GAUSS-MANIN SYSTEMS, BRIESKORN LATTICES AND FROBENIUS STRUCTURES (II)

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Dedicated to Yuri Manin

Abstract. — We give an explicit description of the canonical Frobenius structure attached (by the results of the first part of this article) to the polynomial $f(u_0,\dots,u_n) = w_0u_0 + \cdots + w_nu_n$ restricted to the torus $U = \{(u_0,\dots,u_n) \in \mathbb{C}^{n+1} \mid \prod_i u_i^{w_i} = 1\}$, for any family of positive integers $w_0,\dots,w_n$ such that $\gcd(w_0,\dots,w_n) = 1$.

Résumé (*Systèmes de Gauss-Manin, réseaux de Brieskorn et structures de Frobenius (II)*)

Nous donnons une description explicite de la structure de Frobenius associée (par les résultats de la première partie de cet article) au polynôme $f(u_0,\dots,u_n) = w_0u_0 + \cdots + w_nu_n$ restreint au tore $U = \{(u_0,\dots,u_n) \in \mathbb{C}^{n+1} \mid \prod_i u_i^{w_i} = 1\}$ pour toute famille de poids $w_0,\dots,w_n$ tels que $\mathrm{pgcd}(w_0,\dots,w_n) = 1$.

1. Introduction

1.a. This article explains a detailed example of the general result developed in the first part [3]. We were motivated by [1], where S. Barannikov describes a Frobenius structure attached to the Laurent polynomial $f(u_0,\dots,u_n) = u_0 + \cdots + u_n$ restricted to the torus $U = \{(u_0,\dots,u_n) \in \mathbb{C}^{n+1} \mid \prod_i u_i = 1\}$, and

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shows that it is isomorphic to the Frobenius structure attached to the quantum cohomology of the projective space $\mathbb{P}^n(\mathbb{C})$ (as defined *e.g.*, in [5]).

We will freely use the notation introduced in the first part [3]. A reference like "§ I.3.c" will mean [3, § 3.c].

In the following, we fix an integer $n \geqslant 2$ and positive integers $w_0, \dots, w_n$ such that $\gcd(w_0, \dots, w_n) = 1$. It will be convenient to assume that $w_0 \leqslant \cdots \leqslant w_n$. We put

$$\mu := \sum_{i=0}^{n} w_i. \tag{1.1}$$

We will analyze the Gauss-Manin system attached to the Laurent polynomial

$$f(u_1, \dots, u_n) = w_0 u_0 + w_1 u_1 + \cdots + w_n u_n$$

restricted to the subtorus $U \subset (\mathbb{C}^*)^{n+1}$ defined by the equation

$$u_0^{w_0} \cdots u_n^{w_n} = 1.$$

The case $\mu = n + 1$ (and all w_i equal to 1) was considered in [1]. We will not need any explicit use of Hodge Theory, as the whole computation can be made "by hand". We will use the method of § I.3.c to obtain information concerning the Frobenius structure on any germ of universal deformation space of f. As we have seen in [3], we have to analyze with some details the structure of the Gauss-Manin system and the Brieskorn lattice of f.

1.b. Fix a $\mathbb{Z}$-basis of $\{\sum_i w_i x_i = 0\} \subset \mathbb{Z}^{n+1}$. It defines a $(n+1) \times n$ matrix M. Denote by $m_0, \dots, m_n$ the lines of this matrix. We thus get a parametrization of U by $(\mathbb{C}^*)^n$ by putting $u_i = v^{m_i}$ for $i = 0, \dots, n$ and $v = (v_1, \dots, v_n)$. The vectors $m_0, \dots, m_n$ are the vertices of a simplex $\Delta \subset \mathbb{Z}^n$, which is nothing but the Newton polyhedron of f when expressed in the coordinates v. Notice that the determinant of the $n \times n$ matrix $(m_0, \dots, \widehat{m_i}, \dots, m_n)$ is $\pm w_i$.

Lemma 1.2. — *The Laurent polynomial f is convenient and nondegenerate with respect to its Newton polyhedron.*

Proof. — The nondegeneracy follows from the linear independence of any n distinct vectors among $m_0, \dots, m_n$. Clearly, 0 is contained in the interior of Δ. □

We know then that f is M-tame (*cf.* § I.4) and we may therefore apply the results of § I.2 to f. An easy computation shows that f has μ simple critical points, which are the $\zeta(1, \dots, 1)$ with $\zeta^\mu = 1$, and thus μ distinct critical values $\mu\zeta$. We hence have $\mu(f) = \mu$.

1.c. Denote by $\mathcal{S}_w$ the *disjoint* union of the sets

$$\{\ell\mu/w_i \mid \ell = 0, \dots, w_i - 1\} \subset \mathbb{Q}.$$

Hence $\#\mathcal{S}_w = \mu$. Number the elements of $\mathcal{S}_w$ from 0 to $\mu - 1$ in an *increasing* way, with respect to the usual order on $\mathbb{Q}$. We therefore have $\mathcal{S}_w = \{s_w(0), \dots, s_w(\mu-1)\}$ with $s_w(k) \leqslant s_w(k+1)$. In particular, we have

$$s_w(0) = \dots = s_w(n) = 0, \quad s_w(n+1) = \frac{\mu}{\max_i w_i} \leqslant n+1.$$

Moreover, using the involution $\ell \mapsto w_i - \ell$ for $\ell \geqslant 1$, one obtains, for $k \geqslant n+1$, the relation

$$s_w(k) + s_w(\mu + n - k) = \mu. \tag{1.3}$$

We consider the function $\sigma_w : \{0, \dots, \mu-1\} \to \mathbb{Q}$ defined by

$$\sigma_w(k) = k - s_w(k). \tag{1.4}$$

Hence $\sigma_w(k) = k$ for $k = 0, \dots, n$. That $s_w(\bullet)$ is increasing is equivalent to

$$\forall\, k = 0, \dots, \mu-1, \quad \sigma_w(k+1) \leqslant \sigma_w(k) + 1, \tag{1.5}$$

where we use the convention $\sigma_w(\mu) = \sigma_w(0) = 0$. We will prove:

Theorem 1. — *The polynomial* $\prod_{k=0}^{\mu-1}(S + \sigma_w(k))$ *is equal to the spectral polynomial* $\mathrm{SP}_f(S)$ *attached to* f *(cf. § I.2.e).*

For instance, if we take the Laurent polynomial $f(u_0, \dots, u_n)$ on the torus $\prod u_i = 1$, *i.e.*, $w_0 = \dots = w_n = 1$, we get $\mathrm{SP}_f(S) = \prod_{k=0}^{n}(S+k)$.

Notice that the symmetry property (1.3) is a little bit more precise than the symmetry of the spectrum (*cf.* [9]), which would say that, for any $j \in \{0, \dots, n\}$,

$$\#\{k \mid \sigma_w(k) = j\} = \#\{k \mid \sigma_w(k) = n - j\}.$$

Indeed, for $k \in \{n+1, \dots, \mu-1\}$, (1.3) means that $\sigma_w(k) + \sigma_w(\mu + n - k) = n$ and we clearly have $\sigma_w(k) + \sigma_w(n-k) = n$ for $k = 0, \dots, n$.

1.d. Consider now the two $\mu \times \mu$ matrices

$$A_\infty = \mathrm{diag}\big(\sigma_w(0), \dots, \sigma_w(\mu-1)\big), \quad A_0 = \mu \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{1.6}$$

Notice that A_0 is semisimple with distinct eigenvalues $\mu\zeta$, where ζ is a μ-th primitive root of 1. In the canonical basis $(e_0, \dots, e_{\mu-1})$ of the space $\mathbb{C}^\mu$ on which these matrices act, consider the nondegenerate bilinear form g defined by

$$g(e_k, e_\ell) = \begin{cases} 1 & \begin{cases} \text{if } 0 \leqslant k \leqslant n \text{ and } k+\ell = n, \\ \text{or if } n+1 \leqslant k \leqslant \mu - 1 \text{ and } k+\ell = \mu+n, \end{cases} \\ 0 & \text{otherwise,} \end{cases}$$

with respect to which A_∞ satisfies $A_\infty + {}^tA_\infty = n\,\mathrm{Id}$. The data (A_0, A_∞, g, e_0) define (*cf.* [4, Main Theorem p. 188], see also [5, § II.3] or [10, Th. VII.4.2]) a unique germ of semisimple Frobenius manifold at the point $(\mu, \mu\zeta, \dots, \mu\zeta^{\mu-1}) \in \mathbb{C}^\mu$.

The main result of this article is then:

Theorem 2. — *The canonical Frobenius structure on any germ of a universal unfolding of the Laurent polynomial $f(u_0, \dots, u_n) = \sum_i w_i u_i$ on U, as defined in* [3]*, is isomorphic to the germ of universal semisimple Frobenius structure with initial data (A_0, A_∞, g, e_0) at the point $(\mu, \mu\zeta, \dots, \mu\zeta^{\mu-1}) \in \mathbb{C}^\mu$.*

Remark. — It would be interesting to give an explicit description of the Gromov-Witten potential attached to this Frobenius structure.

2. The rational numbers $\sigma_w(k)$

Let us be now more precise on the definition of $s_w(k)$. Define inductively the sequence $(a(k), i(k)) \in \mathbb{N}^{n+1} \times \{0, \dots, n\}$ by

$$a(0) = (0, \dots, 0), \qquad i(0) = 0$$
$$a(k+1) = a(k) + \mathbf{1}_{i(k)}, \quad i(k+1) = \min\{i \mid a(k+1)_i/w_i = \min_j a(k+1)_j/w_j\}.$$

It is immediate that $|a(k)| := \sum_{i=0}^n a(k)_i = k$ and that, for $k \leqslant n+1$, we have $a(k)_i = 1$ if $i < k$ and $a(k)_i = 0$ if $i \geqslant k$. In particular, $a(n+1) = (1, \dots, 1)$.

Lemma 2.1. — *The sequence $(a(k), i(k))$ satisfies the following properties:*

(1) *for all $k \in \mathbb{N}$, $\dfrac{a(k)_{i(k)}}{w_{i(k)}} \leqslant \dfrac{a(k+1)_{i(k+1)}}{w_{i(k+1)}} \leqslant \dfrac{a(k)_{i(k)}+1}{w_{i(k)}}$,*

(2) *$a(\mu) = (w_0, \dots, w_n)$ and for all $k \in \{0, \dots, \mu-1\}$, we have $a(k)_{i(k)} \leqslant w_{i(k)} - 1$,*

(3) *the map $\{0, \dots, \mu-1\} \to \coprod_{i=0}^n \{0, \dots, w_i - 1\}$, defined by $k \mapsto [i(k), a(k)_{i(k)}]$ is bijective.*

(4) *For $\ell \in \mathbb{N}$, we have $i(k+\ell\mu) = i(k)$ and $a(k+\ell\mu)_{i(k)} = \ell w_{i(k)} + a(k)_{i(k)}$.*

We will then put $s_w(k) := \mu a(k)_{i(k)}/w_{i(k)}$. We have $s_w(k+\ell\mu) = \ell\mu + s_w(k)$ for $\ell \in \mathbb{N}$.

Proof

(1) By induction on k. If $i(k+1) = i(k)$, the result is clear. Otherwise, we have $a(k+1)_{i(k+1)}/w_{i(k+1)} = a(k)_{i(k+1)}/w_{i(k+1)}$ and the first inequality follows from the definition of $i(k)$. Similarly, the second inequality is given by the definition of $i(k+1)$.

(2) Let us first remark the implication

$$a(k)_j \leqslant w_j \ \forall\, j \text{ and } \{j \mid a(k)_j < w_j\} \neq \varnothing \Longrightarrow a(k+1)_j \leqslant w_j \ \forall\, j.$$

[Indeed, from the assumption we have $a(k)_{i(k)} < w_{i(k)}$, hence $a(k+1)_{i(k)} = a(k)_{i(k)} + 1 \leqslant w_{i(k)}$. For $j \neq i(k)$, $a(k+1)_j = a(k)_j \leqslant w_j$.] Therefore, there exists k_0 such that $a(k_0) = (w_0, \dots, w_n)$. Then $k_0 = |a(k_0)| = \mu$. Moreover, by what we have just seen, we have $a(k)_{i(k)} < w_{i(k)}$ for $k < \mu$.

(3) The map does exist, after (2), is clearly injective, therefore bijective as the two sets have the same number of elements.

(4) We have $a(\mu) = (w_0, \dots, w_n)$, so that $i(\mu) = 0$, and we may apply the reasoning of (2) for $k = \mu, \dots, 2\mu - 1$, etc. □

REMARK 2.2. — In general, the numbers $s_w(k)$ are rational. These are integers (hence the spectrum of f is integral) if and only if the following condition holds:

$$\forall\, i, \quad w_i \mid \mu = w_0 + \cdots + w_n. \tag{2.3}$$

Consider the simplex $\Delta(w)$ in $\mathbb{R}^n$ obtained as the intersection of the hyperplane $\mathcal{H} = \left\{ \sum_{i=0}^n w_i x_i = 0 \right\} \subset \mathbb{R}^{n+1}$ with the half spaces $x_i \geqslant -1$. Fix also the lattice $\mathcal{H}_{\mathbb{Z}} = H \cap \mathbb{Z}^{n+1}$. Then Condition (2.3) is equivalent to the condition that the vertices of $\Delta(w)$ are contained in the lattice $\mathcal{H}_{\mathbb{Z}}$. In other words, $\Delta(w)$ is a *reflexive simplex* in the sense of Batyrev [2]. For instance, if $n = 3$, one finds the following possibilities for w_i (up to a permutation):

w_0	w_1	w_2	w_3	μ
1	1	1	1	**4**
1	1	1	3	**6**
1	1	2	2	**6**
1	1	2	4	**8**
1	2	2	5	**10**
1	1	4	6	**12**

w_0	w_1	w_2	w_3	μ
1	2	3	6	**12**
1	3	4	4	**12**
1	2	6	9	**18**
1	4	5	10	**20**
1	3	8	12	**24**
2	3	10	15	**30**
1	6	14	21	**42**

For $n=4$, here are some examples (maybe not complete):

w_0	w_1	w_2	w_3	w_4	μ
1	1	1	1	2	**6**
1	1	2	2	2	**8**
1	1	1	1	4	**8**
1	1	1	3	3	**9**
1	1	1	2	5	**10**
2	2	2	3	3	**12**
1	1	3	3	4	**12**
1	1	2	2	6	**12**
1	1	1	3	6	**12**
1	1	3	5	5	**15**

w_0	w_1	w_2	w_3	w_4	μ
1	1	2	4	8	**16**
1	1	4	4	10	**20**
1	1	4	6	12	**24**
1	1	2	8	12	**24**
1	1	3	10	15	**30**
1	1	4	12	18	**36**
1	1	8	10	20	**40**
1	1	6	16	24	**48**
1	1	8	20	30	**60**
1	2	12	15	30	**60**

3. The Gauss-Manin system

The Gauss-Manin system G of the Laurent polynomial f is a module over the ring $\mathbb{C}[\tau,\tau^{-1}]$. It is defined as in § I.2.c:

$$G=\Omega^n(U)[\tau,\tau^{-1}]/(d-\tau df\wedge)\Omega^{n-1}(U)[\tau,\tau^{-1}].$$

Put $\theta=\tau^{-1}$. The Brieskorn lattice $G_0=\operatorname{image}(\Omega^n(U)[\theta]\to G)$ is a free $\mathbb{C}[\theta]$-module of rank μ because, by Lemma 1.2, f is convenient and nondegenerate (*loc. cit.*). We will consider the increasing filtration $G_p=\tau^pG_0$ ($p\in\mathbb{Z}$). Let ω_0 be the n-form on U defined by

$$\omega_0=\frac{\frac{du_0}{u_0}\wedge\cdots\wedge\frac{du_n}{u_n}}{d\big(\prod_i u_i^{w_i}\big)}\Big|_{\prod_i u_i^{w_i}=1}.$$

Let $v\mapsto u=v^m$ be a parametrization of U as in § 1.b. The form ω_0 can be written as $\omega_0=\pm\frac{dv_1}{v_1}\wedge\cdots\wedge\frac{dv_n}{v_n}$. The Gauss-Manin system G is then identified with the $\mathbb{C}[\tau,\tau^{-1}]$-module (putting $v=(v_1,\dots,v_n)$)

$$\mathbb{C}[v,v^{-1},\tau,\tau^{-1}]/\{v_j\partial_{v_j}(\varphi_j)-\tau(v_j\partial_{v_j}f)\varphi_j\mid\varphi_j\in\mathbb{C}[v,v^{-1},\tau,\tau^{-1}],\ j=1,\dots,n\}.$$

It comes equipped with an action of ∂_τ: if $\psi\in\mathbb{C}[v,v^{-1}]$, let $[\psi]$ denote its class in G; then $\partial_\tau[\psi]=[-f\psi]$ (this does not depend on the representative of the class). Using the coordinate θ, we have $\theta^2\partial_\theta[\psi]=[f\psi]$; this action is extended in the usual way to Laurent polynomials in τ with coefficients like $[\psi]$.

It is convenient to use the coordinates $u=(u_0,\dots,u_n)$. Then the previous quotient is written as

$$\mathbb{C}[u,u^{-1},\tau,\tau^{-1}]/\big(I_w+\mathbb{C}[u,u^{-1},\tau,\tau^{-1}](g(u)-1)\big),$$

where we have put $g(u)=\prod_i u_i^{w_i}$ and I_w is the $\mathbb{C}[\tau,\tau^{-1}]$-submodule of $\mathbb{C}[u,u^{-1},\tau,\tau^{-1}]$ consisting of the expressions

$$(3.1)\quad \sum_{i=0}^{n} m_{ji}\Big(u_i\frac{\partial}{\partial u_i}-\tau w_i u_i\Big)\varphi_j,\ \text{with } \varphi_j\in\mathbb{C}[u,u^{-1},\tau,\tau^{-1}],\ (j=1,\dots,n).$$

Consider the sequence $(a(k),i(k))$ of Lemma 2.1, and for each $k=0,\dots,\mu$, put

$$\omega_k=u^{a(k)}\omega_0\in G_0,$$

Notice that $\omega_\mu=\omega_0$ and, using (3.5) below, that $f\omega_0=\mu\omega_1$.

Proposition 3.2. — *The classes of $\omega_0,\omega_1,\dots,\omega_{\mu-1}$ form a $\mathbb{C}[\theta]$-basis $\boldsymbol{\omega}$ of G_0. Moreover, they satisfy the equation*

$$-\frac{1}{\mu}(\tau\partial_\tau+\sigma_w(k))\omega_k=\tau\omega_{k+1}\quad (k=0,\dots,\mu-1),$$

and we have Bernstein's relation in G:

$$\prod_{k=0}^{\mu-1}\big[-\tfrac{1}{\mu}(\tau\partial_\tau-s_w(k))\big]\cdot\omega_0=\tau^\mu\omega_0.$$

The V-order of ω_k is equal to $\sigma_w(k)$ and $\boldsymbol{\omega}$ induces a $\mathbb{C}$-basis of $\oplus_\alpha \mathrm{gr}_\alpha^V(G_0/G_{-1})$.

From Theorem I.4.5, Lemma I.4.3(3), and the symmetry (1.3), we get

$$(3.3)\quad \text{for } k=0,\dots,\mu-1,\quad 0\leqslant\sigma_w(k)\leqslant n\quad\text{and}\quad \begin{cases}\sigma_w(k)=0\Rightarrow k=0,\\ \sigma_w(k)=n\Rightarrow k=n.\end{cases}$$

This implies that, for any $\alpha\in\,]0,n[$, the length of a maximal subsequence $\alpha,\alpha+1,\dots,\alpha+\ell$ of $\sigma_w(\bullet)$ is $\leqslant n$, and even $\leqslant n-1$ if α is an integer. In other words:

Corollary 3.4. — *The length of any maximal nonzero integral (resp. non-integral) constant subsequence of $s_w(\bullet)$ is $\leqslant n-1$ (resp. $\leqslant n$).*

The proposition also gives a Birkhoff normal form for G_0:

$$\theta^2\partial_\theta\boldsymbol{\omega}=\boldsymbol{\omega}A_0+\theta\boldsymbol{\omega}A_\infty$$

with A_0,A_∞ as in (1.6). The matrix A_0 is nothing but the matrix of multiplication by f on $G_0/\theta G_0$ in the basis induced by $\boldsymbol{\omega}$. Its eigenvalues are the critical values of f, as expected. In the case where $\mu=n+1$ (and all w_i equal to 1), we find that $A_\infty=\mathrm{diag}(0,1,\dots,n)$ and A_0 is as in (1.6) with size $\mu=n+1$.

Proof of Proposition 3.2. — It will be convenient to select some coordinate, say u_0. Multiplying (3.1) (applied to $\varphi_1 = \cdots = \varphi_n = \varphi$) on the left by the inverse matrix of the matrix formed by the columns of $m_1, \dots, m_n$, one finds that, for any $\varphi \in \mathbb{C}[u, u^{-1}, \tau, \tau^{-1}]$, we have in G

$$\forall\, i = 1, \dots, n \quad (\tfrac{1}{w_i} u_i \partial_{u_i} - \tfrac{1}{w_0} u_0 \partial_{u_0}) \varphi\omega_0 = \tau(u_i - u_0)\varphi\omega_0. \tag{3.5}$$

Applying this to any monomial $\varphi = u^a$ and summing these equalities, we get the following relation for $j = 0$, hence for any $j = 0, \dots, n$ by a similar argument:

$$-\frac{1}{\mu}\big(\tau\partial_\tau + L_j(a)\big) u^a \omega_0 = \tau u^{a+\mathbf{1}_j}\omega_0, \tag{3.6}$$

where we put $L_j(a) = \sum_{i=0}^n a_i - \mu a_j / w_j$. This is nothing but (I.4.12) in the present situation. Apply this for $a = a(k)$ and $j = i(k)$ ($k = 0, \dots, \mu - 1$) to get the first relation in the lemma (remark that $L_{i(k)}(a(k)) = \sigma_w(k)$). Bernstein's relation for ω_0 is then clear. Remark also that ω_k is given by

$$\omega_k = \tau^{-k} \prod_{j=0}^{k-1} \Big[-\frac{1}{\mu}(\tau\partial_\tau - s_w(j))\Big] \cdot \omega_0.$$

It is not difficult to derive from Bernstein's relation for ω_0 a Bernstein relation for each ω_k and conclude that ω_k has V-order $\leqslant \sigma_w(k)$. [Notice also that, as $\sigma_w(k) = L_{i(k)}(a(k)) = \max_j L_j(a(k))$, the order of ω_k with respect to the Newton filtration is $\leqslant \sigma_w(k)$; this is compatible with Theorem I.4.5.]

Let us now show that $\omega_0, \dots, \omega_{\mu-1}$ generate G_0 as a $\mathbb{C}[\theta]$-module. Notice that Bernstein's relation for ω_0 implies that $\partial_\tau^\mu \omega_0 \in \mathbb{C}[\theta]\langle \omega_0, \dots, \partial_\tau^{\mu-1}\omega_0\rangle = \mathbb{C}[\theta]\langle \omega_0, \dots, \omega_{\mu-1}\rangle$, and this also holds for $\partial_\tau^\ell \omega_0$ for $\ell \geqslant \mu$. It is therefore enough to show that $(f^\ell \omega_0)_{\ell \geqslant 0}$ generate G_0 over $\mathbb{C}[\theta]$. Write (3.5) as

$$u^{a+\mathbf{1}_i}\omega_0 = \Big[u^a u_0 + \Big(\frac{a_i}{w_i} - \frac{a_0}{w_0}\Big)\theta u^a\Big]\omega_0. \tag{3.7}$$

The Brieskorn lattice G_0 is generated over $\mathbb{C}[\theta]$ by the $u_0^\ell \omega_0$ with $\ell \in \mathbb{N}$: indeed, it is generated by the $u^a\omega_0$; then,

– if $a_i \geqslant 1$ for some $i \geqslant 1$, one decreases a_i to 0 with (3.7);

– if $a_i \leqslant -1$ for some $i \geqslant 1$, one iterates (3.7) w_0 times and use the relation $u^w \omega_0 = \omega_0$ to express $u^b\omega_0$ (any b) as a sum (with constant coefficients) of terms $\theta^k u_0^\ell u^{b+w'}\omega_0$ and of $u^{b+w'+\mathbf{1}_i}$, with $k, \ell \geqslant 0$ and $w' = (0, w_1, \dots, w_n)$; hence if $b_i < 0$, there exists r such that $b_i + rw_i \geqslant 0$ and one iterates r times the previous process to write $u^b\omega_0$ with terms $\theta^k u^a$, with $a_i \geqslant 1$, to reduce to the previous case;

– notice that, in both previous processes, we never decrease the degree in u_0; now, we are reduced to considering $u_0^\ell\omega_0$ with $\ell<0$; use once more the relation $u^{kw}u^b\omega_0=u^b\omega_0$ (for any $k\geqslant 0$, any b) to replace u_0^ℓ with u^a with $a_0,\dots,a_n\geqslant 0$ and apply the first case.

A similar argument gives the result for the family $(f^\ell\omega_0)_{\ell\geqslant 0}$. As G_0 is $\mathbb{C}[\theta]$-free (*cf.* Remark I.4.8 and § I.2.c), we conclude that $\boldsymbol{\omega}$ is a $\mathbb{C}[\theta]$-basis of G_0. [Instead of using Remark I.4.8, one can directly conclude here that G_0 is $\mathbb{C}[\theta]$-free of rank μ by showing first that $\boldsymbol{\omega}$ generates G as a $\mathbb{C}[\tau,\tau^{-1}]$-module.]

Remark also that $(\omega_0,\dots,f^{\mu-1}\omega_0)$ is another basis, but the differential equation does not take Birkhoff normal form in such a basis.

We will now determine the V-filtration. Put $\omega'_k=\tau^{[\sigma_w(k)]}\omega_k$. Then $\boldsymbol{\omega}'$ is another $\mathbb{C}[\tau,\tau^{-1}]$-basis of G. The V-order of ω'_k is $\leqslant\sigma_w(k)-[\sigma_w(k)]<1$. For $\alpha\in[0,1[$, put

$$\begin{aligned}U_\alpha G&=\mathbb{C}[\tau]\langle\omega'_k\mid\sigma_w(k)-[\sigma_w(k)]\leqslant\alpha\rangle+\tau\mathbb{C}[\tau]\langle\omega'_k\mid\sigma_w(k)-[\sigma_w(k)]>\alpha\rangle\\ U_{<\alpha}G&=\mathbb{C}[\tau]\langle\omega'_k\mid\sigma_w(k)-[\sigma_w(k)]<\alpha\rangle+\tau\mathbb{C}[\tau]\langle\omega'_k\mid\sigma_w(k)-[\sigma_w(k)]\geqslant\alpha\rangle,\end{aligned}$$

and $U_{\alpha+p}G=\tau^pU_\alpha G$ (*resp.* $U_{<\alpha+p}G=\tau^pU_{<\alpha}G$) for any $p\in\mathbb{Z}$. We then have

$$U_\alpha G=\mathbb{C}\langle\omega'_k\mid\sigma_w(k)-[\sigma_w(k)]=\alpha\rangle+U_{<\alpha}G.$$

Notice that, according to the formula for ω_k, the elements ω'_k satisfy

$$\begin{aligned}-\frac{1}{\mu}\big(\tau\partial_\tau+\sigma_w(k)-[\sigma_w(k)]\big)\omega'_k&=\tau^{[\sigma_w(k)]+1-[\sigma_w(k+1)]}\omega'_{k+1}\\&=\tau^{\lceil s_w(k+1)\rceil-\lceil s_w(k)\rceil}\omega'_{k+1},\end{aligned}\tag{3.8}$$

with $\lceil s\rceil:=-[-s]$. Recall that the sequence $(s_w(k))$, hence the sequence $(\lceil s_w(k)\rceil)$, is increasing. If $\lceil s_w(k+1)\rceil>\lceil s_w(k)\rceil$, then

$$\big(\tau\partial_\tau+\sigma_w(k)-[\sigma_w(k)]\big)\omega'_k\in U_{<0}G.$$

Otherwise, we have

$$\lceil s_w(k)\rceil-s_w(k)\geqslant\lceil s_w(k+1)\rceil-s_w(k+1),$$

i.e.,

$$\sigma_w(k)-[\sigma_w(k)]\geqslant\sigma_w(k+1)-[\sigma_w(k+1)],$$

and we conclude that $U_\alpha G$ is stable under $\tau\partial_\tau$ and that $\tau\partial_\tau+\alpha$ is nilpotent on gr_α^UG. The filtration $U_\bullet G$ satisfies then the characterizing properties of $V_\bullet G$, hence is equal to it.

We may now compute $G_p \cap V_\alpha$ for $p \in \mathbb{Z}$ and $\alpha \in [0,1[$. Any element of $G_p \cap V_\alpha$ decomposes uniquely as $\sum_{k=0}^{\mu-1} p_k(\tau)\omega'_k$, with

$$p_k(\tau) \in \begin{cases} \tau^{p-[\sigma_w(k)]}\mathbb{C}[\tau^{-1}] \cap \mathbb{C}[\tau] & \text{if } \sigma_w(k) - [\sigma_w(k)] \leqslant \alpha, \\ \tau^{p-[\sigma_w(k)]}\mathbb{C}[\tau^{-1}] \cap \tau\mathbb{C}[\tau] & \text{if } \sigma_w(k) - [\sigma_w(k)] > \alpha \end{cases}$$

It follows that

$$G_p \cap V_\alpha = \sum_{k \mid \sigma_w(k)=\alpha+p} \mathbb{C}\cdot\omega'_k + G_p \cap V_{<\alpha} + G_{p-1} \cap V_\alpha, \tag{3.9}$$

and therefore $\mathrm{gr}_p^G \mathrm{gr}_\alpha^V G$ is generated by the classes of ω'_k with $\sigma_w(k) = \alpha + p$. These classes form a basis of $\mathrm{gr}_p^G \mathrm{gr}_\alpha^V G$, as $\dim \oplus_p \oplus_{\alpha\in[0,1[} \mathrm{gr}_p^G \mathrm{gr}_\alpha^V G = \mu$. This gives the last statement of the proposition. □

For $\alpha \in [0,1[$, let ω'_k be such that $\lceil s_w(k)\rceil - s_w(k) = \alpha$ and denote by $[\omega'_k]$ the class of ω'_k in $H_\alpha := \mathrm{gr}_\alpha^V G$. After (1.5) we have:

$$-\frac{1}{\mu}(\tau\partial_\tau + \alpha)[\omega'_k] = \begin{cases} 0 & \text{if } s_w(k+1) > s_w(k), \\ [\omega'_{k+1}] & \text{if } s_w(k+1) = s_w(k). \end{cases} \tag{3.10}$$

It follows that the *primitive elements* relative to the nilpotent operator induced by $(-1/\mu)(\tau\partial_\tau + \alpha)$ on H_α are the elements $[\omega'_k]$ such that

$$k \geqslant n+1, \quad \lceil s_w(k)\rceil - s_w(k) = \alpha \quad \text{and} \quad s_w(k-1) < s_w(k)$$

and, if moreover $\alpha = 0$, the element $[\omega'_0] = [\omega_0]$.

Therefore, the Jordan blocks of $(-1/\mu)(\tau\partial_\tau + \alpha)$ on H_α are in one-to-one correspondence with the maximal constant sequences in $\mathcal{S}_w$, and the corresponding sizes are the same. All Jordan blocks, except that of $[\omega_0]$ if $\alpha = 0$, have thus size $\leqslant n$, and even $\leqslant n-1$ if α is an integer (*cf.* Cor. 3.4). Recall also (*cf.* [8, 9]) that $H := \oplus_{\alpha\in[0,1[} H_\alpha$ may be identified with the relative cohomology space $H^n(U, f^{-1}(t))$ for $|t| \gg 0$, that H_α corresponds to the generalized eigenspace of the monodromy corresponding to the eigenvalue $\exp 2i\pi\alpha$, and that the unipotent part of the monodromy operator T is equal to $\exp 2i\pi N$ with $N := -(\tau\partial_\tau + \alpha)$.

Example 3.11. — Take $n = 4$ and $w_0 = 1$, $w_1 = 2$, $w_2 = 12$, $w_3 = 15$ and $w_4 = 30$, so that $\mu = 60$. Then the only possible α is 0 and N has one Jordan block of size 5, 3 blocks of size 3, 13 blocks of size 2 and 20 blocks of size 1. On the other hand, if $\mu = n+1$ (and all w_i equal to 1), the only possible α is 0 and N has only one Jordan block (of size $n+1$).

4. Poincaré duality and higher residue pairings

Consider on $\mathbb{C}[\tau,\tau^{-1}]$ the ring involution induced by $\tau \mapsto -\tau$. We will set $\overline{p(\tau)} := p(-\tau)$ (there is no complex conjugation involved here). Given a $\mathbb{C}[\tau,\tau^{-1}]$-module G, we denote by $\overline{G}$ the $\mathbb{C}$-vector space G equipped with the new module structure $p(\tau)\cdot g = p(-\tau)g$. For convenience, we denote by $\overline{g}$ the elements of $\overline{G}$. The $\mathbb{C}[\tau,\tau^{-1}]$-structure of $\overline{G}$ is therefore given by the rule: $\overline{p(\tau)}\overline{g} = \overline{p(\tau)g}$.

If G is moreover equipped with a connection, *i.e.*, with a compatible action of ∂_τ, then so is $\overline{G}$ and we have $\partial_\tau\overline{g} := \overline{-\partial_\tau g}$. Notice that $\tau\partial_\tau\overline{g} = \overline{\tau\partial_\tau g}$.

Duality for $\mathcal{D}$-modules gives (*cf.* [9]) the existence of a nondegenerate $\mathbb{C}[\tau,\tau^{-1}]$-bilinear pairing

$$S : G \underset{\mathbb{C}[\tau,\tau^{-1}]}{\otimes} \overline{G} \longrightarrow \mathbb{C}[\tau,\tau^{-1}]$$

satisfying the following properties:

(1) $\dfrac{dS(g',\overline{g''})}{d\tau} = S(\partial_\tau g',\overline{g''}) + S(g',\partial_\tau\overline{g''}) = S(\partial_\tau g',\overline{g''}) - S(g',\overline{\partial_\tau g''})$,
(equivalently, $\tau\partial_\tau S(g',\overline{g''}) = S(\tau\partial_\tau g',\overline{g''}) + S(g',\overline{\tau\partial_\tau g''})$),

(2) S sends $V_0\otimes\overline{V_{<1}}$ in $\mathbb{C}[\tau]$,

(3) S sends $G_0\otimes\overline{G_0}$ in $\theta^n\mathbb{C}[\theta] = \tau^{-n}\mathbb{C}[\tau^{-1}]$,

(4) $S(g'',\overline{g'}) = (-1)^n\overline{S(g',\overline{g''})}$ (this reflects the $(-1)^n$-symmetry of the Poincaré duality on U).

Notice that (1) means that S is a horizontal section of the $\mathbb{C}[\tau,\tau^{-1}]$-module $\mathrm{Hom}_{\mathbb{C}[\tau,\tau^{-1}]}(G\otimes\overline{G},\mathbb{C}[\tau,\tau^{-1}])$ equipped with its natural connection, or also that S is a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-linear morphism $\overline{G}\to G^*$, if one endows $G^* = \mathrm{Hom}_{\mathbb{C}[\tau,\tau^{-1}]}(G,\mathbb{C}[\tau,\tau^{-1}])$ with its natural connection. Therefore, (2) follows from (1) because any $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-linear morphism is strict with respect to the Malgrange-Kashiwara filtrations V and we have

$$V_\beta(G^*) = \mathrm{Hom}_{\mathbb{C}[\tau]}(V_{<-\beta+1}G,\mathbb{C}[\tau])$$

(*cf.* [9]).

In the case of singularities, this corresponds to the "higher residue pairings" of K. Saito [11]. The link with Poincaré duality is explained in [12].

For our Laurent polynomial f, we will recover in an elementary way the existence of such a pairing S satisfying the previous properties. More precisely, we have:

Lemma 4.1. — *There exists a unique (up to a nonzero constant) nondegenerate pairing S satisfying Properties* (1), (2), (3). *It is given by the formula:*

$$S(\omega_k,\overline{\omega_\ell})=\begin{cases}S(\omega_0,\overline{\omega_n}) & \begin{cases}\text{if } 0\leqslant k\leqslant n \text{ and } k+\ell=n,\\ \text{or if } n+1\leqslant k\leqslant \mu-1 \text{ and } k+\ell=\mu+n,\end{cases}\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, for any k,ℓ, $S(\omega_k,\overline{\omega_\ell})$ belongs to $\mathbb{C}\tau^{-n}$ and S satisfies (4).

Proof. — Assume that a pairing S satisfying (1), (2), (3) exists. For $k,\ell=0,\dots,\mu-1$, we have $S(\omega_k,\overline{\omega_\ell})\in\tau^{-n}\mathbb{C}[\tau^{-1}]$ by (3) and $S(\omega_0,\overline{\omega_\ell})\in\tau^{-[\sigma_w(\ell)]}\mathbb{C}[\tau]$ by (2). Therefore, $S(\omega_0,\overline{\omega_\ell})\neq 0$ implies $[\sigma_w(\ell)]\geqslant n$, and if $[\sigma_w(\ell)]=n$, we have $S(\omega_0,\overline{\omega_\ell})\in\mathbb{C}\tau^{-n}$. But we know by (3.3) that

$$[\sigma_w(\ell)]\begin{cases}<n & \text{if } \ell\neq n,\\ =n & \text{if } \ell=n.\end{cases}$$

Therefore, $S(\omega_0,\overline{\omega_\ell})=0$ if $\ell\neq n$ and $S(\omega_0,\overline{\omega_n})\in\mathbb{C}\tau^{-n}$.

Notice also that we have by (1) and Proposition 3.2:

$$\begin{aligned}(4.2)\qquad -\frac{1}{\mu}(\tau\partial_\tau+n)S(\omega_k,\overline{\omega_\ell})\\ =\tau\big[S(\omega_{k+1},\overline{\omega_\ell})-S(\omega_k,\overline{\omega_{\ell+1}})\big]+\frac{\sigma_w(k)+\sigma_w(\ell)-n}{\mu}S(\omega_k,\overline{\omega_\ell}),\end{aligned}$$

if we put as above $\omega_\mu=\omega_0$.

Argue now by induction for $k<n$: as $S(\omega_k,\overline{\omega_\ell})\in\mathbb{C}\tau^{-n}$, the LHS in (4.2) vanishes. This shows that $S(\omega_{k+1},\overline{\omega_\ell})=0$ if $\ell\neq n-k,n-1-k$. Moreover, if $\ell=n-k$, we have $\sigma_w(k)+\sigma_w(\ell)-n=0$, hence $S(\omega_{k+1},\overline{\omega_{n-k}})=0$. Last, we have $S(\omega_{k+1},\overline{\omega_{n-1-k}})=S(\omega_k,\overline{\omega_{n-k}})$.

Argue similarly for $k\geqslant n+1$. □

Notice that, if A_∞^* denotes the adjoint of A_∞ with respect to S, then $A_\infty+A_\infty^*=n\,\mathrm{Id}$, *i.e.*, $A_\infty-(n/2)\,\mathrm{Id}$ is skewsymmetric with respect to S.

5. M. Saito's solution to the Birkhoff problem

One step in constructing the Frobenius structure associated to f consists in solving Birkhoff's problem for the Brieskorn lattice G_0 in the Gauss-Manin system G, that is, in finding a $\mathbb{C}[\tau]$-lattice E of G, which glues with G_0 to a trivial vector bundle on $\mathbb{P}^1$. Recall (*cf.* [3, App. B] for what follows) that there is a one-to-one correspondence between such lattices E which are *logarithmic*, and decreasing filtrations $\oplus_{\alpha\in[0,1[}H^\bullet_\alpha$ of $H=\oplus_{\alpha\in[0,1[}H_\alpha$ which are stable under N and which are *opposite* to the filtration

$$G_p(H) := \mathop{\oplus}_{\alpha\in[0,1[}(G_p\cap V_\alpha)/(G_p\cap V_{<\alpha}) = \mathbb{C}\langle[\omega'_k] \mid [\sigma_w(k)]\leqslant p\rangle \quad \text{after (3.9).}$$

This is analogous to [12, Th. 3.6].

In [12, Lemma 2.8], M. Saito defines a canonical decreasing filtration $H^\bullet_{\mathrm{Saito}}$ in terms of the monodromy filtration $M_\bullet$ of the nilpotent endomorphism $2i\pi N$ of H and of the filtration conjugate to $G_p(H)$, the conjugation being taken with respect to the real structure on H coming from the identification with $H^n(U,f^{-1}(t))$. This defines therefore a canonical solution to Birkhoff's problem for G_0.

Consider now the decreasing filtration $H^\bullet$ of H explicitly defined by

$$H^p = \mathbb{C}\langle[\omega'_k] \mid [\sigma_w(k)]\geqslant p\rangle, \tag{5.1}$$

where $[\,]$ denotes the class in $H=\oplus_{\alpha\in[0,1[}V_\alpha G/V_{<\alpha}G$. Then $H^\bullet$ is opposite to $G_\bullet(H)$. It satisfies

$$H^0=H,\ H^{n+1}=0,\quad NH^p\subset H^{p+1}$$

and, for $k=0,\dots,\mu-1$ and $\alpha\in[0,1[$,

$$(H^p_\alpha)^\perp = \begin{cases} H^{n-p}_{1-\alpha} & \text{if } \alpha\neq 0,\\ H^{n+1-p}_0 & \text{if } \alpha=0,\end{cases}$$

where $^\perp$ means taking the orthogonal with respect to the symmetric bilinear form g on H induced by S. If $\mu=n+1$ (and all w_i equal to 1), then $H^p=M_{n-2p}$ (this implies that the mixed Hodge structure on H is "Hodge-Tate").

Proposition 5.2. — *The filtration $H^\bullet$ is equal to the opposite filtration $H^\bullet_{\mathrm{Saito}}$. The associated logarithmic lattice is $E:=\mathbb{C}[\tau]\langle\omega_0,\dots,\omega_{\mu-1}\rangle$.*

Proof. — Let us begin with the second statement. The lattice E introduced in the proposition is logarithmic, by Proposition 3.2. A computation analogous to that of $G_p\cap V_\alpha$ shows that the filtration $\tau^pE\cap V_\alpha/\tau^pE\cap V_{<\alpha}$ of H_α is equal to H^p_α. Therefore, E is the logarithmic lattice corresponding to $H^\bullet$ by the correspondence recalled above.

For the first statement, put $F^\bullet(H)=G_{n-\bullet}(H)$. This is a decreasing filtration. Consider also the increasing filtration

$$W_\bullet(H_\alpha) = \begin{cases} M_{\bullet-n-1}(H_\alpha) & \text{if } \alpha\neq 0,\\ M_{\bullet-n}(H_0) & \text{if } \alpha=0,\end{cases}$$

where $M_\bullet(H)$ denotes the monodromy filtration of the nilpotent endomorphism $2i\pi N$ on H. Recall that $W_\bullet(H)$ is defined over $\mathbb{R}$ (even over $\mathbb{Q}$) as $2i\pi N$ is so. Then the opposite filtration given by M. Saito is

$$H^\bullet_{\text{Saito}} = \sum_q \overline{F}^q \cap W_{n+q-\bullet}(H),$$

where $\overline{E}$ denotes the conjugate of the subspace E of H with respect to the complex conjugation coming from the identification

$$H \xrightarrow{\sim} H^n(U, f^{-1}(t), \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} H^n(U, f^{-1}(t), \mathbb{R}).$$

We therefore need to give a description of the conjugation in term of the basis $[\omega'_k]$.

Let k_0 be such that $[\omega'_{k_0}]$ is a primitive element with respect to N, and denote by ν_{k_0} its weight. Then $N^{\nu_{k_0}+1}[\omega'_{k_0}] = 0$. For $j = 0, \dots, \nu_{k_0}$, put $k = k_0 + j$. Then $[\omega'_k] = (\frac{1}{\mu}N)^j[\omega'_{k_0}]$ has order $\nu_{k_0} + 1 - j$ with respect to N, and weight $\nu_k := \nu_{k_0} - 2j$. Moreover, we have $\sigma_w(k) = \sigma_w(k_0) + j$, as $j \mapsto s_w(k_0 + j)$ is constant. The space $B_{k_0} := \langle N^j[\omega'_{k_0}] \mid j = 0, \dots, \nu_{k_0}\rangle$ is a Jordan block of N.

Assume that $k_0 \geqslant n + 1$. Then $[\omega'_{\mu+n-k_0}]$ is primitive with respect to ${}^tN$, hence $[\omega'_{\mu+n-k_0-\nu_{k_0}}]$ is primitive with respect to N. It will be convenient to put $\overline{k}_0 = \mu+n-k_0-\nu_{k_0}$ and, for $k = k_0+j$ with $j = 0, \dots, \nu_{k_0}$, $\overline{k} = \overline{k}_0+j$. We therefore have $\overline{k} = \mu + n - k - \nu_k$. Notice that, for such a k, we have $s_w(k) = s_w(k_0) = \mu - s_w(\overline{k}_0) = \mu - s_w(\overline{k})$. We also have $\sigma_w(\mu + n - k) - \nu_k = \sigma_w(\overline{k})$ if $k \geqslant n + 1$.

For $k \in [0, n]$, we simply put $\overline{k} = k$.

The proof of the following lemma will be given in § 6.

Lemma 5.3. — *For $k_0 \geqslant n + 1$, the conjugate of the Jordan block B_{k_0} is the Jordan block $B_{\overline{k}_0}$, and B_0 is self-conjugate.*

It follows from this lemma that, for k as above, we have

$$\overline{[\omega'_k]} = \sum_{\ell=k}^{k_0+\nu_0} a_\ell [\omega'_{\overline{\ell}}] \tag{5.4}$$

with $a_k \neq 0$.

Let us now end the proof of Proposition 5.2. We have

$$F^q \cap W_{n+q-p} = G_{n-q} \cap M_{q-p(-1)} = \langle [\omega'_k] \mid [\sigma_w(k)] \leqslant n-q \text{ and } \nu_k \leqslant q-p(-1)\rangle,$$

where (-1) is added if $\sigma_w(k) \notin \mathbb{Z}$ and not added otherwise. Therefore,

$$\sum_q F^q \cap W_{n+q-p} = \langle [\omega'_k] \mid [\sigma_w(k)] + \nu_k \leqslant n - p(-1)\rangle.$$

Remark now that, if $k \geqslant n+1$,

$$\begin{aligned}[][\sigma_w(k)] + \nu_k \leqslant n - p(-1) &\iff [n - \sigma_w(\mu + n - k)] + \nu_k \leqslant n - p(-1)\\ &\iff \lceil \sigma_w(\mu + n - k - \nu_k) \rceil \geqslant p(+1)\\ &\iff [\sigma_w(\mu + n - k - \nu_k)] \geqslant p\\ &\iff [\sigma_w(\overline{k})] \geqslant p.\end{aligned}$$

Arguing similarly for $k \leqslant n$, we conclude from Lemma 5.3 and (5.1) that

$$\begin{aligned}\sum_q \overline{F}^q \cap W_{n+q-p} &= \langle \overline{[\omega'_k]} \mid [\sigma_w(k)] + \nu_k \leqslant n - p(-1) \rangle\\ &= \langle [\omega'_{\overline{k}}] \mid [\sigma_w(\overline{k})] \geqslant p \rangle\\ &= H^p.\end{aligned} \tag{5.5}$$

Notice that (5.5) follows from (5.4), as σ_w is increasing on each $B_{\overline{k}_0}$. □

6. Some topology of f and proof of Lemma 5.3

6.a. Lefschetz thimbles. — Denote by Δ the subset $(\mathbb{R}_+^*)^{n+1} \cap U \subset U$, defined by $u_i > 0$ for $i = 0, \dots, n$. The restriction $f_{|\Delta}$ of f to Δ takes values in $[\mu, +\infty[$ and has only one critical point (which is a Morse critical point of index 0), namely $(1, \dots, 1)$, with critical value equal to μ. Notice also that $f_{|\Delta}$ is proper. Therefore, Δ is a Lefschetz thimble for f with respect to the critical point $(1, \dots, 1)$. Other Lefschetz thimbles at $\zeta^\ell(1, \dots, 1)$ are $\zeta^\ell \Delta$ $(\ell = 0, \dots, \mu - 1)$.

Fix $\tau \neq 0$. The morphisms

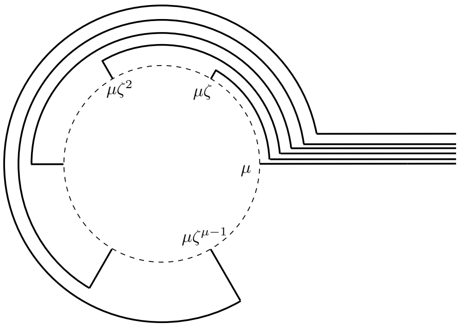
$$H_n(U, \operatorname{Re} \tau f > C'; \mathbb{Q}) \longrightarrow H_n(U, \operatorname{Re} \tau f > C; \mathbb{Q})$$

for $C' > C$ are isomorphisms if C is big enough. We denote by $H_n(U, \operatorname{Re} \tau f \gg 0)$ the limit of this inverse system. This is the germ at τ of a local system $\mathcal{H}$ of rank μ on $\mathbb{C}^* = \{\tau \neq 0\}$. Notice that Δ defines a nonzero element of the germ $\mathcal{H}_\tau$ at any τ with $\operatorname{Re} \tau > 0$, *i.e.*, a section $\Delta(\tau)$ of $\mathcal{H}$ on $\{\operatorname{Re} \tau > 0\}$. Therefore, it defines in a unique way a multivalued section of $\mathcal{H}$ on $\mathbb{C}^*$.

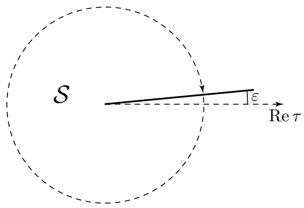
Let $\varepsilon > 0$ be small enough. As f is a C^∞ fibration over the open set $\mathbb{C} \smallsetminus \{\mu\zeta^\ell \mid \ell = 0, \dots, \mu - 1\}$, it is possible to find a basis of sections $\Delta_0(\tau), \dots, \Delta_{\mu-1}(\tau)$ of $\mathcal{H}$ on the open set

$$\mathcal{S} = \{\tau = |\tau|\, e^{2i\pi\theta} \mid \theta \in]\varepsilon - 1, \varepsilon[\}$$

in such a way that, for any $\ell \in \{0, \dots, \mu-1\}$ and $|\tau| > 0$, we have $\Delta_\ell(\zeta^{-\ell}|\tau|) = \zeta^\ell \Delta$. Of course, this basis extends as a basis of multivalued sections of $\mathcal{H}$ on $\mathbb{C}^*$.



The images $f(\Delta_\ell(|\tau|))$



The domain $\mathcal{S}$

6.b. Integrals along Lefschetz thimbles. — Let $\eta \in G$ and let $\widetilde{\eta}$ be a representative of η in $\Omega^n(U)[\tau, \tau^{-1}]$. Then the function

$$\mathcal{S} \ni \tau \longmapsto \int_{\Delta_\ell(\tau)} e^{-\tau f} \widetilde{\eta}$$

only depends on η and is holomorphic on $\mathcal{S}$. It is denoted by $\varphi_{\ell,\eta}(\tau)$. Moreover, we have

$$\frac{\partial \varphi_{\ell,\eta}(\tau)}{\partial \tau} = \varphi_{\ell,\partial_\tau \eta}(\tau).$$

It follows that, if $\eta \in V_\alpha G$, then

$$\varphi_{\ell,\eta}(\tau) = \tau^{-\alpha} \Big[\sum_{m=0}^{m_\eta} c_{\eta,\ell}^{(m)} \frac{(-\frac{1}{2i\pi} \log \tau)^m}{m!} + o(1) \Big] := \tau^{-\alpha} \big[\psi_{\eta,\ell}(\tau) + o(1) \big], \tag{6.1}$$

where $\tau^{-\alpha} = |\tau|^{-\alpha} e^{-2i\pi\alpha\theta}$, $\frac{1}{2i\pi} \log \tau = (\frac{1}{2i\pi} \log |\tau|) + \theta$, with $\theta \in \,]\varepsilon - 1, \varepsilon[$ and $c_{\eta,\ell}^{(m)} \in \mathbb{C}$. The coefficients $c_{\eta,\ell}^{(m)}$ only depend on the class $[\eta]$ of η in $\mathrm{gr}_\alpha^V G$, so we will denote them by $c_{[\eta],\ell}^{(m)}$, and we have

$$c_{[\eta],\ell}^{(m)} = c_{(2i\pi N)^m [\eta],\ell}^{(0)}. \tag{6.2}$$

We will now characterize the Jordan blocks B_{k_0} in H_α. Such a Jordan block is characterized by the constant value s of $s_w(\bullet)$, so that it will be convenient to denote such a block by $B_{\alpha,s}$.

Lemma 6.3. — *For $[\eta] \in H_\alpha$, we have $[\eta] \in B_{\alpha,s}$ if and only if, for any $\ell = 0, \ldots, \mu - 1$ and any $j \geqslant 0$, we have*

$$c_{(2i\pi N)^j [\eta],\ell}^{(0)} = \zeta^{\ell s} \sum_m c_{(2i\pi N)^{j+m} [\eta],0}^{(0)} \frac{(-\ell/\mu)^m}{m!}.$$

Proof. — For $\eta = \omega_k$, denote $c^{(m)}_{[\omega_k],\ell} = c^{(m)}_{[\omega'_k],\ell}$ by $c^{(m)}_{k,\ell}$. Then we have

$$\int_{\Delta_\ell(\tau)} e^{-\tau f}\omega_k = \tau^{-\sigma_w(k)}\Big(\Big[c^{(m_k)}_{k,\ell}\frac{(-\frac{1}{2i\pi}\log\tau)^{m_k}}{m_k!}+\cdots+c^{(0)}_{k,\ell}\Big]+o(1)\Big), \tag{6.4}$$

where m_k+1 denotes the order of $[\omega_k]$ in $\mathrm{gr}^V_{\sigma_w(k)}G$ with respect to $2i\pi N$. Remark now that, as $\omega_k = u^{a(k)}\omega_0$ and $|a(k)| = k$, we have

$$\int_{\Delta_\ell(\zeta^{-\ell}|\tau|)} e^{-\zeta^{-\ell}|\tau|f}\omega_k = \int_{\zeta^\ell\Delta} e^{-\zeta^{-\ell}|\tau|f}\omega_k = \zeta^{k\ell}\int_\Delta e^{-|\tau|f}\omega_k.$$

Hence, we get

$$\begin{aligned} c^{(m_k)}_{k,\ell}&\frac{(-\frac{1}{2i\pi}\log|\tau|+\ell/\mu)^{m_k}}{m_k!}+\cdots+c^{(0)}_{k,\ell}\\ &\qquad= \zeta^{\ell s_w(k)}\Big[c^{(m_k)}_{k,0}\frac{(-\frac{1}{2i\pi}\log|\tau|)^{m_k}}{m_k!}+\cdots+c^{(0)}_{k,0}\Big],\end{aligned}$$

and in particular

$$c^{(0)}_{k,\ell} = \zeta^{\ell s_w(k)}\cdot\sum_{m=0}^{m_k}\frac{(-\ell/\mu)^m}{m!}\,c^{(m)}_{k,0}. \tag{6.5}$$

Therefore, any element $[\eta]$ in $B_{\alpha,s}$ satisfies the equality of Lemma 6.3 for $j=0$, hence for any j.

Conversely, remark first that, if $[\eta]$ is fixed, then the equality of Lemma 6.3 for any $j\geqslant 0$ is equivalent to

$$\psi_{[\eta],\ell}(\zeta^{-\ell}|\tau|) = \zeta^{\ell s}\psi_{[\eta],0}(|\tau|),$$

where $\psi_{[\eta],\ell}$ is defined by (6.1) (two polynomials are equal iff all the corresponding derivatives at 0 are equal).

Write $[\eta] = \sum\lambda_k[\omega'_k]$ in H_α, denote $m'_{[\eta]} = \max_{k|\lambda_k\neq 0} m_k$ and put $K_{[\eta]} = \{k \mid m_k = m'_{[\eta]}\}$. Notice that, for $k,k'\in K_{[\eta]}$, we have $s_w(k)\neq s_w(k')$. If $m'_{[\eta]} > m_{[\eta]}$, we have $\sum_{k\in K_{[\eta]}}\zeta^{\ell s_w(k)}\lambda_k c^{(m_k)}_{k,0} = 0$ for any $\ell = 0,\dots,\mu-1$. It follows that $\lambda_k c^{(m_k)}_{k,0} = 0$, hence $\lambda_k = 0$, for any $k\in K_{[\eta]}$, a contradiction. Therefore $m'_{[\eta]} = m_{[\eta]}$. Argue similarly to show that $K_{[\eta]}$ is reduced to one element, denoted by $k_{[\eta]}$, and that $s_w(k_{[\eta]}) = s$. Apply the lemma by induction on $m_{[\eta']}$ to $[\eta'] = [\eta]-\lambda_{k_{[\eta]}}[\omega'_{k_{[\eta]}}]$. □

6.c. Isomorphism between nearby cycles. — The multivalued cycles $\Delta_\ell(\tau)$ form a basis of the space of multivalued global sections of $\mathcal{H}$, that we denote by $\psi_\tau\mathcal{H}$. This basis defines the integral (hence the real) structure on $\psi_\tau\mathcal{H}$.

Denote by $\mathcal{N}_{\alpha,p}$ the space of linear combinations with meromorphic coefficients of germs at $\tau=0$ of the multivalued functions $e_{\alpha,q}=\tau^\alpha(-\frac{1}{2i\pi}\log\tau)^q/q!$ ($q\leqslant p$). For p large enough (here $p\geqslant n+1$ is enough), the map

$$\begin{aligned}\varphi: V_\alpha G &\longrightarrow V_0(G\otimes\mathcal{N}_{\alpha,p})\\ \eta &\longmapsto \sum_{j=0}^{p}[2i\pi(\tau\partial_\tau+\alpha)]^j\eta\otimes e_{\alpha,j}\end{aligned}$$

induces an isomorphism

$$\mathrm{gr}_\alpha^V G \xrightarrow{\sim} \mathrm{Ker}\big[\tau\partial_\tau:\mathrm{gr}_0^V(G\otimes\mathcal{N}_{\alpha,p})\longrightarrow \mathrm{gr}_0^V(G\otimes\mathcal{N}_{\alpha,p})\big].$$

As G is regular at $\tau=0$, there exists a perturbation $\eta\mapsto\psi(\eta)\in V_{<0}(G\otimes\mathcal{N}_{\alpha,p})$ such that $\varphi(\eta)+\psi(\eta)\in\mathrm{Ker}\big[\tau\partial_\tau: G\otimes\mathcal{N}_{\alpha,p}\to G\otimes\mathcal{N}_{\alpha,p}\big]$.

Recall (see, *e.g.*, [7]) that $\mathrm{Ker}\big[\tau\partial_\tau: G\otimes\mathcal{N}_{\alpha,p}\to G\otimes\mathcal{N}_{\alpha,p}\big]$ is identified with H_α. Set $\mathcal{N}=\oplus_{\alpha\in[0,1[}\mathcal{N}_{\alpha,n+1}$. Given a section λ of $H=\mathrm{Ker}\big[\tau\partial_\tau: G\otimes\mathcal{N}\to G\otimes\mathcal{N}\big]$ and a section δ of $\psi_\tau\mathcal{H}$, choose a representative $\widetilde{\lambda}$ of λ in $\Omega^n(U)\otimes_{\mathbb{C}}\mathcal{N}$. Then $\int_\delta e^{-\tau f}\widetilde{\lambda}\in\mathbb{C}$. Then (see Appendix) λ belongs to $H_{\mathbb{Q}}$ if and only if, for any $\ell=0,\dots,\mu$ and some nonzero τ, we have

$$\int_{\Delta_\ell(\tau)} e^{-\tau f}\widetilde{\lambda}\in\mathbb{Q}. \tag{6.6}$$

For $\eta\in V_\alpha G$ and $\widetilde{\lambda}=\varphi(\eta)+\psi(\eta)$, and using (6.1), one finds

$$\int_{\Delta_\ell(\tau)} e^{-\tau f}\varphi(\eta)=c^{(0)}_{\eta,\ell}+o(1).$$

As a consequence, the conjugate $\overline{[\eta]}$ of $[\eta]$ satisfies

$$c^{(0)}_{\overline{[\eta]},\ell}=\overline{c^{(0)}_{[\eta],\ell}}.$$

It follows now from Lemma 6.3 that

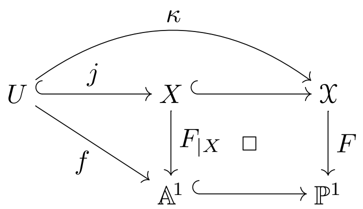
$$\overline{B_{\alpha,s}}=\begin{cases}B_{1-\alpha,\mu-s} & \text{if } \alpha\in\,]0,1[,\\ B_{0,\mu-s} & \text{if } \alpha=0.\end{cases}$$

As $s_w(\overline{k}_0)=\mu-s_w(k_0)$, this ends the proof of Lemma 5.3. □

Appendix

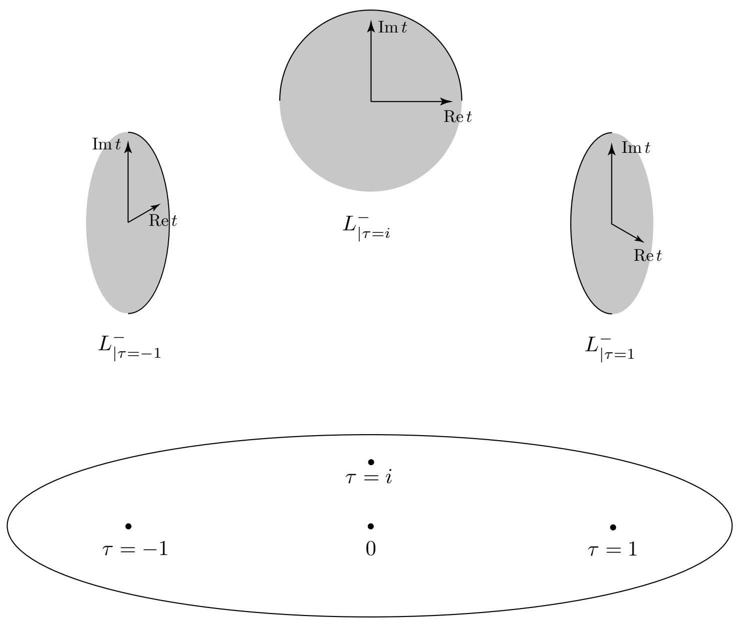
In this appendix, we explain with some details why the real structure on H as defined by (6.6) is indeed the real structure used in [8] to define the Hodge structure on H. We will need to recall some notation and results of [8].

We will denote by U a smooth quasi-projective variety and by $f:U\to\mathbb{A}^1$ a regular function on U. We denote by t the coordinate on the affine line $\mathbb{A}^1$. We also fix an embedding $\kappa : U \hookrightarrow \mathfrak{X}$ into a smooth projective variety such that there exists an algebraic map $F : \mathfrak{X} \to \mathbb{P}^1$ extending f. We have a commutative diagram, where the right part is Cartesian, thus defining X as a fibred product,



Denote by $\varepsilon : \widetilde{\mathbb{P}}^1 \to \mathbb{P}^1$ the real blow-up of $\mathbb{P}^1$ centered at ∞ ($\widetilde{\mathbb{P}}^1$ is diffeomorphic to a closed disc) and by $\widetilde{F} : \widetilde{\mathfrak{X}} \to \widetilde{\mathbb{P}}^1$ the fibre-product of F with the blowing-up ε. Denote by $\widetilde{\kappa}$ the inclusion $U \hookrightarrow \widetilde{\mathfrak{X}}$.

Denote by S^1 the inverse image of ∞ by the blowing-up ε. Let $\widehat{\mathbb{A}}^1$ be an affine line with coordinate τ. Denote by L'^+ the closed set of $S^1 \times \widehat{\mathbb{A}}^1 \subset \widetilde{\mathbb{P}}^1 \times \widehat{\mathbb{A}}^1$ defined by $\mathrm{Re}(e^{i\theta}\tau) \geqslant 0$, with $\theta = \arg t$ and where t is the coordinate on $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \{\infty\}$, and set $L^- = \widetilde{\mathbb{P}}^1 \times \widehat{\mathbb{A}}^1 \smallsetminus L'^+$. For $\tau \neq 0$, denote by $L'^+_\tau, L^-_\tau \subset \widetilde{\mathbb{P}}^1$ the fibre of L'^+, L^- over τ.



Affine line $\widehat{\mathbb{A}}^1$ with coordinate τ

Denote similarly by $L'^{+}_{\widetilde{\mathfrak{X}}}, L^{-}_{\widetilde{\mathfrak{X}}} \subset \widetilde{\mathfrak{X}} \times \widehat{\mathbb{A}}^1$ (*resp.* $L'^{+}_{\widetilde{\mathfrak{X}},\tau}, L^{-}_{\widetilde{\mathfrak{X}},\tau} \subset \widetilde{\mathfrak{X}}$) the inverse image of the corresponding sets by $\widetilde{F} \times \mathrm{Id}_{\widehat{\mathbb{A}}^1}$ (*resp.* by $\widetilde{F}$).

We denote by $\alpha : \mathbb{A}^1 \times \widehat{\mathbb{A}}^1 \hookrightarrow L^-$ and $\beta : L^- \hookrightarrow \widetilde{\mathbb{P}}^1 \times \widehat{\mathbb{A}}^1$ (*resp.* $\alpha_\tau : \mathbb{A}^1 \hookrightarrow L^-_\tau$ and $\beta_\tau : L^-_\tau \hookrightarrow \widetilde{\mathbb{P}}^1$) the inclusions, and by the same letters the corresponding inclusions

$$\alpha : X \times \widehat{\mathbb{A}}^1 \longrightarrow L^-_{\widetilde{\mathfrak{X}}} \quad \text{and} \quad \beta : L^-_{\widetilde{\mathfrak{X}}} \longrightarrow \widetilde{\mathfrak{X}} \times \widehat{\mathbb{A}}^1,$$

resp.

$$\alpha_\tau : X \longrightarrow L^-_{\widetilde{\mathfrak{X}},\tau} \quad \text{and} \quad \beta_\tau : L^-_{\widetilde{\mathfrak{X}},\tau} \longrightarrow \widetilde{\mathfrak{X}}.$$

Therefore we have $\beta_\tau \circ \alpha_\tau \circ j = \widetilde{\kappa}$.

In [8, (1.8)], we have defined the Fourier transform $\mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U)$ as the following complex on $\mathfrak{X} \times \widehat{\mathbb{A}}^1$ (there is a shift by 1 in *loc. cit.*, that we do not introduce here):

$$\mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U) := \boldsymbol{R}\varepsilon_*\, \beta_!\, \boldsymbol{R}\alpha_*\, \boldsymbol{R}j_*\mathbb{C}_{U\times\widehat{\mathbb{A}}^1},$$

where we still denote by j (*resp.* κ) the inclusion $U \times \widehat{\mathbb{A}}^1 \hookrightarrow X \times \widehat{\mathbb{A}}^1$ (*resp.* $U \times \widehat{\mathbb{A}}^1 \hookrightarrow \mathfrak{X} \times \widehat{\mathbb{A}}^1$). This complex has a natural $\mathbb{Q}$-structure (replace $\mathbb{C}_U$ with $\mathbb{Q}_U$). This induces a $\mathbb{Q}$-structure on the nearby cycle complex $\psi_\tau \mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U) = \psi_\tau \mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{Q}_U) \otimes_{\mathbb{Q}} \mathbb{C}$.

Denote by $\mathcal{E}^{-\tau f}$ the algebraic $\mathcal{D}_{U\times\widehat{\mathbb{A}}^1}$-module $\mathcal{O}_{U\times\widehat{\mathbb{A}}^1} e^{-\tau f}$ (*i.e.*, the $\mathcal{O}_{U\times\widehat{\mathbb{A}}^1}$-module $\mathcal{O}_{U\times\widehat{\mathbb{A}}^1}$ with connection $e^{\tau f} \circ d \circ e^{-\tau f}$). The quasi-isomorphism

$$\mathrm{DR}^{\mathrm{an}}_{\mathfrak{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f}) \xrightarrow{\ \sim\ } \mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U) \tag{A.1}$$

constructed in [8, Th. 2.2] is then used to define the $\mathbb{Q}$-structure on the complex (of sheaves on $\mathfrak{X}$) $\psi_\tau \mathrm{DR}^{\mathrm{an}}_{\mathfrak{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f})$ [on the other hand one uses the V-filtration relative to $\tau = 0$ on $\kappa_+\mathcal{E}^{-\tau f}$ to construct the Hodge filtration on this complex]. By DR we mean the usual de Rham complex, starting in degree 0.

Denote by $\widetilde{\mathcal{O}}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1}$ the sheaf of multivalued holomorphic functions on $\widehat{\mathbb{A}}^1 \smallsetminus \{0\}$. Then

$$\psi_\tau \mathrm{DR}^{\mathrm{an}}_{\mathfrak{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f}) = i^{-1}_{\tau=0} \mathrm{DR}^{\mathrm{an}}_{\mathfrak{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f} \otimes_{\widehat{p}^{-1}\mathcal{O}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1}} \widetilde{\mathcal{O}}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1}),$$

where $\widehat{p} : \mathfrak{X} \times \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{A}}^1$ denotes the projection and $i_{\tau=0} : \mathfrak{X} \times \{0\} \hookrightarrow \mathfrak{X} \times \widehat{\mathbb{A}}^1$ denotes the inclusion (see, *e.g.*, [7, (4.9.4)]). We are interested in analyzing the $\mathbb{Q}$-structure on the cohomology of $\boldsymbol{R}\Gamma\big(\mathfrak{X}, \psi_\tau \mathrm{DR}^{\mathrm{an}}_{\mathfrak{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f})\big)$. Use C^∞ forms on $\mathfrak{X}$ to identify it with

$$\Gamma\big(\mathfrak{X}, i^{-1}_{\tau=0}\mathcal{E}^{\bullet}_{\mathfrak{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f} \otimes_{\widehat{p}^{-1}\mathcal{O}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1}} \widetilde{\mathcal{O}}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1})\big),$$

with $n = \dim \mathcal{X}$. Similarly, denote by $\widetilde{\mathbb{C}}_{\widehat{\mathbb{A}}^1}$ the sheaf of multivalued local sections of $\mathbb{C}_{\widehat{\mathbb{A}}^1}$ (*i.e.*, local sections on the universal covering of $\widehat{\mathbb{A}}^1 \smallsetminus \{0\}$). Then $\psi_\tau \mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U)$ is equal to $i_{\tau=0}^{-1}\big(\mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U) \otimes \widehat{p}^{-1}\widetilde{\mathbb{C}}_{\widehat{\mathbb{A}}^1}\big)$.

In order to know that the cohomology class of a closed multivalued section of $\widehat{p}_*\mathcal{E}^\bullet_{\mathcal{X}\times\widehat{\mathbb{A}}^1}(\kappa_+\mathcal{E}^{-\tau f} \otimes_{\widehat{p}^{-1}\mathcal{O}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1}} \widetilde{\mathcal{O}}^{\mathrm{an}}_{\widehat{\mathbb{A}}^1})$ is rational, one has to compute its image in $\boldsymbol{R}\widehat{p}_*\mathfrak{F}_F(\boldsymbol{R}j_*\mathbb{C}_U) \otimes \widetilde{\mathbb{C}}_{\widehat{\mathbb{A}}^1}$ and decide whether its class is rational or not. As the section is closed, it is enough to verify this after restricting to some (or any) $\tau \neq 0$. Therefore, we need to compute the map (A.1) after restricting to some fixed nonzero τ. In (6.6), we apply this computation to the multivalued form $e^{-\tau f}\widetilde{\lambda}$.

Denote by $\mathcal{E}_{\widetilde{\mathcal{X}}}$ the sheaf of C^∞ functions (in the sense of Whitney) on $\widetilde{\mathcal{X}}$, by $\mathcal{E}^{\mathrm{mod}}_{\widetilde{\mathcal{X}}}$ the sheaf on $\widetilde{\mathcal{X}}$ of C^∞ functions on U which have moderate growth along $\widetilde{\mathcal{X}} \smallsetminus U$, and by $\mathcal{E}^{\mathrm{mod},-\tau}_{\widetilde{\mathcal{X}}}$ the subsheaf of functions which moreover are infinitely flat along $L'^{+}_{\widetilde{\mathcal{X}},\tau}$.

On the other hand, denote by $\mathcal{C}^\bullet_{U\cup L'^{+}_{\widetilde{\mathcal{X}},\tau},L'^{+}_{\widetilde{\mathcal{X}},\tau}}$ the complex of sheaves on $\widetilde{\mathcal{X}}$, consisting of germs on $\widetilde{\mathcal{X}}$ of relative singular cochains (*i.e.*, germs of singular cochains in $U \cup L'^{+}_{\widetilde{\mathcal{X}},\tau}$ with boundary in $L'^{+}_{\widetilde{\mathcal{X}},\tau}$).

By the de Rham theorem, the integration of forms induces a quasi-isomorphism of complexes $\int : \mathcal{E}^\bullet_U \to \mathcal{C}^\bullet_U \otimes_{\mathbb{Z}} \mathbb{C}$; moreover, the natural morphism $\mathcal{E}^{\mathrm{mod},\bullet}_{\widetilde{\mathcal{X}}} \to (\alpha_\tau \circ j)_*\mathcal{E}^\bullet_U$ is a quasi-isomorphism, so the integration morphism $\int : \mathcal{E}^{\mathrm{mod},\bullet}_{\widetilde{\mathcal{X}}} \to (\alpha_\tau \circ j)_*\mathcal{C}^\bullet_U \otimes_{\mathbb{Z}} \mathbb{C}$, which is obtained by composing both morphisms, is a quasi-isomorphism.

Similarly, we have a commutative diagram

$$\begin{array}{ccccc}
\beta_{\tau,!}\mathcal{E}^{\mathrm{mod},\bullet}_{\widetilde{\mathcal{X}}} & \xrightarrow{\ \sim\ } & \mathcal{E}^{\mathrm{mod},-\tau,\bullet}_{\widetilde{\mathcal{X}}} & & \\
\wr\downarrow & & & \searrow{\scriptstyle\int} & \\
\beta_{\tau,!}(\alpha_\tau \circ j)_*\mathcal{E}^\bullet_U & \xrightarrow[\int]{\ \sim\ } & \beta_{\tau,!}(\alpha_\tau \circ j)_*\mathcal{C}^\bullet_U \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\ \sim\ } & \mathcal{C}^\bullet_{U\cup L'^{+}_{\widetilde{\mathcal{X}},\tau},L'^{+}_{\widetilde{\mathcal{X}},\tau}} \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

Hence we get:

Proposition A.2 (A variant of the de Rham theorem)
Both complexes $\mathcal{C}^\bullet_{U\cup L'^{+}_{\widetilde{\mathcal{X}},\tau},L'^{+}_{\widetilde{\mathcal{X}},\tau}} \otimes_{\mathbb{Z}} \mathbb{C}$ *and* $\mathcal{E}^{\mathrm{mod},-\tau,\bullet}_{\widetilde{\mathcal{X}}}$ *are quasi-isomorphic to* $\beta_{\tau,!}\boldsymbol{R}\alpha_{\tau,*}\boldsymbol{R}j_*\mathbb{C}_U$. *Moreover, the integration of forms induces a natural quasi-isomorphism of complexes*

$$\int : \mathcal{E}^{\mathrm{mod},-\tau,\bullet}_{\widetilde{\mathcal{X}}} \xrightarrow{\ \sim\ } \mathcal{C}^\bullet_{U\cup L'^{+}_{\widetilde{\mathcal{X}},\tau},L'^{+}_{\widetilde{\mathcal{X}},\tau}} \otimes_{\mathbb{Z}} \mathbb{C}. \qquad \square$$

Now, given a section of $\mathcal{E}_{\widetilde{X}}^{\mathrm{mod},\bullet} \otimes j_+\mathcal{E}^{-\tau f}$, *i.e.*, a section of $\mathcal{E}_{\widetilde{X}}^{\mathrm{mod},\bullet}$ multiplied by $e^{-\tau f}$, it is also a section of $\mathcal{E}_{\widetilde{X}}^{\mathrm{mod},-\tau,\bullet}$, and its image by (A.1) is nothing but its integral, according to the previous commutative diagram.

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