcal{O}_\Delta v_0$, and the result follows from the property that $(t\partial_t + 1) : \mathcal{O} \to \mathcal{O}$ is an isomorphism, proved as above.

For the last statement, we first note that the morphism is functorial in $(\mathcal{V}_*, \nabla)$. We can therefore reduce to the case of rank 1 by the argument of Exercise 6.1. If $\gamma \neq 0$, the isomorphism is obvious since both complexes are quasi-isomorphic to zero. If $\gamma = 0$, the isomorphism property is checked in a straightforward way. □

6.12.b. The de Rham complex of the middle extension. This de Rham complex will be the main object for the Hodge-Zucker theorem 6.11.1. We first introduce the middle extension $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. We know that $\mathcal{V}_*$ is generated by $\mathcal{V}_*^{>-1}$ as an $\mathcal{O}_\Delta(*0)$-module (with connection). On the other hand, we define $\mathcal{V}_{\mathrm{mid}}$ as the $\mathcal{O}_\Delta$-submodule of $\mathcal{V}_*$ generated by $\mathcal{V}_*^{>-1}$ through the iterated action of ∇_{∂_t} (and not t^{-1}). In other words,

$$\mathcal{V}_{\mathrm{mid}} := \sum_{j\geqslant 0} (\nabla_{\partial_t})^j \, \mathcal{V}_*^{>-1} \subset \mathcal{V}_*. \tag{6.12.3}$$

(See Exercise 6.2(6).) The main properties of $\mathcal{V}_{\mathrm{mid}}$ are developed in Exercise 6.14.

We now compute the de Rham complex of the middle extension $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. For $\beta \in \mathbb{R}$, let us denote by $\lceil\beta\rceil = -[-\beta]$ the smallest integer bigger than or equal to β. We have $\gamma := \beta - \lceil\beta\rceil \in (-1, 0]$. We set, for any $\beta \in \mathbb{R}$ (inductively if $\beta \leqslant -1$),

$$\mathcal{V}_{\mathrm{mid}}^\beta = \begin{cases} \mathcal{V}_*^\beta & \text{if } \beta > -1, \\ (\nabla_{\partial_t})^k \, \mathcal{V}_*^\gamma + \mathcal{V}_*^{>\beta} & \text{if } \beta \leqslant -1, \\ & \text{with } k = -\lceil\beta\rceil = [-\beta],\ \gamma = \beta - \lceil\beta\rceil, \end{cases} \tag{6.12.4}$$

where $>\beta$ is the next β' such that $\mathrm{gr}^{\beta'}\mathcal{V}_* \neq 0$. For $\beta \leqslant -1$, the formula also reads

$$\mathcal{V}_*^\beta = (\nabla_{\partial_t})^k \, \mathcal{V}_*^\gamma + \sum_{j=0}^{k-1} (\nabla_{\partial_t})^j \, \mathcal{V}_*^{>-1}. \tag{6.12.5}$$

For example, $\mathcal{V}_{\mathrm{mid}}^{-1} = \partial_t \mathcal{V}_*^0 + \mathcal{V}_*^{>-1}$. We also set $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} := \mathcal{V}_{\mathrm{mid}}^\beta / \mathcal{V}_{\mathrm{mid}}^{>\beta}$. We note that, by Exercise 6.14(4), $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}}$ is naturally included in $\mathrm{gr}^\beta \mathcal{V}_*$ for each β and is preserved by the nilpotent endomorphism N.

(2) This is obviously *not true* away from the origin.

***6.12.6. Definition* (The morphisms can and var).** We define can : $\mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}$ as the homomorphism induced by $-\partial_t$ and var : $\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}$ as that induced by t, so that

$$\mathrm{var}\circ\mathrm{can} = \mathrm{N} : \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \longrightarrow \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \quad\text{and}\quad \mathrm{can}\circ\mathrm{var} = \mathrm{N} : \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}.$$

By the definition of $\mathcal{V}_{\mathrm{mid}}$, can is onto and var is injective. In other words, the corresponding quiver $(\mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}, \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}, \mathrm{can}, \mathrm{var})$ is a middle extension quiver, in the sense of Definition 3.3.10.

In a way similar to (6.12.1), the complex $\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ is filtered by the subcomplexes $V^\beta\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ whose terms are thus $\mathcal{O}_\Delta$-free of finite rank.

***6.12.7. Lemma* (The de Rham complex of the middle extension)**

The inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \hookrightarrow \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ *is a quasi-isomorphism provided* $\beta \leqslant 0$. *Moreover, the germs at the origin of these complexes can be computed as the complex of finite dimensional vector spaces*

$$0 \longrightarrow \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \xrightarrow{\ \partial_t\ } \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0.$$

As a consequence, $H^1\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) = 0$ *and the natural morphism*

$$H^0\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \longrightarrow j_*\mathcal{V}^\nabla$$

is an isomorphism.

Proof. For the first statement, we argue as in Lemma 6.12.2, together with Exercise 6.14(5). The second statement is obtained similarly by using Exercise 6.14(6). The last statement follows then from that of Lemma 6.12.2. □

In particular, since $t : \mathcal{V}_*^{-1} \to \mathcal{V}_*^0$ is injective, it induces an isomorphism

$$(\partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>-1}) \xrightarrow{\ \sim\ } (t\partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>0})$$

and we have

$$\{0 \to \mathcal{V}_*^0 \xrightarrow{\ t\partial_t\ } (t\partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>0}) \to 0\} \simeq V^0\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \xrightarrow{\ \sim\ } \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla). \tag{6.12.8}$$

We can refine the presentation (6.12.8) by using the lifted monodromy filtration $\mathrm{M}_\bullet\mathcal{V}_*^0$. Indeed, the finite dimensional vector space $\mathrm{gr}^0\mathcal{V}_*$ is equipped with the nilpotent endomorphism induced by $\mathrm{N} = -t\partial_t$, hence is equipped with the corresponding monodromy filtration $\mathrm{M}_\bullet\mathrm{gr}^0\mathcal{V}_*$ (see Lemma 3.3.1). We can then consider the lifted monodromy filtration $\mathrm{M}_\ell\mathcal{V}_*^0$ (see Definition 6.3.4).

6.12.9. Lemma. *The complex* $\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ *is quasi-isomorphic to*

$$\{0 \longrightarrow \mathrm{M}_0\mathcal{V}_*^0 \xrightarrow{\ t\partial_t\ } \mathrm{M}_{-2}\mathcal{V}_*^0 \longrightarrow 0\}.$$

Proof. Clearly, the complex in the lemma is a subcomplex of (6.12.8). Let us consider the quotient complex. This is

$$0 \longrightarrow (\mathrm{gr}^0\mathcal{V}_*/\mathrm{M}_0\mathrm{gr}^0\mathcal{V}_*) \xrightarrow{\ t\partial_t\ } (\mathrm{image}\, t\partial_t/\mathrm{M}_{-2}\mathrm{gr}^0\mathcal{V}_*) \longrightarrow 0. \tag{6.12.10}$$

Applying Lemma 3.3.7, we find that this complex is quasi-isomorphic to 0 (i.e., the middle morphism is an isomorphism). □

6.12.c. The holomorphic L^2 de Rham complex. The Hodge-Zucker theorem 6.11.1 relies on the L^2 computation of the hypercohomology of a de Rham complex, since this L^2 approach naturally furnishes a Hermitian form on the hypercohomology spaces (see Section 4.2.e). In order to analyze the global L^2 condition on a Riemann surface, it is convenient to introduce it in a local way, in the form of an L^2 de Rham complex. We will find in Theorem 6.12.15 the justification for focusing on the de Rham complex of the middle extension.

Hermitian bundle and volume form. Assume that the holomorphic vector bundle $\mathcal{V}$ on Δ^* is equipped with a metric h (equivalently, the C^∞ bundle $\mathcal{H} = \mathcal{C}^\infty_{\Delta^*} \otimes_{\mathcal{O}_{\Delta^*}} \mathcal{V}$ is equipped with such a metric). If we fix a metric on the punctured disc, with volume element vol, we can define the L^2-norm of a section v of $\mathcal{V}$ on an open set $U \subset \Delta^*$ by the formula

$$\|v\|_2^2 = \int_U \mathrm{h}(v, \overline{v}) \ \mathrm{vol}\,.$$

In order to be able to apply the techniques of Section 4.2.e, we choose a metric on Δ^* which is complete in the neighbourhood of the puncture. We will assume that, near the puncture, the volume form is given by

$$\text{(6.12.11)}\quad \mathrm{vol} = \frac{dx^2 + \mathrm{d}y^2}{|t|^2 \mathrm{L}(t)^2}, \quad \text{with } x = \operatorname{Re} t,\ y = \operatorname{Im} t,\ \mathrm{L}(t) := \bigl|\log |t|^2\bigr| = -\log t\bar{t}.$$

Let us be more explicit concerning the *Poincaré metric.* Working in polar coordinates $t = re^{i\theta}$ and volume element $\mathrm{d}\theta\,\mathrm{d}r/r$, vol can also be written as

$$\mathrm{vol} = \mathrm{L}(r)^{-2} \cdot \mathrm{d}\theta\,\mathrm{d}r/r$$

and the metric on $\mathcal{E}^1_{\Delta^*}$ is given by

$$\|\mathrm{d}r/r\| = \|\mathrm{d}\theta\| = \mathrm{L}(r).$$

We thus get a characterization of the L^2 behaviour of forms near the puncture:

$$(6.12.12)_0 \qquad f \in L^2(\mathrm{vol}) \iff |\log r|^{-1} f \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r);$$

$$(6.12.12)_1 \qquad \omega = f\,\mathrm{d}r/r + g\,\mathrm{d}\theta \in L^2(\mathrm{vol}) \iff f \text{ and } g \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r);$$

$$(6.12.12)_2 \qquad \eta = h\,\mathrm{d}\theta\,\mathrm{d}r/r \in L^2(\mathrm{vol}) \iff |\log r|\, h \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r).$$

For example, given a section $\omega \otimes v$ of $\Omega^1_{\Delta^*} \otimes \mathcal{V}$ on an open subset of Δ^*, where ω is written in polar coordinates as $f\,\mathrm{d}r/r + g\,\mathrm{d}\theta$, its L^2-norm with respect to the metric h and the volume vol is

$$\text{(6.12.13)} \qquad \|\omega \otimes v\|_2^2 = \|fv\|_2^2 + \|gv\|_2^2\,.$$

On the other hand, by Exercise 6.6, we have

$$\text{(6.12.14)} \qquad r^\beta\, |\log r|^{\ell/2} \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r) \iff \beta > 0 \text{ or } (\beta = 0 \text{ and } \ell \leqslant -2).$$

The holomorphic L^2 de Rham complex. We will consider the *holomorphic L^2 de Rham complex*

$$\mathrm{DR}(\mathcal{V}_*, \nabla)_{(2)} = \{0 \to \mathcal{V}_{*(2)} \xrightarrow{\nabla} (\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)} \to 0\},$$

which is the subcomplex of the meromorphic de Rham complex $\mathrm{DR}(\mathcal{V}_*, \nabla)$ defined in the following way:

- $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)}$ is the subsheaf of $\Omega^1_\Delta \otimes \mathcal{V}_*$ consisting of sections whose restriction to Δ^* is L^2 (with respect to the metric h on $\mathcal{V}$ and the volume vol on Δ^*),
- $\mathcal{V}_{*(2)}$ is the subsheaf of $\mathcal{V}_*$ consisting of sections v whose restriction to Δ^* is L^2 *and* such that ∇v belongs to $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)}$ defined above.

Let us note that, by the very definition, we get a complex. The following theorem is the first step toward an L^2 computation of $j_*\mathcal{V}^\nabla$.

6.12.15. Theorem. *If $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, we have $(\mathrm{DR}\,\mathcal{V}_{*(2)}) \simeq \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} = j_*\mathcal{V}^\nabla$.*

Proof. We start by identifying the terms in degree one, since the L^2 condition is simpler for them.

6.12.16. Lemma. *We have $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)} = (\mathrm{d}t/t) \otimes \mathrm{M}_{-2}\mathcal{V}^0_*$ and $\mathcal{V}_{*(2)} = \mathrm{M}_0\mathcal{V}^0_*$.*

Proof. Let v be a section of $\mathcal{V}_*$ such that $(\mathrm{d}t/t)\otimes v$ is L^2. Equivalently, both $(\mathrm{d}r/r)\otimes v$ and $\mathrm{d}\theta \otimes v$ are L^2, that is, v is L^2, according to $(6.12.12)_1$. If v is a section of $\mathrm{M}_\ell\mathcal{V}^\beta_*$, its norm behaves like $r^\beta \mathrm{L}(r)^{\ell/2}$ near the origin, and (6.12.14) implies that the L^2 condition is achieved iff $\beta > 0$ or $\beta = 0$ and $\ell \leqslant -2$.

Similarly, one checks that the holomorphic sections of $\mathcal{V}$ which are L^2 near the origin are the sections of $\mathrm{M}_0\mathcal{V}^0_*$, since one is led to test whether $\mathrm{L}(r)^{-1}\,\|v\|_{\mathrm{h}}$ is L^2 or not. In order to conclude that $\mathcal{V}_{*(2)} = \mathrm{M}_0\mathcal{V}^0_*$, it is enough to check that $t\partial_t(\mathrm{M}_0\mathcal{V}^0_*) \subset \mathrm{M}_{-2}\mathcal{V}^0_*$. This immediately follows from the definition of the monodromy filtration $\mathrm{M}_\bullet$. □

This conclude the proof of Theorem 6.12.15, since $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is expressed by the formula of Lemma 6.12.9. □

6.13. The L^2 de Rham complex and the L^2 Poincaré lemma

We take up the definitions of Section 4.2.d. The role of the complex manifold X is played by Δ^* with its Poincaré metric, which induces a metric on the sheaves of C^∞ differential forms on Δ^*, and the value of the L^2-norm of forms up to a positive constant is given by the formulas (6.12.12).

Let $\mathcal{H}$ be a C^∞ bundle $\mathcal{H}$ on Δ^*, equipped with a Hermitian metric h. Correspondingly, the sheaf $\mathcal{E}^i_{\Delta^*} \otimes \mathcal{H}$ is equipped with a metric, and the L^2 norm of a section of this sheaf is given by a formula like (6.12.13). The various L^2 sheaves are thus defined on Δ^*, and we can use the notion of L^2-adapted basis (see Definition 4.2.21).

6.13.1. Examples **(of L^2-adapted frames).**

(1) The frame $(\mathrm{d}r/r, \mathrm{d}\theta)$ is an L^2-adapted frame of $\mathcal{E}^1_{\Delta^*}$. If $\boldsymbol{v}$ is an L^2-adapted frame of $\mathcal{H}$, then $(\mathrm{d}r/r\otimes\boldsymbol{v}, \mathrm{d}\theta\otimes\boldsymbol{v})$ is an L^2-adapted frame of $\mathcal{E}^1_{\Delta^*}\otimes\mathcal{H}$.

(2) In the setting of the model of Section 6.3.c, the frame $\boldsymbol{v}$ is L^2-adapted. Indeed, the frame $\boldsymbol{\varepsilon}\cdot\mathrm{e}^{\mathsf{X}}$ is L^2-adapted by 4.2.22(4), and $\boldsymbol{v}$ is obtained by a rescaling of the latter, so 4.2.22(3) gives the assertion.

Let us group (with respect to $\beta\in(-1,0]$) the model frames of Section 6.3.c to get a frame $\boldsymbol{v}=(\boldsymbol{v}_\beta)_\beta$ of $\mathcal{V}_*^{>-1}$ and let $\boldsymbol{v}^o$ denote its restriction to $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$. It corresponds, via the canonical decomposition $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}=\bigoplus_{\beta>-1}\mathrm{gr}^\beta\mathcal{V}_*$, to grouping of the bases $\boldsymbol{v}^o$ of Section 6.3.c.

Assume that $(\mathcal{V}_*,\nabla)$ underlies a polarized variation of Hodge structure (H,S) on Δ^*. By Theorem 6.8.7, $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ underlies a polarized Hodge-Lefschetz structure, and we can define on it the model basis $\boldsymbol{v}^o$ as above.

6.13.2. Proposition **(A criterion for L^2-adaptedness).** *With these assumptions, let $\boldsymbol{v}'$ be any holomorphic frame of $\mathcal{V}_*^{>-1}$ such that its restriction to $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ is equal to $\boldsymbol{v}^o$. Then $\boldsymbol{v}'$ is L^2-adapted with respect to the Hodge metric.*

Proof. According to Theorem 6.3.11 and Lemma 4.2.22(2), we can replace the Hodge metric by the model metric, that we still denote by h. Then the model frame $\boldsymbol{v}$ is expressed as

$$\boldsymbol{v}=\boldsymbol{\varepsilon}\cdot\mathrm{e}^{\mathsf{X}}\,\mathsf{P}_{\mathrm{diag}}(t),$$

where now X denotes the diagonal bloc matrix with diagonal β-bloc corresponding to that of Section 6.3.c, and similarly $\mathsf{P}_{\mathrm{diag}}$ has diagonal blocs P_β. On the other hand, we can write $\boldsymbol{v}'=\boldsymbol{v}\cdot(\mathrm{Id}+t\,\mathsf{A}(t))$ for some holomorphic matrix $\mathsf{A}(t)$. Then

$$\boldsymbol{v}'=\boldsymbol{\varepsilon}\cdot\mathrm{e}^{\mathsf{X}}\,\mathsf{P}_{\mathrm{diag}}(t)(\mathrm{Id}+t\,\mathsf{A}(t))=\boldsymbol{\varepsilon}\cdot\mathrm{e}^{\mathsf{X}}(\mathrm{Id}+t\,\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1})\,\mathsf{P}_{\mathrm{diag}}(t).$$

An entry of $\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1}$ is obtained from the corresponding one of A by multiplying it by a term of the form $|t|^{\beta'-\beta}\mathrm{L}(t)^{k/2}$ for some suitable $\beta,\beta'\in(-1,0]$ and $k\in\mathbb{Z}$. Since $|\beta'-\beta|<1$, it follows that $\mathrm{Id}+t\,\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1}$ is bounded as well as its inverse matrix, so that $\boldsymbol{\varepsilon}\cdot\mathrm{e}^{\mathsf{X}}(\mathrm{Id}+t\,\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1})$ is L^2-adapted, according to Lemma 4.2.22(4). Since $\boldsymbol{v}'$ is obtained from the latter by applying a rescaling, it is also L^2-adapted (Lemma 4.2.22(3)). □

The L^2 sheaves $\mathcal{L}_{(2)}(\mathcal{E}^i_{\Delta^*}\otimes\mathcal{H},\mathrm{h})$ can be extended as sheaves on Δ by the assignment $U\mapsto L^2(U\cap\Delta^*,\mathcal{E}^i_{\Delta^*}\otimes\mathcal{H},\mathrm{h})$. We simply denote them by $\mathcal{L}^i_{(2)}(\mathcal{H},\mathrm{h})$.

Assume moreover that $\mathcal{H}$ is equipped with a flat connection

$$D=D'+D'':\mathcal{H}\longrightarrow\mathcal{E}^1_{\Delta^*}\otimes\mathcal{H}.$$

By flatness, the bundle $\mathcal{V}=\operatorname{Ker}D''$ equipped with the connection ∇ induced by D' is a holomorphic bundle with holomorphic connection on Δ^*. Moreover, $\underline{\mathcal{H}}:=\operatorname{Ker}D=\mathcal{V}^\nabla:=\operatorname{Ker}\nabla$ is a locally constant sheaf on Δ^*. The sheaf $\mathcal{L}_{(2)}(\mathcal{H},\mathrm{h},D)$ on Δ^* (see Definition 4.2.26) can similarly be extended as a sheaf on Δ. If $U\subset\Delta$ is an open subset containing the origin, a section $u\in L^2(U,\mathcal{H},\mathrm{h})$ belongs to $\Gamma(U,\mathcal{L}_{(2)}(\mathcal{H},\mathrm{h},D))$

if its restriction to $U\cap\Delta^*$ belongs to $\Gamma(U\cap\Delta^*,\mathcal{L}_{(2)}(\mathcal{H},\mathrm{h},D))$ and if $Du\in L^2(U,\mathcal{H},\mathrm{h})$. One can use the approximation lemma 4.2.24.

The L^2 de Rham complex (4.2.27) reads

$$0\longrightarrow \mathcal{L}^0_{(2)}(\mathcal{H},\mathrm{h},D)\xrightarrow{\;D\;}\mathcal{L}^1_{(2)}(\mathcal{H},\mathrm{h},D)\xrightarrow{\;D\;}\mathcal{L}^2_{(2)}(\mathcal{H},\mathrm{h},D)\longrightarrow 0, \tag{6.13.3}$$

where the upper index refers to the degree of forms, as a complex of sheaves on Δ. Clearly, $\mathcal{L}^2_{(2)}(\mathcal{H},\mathrm{h},D)=\mathcal{L}^2_{(2)}(\mathcal{H},\mathrm{h})$ since the latter condition is tautologically satisfied. When restricted to Δ^* the L^2 Poincaré lemma 4.2.28 shows that the complex (6.13.3) is a resolution of the locally constant sheaf $\underline{\mathcal{H}}$.

Without further conditions on $(\mathcal{H},\mathrm{h},D)$, one cannot give much information on (6.13.3) near the origin. The polarized Hodge property provides the formula we expect.

***6.13.4. Theorem* (L^2 Poincaré lemma).** *If $(\mathcal{V},\nabla,\mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, the natural inclusion of complexes $(\mathrm{DR}\,\mathcal{V}_{*(2)})\hookrightarrow\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$ is a quasi-isomorphism. Equivalently (see Theorem 6.12.15),*

1. *the L^2 complex $\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$ has nonzero cohomology in degree zero at most,*
2. *the inclusion $j_*\underline{\mathcal{H}}\simeq H^0(\mathrm{DR}\,\mathcal{V}_{*(2)})\hookrightarrow H^0\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$ is an isomorphism.*

By Lemma 4.2.28, its suffices to prove the theorem for the germ of the L^2 de Rham complex at the origin. The assertions amount then to

1. $H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)_0)=(j_*\underline{\mathcal{H}})_0$,
2. $H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)_0)$ and $H^2(\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)_0)$ are zero.

Applying the hypercohomology functor to Theorems 6.13.4 and 6.12.15, we obtain:

6.13.5. Theorem. *Let $j:X^*\hookrightarrow X$ be the inclusion of the complement of a finite set in a compact Riemann surface X. If $(\mathcal{V},\nabla,\mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure on X^*, the cohomology $H^\bullet(X,j_*\mathcal{V}^\nabla)$ is equal to the L^2 cohomology of the C^∞-bundle with flat connection $(\mathcal{H},D)$ associated with the holomorphic bundle $(\mathcal{V},\nabla)$, the L^2 condition being taken with respect to the Hodge metric h on $\mathcal{H}$ and a complete metric on X^*, locally equivalent near each puncture to the Poincaré metric.* □

The L^2 Poincaré pairing. For $i,j\geqslant 0$ with $i+j=2$, we have a natural pairing of sheaves

$$\mathcal{L}^i_{(2)}(\mathcal{H})\otimes\mathcal{L}^j_{(2)}(\mathcal{H})\longrightarrow\mathcal{L}^2_{(1)}(\mathcal{H}), \tag{6.13.6}$$

where $\mathcal{L}^2_{(1)}(\mathcal{H})$ denotes the sheaf of L^1_{loc} 2-forms (i.e., $(1,1)$-forms) on X, which can thus be integrated. This pairing is compatible with the differential, and induces therefore a pairing of graded complexes, which in turn produces, by taking global sections, a pairing on cohomology.

Proof of Theorem* 6.13.4*: first reduction. We consider the decomposition (6.2.5 ∗∗) and we work with the corresponding decomposition (6.3.28) of $(\mathcal{H}, D)$. According to Theorem 6.3.11, we can replace the metric h with the model metric $\mathrm{h}^{\mathrm{Del}}$ without changing the L^2 de Rham complex. We now simply denote by h the model metric on each $\mathcal{H}_\beta$. We thus have

$$\mathcal{L}^\bullet_{(2)}(\mathcal{H}, \mathrm{h}, D) \simeq \bigoplus_{\beta\in(-1,0]} \mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D). \tag{6.13.7}$$

The assertions (1) and (2) above can thus be shown for each $\mathcal{H}_\beta$ separately. We notice that, if $\beta \neq 0$, (1) is also a vanishing assertion.

Let us fix $\beta \in (-1, 0]$ and let us work with the model frame $(\boldsymbol{v}_{\beta,\ell})_\ell$ of Section 6.3.c (we now do not distinguish the components in the Lefschetz decomposition and set $\boldsymbol{v}_{\beta,\ell} = (\boldsymbol{v}_{\beta,\ell',j})_{\ell'-2j=\ell}$ so that $\boldsymbol{v}^o_{\beta,\ell}$ is a basis of $\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^\beta \mathcal{V}_*$). We have seen in Example 6.13.1(2) that this frame is L^2-adapted. Denoting by $\mathcal{H}_{\beta,\ell}$ the subbundle framed by $(\boldsymbol{v}_{\beta,\ell})$, and setting $\mathrm{M}_\ell\mathcal{H}_\beta = \bigoplus_{\ell'\leqslant\ell} \mathcal{H}_{\beta,\ell'}$ that we equip with the induced metric, L^2-adaptedness implies an exact sequence for each i

$$0 \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_{\ell-1}\mathcal{H}_\beta, \mathrm{h}) \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}) \longrightarrow \mathcal{L}^i_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}) \longrightarrow 0.$$

On the other hand, since $\mathrm{M}_\ell\mathcal{H}_\beta$ is preserved by the connection, we can equip $\mathcal{H}_{\beta,\ell}$ with the quotient connection by means of the identification with $\mathrm{gr}^{\mathrm{M}}_\ell\mathcal{H}_\beta$, that is, $D\boldsymbol{v}_{\beta,\ell} = \beta(\mathrm{d}t/t) \otimes \boldsymbol{v}_{\beta,\ell}$. We thus have an exact sequence

$$0 \longrightarrow (\mathrm{M}_{\ell-1}\mathcal{H}_\beta, D) \longrightarrow (\mathrm{M}_\ell\mathcal{H}_\beta, D) \longrightarrow (\mathcal{H}_{\beta,\ell}, D) \longrightarrow 0.$$

Then one checks that the sequence

$$0 \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_{\ell-1}\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^i_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}, D) \longrightarrow 0$$

is exact, leading to an exact sequence of complexes

$$0 \longrightarrow \mathcal{L}^\bullet_{(2)}(\mathrm{M}_{\ell-1}\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^\bullet_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}, D) \longrightarrow 0.$$

We can thus regard $\mathcal{L}^\bullet_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}, D)$ as defining an increasing filtration of the complex $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D)$ with associated graded complexes $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}, D)$. Since this filtration is finite, $H^k(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D)_0)$ is the abutment of a spectral sequence with E_1 term defined as (taking into account that M_ℓ is increasing, that we make decreasing by setting $\mathrm{M}^p = \mathrm{M}_{-p}$)

$$E_1^{p,q} = H^{p+q}(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,-p}, \mathrm{h}, D)_0) \implies H^{p+q}(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D)_0). \tag{6.13.8}$$

We first aim at computing $E_1^{p,q}$. The main tool will be Hardy's inequalities.

Proof of Theorem* 6.13.4*: Hardy's inequalities and an application. We will make use of the following type of inequalities, called Hardy's inequalities.

***6.13.9. Theorem* (L^2 Hardy inequalities, see e.g. [OK90, Th. 1.14])**

Let R be a real number in $(0,1)$ and let v, w be two functions (weights) on $I_R = (0, R)$, which are measurable and almost everywhere positive and finite. Let f be a C^1

function[3] *on I. Then the following inequality holds between L^2 norms with respect to the measure* $\mathrm{d}r$*:*

$$\|f\cdot w\|_2 \leqslant C\|f'\cdot v\|_2,$$

with

$$C=\begin{cases}\sup_{r\in I}\int_r^R w(\rho)^2\,\mathrm{d}\rho\cdot\int_0^r v(\rho)^{-2}\,\mathrm{d}\rho & \text{if } \lim_{r\to 0_+} f(r)=0,\\ \sup_{r\in I}\int_0^r w(\rho)^2\,\mathrm{d}\rho\cdot\int_r^R v(\rho)^{-2}\,\mathrm{d}\rho & \text{if } \lim_{r\to R_-} f(r)=0.\end{cases}$$

□

6.13.10. Corollary. *Let $(b,k)\in\mathbb{R}\times\mathbb{Z}$ with $(b,k)\neq(0,1)$. Given $g(r)$ continuous and integrable on I_R, let us set*

$$f(r)=\begin{cases}\int_0^r g(\rho)\,\mathrm{d}\rho & \text{if } b<0 \text{ or if } (k\geqslant 2 \text{ and } b=0),\\ \int_{\min(R,\mathrm{e}^{-k/2b})}^r g(\rho)\,\mathrm{d}\rho & \text{if } b>0 \text{ or if } (k\leqslant 0 \text{ and } b=0).\end{cases}$$

(In the second case, we replace $\mathrm{e}^{-k/2b}$ with its limit $+\infty$ when $b\to 0_+$.) Then there exists a constant $C=C(k,b)>0$ such that the following inequality holds (we consider $L^2(I_R;\mathrm{d}r/r)$ norms)

$$\begin{aligned}\|f(r)\cdot r^b\mathrm{L}(r)^{k/2-1}\|_{2,\mathrm{d}r/r} &\leqslant C\|g(r)\cdot r^b\mathrm{L}(r)^{k/2-1}\cdot r\mathrm{L}(r)\|_{2,\mathrm{d}r/r}\\ &= C\|g(r)\cdot r^{b+1}\mathrm{L}(r)^{k/2}\|_{2,\mathrm{d}r/r}.\end{aligned}$$

Moreover, for k fixed, there exists $b_o=b_o(R)>0$ such that, for $|b|\geqslant b_o$, the constant $C(k,b)$ can be chosen equal to 1.

The case where $b=0$ and $k=1$ is missing. This leads to the following definition, where we are only interested in germs at the origin, so that $R\in(0,1)$ can be arbitrary small.

6.13.11. Definition. The "Hardy bad space" $\mathfrak{H}$ is the quotient of the space of measurable functions g on I_R for some $R\in(0,1)$ such that $\|g(r)\cdot r\mathrm{L}(r)^{1/2}\|_{2,\mathrm{d}r/r}<\infty$, modulo the space of such g's which can be realized (maybe with a smaller R) as the weak derivative f' of functions f which are L^1_{loc} on I_R and satisfy $\|f(r)\cdot\mathrm{L}(r)^{-1/2}\|_{2,\mathrm{d}r/r}<\infty$.

***Proof of Corollary* 6.13.10.** We will choose the following weight functions with respect to the measure dr:

$$w(r)=r^{b-1/2}\mathrm{L}(r)^{k/2-1} \quad\text{and}\quad v(r)=r^{b+1/2}\mathrm{L}(r)^{k/2}.$$

[3] A weaker property (absolute continuity on every closed subinterval) is sufficient.

The case $b > 0$***, and the case*** $b = 0$ ***with*** $k \leqslant 0$***.***

(1) If $b > 0$ and $R \leqslant \mathrm{e}^{-k/2b}$, i.e., $k/2b \leqslant \mathrm{L}(R)$, we set $b_o = |k|/2\mathrm{L}(R)$ and we have

$$f(r) = -\int_r^R g(\rho)\,\mathrm{d}\rho$$

and $\lim_{r\to R_-} f(r) = 0$. We will show the finiteness of

$$\sup_{r\in[0,R]}\Big(\int_0^r \rho^{2b-1}\mathrm{L}(\rho)^{k-2}\,\mathrm{d}\rho\cdot\int_r^R \rho^{-2b-1}\mathrm{L}(\rho)^{-k}\,\mathrm{d}\rho\Big).$$

After the change of variable $y = \mathrm{L}(\rho)$ and setting $x = \mathrm{L}(r)$, we have to estimate

$$\sup_{x\in(\mathrm{L}(R),+\infty)}\Big(\int_x^{+\infty} \mathrm{e}^{-2by}y^k\,\frac{\mathrm{d}y}{y^2}\cdot\int_{\mathrm{L}(R)}^x \mathrm{e}^{2by}y^{-k}\,\mathrm{d}y\Big).$$

The function $y \mapsto \mathrm{e}^{-2by}y^k$ is decreasing on $(\mathrm{L}(R), +\infty)$, hence the first integral is bounded by $\mathrm{e}^{-2bx}x^{k-1}$, and the second one by $\mathrm{e}^{2bx}x^{-k}(x - \mathrm{L}(R))$, so the sup is bounded by one. Hardy's inequality holds with $C = 1$.

If $k \leqslant 0$ and $b = 0$, the same argument applies and gives the same constant $C = 1$.

(2) Assume now $b > 0$ and $k/2b > \mathrm{L}(R) > 0$, so that $k \geqslant 1$. We have

$$f(r) = \int_{\mathrm{e}^{-k/2b}}^r g(\rho)\,\mathrm{d}\rho \quad\text{and}\quad \lim_{r\to 0+} f(r) = 0.$$

We will show the finiteness of

$$\sup_{r\in[0,R]}\Big(\int_r^R \rho^{2b-1}\mathrm{L}(\rho)^{k-2}\,\mathrm{d}\rho\cdot\int_0^r \rho^{-2b-1}\mathrm{L}(\rho)^{-k}\,\mathrm{d}\rho\Big).$$

We decompose the argument following whether $r \in (0, \mathrm{e}^{-k/2b})$ or $r \in (\mathrm{e}^{-k/2b}, R)$.

(a) If $r \in (0, \mathrm{e}^{-k/2b})$, we can apply the same argument as in (1) after replacing $\mathrm{L}(R)$ with $k/2b$, and we can therefore choose $C = 1$.

(b) If $r \in (\mathrm{e}^{-k/2b}, R)$, we want show the finiteness of

$$\int_{\mathrm{L}(R)}^x \mathrm{e}^{-2by}y^k\,\frac{\mathrm{d}y}{y^2}\cdot\int_x^{k/2b} \mathrm{e}^{2by}y^{-k}\,\mathrm{d}y,$$

with $x \in (k/2b, +\infty)$. The function $\mathrm{e}^{-2by}y^k$ is increasing, and the second integral is bounded by $\mathrm{e}^k(k/2b)^{-k}(k/2b - x)$, hence by $\mathrm{e}^k(k/2b)^{-k+1}$. Similarly, the first one is bounded by $\mathrm{e}^{-2bx}x^k(1/\mathrm{L}(R) - 1/x) = \mathrm{e}^{-2bx}x^{k-1}(x/\mathrm{L}(R) - 1)$ which has limit zero when $x \to \infty$.

The case $b < 0$ ***and the case*** $b = 0$ ***with*** $k \geqslant 2$***.*** If $b < 0$, we have

$$f(r) = \int_0^r g(\rho)\mathrm{d}\rho \quad\text{and}\quad \lim_{r\to 0_+} f(r) = 0.$$

(1) We assume that $\mathrm{e}^{(k-2)/2|b|} \geqslant R$, i.e., $k \geqslant 2(1 - |b|\,\mathrm{L}(R))$, which is satisfied in particular whenever $k \geqslant 2$. We also set $b_o = |2 - k|/\mathrm{L}(R)$. Then the function $\mathrm{e}^{-2by}y^{k-2}$ is increasing on $(\mathrm{L}(R), +\infty)$. An upper bound of

$$\int_{\mathrm{L}(R)}^x \mathrm{e}^{-2by}y^{k-2}\,\mathrm{d}y\cdot\int_x^{+\infty} \mathrm{e}^{2by}y^{2-k}\,\frac{\mathrm{d}y}{y^2}$$

is given by

$$(x-\mathrm{L}(R))\mathrm{e}^{-2bx}x^{k-2}\cdot \mathrm{e}^{2bx}x^{-k+2}x^{-1}=(1-\mathrm{L}(R)/x)\leqslant 1.$$

The case when $b=0$ and $k\geqslant 2$ can be treated in a similar way.

(2) If $\mathrm{e}^{(k-2)/2|b|}<R$, i.e., $k<2(1-|b|\,\mathrm{L}(R))$, the function $\mathrm{e}^{-2by}y^{k-2}$ is decreasing on $(\mathrm{L}(R),+\infty)$. The first integral is bounded by $\mathrm{e}^{2|b|\mathrm{L}(R)}\mathrm{L}(R)^{k-2}(x-\mathrm{L}(r))$. For the second one, we can choose $\varepsilon>0$ small enough such that $\mathrm{e}^{2by}y^{-k}$ is bounded by $C_\varepsilon \mathrm{e}^{-2(|b|-\varepsilon)y}$ on $[\mathrm{L}(R),+\infty)$ and we bound the second integral by $C_\varepsilon\mathrm{e}^{-2(|b|-\varepsilon)x}/2(|b|-\varepsilon)$. Hence, the product of both integrals tends to 0 when $x\to+\infty$. □

Proof of Theorem* 6.13.4*: computation of of the* E_1*-term of the spectral sequence (6.13.8)

We will prove the following precise result as a consequence of Hardy's inequalities.

6.13.12. Lemma.

(1) *If* $\beta\neq 0$*, the cohomology spaces of* $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0$ *all vanish.*

(2) *If* $\beta=0$*, the cohomology spaces of* $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0$ *are given by the following formulas.*

$$(6.13.12)_0\qquad H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0)=\begin{cases}\mathcal{H}^o_{0,\ell} & \text{if } \ell\leqslant 0,\\ 0 & \text{if } \ell\geqslant 1,\end{cases}$$

$$(6.13.12)_1\qquad H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0)=\begin{cases}\mathcal{H}^o_{0,\ell}\otimes \mathrm{d}\theta & \text{if } \ell\leqslant -2,\\ \mathfrak{H}\otimes_{\mathbb{C}}\mathcal{H}^o_{0,2}\otimes(\mathrm{d}r/r) & \text{if } \ell=1,\\ 0 & \text{otherwise},\end{cases}$$

$$(6.13.12)_2\qquad H^2(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0)=\begin{cases}0 & \text{if } \ell\neq -1,\\ \mathfrak{H}\otimes_{\mathbb{C}}\mathcal{H}^o_{0,0}\otimes((\mathrm{d}r/r)\wedge \mathrm{d}\theta) & \text{if } \ell=-1.\end{cases}$$

Proof. Recall that $\mathfrak{H}$ is introduced in Definition 6.13.11. Since D is diagonal with respect to the frame $\boldsymbol{v}_{\beta,\ell}$ of $\mathcal{H}_{\beta,\ell}$, and by L^2-adaptedness, we may, and will, assume during the proof that $\mathcal{H}_{\beta,\ell}$ has rank 1 with frame $v_{\beta,\ell}$. We will use the following lemma.

6.13.13. Lemma.

(1) *Let* $f(r)\in L^2(I_R,r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$*. Then there exists a sequence* $f_m\in C^0(I_R)$ *such that* $f_m\to f$ *in* $L^2(I_R,r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$*.*

(2) *Let* $f(r)\in L^2(I_R,r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$ *be such that* $f'(r)\in L^2(I_R,r^{2\beta}\mathrm{L}(r)^{k+2}\mathrm{d}r/r)$*. Then there exists a sequence* $f_m\in C^1(I_R)$ *such that* $f_m\to f$ *in* $L^2(I_R,r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$ *and* $f'm\to f'$ *in* $L^2(I_R,r^{2\beta}\mathrm{L}(r)^{k+2}\mathrm{d}r/r)$*.*

Computation of $H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0)$**.** If $\beta\in(-1,0)$, there is no nonzero germ of horizontal section of $(j_*\mathcal{L}^1_{\mathrm{loc}}\otimes\mathcal{H}_{\beta,\ell})_0$, a fortiori no nonzero L^2 section. Let us thus assume $\beta=0$, so that the connection is simply d. Then $H^0(j_*\mathcal{L}^1_{\mathrm{loc}}\otimes\mathcal{H}_{\beta,\ell})_0=\mathbb{C}v_{0,\ell}$ and the question reduces to whether $\mathrm{L}(t)^{-1} \cdot \mathrm{L}(t)^{\ell/2} \in L^2(\mathrm{d}\theta\, \mathrm{d}r/r)$, according to $(6.12.12)_0$. In conclusion, due to (6.12.14),

$$H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, D)_0) = \begin{cases} \mathbb{C}v_{0,\ell} & \text{if } \beta = 0 \text{ and } \ell \leqslant 0, \\ 0 & \text{if } \beta \neq 0 \text{ or if } (\beta = 0 \text{ and } \ell \geqslant 1). \end{cases}$$

Computation of $H^2(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, D)_0)$**.** As a prelude to the Dolbeault case, we wish to solve $D(f(r,\theta)(\mathrm{d}t/t) \otimes v_{\beta,\ell}) = \eta$ weakly, that is, $\bar{t}\partial_{\bar{t}} f = -h$ weakly, or equivalently $\frac{1}{2}(r\partial_r + \mathrm{i}\,\partial_\theta) f = -h$ weakly on Δ^*_R with $R < 1$ small enough, with $f \cdot (\mathrm{d}t/t) \otimes v_{\beta,\ell}$ in $\Gamma(\Delta^*_R, \mathcal{L}^1_{(2)}(\mathcal{H}_{\beta,\ell}, D))$.

Let us develop a section $\eta = h(r,\theta)(\mathrm{d}t/t) \wedge (\mathrm{d}\bar{t}/\bar{t})$ of $j_*\mathcal{E}^2_{\Delta^*_R}$ in Fourier series, with $h(r,\theta) = \sum_{n\in\mathbb{Z}} h_n(r) e^{\mathrm{i}\,n\theta}$. The L^2 condition $(6.12.12)_2$ twisted by $v_{\beta,\ell}$ reads

$$\sum_n \|h_n(r) \cdot r^\beta \mathrm{L}(r)^{1+\ell/2}\|^2_{2,\mathrm{d}r/r} < +\infty.$$

Solving termwise the above differential equation amounts to solving in the weak sense

$$(r^{-n} f_n(r))' = -2r^{-n-1} h_n(r), \tag{6.13.14}$$

with $f_n(r) \in L^1_{\mathrm{loc}}(I_R)$ and

$$\|f_n(r) r^\beta \mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r} \leqslant C \|h_n(r) \cdot r^\beta \mathrm{L}(r)^{1+\ell/2}\|_{2,\mathrm{d}r/r} \tag{6.13.15}$$

for each n and a constant C independent of n. According to Lemma 6.13.13(1), for n fixed, we can choose a sequence $h_{n,m} \in C^0(I_R)$ such that

$$h_{n,m} \longrightarrow h_n \quad \text{in } L^2(I_R, r^{2\beta}\mathrm{L}(r)^{\ell+2}\mathrm{d}r/r) \text{ when } m \longrightarrow \infty.$$

In particular, for m large, $h_{n,m} \in L^2(I_R, r^{2\beta}\mathrm{L}(r)^{\ell+2}\mathrm{d}r/r)$. Assume that we have solved (6.13.14) for $h_{n,m}$ with $f_{n,m}$ being C^1 on I_R and satisfying (6.13.15) for a constant C independent of n, m. Then, by arguing with Cauchy sequences, $f_n = \lim_{m\to\infty} f_{n,m}$ exists in $L^2(I_R, r^{2\beta}\mathrm{L}(r)^\ell \mathrm{d}r/r)$ and solves (6.13.14) for h_n in the weak sense.

According to Lemma 6.13.13, we can thus assume that h_n is continuous on I_R. Let us set $b = \beta + n$. Due to Corollary 6.13.10 we can solve (6.13.14) with

$$\|f_n(r) r^\beta \mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r} \leqslant C \|h_n(r) \cdot r^\beta \mathrm{L}(r)^{1+\ell/2}\|_{2,\mathrm{d}r/r}$$

- for any ℓ, if $\beta \in (-1, 0)$, or if $\beta = 0$ and $n \neq 0$,
- for $\ell \neq -1$, if $\beta = 0$ and $n = 0$.

Notice that the constant C can be chosen independent of n since, for $|n|$ large, i.e., $|b|$ large, it can be chosen equal to 1. Therefore, we obtain the first line of $(6.13.12)_2$, as well as the second line by definition of $\mathfrak{H}$.

Computation of $H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, D)_0)$**.** As in the previous case, we start with $\omega \otimes v_{\beta,\ell} = [f\mathrm{d}r/r + g\mathrm{d}\theta] \otimes v_{\beta,\ell}$ with f and g expanded as Fourier series with coefficients f_n, g_n satisfying

$$\sum_n \|f_n(r) \cdot r^\beta \mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r} + \sum_n \|g_n(r) \cdot r^\beta \mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r} < +\infty.$$

The closedness of $D(\omega\otimes v_{\beta,\ell})$ reads $(r\partial_r+\beta)g=(\partial_\theta+i\beta)f$ weakly, and we wish to solve $D(h\otimes v_{\beta,\ell})=\omega\otimes v_{\beta,\ell}$ weakly, that is, $(r\partial_r+\beta)h=f$ and $(\partial_\theta+i\beta)h=g$ weakly, with appropriate L^2 conditions. Written on the Fourier coefficients, the closedness condition reads

$$rg_n'(r)+\beta g_n(r)=i(n+\beta)f_n(r)\quad\text{weakly.}$$

We look for h_n such that

$$\sum_n \|h_n(r)\cdot r^\beta \mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}<+\infty$$

and
$$(rh_n'+\beta h_n)=f_n,\quad i(n+\beta)h_n=g_n\quad\text{weakly.}$$

If $n+\beta\neq 0$, then h_n is given by $g_n/i(n+\beta)$ and satisfies the left equation above, by the integrability property. The only point is to bound $\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}$. We have

$$\begin{aligned}\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}&=|n+\beta|^{-1}\|g_n(r)\mathrm{L}(r)^{-1}\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}\\&\leqslant |n+\beta|^{-1}\mathrm{L}(R)^{-1}\|g_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r},\end{aligned}$$

so there exists $C>0$ such that

$$\sum_{n|n+\beta\neq0}\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}\leqslant C\sum_{n|n+\beta\neq0}\|g_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}.$$

If $\beta\neq0$, there is no restriction on n and thus

$$H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0)=0\quad\text{if }\beta\neq0.$$

If $\beta=0$, any class in $H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0)$ has a representative $f_0(r)(\mathrm{d}r/r)+g_0(r)\mathrm{d}\theta$, with $g_0(r)$ constant. This constant may be nonzero only if $\mathrm{L}(r)^{\ell/2}\in L^2(I_R,\mathrm{d}r/r)$, that is, $\ell\leqslant-2$ (Exercise 6.6). On the other hand, we look for h_0 such that $h_0'=r^{-1}f_0$. By the reasoning done for H^2, this equation has a solution if $\ell\neq1$. This concludes the proof of $(6.13.12)_1$. ☐

End of the proof of Theorem* 6.13.4*: analysis of the spectral sequence. In the decomposition (6.13.7), we immediately conclude by induction on ℓ from Lemma 6.13.12 that $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta,D)_0$ is quasi-isomorphic to 0 if $\beta\neq0$. We are thus left with computing the cohomology of $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_0,D)_0$, a complex which is filtered by $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0$. We will analyze the spectral sequence (6.13.8) when $\beta=0$, whose nonzero terms $E_1^{p,q}$ are given by Lemma 6.13.12:

$$\begin{aligned}E_1^{p,-p}&=\mathcal{H}^o_{0,-p}\quad\text{for any }p\geqslant0,\\E_1^{p,1-p}&=\mathcal{H}^o_{0,-p}\otimes\mathrm{d}\theta\quad\text{for any }p\geqslant2,\\E_1^{-1,2}&=\mathfrak{H}\otimes\mathcal{H}^o_{0,1}\otimes(\mathrm{d}r/r),\\E_1^{1,1}&=\mathfrak{H}\otimes\mathcal{H}^o_{0,-1}\otimes((\mathrm{d}r/r)\wedge\mathrm{d}\theta).\end{aligned}$$

The only possible nonzero d_1's are $d_1:E_1^{p,-p}\to E_1^{p+1,-p}$ for $p\geqslant0$, induced by D. The only term in D which does not preserve the filtration is $-\mathrm{N}\mathrm{d}t/t$, and it shifts the filtration by -2, so $d_1=0$ and the previous equalities also hold for the corresponding E_2's.

Now, for $p \geqslant 0$, $d_2 : E_2^{p,-p} \to E_2^{p+2,-p-1}$ is induced by $-\mathrm{N} : \mathcal{H}^o_{0,-p} \to \mathcal{H}^o_{0,-p-2}$, which is surjective (see (1) in the proof of Lemma 3.3.7). On the other hand, $d_2 : E_2^{-1,2} \to E_2^{1,1}$ is equivalent to $\mathrm{N} : \mathcal{H}^o_{0,1} \to \mathcal{H}^o_{0,-1}$. Since $\mathrm{N} : \mathrm{gr}_1^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_* \to \mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_*$ is an isomorphism, we conclude that $E_3^{p,q} = 0$ except possibly $E_3^{p,-p}$ with $p \geqslant 0$, and we have

$$E_3^{-\ell,\ell} \simeq \mathrm{Ker}\big[\mathrm{N} : \mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_*\big] \quad \text{for } \ell \leqslant 0.$$

The spectral sequence (6.13.8) degenerates thus at E_3, and $H^i(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_0, D)_0) = 0$ if $i = 1, 2$. Moreover, the inclusion $(j_*\underline{\mathcal{H}})_0 \hookrightarrow H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_0, D)_0)$ is an isomorphism, since both spaces have the same dimension

$$\begin{aligned}\dim \mathrm{Ker}\big[\mathrm{N} : \mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{gr}^0\mathcal{V}_*\big] &= \dim \mathrm{Ker}\big[\mathrm{N} : \mathrm{M}_0\mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{M}_{-2}\mathrm{gr}^0\mathcal{V}_*\big]\\ &= \sum_{\ell\leqslant 0} \dim \mathrm{Ker}\big[\mathrm{N} : \mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_*\big].\end{aligned}$$

This concludes the proof of Theorem 6.13.4. □

6.14. The Hodge filtration

In this section, we assume that $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of Hodge structure. Our aim is to define a Hodge filtration on the cohomology $H^\bullet(X, j_*\mathcal{V}^\nabla)$, and to prove that it endows this cohomology with a polarizable Hodge structure. We will also make precise the polarization. The method will be of a local nature, in a way similar to the computation of the L^2 cohomology.

6.14.a. The Hodge filtration on $\mathcal{V}_{\mathrm{mid}}$. We first define the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ from that on $\mathcal{V}^{>-1}$ by the formula

$$F^p\mathcal{V}_{\mathrm{mid}} = \sum_{j\geqslant 0} (\nabla_{\partial_t})^j F^{p+j}\mathcal{V}_*^{>-1}, \tag{6.14.1}$$

in order to obtain Griffiths transversality (recall that $\mathcal{V}_*^{>-1} = \mathcal{V}_{\mathrm{mid}}^{>-1}$, see (6.12.4)). One first checks that this formula defines an $\mathcal{O}_\Delta$-module by using he standard commutation rule. For example, for a local section m of $F^{p+1}\mathcal{V}_*^{>-1}$,

$$\begin{aligned} g(t)\nabla_{\partial_t} m = \nabla_{\partial_t} g(t)m - g'(t)m \in \nabla_{\partial_t} F^{p+1}\mathcal{V}_*^{>-1} + F^{p+1}\mathcal{V}_*^{>-1}\\ \subset \nabla_{\partial_t} F^{p+1}\mathcal{V}_*^{>-1} + F^{p}\mathcal{V}_*^{>-1}.\end{aligned}$$

With this definition, the relation $\nabla_{\partial_t} F^p\mathcal{V}_{\mathrm{mid}} \subset F^{p-1}\mathcal{V}_{\mathrm{mid}}$ is clearly satisfied. We now give more properties of the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$. For $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, we set $F^p\mathcal{V}^\beta_{\mathrm{mid}} := F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}^\beta_{\mathrm{mid}}$ and $F^p\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}} := F^p\mathcal{V}^\beta_{\mathrm{mid}}/F^p\mathcal{V}^{>\beta}_{\mathrm{mid}}$.

6.14.2. Proposition **(Properties of the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$).**

(1) *The filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ is exhaustive, that is,* $\bigcup_p F^p\mathcal{V}_{\mathrm{mid}} = \mathcal{V}_{\mathrm{mid}}$.

(2) *For every $\beta > -1$, we have*

$$F^p\mathcal{V}^\beta_{\mathrm{mid}} = j_*F^p\mathcal{V} \cap \mathcal{V}^\beta_{\mathrm{mid}} = j_*F^p\mathcal{V} \cap \mathcal{V}^\beta_*.$$

(3) *Moreover,*

(a) *for every* $\beta > -1$, $t(F^p\mathcal{V}^{\beta}_{\mathrm{mid}}) = F^p\mathcal{V}^{\beta+1}_{\mathrm{mid}}$*;*

(b) *for every* $\beta < 0$, $\partial_t F^p \mathrm{gr}^{\beta}\mathcal{V}_{\mathrm{mid}} = F^{p-1}\mathrm{gr}^{\beta-1}\mathcal{V}_{\mathrm{mid}}$*;*

(c) *The latter property also holds for* $\beta = 0$.

(4) *Conversely, a filtration* $F^\bullet\mathcal{V}_{\mathrm{mid}}$ *by* $\mathcal{O}_\Delta$*-submodules which satisfies* (6.7.1)*,* (3b) *and* (3c) *also satisfies* (6.14.1).

The inclusions $\subset$ in (3a) and (3b) are easy; the remarkable property is the existence of inclusions $\supset$; we will call the conjunction of (3a) and (3b) the property of *strict* $\mathbb{R}$*-specializability.* Property (3c) involves a Hodge-theoretical argument. we will call the conjunction of (3a)–(3c) the property of filtered middle extension (see Section 9.3.c).

Proof. The statement (1) is clear by (6.12.3).

For (2), it is enough to prove the assertion with $\beta = >-1$ and we start by showing that for any $k \geqslant 0$,

$$F^p\mathcal{V}^{>-k-1}_{\mathrm{mid}} = \sum_{j=0}^{k} \partial_t^j F^{p+j}\mathcal{V}^{>-1}_{*}, \tag{6.14.3}$$

which will give the conclusion in case $k = 0$. It is enough to prove that, for any $\ell \geqslant k+1$,

$$\Big(\sum_{j=k+1}^{\ell} \partial_t^j F^{p+j}\mathcal{V}^{>-1}_{*}\Big) \cap \mathcal{V}^{>-\ell}_{\mathrm{mid}} \subset \Big(\sum_{j=k+1}^{\ell-1} \partial_t^j F^{p+j}\mathcal{V}^{>-1}_{*}\Big),$$

and this reduces to

$$(\partial_t^\ell F^{p+\ell}\mathcal{V}^{>-1}_{*}) \cap \mathcal{V}^{>-\ell}_{\mathrm{mid}} \subset \partial_t^{\ell-1} F^{p+\ell-1}\mathcal{V}^{>-1}_{*} \quad \text{for } \ell \geqslant 1.$$

Let $m \in \mathcal{V}^{>-1}_{*}$ be such that $\partial_t^\ell m \in \mathcal{V}^{>-\ell}_{\mathrm{mid}}$. Let β be such that $\partial_t m \in \mathcal{V}^{\beta}_{\mathrm{mid}}$ with $[\partial_t m] \neq 0$ in $\mathrm{gr}^{\beta}\mathcal{V}_{\mathrm{mid}}$. If $\beta \geqslant -1$, $\partial_t^{\ell-1} : \mathrm{gr}^{\beta}\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\beta-\ell+1}\mathcal{V}_{\mathrm{mid}}$ is an isomorphism and $\partial_t^{\ell-1}(\partial_t m) \notin \mathcal{V}^{>\beta-\ell+1}_{\mathrm{mid}} \supset \mathcal{V}^{>-\ell}_{\mathrm{mid}}$, a contradiction. We must then have $\beta > -1$. Therefore, $\partial_t m \in F^{p+\ell-1}\mathcal{V}^{>-1}_{*}$, as wanted.

(3a) follows from (2) since t acts in an invertible way on $j_*F^p\mathcal{V}$. Let us check (3c), which amounts to

$$F^{p-1}\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \subset \partial_t F^p \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}.$$

Since $t : \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} = \mathrm{gr}^0\mathcal{V}_*$ is injective, this is implied by

$$tF^{p-1}\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \subset t\partial_t F^p \mathrm{gr}^0\mathcal{V}_*.$$

The left-hand side is included in $F^{p-1}\mathrm{gr}^0\mathcal{V}_* \cap \mathrm{Im}(t\partial_t)$. By Theorem 6.8.7, $\mathrm{N} = -t\partial_t : (\mathrm{gr}^0\mathcal{V}_*, F^\bullet) \to (\mathrm{gr}^0\mathcal{V}_*, F^\bullet)(-1)$ is a *morphism of Hodge structure, hence is* F*-strict*, which amounts to $F^{p-1}\mathrm{gr}^0\mathcal{V}_* \cap \mathrm{Im}(t\partial_t) \subset t\partial_t F^p\mathrm{gr}^0\mathcal{V}_*$, as wanted.

Let us now check (3b), which amounts to

$$F^{p-1}\mathcal{V}^{\beta-1}_{\mathrm{mid}} \subset \partial_t(F^p\mathcal{V}^{\beta}_{\mathrm{mid}}) + \mathcal{V}^{>\beta-1}_{\mathrm{mid}} \quad \text{if } \beta < 0.$$

For example, let us assume $\beta \in (-1,0)$. Then

$$\begin{aligned} F^{p-1}\mathcal{V}^{\beta-1}_{\mathrm{mid}} &= F^{p-1}\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}^{>-2}_{\mathrm{mid}} \cap \mathcal{V}^{\beta-1}_{\mathrm{mid}} \\ &= (F^{p-1}\mathcal{V}^{>-1}_{*} + \partial_t F^p\mathcal{V}^{>-1}_{*}) \cap \mathcal{V}^{\beta-1}_{\mathrm{mid}} \quad \text{(after (6.14.3))} \\ &\subset \mathcal{V}^{>\beta-1}_{\mathrm{mid}} + (\partial_t F^p\mathcal{V}^{>-1}_{*}) \cap \mathcal{V}^{\beta-1}_{\mathrm{mid}} \quad (\beta > 0). \end{aligned}$$

Since $\partial_t : \mathrm{gr}^\gamma\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\gamma-1}\mathcal{V}_{\mathrm{mid}}$ is an isomorphism for $\gamma < 0$ (see Exercise 6.14(5)), we have

$$(\partial_t F^p\mathcal{V}^{>-1}_{*}) \cap \mathcal{V}^{\beta-1}_{\mathrm{mid}} = (\partial_t F^p\mathcal{V}^{\beta}_{*}) + \mathcal{V}^{>\beta-1}_{\mathrm{mid}}.$$

The general case of $\beta > 0$ is treated similarly.

Let us end with (4). One first easily checks that (6.7.1) implies (2) and (3a). Then, by a simple induction on k, (3b) and (3c) imply (6.14.3), hence (6.14.1) by passing to the limit on k. □

***6.14.4. Corollary* (of Theorem 6.7.3).** *The $\mathcal{O}_\Delta$-modules*

$$F^p\mathcal{V}_{\mathrm{mid}}, \quad F^p\mathcal{V}^{\beta}_{\mathrm{mid}} := F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}^{\beta}_{\mathrm{mid}}, \quad F^p\mathrm{M}_\ell\mathcal{V}^{\beta}_{\mathrm{mid}} := F^p\mathcal{V}_{\mathrm{mid}} \cap \mathrm{M}_\ell\mathcal{V}^{\beta}_{\mathrm{mid}}$$

are $\mathcal{O}_\Delta$-locally free, hence free, of finite rank.

***Proof*.** Since these sheaves are contained in $\mathcal{V}_{\mathrm{mid}}$, it is enough to prove that they are locally finitely generated. For $\beta > -1$, we simply use Schmid's theorem 6.7.3 and that $F^p\mathcal{V}^{\beta}_{\mathrm{mid}} = F^p\mathcal{V}^{\beta}_{*}$. For $\beta = -1$, we have $F^p\mathcal{V}^{-1}_{\mathrm{mid}} = \partial_t F^{p+1}\mathcal{V}^0_{\mathrm{mid}} + F^p\mathcal{V}^{>-1}$ according to 6.14.2(3c), which implies the desired finiteness. The argument for $\beta < -1$ is similar, by using 6.14.2(3b) instead. Lastly, the finiteness for $F^p\mathcal{V}_{\mathrm{mid}} \cap \mathrm{M}_\ell\mathcal{V}^{\beta}_{\mathrm{mid}}$ is obtained by induction on ℓ, due to the fact that $\mathrm{gr}^{\mathrm{M}}_\ell\mathcal{V}^{\beta}_{\mathrm{mid}}$ is a finite-dimensional vector space. □

6.14.b. The filtered de Rham complex. The de Rham complex $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ has various presentations (Lemmas 6.12.7 and 6.12.9), the latter being linked with the holomorphic L^2 de Rham complex (Theorem 6.12.15). Each of these complexes can naturally be filtered by the usual procedure as in (2.4.3). For $\mathcal{V}_{\mathrm{mid}}$, starting from the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$, we define

(6.14.5)
$$\begin{aligned} F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &:= \{0 \longrightarrow F^p\mathcal{V}_{\mathrm{mid}} \xrightarrow{\nabla} \Omega^1_\Delta \otimes F^{p-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0\} \\ &\simeq \{0 \longrightarrow F^p\mathcal{V}_{\mathrm{mid}} \xrightarrow{\partial_t} F^{p-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0\}. \end{aligned}$$

We also define

(6.14.6)
$$\begin{aligned} F^pV^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &:= \{0 \longrightarrow F^p\mathcal{V}^0_{*} \xrightarrow{\nabla} \Omega^1_\Delta \otimes F^{p-1}\mathcal{V}_{\mathrm{mid}}{-1} \longrightarrow 0\} \\ &\simeq \{0 \longrightarrow F^p\mathcal{V}^0_{\mathrm{mid}} \xrightarrow{\partial_t} F^{p-1}\mathcal{V}^{-1}_{\mathrm{mid}} \longrightarrow 0\}. \end{aligned}$$

Lastly, taking advantage of Theorem 6.12.15, we define

(6.14.7)
$$F^p\,\mathrm{DR}\,\mathcal{V}_{*(2)} := \{0 \longrightarrow F^p\mathrm{M}_0\mathcal{V}^0_{*} \xrightarrow{t\partial_t} F^{p-1}\mathrm{M}_{-2}\mathcal{V}^0_{*} \longrightarrow 0\}.$$

6.14.8. Proposition. *The inclusions of filtered complexes*

$$F^\bullet \operatorname{DR} \mathcal{V}_{*(2)} \hookrightarrow F^\bullet V^0 \operatorname{DR} \mathcal{V}_{\mathrm{mid}} \hookrightarrow F^\bullet \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$$

are filtered quasi-isomorphisms.

Proof. For the second inclusion, we are reduced to proving that, when $\beta < 0$, the complex

$$0 \longrightarrow F^p \mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \xrightarrow{\partial_t} F^{p-1} \mathrm{gr}^{\beta-1} \mathcal{V}_{\mathrm{mid}} \longrightarrow 0$$

is quasi-isomorphic to zero. This is precisely 6.14.2(3b), since we know that $\partial_t : \mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\beta-1} \mathcal{V}_{\mathrm{mid}}$ is an isomorphism for such β's.

For the first inclusion, we first argue as for (6.12.8) (by using 6.14.2(3a)) to identify $F^p V^0 \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$ with the complex

$$0 \longrightarrow F^p \mathcal{V}^0_* \xrightarrow{t\partial_t} (t\partial_t F^p \mathcal{V}^0_* + F^{p-1} \mathcal{V}^{>0}) \longrightarrow 0.$$

The cokernel complex of the first inclusion is then isomorphic to the complex

$$0 \longrightarrow F^p(\mathrm{gr}^0 \mathcal{V}_* / \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^*) \xrightarrow{-\mathrm{N}} \big(\mathrm{N} F^p \mathrm{gr}^0 \mathcal{V}_* / F^{p-1} \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*\big) \longrightarrow 0,$$

and we wish to prove that the middle arrow is an isomorphism. Surjectivity is clear, and injectivity amounts to the equality

$$\mathrm{N} F^p \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^* = F^{p-1} \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*.$$

We know that $\mathrm{N} : \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^* \to \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*$ is surjective, but we need a supplementary argument for the compatibility with the Hodge filtration. This argument is furnished by the Hodge-Lefschetz property provided by Theorem 6.8.7. Indeed, we know that

$$\mathrm{N} : (\mathrm{gr}^0 \mathcal{V}^*, F^\bullet, \mathrm{M}_{w+\bullet}) \longrightarrow (\mathrm{gr}^0 \mathcal{V}^*, F^\bullet, \mathrm{M}_{w+\bullet})(-1)$$

is a morphism of mixed Hodge structures (see Remark 3.2.1), hence it is strictly compatible with both $F^\bullet$ and $\mathrm{M}_{w+\bullet}$ (see Proposition 2.6.8), hence $\mathrm{N} : F^p \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^* \to F^{p-1} \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*$ is surjective too. □

6.14.9. Remarks.

(1) We *do not* claim that the filtered complex $F^\bullet \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$ is *strict*, that is, that $H^1 F^p \operatorname{DR} \mathcal{V}_{\mathrm{mid}} = 0$ for any p.

(2) The graded complex $\mathrm{gr}^p_F \operatorname{DR} \mathcal{V}_{*(2)} \simeq \mathrm{gr}^p_F \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$ is a complex in the category of $\mathcal{O}_\Delta$-modules whose terms are $\mathcal{O}_\Delta$-coherent. Reading this property on a compact Riemann surface X, this implies that the hypercohomology spaces $H^q(X, \mathrm{gr}^p_F \operatorname{DR} \mathcal{V}_{*(2)})$ are finite-dimensional vector spaces.

6.14.c. The L^2 Dolbeault lemma. One of the important points in order to prove E_1-degeneration of the Hodge-to-de Rham spectral sequence in the context of the Hodge-Zucker theorem 6.11.1 is the Dolbeault lemma, making the bridge between the holomorphic world and the L^2 world of harmonic sections. It will ensure finite dimensionality needed in the proof of the Hodge-Deligne theorem 4.2.33 in the case of a complex manifold with a complete metric, here a Riemann surface.

Let us now come back to the Dolbeault lemma in the context of the Hodge-Zucker theorem 6.11.1, where X is a compact Riemann surface and X^* is the same surface with isolated punctures. Given a polarized variation of Hodge structure (H,S) of weight w on X^*, we consider the associated flat bundle with metric $(\mathcal{H},\mathrm{h},D)$ and the associated flat holomorphic bundle $(\mathcal{H}',\nabla)$, also denoted $(\mathcal{V},\nabla)$.

The L^2 Dolbeault complex (4.2.30) reads

$$0\to L^2(X^*,\mathrm{gr}_F^p\mathcal{H},\mathrm{h},\mathcal{D}'')\xrightarrow{\mathcal{D}''}L^2(X^*,\mathrm{gr}_F^p(\mathcal{E}^1_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')\xrightarrow{\mathcal{D}''}L^2(X^*,\mathrm{gr}_F^p(\mathcal{E}^2_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')\to 0.$$

It will be useful to regard $L^2(X^*,\mathrm{gr}_F^p\mathcal{H},\mathrm{h},\mathcal{D}'')$, as well as its relatives, as the space of global sections of a flabby sheaf $\mathcal{L}_{(2)}(\mathrm{gr}_F^p\mathcal{H},\mathrm{h},\mathcal{D}'')$ on X, defined by the assignment

$$X\supset U\longmapsto L^2(U\cap X^*,\mathrm{gr}_F^p\mathcal{H},\mathrm{h},\mathcal{D}'').$$

This gives rise to a complex of sheaves $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^\bullet_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')$ on X with differential $\mathcal{D}''$.

On the holomorphic side, we regard the holomorphic Dolbeault complex (4.2.13) not only on X^* but its extension to X with the L^2 condition. Namely, $\mathrm{gr}_F^p\,\mathrm{DR}(\mathcal{V},\nabla)=\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{V},\theta)$ on X^* is extended to X as $\mathrm{gr}_F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$, that we now can write as $\mathrm{gr}_F^p(\mathrm{DR}\,\mathcal{V}_{*(2)})$, a form which will help us to compare with the L^2 side.

6.14.10. Theorem **(L^2 Dolbeault lemma).** *With the assumptions of Theorem* 6.13.5, *there is a natural inclusion of complexes*

$$\mathrm{gr}_F^p(\mathrm{DR}\,\mathcal{V}_{*(2)})\hookrightarrow\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^\bullet_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')$$

which is a quasi-isomorphism.

Away from the punctures, Lemma 4.2.32 shows that the inclusion is a quasi-isomorphism. We are thus reduced to analyzing the germ $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^\bullet_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$ of the sheaf L^2 complex at the origin of the disc Δ.

Proof of Theorem* 6.14.10*: choice of an L^2-adapted basis. As in the proof of the L^2 Poincaré lemma, we can replace the Hodge metric h by an equivalent one, and we can work with an L^2-adapted frame with respect to this metric. However, we cannot use anymore the decomposition (6.2.5 $**$), which much simplified the expression of the connection when analyzing the L^2 de Rham complex, since it is a priori not compatible with the Hodge filtration. We will use Proposition 6.13.2 instead, in a way compatible with the Hodge filtration.

For that purpose, we specify that the basis $(\boldsymbol{v}^o_{\beta,\ell})_{\beta,\ell}$ of $\mathcal{V}^{-1}_*/t\mathcal{V}^{-1}_*\simeq\bigoplus_{\beta\in(-1,0]}\mathrm{gr}^\beta\mathcal{V}_*$ is compatible with the filtration induced on each $\mathrm{gr}^\beta\mathcal{V}_*$ by the Hodge filtration, which is the Hodge filtration of the polarized Hodge-Lefschetz structure $\bigoplus_\beta(\mathrm{gr}^\beta H,\mathrm{N},\mathrm{gr}^\beta\mathrm{S})$ (Theorem 6.8.7). We thus decompose each $\boldsymbol{v}^o_{\beta,\ell}$ as $\boldsymbol{v}^{o,p}_{\beta,\ell}$ (recall that we now set $\boldsymbol{v}^o_{\beta,\ell}=(\boldsymbol{v}^o_{\beta,\ell',j})_{\ell'-2j=\ell}$ in order to obtain a basis of $\mathrm{gr}^\mathrm{M}_\ell\mathrm{gr}^\beta\mathcal{V}_*$), so that $\boldsymbol{v}^{o,p}_{\beta,\ell}$ is a basis of $\mathrm{gr}_F^p\mathrm{gr}^\mathrm{M}_\ell\mathrm{gr}^\beta\mathcal{V}_*$.

Let us fix p. Since $\mathrm{gr}_F^p \mathcal{V}_*^{>-1}$ is locally free (Theorem 6.7.3), we can lift $\boldsymbol{v}^{o,p}_{\beta,\ell}$ as a family $\boldsymbol{v}^p_{\beta,\ell}$ in $\mathrm{gr}_F^p \mathrm{M}_\ell \mathcal{V}_*^\beta$ so that $(\boldsymbol{v}^p_{\beta,\ell})_{\beta,\ell}$ is a frame of $\mathrm{gr}_F^p \mathcal{V}_*^{>-1}$. By Proposition 6.13.2, (the restriction to Δ^* of) $(\boldsymbol{v}^p_{\beta,\ell})_{\beta,\ell}$ is L^2-adapted with respect to the metric induced by the Hodge metric h on $\mathcal{H}^{p,w-p} \simeq \mathrm{gr}_F^p \mathcal{V}$.

Proof of Theorem* 6.14.10*: simplification of the L^2 complex. We present the L^2 Dolbeault complex as the simple complex associated with a double complex, by decoupling d'' and θ'. This relies on the following lemma.

6.14.11. Lemma. *For $q = 0, 1$, the morphism $\theta' : \mathcal{E}^{0,q}_\Delta \otimes \mathrm{gr}_F^p \mathcal{V} \to \mathcal{E}^{1,q}_\Delta \otimes \mathrm{gr}_F^{p-1} \mathcal{V}$ has bounded L^2-norm.*

Proof. The morphism θ' is the C^∞ morphism associated with the holomorphic morphism $\theta : \mathrm{gr}_F^p \mathcal{V} \to \Omega^1_{\Delta^*} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}$, which is itself induced by

$$\theta : \mathrm{gr}_F^p \mathcal{V}_*^{>-1} \longrightarrow \Omega^1_\Delta \otimes \mathrm{gr}_F^{p-1} \mathcal{V}_*^{>-1}.$$

The restriction of θ at $t = 0$ being that of $\mathrm{gr}_F^{-1} \nabla$, it has matrix $-\mathrm{gr}_F^p \mathsf{N}$ in the bases $\boldsymbol{v}^{o,p}, \boldsymbol{v}^{o,p-1}$. The image by θ of a section $u = \sum_{\beta,\ell,k} u_{\beta,\ell,k}\, v^p_{\beta,\ell,k}$ reads thus

$$\sum_{\beta,\ell,k} u_{\beta,\ell+2,k}\, v^p_{\beta,\ell,k} \frac{\mathrm{d}t}{t} + t \sum_{\beta,\ell,k} \widetilde{u}_{\beta,\ell,k}\, v^p_{\beta,\ell,k} \frac{\mathrm{d}t}{t},$$

where $\widetilde{u}_{\beta,\ell,k}$ belongs to $\sum_{\beta',\ell',k'} \mathcal{O}_\Delta \cdot u_{\beta',\ell',k'}$. Therefore,

$$\|\theta u\|_2 \leqslant \sum_{\beta,\ell,k} \|(u_{\beta,\ell+2,k} + t\widetilde{u}_{\beta,\ell,k}) \mathrm{L}(t) v^p_{\beta,\ell,k}\|_2 \sim \sum_{\beta,\ell,k} \|(u_{\beta,\ell+2,k} + t\widetilde{u}_{\beta,\ell,k}) |t|^\beta \mathrm{L}(t)^{1+\ell/2}\|_2,$$

according to Theorem 6.3.5. On the other hand, by the argument already used in the proof of Proposition 6.13.2, we have

$$\|(u_{\beta,\ell+2,k} + t\widetilde{u}_{\beta,\ell,k}) |t|^\beta \mathrm{L}(t)^{1+\ell/2}\|_\mathrm{h} \sim \|u_{\beta,\ell+2,k} |t|^\beta \mathrm{L}(t)^{1+\ell/2}\|_\mathrm{h}.$$

Since $\boldsymbol{v}^p$ is L^2-adapted we have (see Definition 4.2.21), still using Theorem 6.3.5,

$$\|u_{\beta,\ell+2,k} |t|^\beta \mathrm{L}(t)^{1+\ell/2}\|_2 \sim \|u_{\beta,\ell+2,k} v^p_{\beta,\ell+2,k}\|_2 \leqslant C_{\boldsymbol{v}} \|a\|_2.$$

We conclude that there exists $C > 0$ such that $\|\theta u\|_2 \leqslant C \|u\|_2$. □

This lemma implies that

$$\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^k_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0 = \mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^k_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathrm{d}'')_0 \quad k = 0, 1, 2. \tag{6.14.12}$$

Moreover, we claim that

$$\theta'\big[\mathcal{L}_{(2)}((\mathcal{E}^{0,q}_{X^*} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0\big] \subset \mathcal{L}_{(2)}((\mathcal{E}^{1,q}_{X^*} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0.$$

Indeed, this also follows from the lemma if $q = 1$ since, in that case,

$$\mathcal{L}_{(2)}((\mathcal{E}^{1,1}_{X^*} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0 = \mathcal{L}_{(2)}((\mathcal{E}^{1,1}_{X^*} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h})_0.$$

We need to prove that, given a germ $u \in \mathcal{L}_{(2)}((\mathcal{E}^{0,0}_{X^*} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0$, its image $\theta' u$ belongs to $\mathcal{L}_{(2)}((\mathcal{E}^{1,0}_{X^*} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0$, that is, $\mathrm{d}''(\theta' u) \in \mathcal{L}_{(2)}((\mathcal{E}^{1,1}_{X^*} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0$. But we have, in the weak sense, $\mathrm{d}''(\theta' u) = -\theta'(\mathrm{d}'' u)$, and since

$$\mathrm{d}'' u \in \mathcal{L}_{(2)}((\mathcal{E}^{0,1}_{X^*} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0 = \mathcal{L}_{(2)}((\mathcal{E}^{0,1}_{X^*} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h})_0$$

by assumption, the lemma allows us to conclude.

We can now regard (up to sign) the complex $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{\bullet}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')$ as the simple complex associated with the double complex

$$\begin{array}{ccc} \mathcal{L}_{(2)}((\mathcal{E}^{0,0}_{X^*}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h},\mathrm{d}'')_0 & \xrightarrow{\theta'} & \mathcal{L}_{(2)}((\mathcal{E}^{1,0}_{X^*}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h},\mathrm{d}'')_0 \\ \mathrm{d}''\big\downarrow & & \big\downarrow\mathrm{d}'' \\ \mathcal{L}_{(2)}((\mathcal{E}^{0,1}_{X^*}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h})_0 & \xrightarrow{\theta'} & \mathcal{L}_{(2)}((\mathcal{E}^{1,1}_{X^*}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h})_0 \end{array} \tag{6.14.13}$$

and the inclusion $\mathrm{gr}_F^p(\mathrm{DR}\,\mathcal{V}_{*(2)})_0\hookrightarrow\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{\bullet}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$ is obtained by means of the inclusions

$$\begin{aligned}(\mathrm{gr}_F^p\mathcal{V}_{*(2)})_0 &\subset \mathcal{L}_{(2)}((\mathcal{E}^{0,0}_{X^*}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h},\mathrm{d}'')_0\\ (\mathrm{gr}_F^{p-1}(\Omega^1_\Delta\otimes\mathcal{V}_{*(2)}))_0 &\subset \mathcal{L}_{(2)}((\mathcal{E}^{1,0}_{X^*}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h})_0.\end{aligned}$$

***Proof of Theorem* 6.14.10*: analysis of the vertical morphisms* d'' *in* (6.14.13).** Since these morphisms are diagonal with respect to the L^2-adapted basis $\boldsymbol{v}^p$, the question of the surjectivity of these morphisms will reduce to checking Hardy's inequalities. Let us fix p,β,ℓ. In polar coordinates, we wish to check the surjectivity (or not) of $\bar t\partial_{\bar t}=\frac12(r\partial_r+\mathrm{i}\,\partial_\theta)$:

$$\bar t\partial_{\bar t}:\begin{cases}\mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell-2}\mathrm{d}\theta\,\mathrm{d}r/r;(r\partial_r+\mathrm{i}\,\partial_\theta)\big)_0\to\mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell}\mathrm{d}\theta\,\mathrm{d}r/r\big)_0,\\ \mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell}\mathrm{d}\theta\,\mathrm{d}r/r;(r\partial_r+\mathrm{i}\,\partial_\theta)\big)_0\to\mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell+2}\mathrm{d}\theta\,\mathrm{d}r/r\big)_0.\end{cases}$$

The result has already been obtained in the proof of $(6.13.12)_2$: the first (resp. the second) morphism is onto if $(\beta,\ell)\neq(0,1)$ (resp. $(\beta,\ell)\neq(0,-1)$). Moreover, if $(\beta,\ell)=(0,1)$ (resp. $(\beta,\ell)=(0,-1)$), the subspace $\mathcal{L}_{(2)}\big(\mathrm{L}(r)\mathrm{d}r/r\big)_0$, i.e., consisting of functions only depending on r, surjects to the cokernel.

***Proof of Theorem* 6.14.10*: vanishing of* $H^2\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{\bullet}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$.** The previous analysis shows that only combinations of terms $u(r)v^p_{0,-1}(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)$, where $v^p_{0,-1}$ is any element of the subfamily $\boldsymbol{v}^p_{0,-1}$ (i.e., $\beta=0$ and $\ell=-1$) may not belong to $\mathrm{Im}\,\mathrm{d}''$. However, one then checks that $u(r)v^p_{0,1}(\mathrm{d}\bar t/\bar t)$ belongs to $\mathcal{L}_{(2)}((\mathcal{E}^{0,1}_{X^*}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h})_0$ and, by the previous analysis,

$$\theta'\big(u(r)v^p_{0,1}(\mathrm{d}\bar t/\bar t)\big)\equiv u(r)v^p_{0,-1}(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)\mod \mathrm{Im}\,\mathrm{d}''.$$

This implies the vanishing of $H^2\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{\bullet}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$. □

***End of the proof of Theorem* 6.14.10.** The previous step identifies, up to a quasi-isomorphism, the complex $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{\bullet}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$ with its subcomplex

$$0\longrightarrow\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{H}),\mathrm{h},\mathcal{D}'')_0\xrightarrow{\mathcal{D}''}\mathrm{Ker}\,\mathcal{D}''\longrightarrow0,$$

where

$$\mathrm{Ker}\,\mathcal{D}''=\mathrm{Ker}\Big[\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^1_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0\xrightarrow{\mathcal{D}''}\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^2_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0\Big].$$

6.14.14. Lemma. *Any local section* $u'\cdot(\mathrm{d}t/t)+u''\cdot(\mathrm{d}\bar t/\bar t)$ *of* $\mathrm{Ker}\,\mathcal{D}''$ *is equivalent, modulo* $\mathrm{Im}\,\mathcal{D}''$*, to a local section of* $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0\cap\mathrm{Ker}\,\mathcal{D}''$.

Proof. Since $u'\cdot(\mathrm{d}t/t)+u''\cdot(\mathrm{d}\bar t/\bar t)$ is assumed to belong to $\operatorname{Ker}\mathcal{D}''$, it is enough to show that it belongs to $\operatorname{Im}\mathcal{D}''+\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$, and it is also enough to show that such is the case for $u''\cdot(\mathrm{d}\bar t/\bar t)$.

First, we write $u''=u''_{\neq(0,1)}+u''_{(0,1)}$, where $u''_{\neq(0,1)}$ resp. $u''_{(0,1)}$ is a combination of basis sections $v^p_{\beta,\ell}$ with $(\beta,\ell)\neq(0,1)$ resp. $(\beta,\ell)=(0,1)$. Since $u''_{\neq(0,1)}\in\operatorname{Im}\mathrm{d}''$ by the previous analysis, it belongs to $\operatorname{Im}\mathcal{D}''+\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$. We can thus write our original section (up to changing notation for u') as $u'\cdot(\mathrm{d}t/t)+u''_{(0,1)}\cdot(\mathrm{d}\bar t/\bar t)$, and as such it still belongs to $\operatorname{Ker}\mathcal{D}''$.

Let us denote by $u''_{(0,-1)}$ the combination of basis sections $v^p_{0,-1}$ where the coefficient of $v^p_{0,-1}$ is that of $u''_{(0,1)}$ on $v^p_{0,1}$. Arguing as in the proof of the vanishing of H^2, we find

$$\theta'(u''_{(0,1)}(\mathrm{d}\bar t/\bar t))\equiv u''_{(0,-1)}(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)\mod\operatorname{Im}\mathrm{d}''.$$

On the other hand, by assumption, $\theta'(u''_{(0,1)}(\mathrm{d}\bar t/\bar t))=\mathrm{d}''\big(u'\cdot(\mathrm{d}t/t)\big)$, so that $u''_{(0,-1)}(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)\in\operatorname{Im}\mathrm{d}''$. But the preliminary analysis of $\operatorname{Im}\mathrm{d}''$ done above shows that this is equivalent to $u''_{(0,1)}(\mathrm{d}\bar t/\bar t)\in\operatorname{Im}\mathrm{d}''$. As a consequence, $u''_{(0,1)}(\mathrm{d}\bar t/\bar t)$ belongs to $\operatorname{Im}\mathcal{D}''+\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$, as wanted. □

We note that, because of (6.14.12) and by considering types,

$$\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0\cap\operatorname{Ker}\mathcal{D}''=\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\cap\operatorname{Ker}\mathrm{d}''.$$

Then the L^2 Dolbeault complex $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{\bullet}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$ is now seen to be quasi-isomorphic to its subcomplex

$$0\to\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\xrightarrow{\mathcal{D}''}\big[\operatorname{Im}\mathcal{D}''+\big(\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\cap\operatorname{Ker}\mathrm{d}''\big)\big]\to0.$$

Besides, by considering types, the latter is isomorphic to its subcomplex (up to sign)

$$0\longrightarrow\operatorname{Ker}\mathrm{d}''\xrightarrow{\theta'}\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^{1,0}_{X^*}\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\cap\operatorname{Ker}\mathrm{d}''.\tag{6.14.15}$$

Extending the germs to a small disc Δ, the restriction of the above complex to Δ^* is isomorphic to the holomorphic Dolbeault complex

$$0\longrightarrow\mathrm{gr}^p\mathcal{V}\xrightarrow{\theta}\Omega^1_{\Delta^*}\otimes\mathrm{gr}^p\mathcal{V},$$

as already mentioned. Then, by definition of the L^2 condition, (6.14.15) is nothing but $\mathrm{gr}_F^p\operatorname{DR}\mathcal{V}_{*(2)}$, and this ends the proof of Theorem 6.14.10. □

6.14.d. Conclusion: proof of the Hodge-Zucker theorem. We are now in position to apply Hodge theory on complete non-compact complex manifolds as in Section 4.2.e. Starting from a polarized variation of Hodge structure (H,S) on the punctured Riemann surface X^* equipped with a complete metric locally like the Poincaré metric near each puncture, we consider the corresponding L^2 de Rham complex $\mathcal{L}^{\bullet}_{(2)}(\mathcal{H},\mathrm{h},D)$. By Theorem 6.13.5, the cohomology of the complex $\Gamma(X,\mathcal{L}^{\bullet}_{(2)}(\mathcal{H},\mathrm{h},D))$ is isomorphic to $H^*(X,j_*\underline{\mathcal{H}})$, hence is finite dimensional. On the other hand, by the L^2 Dolbeault lemma 6.14.10, each cohomology space $H^k\big(\Gamma(X,\mathcal{L}^{\bullet}_{(2)}(\mathrm{gr}^F_p\mathcal{H},\mathrm{h},\mathcal{D}''))\big)$ is finite-dimensional, being isomorphic to the cohomology on X of a complex whose terms are $\mathcal{O}_X$-coherent (see Remark 6.14.9(2)). The finiteness conditions in Theorem 4.2.33 are thus fulfilled, and we obtain the desired Hodge decomposition. It is important to remark that, according to Theorems 6.13.5 and 6.14.10 read in the reverse direction, we can express the Hodge structure on $H^*(X, j_*\underline{\mathcal{H}})$ only in terms of the algebraic object $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}}, \nabla)$.

Let us now consider the polarization. The cohomology $H^1(X, j_*\underline{\mathcal{H}})$ is primitive, so the polarization on it can be expressed without referring to an ample line bundle. The positivity property of the polarization on H^0 and H^1 is proved exactly as in Theorem 4.2.16 in the case of compact Riemann surfaces, by replacing sections of the C^∞ de Rham complex on X with sections of the L^2 complex, with respect to the complete metric fixed on X^*, and using the pairing (6.13.6). There is no need here to argue on primitivity of L^2 sections.

6.14.16. Remarks.

(1) As in Remark 4.2.18(4), a consequence of the Hodge-Zucker theorem 6.11.1 is that the maximal constant subsheaf of $\underline{\mathcal{H}}$ has stalk $H^0(X^*, \underline{\mathcal{H}}) = H^0(X, j_*\underline{\mathcal{H}})$, and thus underlies a constant polarizable variation of Hodge structure of weight w whose restriction at any point of X is a direct summand in H on which the polarization of H induces a polarization (see Exercise 2.12). Poincaré duality enables us to transport this polarized Hodge structure to $H^2(X, j_*\underline{\mathcal{H}})$.

(2) (Degeneration at E_1 of the Hodge-to-de Rham spectral sequence) One checks that the filtered complex $\boldsymbol{R}\Gamma\big(X, F^\bullet(\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})_{(2)}\big)$ is *strict*, exactly as in Remark 4.2.18(2). This reads here as the injectivity of the natural horizontal morphisms

$$\begin{array}{ccc}\boldsymbol{H}^k(X, F^pV^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) & \hookrightarrow & \boldsymbol{H}^k(X, V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) \\ \wr\downarrow & & \downarrow\wr \\ \boldsymbol{H}^k(X, F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) & \hookrightarrow & \boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\end{array}$$

6.14.e. Structure of polarized variations of $\mathbb{C}$-Hodge structure

Let X be a compact Riemann surface, let X^* be the complement of a finite set of point, and let $(\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}, D, \mathrm{S})$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on X^*. By Corollary 6.4.2, the local system $\underline{\mathcal{H}}$ is semi-simple, that we write as $\underline{\mathcal{H}} = \bigoplus_{\alpha\in A}\mathcal{H}^o_\alpha\otimes\underline{\mathcal{H}}_\alpha$, with the same notation as in Section 4.3.c. Moreover, the polarization decomposes as well.

6.14.17. Theorem. *The statements of Lemma* 4.3.11 *and Theorem* 4.3.13 *hold in this setting.*

Proof. Indeed, the reference to Theorem 4.3.3 is replaced with a reference to Corollary 6.4.2, so the new argument needed both for Lemma 4.3.11 and for Theorem 4.3.13 only concerns the existence of a pure Hodge structure on

$$\mathrm{End}(\underline{\mathcal{H}}) = H^0(X^*, \mathcal{E}nd(\underline{\mathcal{H}})) = H^0(X, j_*\,\mathcal{E}nd(\underline{\mathcal{H}})),$$

and similarly on $\mathrm{Hom}(\underline{\mathcal{H}}_\alpha, \underline{\mathcal{H}})$, which is provided by the Hodge-Zucker theorem 6.11.1, according to Remark 6.14.16(1). □

6.15. Exercises

Exercise 6.14. Show the following properties (see (6.12.4) for $\mathcal{V}^{\beta}_{\mathrm{mid}}$, $\beta\in\mathbb{R}$).

(1) $\mathcal{V}^{\beta}_{\mathrm{mid}}$ is an $\mathcal{O}_\Delta$-coherent module, which is free of rank equal to $\operatorname{rk}\mathcal{V}$, since, being included in $\mathcal{V}_*$, it is torsion-free.

(2) For $\gamma\in(-1,0]$ and $k\geqslant 0$, $\partial_t^k:\mathrm{gr}^\gamma\mathcal{V}_{\mathrm{mid}}=\mathrm{gr}^\gamma\mathcal{V}_*\to\mathrm{gr}^{\gamma-k}\mathcal{V}_{\mathrm{mid}}$ is onto.

(3) $\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}}\subset\mathrm{gr}^\beta\mathcal{V}_*$. [*Hint*: Clear if $\beta>-1$; for $\gamma\in(-1,0)$ and $\beta=\gamma-k<-1$, use $\partial_t^k:\mathrm{gr}^\gamma\mathcal{V}_*\xrightarrow{\sim}\mathrm{gr}^{\gamma-k}\mathcal{V}_*$; for $\beta=-1$, show the inclusion directly; for $\beta=-1-k\leqslant-2$, use the inclusion for $\beta=-1$ and the bijectivity of $\partial_t^k:\mathrm{gr}^{-1}\mathcal{V}_*\to\mathrm{gr}^{-1-k}\mathcal{V}_*$.]

(4) $\mathcal{V}^{\beta}_{\mathrm{mid}}=\mathcal{V}_{\mathrm{mid}}\cap\mathcal{V}^{\beta}_*$. [*Hint*: Inclusion $\subset$ is clear; for $\supset$, let $m\in\mathcal{V}_{\mathrm{mid}}\cap\mathcal{V}^{\beta}_*$ with $[m]\neq0$ in $\mathrm{gr}^\beta\mathcal{V}_*$; there exists $\beta'\leqslant\beta$ such that $m\in\mathcal{V}^{\beta'}_{\mathrm{mid}}\smallsetminus\mathcal{V}^{>\beta'}_{\mathrm{mid}}$; then (3) implies $\beta'=\beta$.]

(5) For $\beta\neq0$, $\partial_t:\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}}\to\mathrm{gr}^{\beta-1}\mathcal{V}_{\mathrm{mid}}$ is bijective. Deduce that $\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}}=\mathrm{gr}^\beta\mathcal{V}_*$ for $\beta\neq-1,-2,\dots$ [*Hint*: For the injectivity, use (4) to show that $\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}}\subset\mathrm{gr}^\beta\mathcal{V}_*$.]

(6) $\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}\subset\mathrm{gr}^{-1}\mathcal{V}_*$ is identified with the image of $\partial_t:\mathrm{gr}^0\mathcal{V}_*\to\mathrm{gr}^{-1}\mathcal{V}_*$. Conclude that $\partial_t:\mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}\to\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}$ is onto. Using the isomorphism $t:\mathrm{gr}^{-1}\mathcal{V}_*\xrightarrow{\sim}\mathrm{gr}^0\mathcal{V}_*$ identify also $\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}$ with the image of $t\partial_t:\mathrm{gr}^0\mathcal{V}_*\to\mathrm{gr}^0\mathcal{V}_*$.

Exercise 6.15. The goal of this exercise is to illustrate the degeneration property of Remark 6.14.16(2) in a case where Hodge theory is not needed. The punctured Riemann surface is the Riemann sphere $X=\mathbb{P}^1$ with $r\geqslant3$ punctures $x_1,\dots,x_r$ and $\mathcal{V}$ is a rank 1 bundle with connection on X^*. For each $i=1,\dots,r$, the residue α_i of the connection on $\mathcal{V}^0_{\mathrm{mid}}$ at x_i, is assumed to have its real part in $(0,1)$.

(1) Show that $d:=\sum_i\alpha_i\in(0,r)$ is an integer (hence $1\leqslant d\leqslant r-1$) and that $\mathcal{V}^0_{\mathrm{mid}}=\mathcal{O}_{\mathbb{P}^1}(-d)$. Conclude that $H^0(\mathbb{P}^1,\mathcal{V}^0_{\mathrm{mid}})=0$. [*Hint*: Use the residue theorem for connections.]

(2) Show that $\mathcal{V}^{-1}_{\mathrm{mid}}=\mathcal{V}^{>-1}_{\mathrm{mid}}\simeq\mathcal{O}_{\mathbb{P}^1}(r-d)$. Conclude that $H^1(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}\otimes\mathcal{V}^{-1}_{\mathrm{mid}})=0$. [*Hint*: Compute the residue of the connection on $\mathcal{V}^{-1}_{\mathrm{mid}}$; use that $\Omega^1_{\mathbb{P}^1}\simeq\mathcal{O}_{\mathbb{P}^1}(-2)$.]

(3) Show that the long exact sequence

$$\cdots\longrightarrow\boldsymbol{H}^k(\mathbb{P}^1,V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\longrightarrow H^k(\mathbb{P}^1,\mathcal{V}^0_{\mathrm{mid}})\longrightarrow H^k(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}\otimes\mathcal{V}^{-1}_{\mathrm{mid}})\longrightarrow\cdots$$

reduces to the short exact sequence

$$0\longrightarrow H^0(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}\otimes\mathcal{V}^{-1}_{\mathrm{mid}})\longrightarrow\boldsymbol{H}^1(\mathbb{P}^1,V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\longrightarrow H^1(\mathbb{P}^1,\mathcal{V}^0_{\mathrm{mid}})\longrightarrow0.$$

(4) Interpret this result as the degeneration at E_1 of the spectral sequence associated with the filtration of $V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ defined by

$$\begin{aligned}F^1V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}&=0,\\F^0V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}&=\Omega^1_{\mathbb{P}^1}\otimes\mathcal{V}^{>-1}_{\mathrm{mid}}[-1],\\F^{-1}V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}&=V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}.\end{aligned}$$

(5) If all α_i's are real, relate this result with Remark 6.14.16(2). [*Hint*: The local system $\mathcal{V}^\nabla$ is then unitary.]

Exercise 6.16. Let X be a compact Riemann surface and let $(\mathcal{V},\nabla)$ be any non constant irreducible bundle with connection on X^*. Consider the filtration of $V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}}$ defined by

$$\begin{aligned}F^1V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}}&=0,\\ F^0V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}}&=\Omega^1_{\mathbb{P}^1}\otimes\mathcal{V}^{>-1}_{\mathrm{mid}}[-1],\\ F^{-1}V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}}&=V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}}.\end{aligned}$$

(1) Show that degeneration at E_1 of the associated spectral sequence on hypercohomology both for $\mathcal{V}$ and $\mathcal{V}^\vee$ is equivalent to the property that both $\mathcal{V}^0_*$ and $(\mathcal{V}^\vee)^0_*$ do not have nonzero global sections on X. [*Hint*:

- $\Longrightarrow$ Show first that $H^2(X,j_*\mathcal{V})$ and $H^2(X,j_*\mathcal{V}^\vee)$ are zero; show then that degeneration at E_1 is equivalent to the properties $H^1(X,\Omega^1_X\otimes\mathcal{V}^{>-1})=0$ and $H^0(X,\Omega^1_X\otimes\mathcal{V}^{>-1})\hookrightarrow\boldsymbol{H}^1(X,V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}})$ and their dual analogues; use Serre duality and Remark 6.2.3(2) to get the vanishing of $H^0(X,\mathcal{V}^0_*)$ and $H^0(X,(\mathcal{V}^\vee)^0_*)$.
- $\Leftarrow$ The vanishing of $H^1(X,\Omega^1_X\otimes\mathcal{V}^{>-1})$ and its dual analogue is obtained by Serre duality as above; in order to obtain the inclusion property for H^0, use the exact sequence

$$\cdots\longrightarrow H^0(X,\mathcal{V}^0_*)\longrightarrow H^0(X,\mathcal{V}^{-1}_{\mathrm{mid}})\longrightarrow\boldsymbol{H}^1(X,V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}})\simeq H^1(X,j_*\mathcal{V}^\nabla)\longrightarrow\cdots$$

together with the inclusion $H^0(X,\mathcal{V}^{>-1}_{\mathrm{mid}})\subset H^0(X,\mathcal{V}^{-1}_{\mathrm{mid}})$, and the analogous results for $\mathcal{V}^\vee$.]

(2) Show that for a unitary local system $\mathcal{V}^\nabla$ with no nonzero constant global section on X^*, the vector bundle $\mathcal{V}^0_*$ has no nonzero global section. [*Hint*: Prove that the local system is semi-simple with no constant simple component, and that its dual local system satisfies the same property; show that the Hodge filtration of $V^0\operatorname{DR}\mathcal{V}_{\mathrm{mid}}$ is that considered in (1); use the degeneration property of Remark 6.14.16(2) to conclude.]

6.16. Comments

The Hodge-Zucker theorem [**Zuc79**] makes use of the fundamental results of Schmid (Parts 1 and 2 of this chapter), and is the first occurrence of the purity theorem of the intermediate (or minimal) extension of a polarizable variation of Hodge structure. The proof given here is taken from loc. cit., with a small difference in the proof of the L^2 Dolbeault lemma (Theorem 6.14.10), for which we give a local result, while that of Zucker is global (on the cohomology).

In the approach of M. Saito [**Sai88**] to polarizable Hodge modules, the Hodge-Zucker theorem is the only analytic result that needs to be used. Nevertheless, for the extension of the theory to the mixed case, Zucker's theorem in higher dimensions ([**CK82, CKS86, CKS87, KK87**] and the more recent [**Moc22**]) are needed.