Séminaire BOURBAKI Janvier 2012
64ème année, 2011-2012, n$^{\text{o}}$ 1050

WILD HODGE THEORY AND HITCHIN-KOBAYASHI CORRESPONDENCE [after T. Mochizuki]

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INTRODUCTION: THE HARD LEFSCHETZ THEOREM

Simple holonomic $\mathscr{D}$-modules. — Let Z^o be an irreducible complex smooth quasi-projective variety and (V,∇) be an algebraic vector bundle with an integrable connection (i.e., without curvature), with no proper sub-bundle stable by the connection (i.e., simple). This is the type of object we are interested in in this presentation. Let us choose a projective compactification $j: Z^o \hookrightarrow Z$ and an embedding of Z in a smooth projective variety X. It is known that such a (V,∇) extends uniquely into a $\mathscr{D}_X$-module (i.e., an $\mathscr{O}_X$-module with integrable ∇-connection) simple holonomic module with support in Z and that one thus obtains all $\mathscr{D}_X$-simple holonomic modules with support in Z that are smooth on Z^o.

For such a $\mathscr{D}_X$-module $\mathscr{M}$, the analytic de Rham complex

$$\mathrm{DR}\,\mathscr{M} := (\Omega^{\bullet+\dim X}_{X^{\mathrm{an}}} \otimes \mathscr{M}, (-1)^{\dim X}\nabla)$$

is cohomologically $\mathbb{C}$-constructible, according to a theorem of Kashiwara. More precisely, it is a perverse sheaf.

This text is a translation into English of the Bourbaki talk [Sab13].

What are the perverse sheaves obtained in this way? We do not know the answer to this question. Nevertheless, we know that the Riemann-Hilbert correspondence $\mathscr{M} \mapsto \mathrm{DR}\,\mathscr{M}$ is an equivalence between the category of (analytic) $\mathscr{D}_X$-regular holonomic modules and that of perverse sheaves. This, together with the GAGA theorem for regular holonomic modules [KK81], implies that every *simple perverse sheaf* is obtained in this way. According to [BBD82] and [GM83], such a perverse sheaf is nothing other (up to a shift by $\dim Z$) than the Goresky-MacPherson intersection complex $\mathrm{IC}_Z(\mathscr{L})$ (intermediate extension $j_{!*}\mathscr{L}$) associated to an irreducible locally constant sheaf $\mathscr{L}$ on Z^o (i.e., an irreducible linear representation of $\pi_1(Z^o, \star)$).

It is also easy to obtain complexes $\mathrm{DR}\,\mathscr{M}$ which are not simple perverse sheaves, or even semi-simple perverse sheaves (i.e., direct sums of simple objects), if one accepts that a simple $\mathscr{M}$ has irregular singularities. One way to obtain such examples is to use the Fourier transformation.

Example. — Let $T_1, \dots, T_r$ ($r \geqslant 2$) be elements of $\mathrm{GL}_n(\mathbb{C})$ ($n \geqslant 2$) whose product is equal to the identity and which have no common eigenvector. Suppose also that 1 is not an eigenvalue of T_i (multiply each T_i by $\lambda_i \in \mathbb{C}^*$ general enough, making $\prod \lambda_i = 1$). They define an irreducible representation of the fundamental group of $\mathbb{P}^1$ minus r points, all at finite distance, thus an irreducible locally constant sheaf of rank n on this space. Its intersection complex on $\mathbb{A}^1$ (coordinate z) corresponds to a holonomic module M on the Weyl algebra $\mathbb{C}[z]\langle\partial_z\rangle$, whose singularities are all regular.

The Fourier transform ${}^F\!M$ is M itself on which we see ∂_z operating as the multiplication by a variable ζ and $-z$ as the derivation $\partial_z eta$. If M is simple, ${}^F\!M$ is also simple, but we can show (see for example [Mal91]) that the latter has an irregular singularity in $\zeta = \infty$, a regular singularity in $\zeta = 0$, and no other singularity. It can also be shown that the assumptions made on the T_i imply that, on the open subset $\zeta \neq 0$, the locally constant sheaf of its horizontal sections is of rank nr, and its monodromy has only eigenvalue 1, with r Jordan blocks of size 2 and $(n-2)r$ blocks of size 1. This locally constant sheaf is therefore not semisimple.

Let ${}^F\!\mathscr{M}$ be the unique $\mathscr{D}_{\mathbb{P}^1}$-simple holonomic module whose restriction to $\mathbb{A}^1$ (coordinate ζ) is ${}^F\!M$. If $\mathrm{DR}\,{}^F\!\mathscr{M}$ were semi-simple perverse, it would be a direct sum of point-supported sheaves and intersection complexes of irreducible locally constant sheaves (shifted by 1), among which the above locally constant sheaf, hence a contradiction.

However, we will see that the perverse sheaves $\mathrm{DR}\,\mathscr{M}$, for simple $\mathscr{M}$, all satisfy the hard Lefschetz theorem. In the following, we will work in the complex analytic framework only, contrary to the beginning of this introduction.

The hard Lefschetz theorem. — In various talks in 1996 (see [Kas98]), Kashiwara conjectured a very general version of the hard Lefschetz theorem in complex algebraic geometry, which was recently proved by T. Mochizuki [Moc11a]:

THEOREM 0.1. — *Let X be a smooth complex projective algebraic variety and let L be the cup-product operator by the Chern class of an ample line bundle on X. Then,* *for any simple holonomic $\mathscr{D}_X$-module $\mathscr{M}$ on X and any $k \geqslant 1$, the k-th iterate $L^k : \boldsymbol{H}^{-k}(X, \mathrm{DR}\,\mathscr{M}) \to \boldsymbol{H}^k(X, \mathrm{DR}\,\mathscr{M})$ is an isomorphism.*

Remark 0.2. — With the shift defining DR, $\boldsymbol{H}^k$ can be nonzero only for $k \in [-\dim X, \dim X]$. Moreover, for simple $\mathscr{M}$ different from the trivial connection $(\mathscr{O}_X, \mathrm{d})$, we also have vanishing for $k = -\dim X, \dim X$. Indeed, let us consider first the complex $\Gamma(X, \mathrm{DR}\,\mathscr{M})$: its $H^{-\dim X}$ is the space of sections of $\mathscr{M}$ on X annihilated by ∇; so there is a submodule $H^{-\dim X} \otimes (\mathscr{O}_X, \mathrm{d})$ contained in $\mathscr{M}$, and the assumption "$\mathscr{M}$ simple and $\neq (\mathscr{O}_X, \mathrm{d})$" implies $H^{-\dim X} = 0$. The hypercohomology spectral sequence shows that $\boldsymbol{H}^{-\dim X} \subset H^{-\dim X}$, hence the vanishing of $\boldsymbol{H}^{-\dim X}$. A duality argument also enables us to deduce $\boldsymbol{H}^{\dim X} = 0$. Thus, the statement 0.1 is of little interest when X is a curve.

To obtain such a theorem, we first show the existence of a richer structure, which gives rise to a notion of purity [Del71]. The Hodge theory plays this role in complex algebraic geometry (see the excellent overview [dM09]):

(a) If (V, ∇) is a holomorphic bundle with integrable connection on X, the complex $\mathrm{DR}(V, \nabla) = (\Omega_X^{\bullet+\dim X} \otimes V, (-1)^{\dim X}\nabla)$ has cohomology only in degree $-\dim X$, it is the local system V^∇ of horizontal sections of ∇ (Cauchy-Kowalewski theorem and holomorphic Poincaré lemma); (a′) if moreover (V, ∇) underlies *a variation of* $\mathbb{Q}$*-polarizable Hodge structure*, the hard Lefschetz theorem $L^k : H^{dimX-k}(X, V^\nabla) \xrightarrow{\sim} H^{dimX+k}(X, V^\nabla)$ was shown by Deligne (see [Zuc79, Th. 2. 9]), the hard Lefschetz theorem as proved by Hodge (see [Hod52]) being the case $(V, \nabla) = (\mathscr{O}_X, d)$.

(b) The extension of this result to the case where (V, ∇) is a holomorphic bundle with integrable connection on the complementary X^o of a hypersurface D of X and satisfies (a′) has been the subject of much work ([Sch73, Zuc79, CK82, CKS86, CKS87, Kas85, KK87]), resulting in the hard Lefschetz theorem for the intersection cohomology on X of the local system V^∇, when $D = X \smallsetminus X^o$ is a divisor with normal crossings.

(c) M. Saito [Sai88] removed the assumption X smooth and D with normal crossings by introducing the category of polarizable Hodge modules. The theorem 0.1 applies to the complexes $\mathrm{DR}\,\mathscr{M} = \mathrm{IC}_Z(\mathscr{L})[\dim Z]$ provided that $\mathscr{L}$ underlies a variation of $\mathbb{Q}$-polarizable Hodge structure (see also [BBD82, dM05, dM09] for other approaches in the case of local systems of geometrical origin).

In (b) and (c), we work with the Deligne extension of (V, ∇), which has a regular singularity at infinity (i.e., along D), and this moderate "behavior" is necessary a priori to apply the methods of Hodge theory, according to the Griffiths-Schmid regularity theorem [Sch73, Th. 4. 13]). On the other hand, the notion of *polarized variation of twistor structure*, introduced by Simpson [Sim97] allows irregular singularities at infinity, and enables to approach, by a "wild" Hodge theory, Theorem 0.1 for simple holonomic $\mathscr{D}$-modules with possibly irregular singularities.

This notion of polarized variation of twistor structure already occurs in the absence of singularity (in the form of a harmonic metric, see the dictionary of Section 1). For a polarized variation of Hodge structure, this is the structure that remains when one keeps only the flat connection and the Hermitian metric defining the polarization. Under the sole assumption of semi-simplicity of (V, ∇) or, equivalently, of the locally constant sheaf V^∇, the smooth case (a) of theorem 0.1 comes from the existence, due to [Cor88], of a so-called *harmonic* metric for (V, ∇) (the case where (V, ∇) is unitary or, more generally, a polarized variation of Hodge structure, being a very special case). One can indeed develop in this framework the harmonic theory of the Laplacian and obtain the Kähler identities, which lead to Theorem 0.1 (see [Sim92]). Note that the semi-simplicity assumption of V^∇ is important, and it is easy to give an example where 0.1 is in default without this assumption: on a curve of genus $g \geqslant 2$, any nontrivial extension $\mathscr{L}$ of the constant sheaf $\mathbb{C}$ by a nonconstant local system of rank 1 satisfies $\dim H^0(X, \mathscr{L}) \neq \dim H^2(X, \mathscr{L})$.

Notwithstanding Remark 0.2, Simpson [Sim90] has shown, in the case where X is a curve, the existence of a harmonic metric h for (V, ∇) on $X^o \subset X$, with a moderate behavior at the points of D (see also [Biq91, JZ97] in dimension $\geqslant 1$). The asymptotic analysis he makes of this metric in the neighborhood of D extends to this framework that made by Schmid [Sch73] in the case of polarized variations of Hodge structures, which allows in particular to compute the intersection cohomology $H^1(X, j_* V^\nabla)$ $(j : X^o \hookrightarrow X)$ as a space of L^2-cohomology with respect to h and a Poincaré type metric on X^o, and which extends Zucker's results [Zuc79] ([Biq97], see also [Sab05, § 6.2], [Moc07, § 20.2], [JYZ07]). This leads to a statement analogous to the degeneracy in E_1 of the Hodge $\Rightarrow$ de Rham spectral sequence of Rham, namely the computation of $\dim H^1(X, j_* V^\nabla)$ in terms of the Dolbeault cohomology of the associated parabolic Higgs bundle.

The moderate case of Theorem 0.1 is the one where the $\mathscr{D}_X$-module $\mathscr{M}$ has regular singularities, i.e., $\mathrm{DR}\,\mathscr{M} = \mathrm{IC}_Z(\mathscr{L})[\dim Z]$ with Z irreducible and $\mathscr{L}$ simple on Z^o. T. Mochizuki solved this case in [Moc07] by extending the methods mentioned above. The strategy of the proof, which is also valid for the wild case, i.e., when $\mathscr{M}$ has irregular singularities, will be explained in Section 2. The moderate case was also solved by Drinfeld [Dri01] by a method of reduction to characteristic p reminiscent of [BBD82]. Drinfeld relied however on a conjecture made by de Jong [dJ01], proved since [BK06, Gai07].

Recently, Krämer and Weissauer [KW11] have used the moderate case of 0.1 to show a vanishing theorem, for any perverse sheaf $\mathscr{F}$ on a complex abelian variety X, of the spaces $\boldsymbol{H}^j(X, \mathscr{F} \otimes \mathscr{L})$ for any $j \neq 0$ and almost any local system $\mathscr{L}$ of rank 1.

The rest of the text will insist on the new tools introduced by T. Mochizuki [Moc11a] (after those of [Moc07]) to go from the "moderate" case to the "wild" case.

Acknowledgements. — They go to T. Mochizuki, as well as to M. Saito, Ch. Schnell, Ch. Sevenheck, and C. Simpson for the multiple suggestions that helped me improve the first draft of this text.

1. DICTIONARY

I will make explicit the dictionary

harmonic flat bundle/harmonic Higgs bundle

/pure polarized twistor structure of weight 0

(see [Sim92, Sim97]), as it is essential for the singular case. The equations (1.1) below and the idea of the construction on twistor space of a bundle with λ-connection go back to Hitchin [Hit87]. Drinfeld also pointed me to the work of Zakharov, Mikhailov and Shabat [ZM78, ZS79] where we find this type of equation under the name of "chiral field equations".

Let (V,∇) be a holomorphic bundle with an integrable holomorphic connection, and let $(H, D=\nabla+\overline{\partial})$ be the associated flat C^∞ bundle. To any Hermitian metric h on H we associate (cf. [Sim92]) a unique metric connection $\partial_E+\overline{\partial}_E$ characterized by the fact that, if we consider the two $\mathscr{C}_X^\infty$-linear morphisms $\theta:=\nabla-\partial_E: H\to\mathscr{A}_X^{1,0}\otimes H$ and $\theta^\dagger:=\overline{\partial}-\overline{\partial}_E: H\to\mathscr{A}_X^{0,1}\otimes H$, then $\theta^\dagger$ is the h-adjoint of θ (if D is already compatible to h we have $\theta=0$, $\theta^\dagger=0$). We say that (V,∇,h) is an *flat harmonic bundle* (1) if its *pseudo-curvature* $G(\nabla,h)$ is zero:

$$G(\nabla,h):=-4(\overline{\partial}_E+\theta)^2=0,\quad \text{i.e.,}\quad \overline{\partial}_E^2=0,\quad \overline{\partial}_E(\theta)=0,\quad \theta\wedge\theta=0. \tag{1.1}$$

(These three conditions are redundant, the last two implying the first; on a compact Kähler variety, one can even be satisfied with the second, see [Moc07, Rem. 21.33 & Prop. 21.39]. For a harmonic flat bundle, $E:=\ker\overline{\partial}_E: H\to\mathscr{A}_X^{0,1}\otimes H$ is a holomorphic bundle, and $\theta: E,\to\Omega_X^1\otimes E$ is a holomorphic morphism, which satisfies $\theta\wedge\theta=0$. Thus, (E,θ) is a holomorphic *Higgs bundle*.

Starting now from a holomorphic Higgs bundle (E,θ) (i.e., $\theta\wedge\theta=0$) and a Hermitian metric h on E, we say that (E,θ,h) is a *harmonic Higgs bundle* if, denoting $\partial_E+\overline{\partial}_E$ the Chern connection associated to the metric h on the holomorphic bundle E, and $\theta^\dagger$ the h-adjoint of θ, then the connection $\partial_E+\overline{\partial}_E+\theta+\theta^\dagger$ on $H:=\mathscr{C}_X^\infty\otimes_{\mathscr{O}_X}E$ is *integrable.*

We thus have a one-to-one correspondence

$$\text{harmonic flat bundle}\longleftrightarrow\text{harmonic Higgs bundle}$$

When (V,∇,h) or (E,θ,h) are harmonic, there is in fact a one-parameter family of flat holomorphic bundles which degenerates on the associated Higgs bundle: for any $\lambda\in\mathbb{C}$, we set $V^\lambda=\ker(\overline{\partial}_E+\lambda\theta^\dagger: H\to\mathscr{A}_X^{0,1}\otimes H)$, equipped with the operator $\nabla^\lambda:=\lambda\partial_E+\theta$, which is called an λ-*connection.* If $\lambda\neq0$, the operator $(1/\lambda)\nabla^\lambda$ is an integrable holomorphic connection on V^λ while, if $\lambda=0$, we find the Higgs bundle (E,θ).

1. We follow here the terminology of [Sim92]; Mochizuki uses the term "pluri-harmonic" for the equation (1.1), to distinguish it from the a priori weaker equation on a Kähler manifold, also considered in [Cor88, Sim92], namely $\Lambda G(\nabla,h)=0$.

Example 1.2 (the rank-1 case on a punctured disk). — Let us consider the case of a (trivial) rank-1 bundle on the punctured unit disk Δ^* of coordinate z, with a Hermitian metric h. We will denote $\mathfrak{U}$ the set of equivalence classes of pairs $\mathfrak{u} = (a, \alpha) \in \mathbb{R} \times \mathbb{C}$ modulo $\mathbb{Z} \times \{0\}$. Then:

The set of isomorphism classes of harmonic Higgs bundles (or flat bundles) of rank 1 *on* Δ^* *is in bijective correspondence with the set of pairs* $(\psi, \mathfrak{u})$*, with* $\psi \in \mathscr{O}(\Delta^*)$ *with no constant term and* $(\mathfrak{u} \bmod \mathbb{Z}) \in \mathfrak{U}$.

Proof. — We will do it in the Higgs case, the flat case being similar. Let (E, θ, h) be a harmonic Higgs bundle of rank 1 on Δ^*. We will associate to it a unique pair $(\psi, \mathfrak{u} \bmod \mathbb{Z})$. Let ε be a holomorphic basis of E. We have

$$\theta\varepsilon = \varphi(z)\, \varepsilon, \mathrm{d}z, \quad \varphi(z) \text{ holomorphic on } \Delta^*.$$

Let $\varphi(z) = \partial_z\psi(z) + \alpha/z$ with $\psi \in \mathscr{O}(\Delta^*)$ without constant term and $\alpha \in \mathbb{C}$. Let us also set $\|\varepsilon\|_h = \exp(\eta(z))$, where η is real and C^∞ on Δ^*. We can check that the harmonicity condition of (E, θ, h) is equivalent to the fact that the function η is *harmonic* on Δ^*. It is therefore written $\operatorname{Re}\gamma(z) - a \log|z|$ with γ holomorphic on Δ^* and $a \in \mathbb{R}$. Replacing ε by $e = \exp(-\gamma(z)) \cdot \varepsilon$, we can assume that $\eta(z) = -a \log|z|$ with $a \in \mathbb{R}$, and we then have $\|e\|_h = |z|^{-a}$. We thus obtained a pair $(\psi, \mathfrak{u})$.

Then we compute that $v := |z|^{-2\overline{\alpha}} \exp(\psi - \overline{\psi}) \cdot e$ is a holomorphic basis of the associated flat bundle V, of norm $\|v\|_h = |z|^{-a-2\operatorname{Re}\alpha}$. Moreover,

$$\theta e = (z\partial_z\psi + \alpha)\frac{\mathrm{d}z}{z} \otimes e, \quad \nabla v = (2z\partial_z\psi + \mathfrak{e}(1, \mathfrak{u}))\frac{\mathrm{d}z}{z} \otimes v, \quad \mathfrak{e}(1, \mathfrak{u}) := -a + 2i \operatorname{Im}\alpha.$$

Let ε' be another holomorphic basis of E, and e' constructed as above with $\|e'\|_h = |z|^{-a'}$ for some $a' \in \mathbb{R}$. Hence a pair $(\psi', \mathfrak{u}')$. Then $e' = \nu(z)e$ with $\nu(z)$ is holomorphic and has moderate growth, so is meromorphic, hence $a' - a \in \mathbb{Z}$. The Higgs field has the same expression in the bases e and e', which implies $\psi = \psi'$ and $\alpha = \alpha'$. □

Let us now set, for any fixed $\lambda \in \mathbb{C}$,

(1.2 ∗) $\mathfrak{p}(\lambda, \mathfrak{u}) = a + 2\operatorname{Re}(\overline{\alpha}\lambda), \quad \mathfrak{e}(\lambda, \mathfrak{u}) = \alpha - a\lambda - \overline{\alpha}\lambda^2, \quad \text{and} \quad v^\lambda = e^{\overline{\lambda}\psi - \lambda\overline{\psi}}|z|^{-2\overline{\alpha}\lambda} \cdot e.$

Then v^λ is a holomorphic basis of V^λ, of norm $\|v^\lambda\|_h = |z|^{-\mathfrak{p}(\lambda, \mathfrak{u})}$ and

$$\nabla^\lambda v^\lambda = ((1 + |\lambda|)^2 z\partial_z\psi + \mathfrak{e}(\lambda, \mathfrak{u}))\frac{\mathrm{d}z}{z} \otimes v^\lambda. \tag{1.2 ∗∗}$$

Let us return to the general situation. These two equivalent notions (harmonic flat bundle and harmonic Higgs bundle) are also equivalent to the notion of *variation of pure polarized twistor structure of weight* 0. To define it, let us introduce the projective line $\mathbb{P}^1$ with two affine charts $\mathbb{C}_\lambda, \mathbb{C}_\mu$ of coordinates λ and μ respectively, with $\mu = 1/\lambda$ on the intersection of the two charts. The following presentation is not exactly the one given by Simpson [Sim97], but is equivalent to it and will be more convenient in singular situations. It consists in describing bundles on $X \times \mathbb{P}^1$ which are holomorphic with respect to $\mathbb{P}^1$ and C^∞ with respect to X. This presentation allows us to work only with holomorphic bundles on $X \times \mathbb{C}_\lambda$ and, further (Section 6), with $\mathscr{R}_{X\times\mathbb{C}_\lambda}$-holonomic modules.

Let $\sigma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$ be the anti-holomorphic involution which continuously extends the application $\mathbb{C}_\mu \to \mathbb{C}_\lambda$, $\mu \to \lambda = -1/\overline{\mu}$. If $f(x,\lambda)$ is holomorphic in x and λ, then the function $(\sigma^*\overline{f})(x,\mu) = \overline{f(x,-1/\overline{\mu})}$ is anti-holomorphic in x and holomorphic in μ. If $\mathscr{H}$ is a holomorphic bundle on X, then $\sigma^*\overline{\mathscr{H}}$ is a holomorphic bundle on $\overline{X}$, where $\overline{X}$ is the complex conjugate manifold of X. If $\mathscr{H}', \mathscr{H}''$ are two holomorphic bundles on $X \times \mathbb{C}_\lambda$, a *pre-gluing* (between the dual $\mathscr{H}'^\vee$ and $\sigma^*\overline{\mathscr{H}''}$) is an $\mathscr{O}_{X\times\boldsymbol{S}} \otimes_{\mathscr{O}_{\boldsymbol{S}}} \mathscr{O}_{\overline{X}\times\boldsymbol{S}}$-linear pairing

$$C : \mathscr{H}'_{|X\times\boldsymbol{S}} \otimes_{\mathscr{O}_{\boldsymbol{S}}} \sigma^*\overline{\mathscr{H}''}_{|X\times\boldsymbol{S}} \longrightarrow \mathscr{C}^{\infty,\mathrm{an}}_{X\times\boldsymbol{S}},$$

where $\boldsymbol{S} = \{|\lambda| = 1\}$, $\mathscr{O}_{\boldsymbol{S}}$ is the sheaf along $\boldsymbol{S}$ of $\mathscr{O}_{\mathbb{C}_\lambda}$ and $\mathscr{C}^{\infty,\mathrm{an}}_{X\times\boldsymbol{S}}$ is the sheaf of sheaves along $X \times \boldsymbol{S}$ of C^∞ functions which are holomorphic with respect to λ. If $\mathscr{H}'$ and $\mathscr{H}''$ have λ-connections, we require that C is compatible in a natural sense. These triples naturally form a category. The *adjoint* $(\mathscr{H}', \mathscr{H}'', C)^*$ is the triple $(\mathscr{H}'', \mathscr{H}', C^*)$, with $C^*(u'', \sigma^*\overline{u}') := \overline{C(u', \sigma^*\overline{u}'')}$, and the λ-connections remain compatible to C^*.

A variation of twistor structure is such a datum $(\mathscr{H}', \mathscr{H}'', C)$ with λ-integrable connections such that, for any $x \in X$, the pairing induced by C is nondegenerate and thus defines a holomorphic bundle on $\mathbb{P}^1$ by gluing $\mathscr{H}'^\vee_{|\{x\}\times\mathbb{C}_\lambda}$ and $\sigma^*\overline{\mathscr{H}''}_{|\{x\}\times\mathbb{C}_\mu}$. It is a variation of twistor structure of weight w if, for any x, the resulting bundle is isomorphic to a power of $\mathscr{O}_{\mathbb{P}^1}(w)$. If the weight w is zero, the global sections of this bundle at fixed x form a vector space H_x of dimension equal to the rank of $\mathscr{H}'$ and $\mathscr{H}''$, and the adjoint has as global sections the adjoint space $\overline{H}_x^\vee$. We then define a polarization as an isomorphism $\mathscr{S}$ of $(\mathscr{H}', \mathscr{H}'', C)$ onto its adjoint $(\mathscr{H}', \mathscr{H}'', C)^*$, compatible with λ-connections, such that, for any x, the induced isomorphism $H_x \xrightarrow{\sim} \overline{H}_x^\vee$, seen as a sesquilinear pairing on H_x, is a *positive definite* Hermitian form. The bundle H on X whose fibers are the H_x is then a C^∞ bundle with a Hermitian metric h.

Lemma 1.3 ([Sim97]). — *Let $(\mathscr{H}', \mathscr{H}'', C, \mathscr{S})$ be a variation of polarized twistor structure of weight* 0*. The restriction $\mathscr{H}''$ to $\lambda = 1$ (resp. $\lambda = 0$) equipped with the connection (resp. the Higgs field) induced by the λ-connection is a flat holomorphic (resp. the Higgs field) bundle with underlying C^∞ bundle isomorphic to H, and the metric h makes it a harmonic flat (resp. Higgs) bundle.*

Conversely, the construction V^λ from a harmonic flat (resp. Higgs) bundle allows to define a variation of pure polarized twistor structure of weight 0 *by setting* $\mathscr{H}' = \mathscr{H}'' = \ker(\overline{\partial}_\lambda + \overline{\partial}_E + \lambda\theta^\dagger : \mathscr{C}^infty_{X\times\mathbb{C}_\lambda} \otimes_{\mathscr{C}^\infty_X} H \to \mathscr{C}^{0,1}_{X\times\mathbb{C}_\lambda} \otimes_{\mathscr{C}^\infty_X} H)$*,* $\mathscr{S} = \mathrm{Id}$*, C is naturally induced by h and the λ-connection by $\lambda\partial_E + \theta$.*

Example 1.2*, continued.* — Let us now consider v^λ as depending on λ. Then the formula (1.2 $**$) shows that ∇^λ is holomorphically expressed in λ only if $\psi = 0$. If $\psi \neq 0$, we can consider the holomorphic basis $\widetilde{v}^\lambda = e^{-|\lambda|^2\psi} \cdot v^\lambda$ to correct the problem. We then see that, on the one hand, there is no uniqueness of choice (we could just as well take $e^{c-|\lambda|^2\psi} \cdot v^\lambda$, with $c \in \mathbb{C}$), and on the other hand all the choices lead to a basis whose norm is no longer of moderate growth at the origin, if $\lambda \neq 0$ and ψ is not holomorphic at the origin. But, in this case, by making the metric depend on λ in a suitable way, we find the property of moderate growth with respect to this modified metric (we will find this again at the point (e) of Section 5.5).

2. STRATEGY OF THE PROOF

The strategy used to prove Theorem 0.1 follows that of M. Saito [Sai88].

(1) The first step is to enrich the structure of $\mathscr{D}_X$-holonomic modulus (where X is any complex analytic variety) in order to have a notion of weight.

- The category of $\mathscr{D}_X$-holonomic modules with good filtration and rational structure (i.e., an isomorphism of the de Rham complex and the complexified perverse sheaf of a $\mathbb{Q}$-perverse sheaf) was considered by Saito [Sai88]. Forgetting filtration and rational structure defines a forgetting functor to the category of holonomic $\mathscr{D}$-modules.
- In the present case, we generalize the objects $(\mathscr{H}', \mathscr{H}'', C)$ of Section 1: Denoting $\mathscr{R}_{X\times\mathbb{C}}$ the sheaf of λ-differential operators generated by the functions $\mathscr{O}_{X\times\mathbb{C}}$ and the λ-vector fields $\lambda\partial_{z_i}$, we consider the triples $(\mathscr{M}', \mathscr{M}'', C)$, where $\mathscr{M}', \mathscr{M}''$ are $\mathscr{R}_{X\times\boldsymbol{S}}$-holonomic modules (in a natural sense) and C is a pairing between $\mathscr{M}'_{|X\times\boldsymbol{S}}$ and $\sigma^*\overline{\mathscr{M}''}_{|X\times\boldsymbol{S}}$ with values in the sheaf of the distributions on $X\times\boldsymbol{S}$ which depend continuously on $\boldsymbol{S}$. The restriction $\mathscr{M}''/(\lambda-1)\mathscr{M}''$ defines an forgetful functor with values in the category of $\mathscr{D}$-holonomic modules.

(2) Without further constraints, the above categories are not abelian.

- Saito's idea is essentially to impose additional local conditions: for any holomorphic function seed f on X the functor ψ_f^{mod} of *moderate close cycles* along $f=0$, defined a priori on the category of holonomic $\mathscr{D}$-modules using the V-filtration of Kashiwara-Malgrange, must exist for the filtered objects considered. It therefore imposes on the one hand the existence of such a functor and on the other hand that the result gives an object of the same type (except for the grading by the so-called "monodromic" filtration) with a strictly smaller support dimension. When the dimension of the support is zero, we impose to obtain a polarized Hodge structure. The simplest case of this procedure is the restriction to a point of a variation of Hodge structure. We thus obtain the category of polarizable Hodge modules [Sai88].
- This idea can be transferred quite directly to the triplets $(\mathscr{M}', \mathscr{M}'', C)$, the definition of ψ_f^{mod} on the coupling C being obtained by taking the residue in different values of s of the Mellin transform of the distribution $|f|^{2s}C$. This gives us the category of holonomic $\mathscr{D}$-modules with moderate polarizable twistor structure [Sab05, Moc07].
- For the wild case (irregular singularities), the use of moderate near cycles is not sufficient. The irregular nearby cycles, as defined by Deligne [Del07b]

are on the other hand sufficient ([Sab09], [Moc11a]). We obtain the category of holonomic $\mathscr{D}$-modules with wild polarizable twistor structure [Moc11a] (see Section 6).

(3) We then show Theorem 0.1 for polarizable Hodge modules or with polarizable twistor structure. The hypercohomology spaces are either filtered complex spaces with rational structure and polarization, or triplets $(\mathscr{H}', \mathscr{H}'', C)$ formed by $\mathscr{O}_{\mathbb{C}_\lambda}$-modules and a pairing. We further show that Lefschetz morphisms are strictly compatible with filtrations or remain isomorphisms by restriction to $\lambda = 1$. This is obtained by showing that hypercohomology spaces are pure Hodge structures or pure twistor structures. We conclude that all $\mathscr{D}$-modules obtained by forgetting the filtration from polarizable Hodge modules or by restriction to $\lambda = 1$ of $\mathscr{D}$-modules with polarizable twistor structure satisfy the hard Lefschetz theorem.

(4) The results of (3) are obtained by the method of Lefschetz pencils. This brings us back to showing them in the case of curves.

- For polarizable Hodge modules [Sai88], we rely on the theorems computing the L^2-cohomology obtained by Zucker [Zuc79], since the restriction of such a $\mathscr{D}$-filtered module to a dense Zariski open of the curve is none other than a variation of polarizable $\mathbb{Q}$-Hodge structure.
- For $\mathscr{D}$-modules with polarizable twistor structure, we start by noticing that the restriction to a dense Zariski open provides, by the dictionary of Section 1, a flat holomorphic bundle with harmonic metric. Moreover, this metric is moderate, in the sense of [Sim90], or wild, in the sense explained in Section 3. The results of [Sim90] and then of [Moc11a] in the case of curves for such metrics allow to adapt Zucker's method, as indicated in the introduction (a slightly different approach in the moderate case is used in [Biq97] and [Sab05]; see also [Sab99] for a Poincaré lemma L^2 in the wild case).

(5) We now come to the second part of the program, which is more analytical. We have to identify exactly the $\mathscr{D}$-holonomic modules produced at the point (3). Insofar as we have set up, during the proof of Theorem 0.1 for categories of polarisable Hodge modules or modules with polarisable twistor structure, tools such as the direct image by a projective morphism and the decomposition theorem analogous to that of [BBD82], we can reduce this identification to the case where the support Z is smooth and where the Zariski open smooth locus Z^o of the holonomic $\mathscr{D}$-module has as its complement a divisor with normal crossings.

- For polarizable Hodge modules, M. Saito [Sai90, Th. 3.21] identifies them, via the Riemann-Hilbert correspondence, to those corresponding to intersection complexes of polarizable variations of Hodge structure. A delicate point is the reconstruction, from such a variation, of a polarizable Hodge module, and the essential ingredient is the existence theorem of a limit mixed Hodge structure, due to Cattani, Kaplan and Schmid on the one hand, and Kashiwara and

Kawaï on the other hand, as well as the description of the latter, generalizing in any dimension Schmid's theorem in dimension 1 (see the introduction).

- For $\mathscr{D}$-modules with *moderate* polarizable twistor structure, Mochizuki identifies the holonomic $\mathscr{D}$-modules produced at the point (3) with those corresponding, by Riemann-Hilbert, to the intersection complexes of flat bundles equipped with a moderate harmonic metric (see Section 3 below). The reconstruction (see Section 5) is also based, *in fine*, on the asymptotic theory of variations of polarizable Hodge structure.
- In the wild case, we first come back to consider a better controlled situation (wild and good case, see Section 3). Mochizuki has shown, as a consequence of his proof of the wild Hitchin-Kobayashi correspondence (Section 4), the existence of a suitable compactification (Z, D) of Z^o for which this property is satisfied. One can now disconnect this argument from the whole proof, and use here the results of Kedlaya [Ked10, Ked11] (see Section 3.2), which I will do to simplify the exposition.

(6) The Hitchin-Kobayashi correspondence, explained in this framework in section 4, finally allows Mochizuki to put in bijective correspondence the moderate or wild and good harmonic flat bundles on (Z, D) with the smooth flat semisimple *meromorphic* bundles on Z^o, thus generalizing Corlette's theorem [Cor88] from the projective case (see also [JZ97] for the quasi-projective case). These are themselves in bijective correspondence with the smooth semisimple holonomic $\mathscr{D}_Z$-modules on Z^o. The proof in Sections 4 and 5 will insist on the "reconstruction" aspect (essential surjectivity in Theorem 6.2 below). It should be noted, however, that in the wild case, the direct aspect (the $\mathscr{D}$-modules obtained at the point (3) are semisimple) and the full faithfulness are non-trivial points because, in particular, we do not have the canonical Deligne meromorphic extension (cf. § 3.2). This question is treated in § 19.3 of [Moc11a].

In conclusion, the holonomic $\mathscr{D}$-modules produced at point (3) are exactly the holonomic semisimple $\mathscr{D}$-modules on X, which completes the proof of Theorem 0.1.

Remark 2.1. — We have lost here some symmetry between Higgs bundles and flat bundles. Indeed, the good wild harmonic Higgs bundles on (Z, D) are in bijective correspondence with the poly-stable Higgs bundles with zero parabolic characteristic numbers (see Section 4.1). But we do not know how to identify the Higgs objects obtained at the point (3) by restriction to $\lambda = 0$, as $\mathscr{O}_{T^*X}$-coherent modules.

3. WILD HIGGS BUNDLES AND FLAT MEROMORPHIC BUNDLES WITH IRREGULAR SINGULARITIES

3.0. Convention

In all this text, we will call "global setting" the data of a smooth projective variety X and a divisor D with normal crossings in X. Let us denote $j : X^o := X \smallsetminus D \hookrightarrow X$ the inclusion. The components of the divisor, assumed to be smooth, are indexed by a finite set $\mathcal{I}$. We also give ourselves an ample line bundle L on X.

By "local setting", we mean rather that X is a product Δ^n of disks with coordinates $(z_1, \ldots, z_n)$, and D has the equation $z_1 \cdots z_\ell = 0$, so that the finite set $\mathcal{I}$ is here equal to $\{1, \ldots, \ell\}$.

3.1. Wild Higgs bundles

Let us consider the local setting. Let (E, θ) be a holomorphic Higgs bundle on X^o. We write the Higgs field as $\theta = \sum_{i=1}^{\ell} F_i \mathrm{d}z_i/z_i + \sum_{j=\ell+1}^{n} G_j \mathrm{d}z_j$, where F_i, G_j are holomorphic endomorphisms of E. The coefficients $f_{i,k}, g_{j,k}$ of the characteristic polynomials of F_i and G_j are holomorphic functions on X^o.

The Higgs bundle (E, θ) is *moderate* (in the considered chart) if $f_{i,k}, g_{j,k}$ extend into *holomorphic* functions on X (we will denote these extensions in the same way) and $f_{i,k|D_i}$ is *constant* for all i, k. In particular, the eigenvalues (and their multiplicity) of $F_{i|D_i}$ are constant and equal to those of $F_i(0)$. Because of the Higgs condition, the endomorphisms F_i, G_j commute. We deduce that (E, θ) decomposes locally along the set $\mathrm{Sp}(\theta)$ of eigenvalues $(\alpha_1, \ldots, \alpha_\ell)$ of $(F_1(0), \ldots, F_\ell(0))$ into a direct sum of moderate Higgs subbundles (see [Moc07, § 8.2.1]):

$$(E, \theta) \simeq \bigoplus_{\boldsymbol{\alpha} \in \mathrm{Sp}(\theta)} (E_{\boldsymbol{\alpha}}, \theta_{\boldsymbol{\alpha}}). \tag{3.1 $*$}$$

In the "wild" case, the functions $f_{i,k}, g_{j,k}$ are allowed to have poles along D, but in a controlled way. Still working in local coordinates, we will denote by $\mathscr{O}(*D)$ the space of meromorphic functions on X with poles of arbitrary order along D, and we will be interested in the polar parts $\mathscr{O}(*D)/\mathscr{O}$.

The Higgs bundle (E, θ) has a *wild unramified decomposition* if there exists a finite family $\mathrm{Irr}(\theta) \subset \mathscr{O}(*D)/\mathscr{O}$ of polar parts and a decomposition

$$(E, \theta) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}) \tag{3.1 $**$}$$

such that, for each $\mathfrak{a} \in \mathrm{Irr}(\theta)$, the Higgs bundle $(E_{\mathfrak{a}}, \theta_{\mathfrak{a}} - \mathrm{d}\mathfrak{a} \otimes \mathrm{Id})$ is *moderate* (this condition depends only on the polar part $\mathfrak{a}$, and not on a bearing to $\mathscr{O}(*D)$, hence the shortcut of notation). The Higgs bundle (E, θ) has a *wild decomposition with ramification* if it admits a decomposition (3.1 $**$) after inverse image by a finite morphism ramified around D, described in suitable local coordinates by $\rho : (x_1, \ldots, x_n) \mapsto (z_1, \ldots, z_n) = (x_1^{\nu_1}, \ldots, x_\ell^{\nu_\ell}, x_{\ell+1}, \ldots, x_n)$. Finally, (E, θ) is said to be wild if, for any point of D, there exists a projective modification π of a neighborhood of this point such that $\pi^{-1}(D)$ is still with normal crossings and $\pi^*(E, \theta)$ admits a wild decomposition with ramification in the neighborhood of any point of $\pi^{-1}(D)$.

Example 1.2, *continued.* — Since ψ is without constant term, we can write $\psi = \mathrm{d}(\mathfrak{a} + \eta)$ with η holomorphic and $\mathfrak{a}$ holomorphic in z^{-1} without constant term. The Higgs bundle (E, θ) is moderated if and only if $\mathfrak{a} = 0$. It is wild if and only if ψ is meromorphic in $z = 0$ (i.e., $\mathfrak{a} \in z^{-1}\mathbb{C}[z]$).

Remarks 3.2

(i) It will be a constant in the treatment of the moderate/wild case to reduce the wild case to the moderate case by adding such $d\mathfrak{a}$ and direct sums, possibly with ramification. The case of flat bundles will be complicated by the introduction of Stokes structures, which do not appear for Higgs bundles.

(ii) The above conditions do not depend on the choice of coordinates adapted to D, and if $\dim X = 1$ they reduce to the holomorphy (resp. the meromorphy) on X of the coefficients of the characteristic polynomial of F_1. In dimension $\geqslant 2$, the property of "wild decomposition with ramification" implies that, after ramification, the characteristic polynomial $\chi_{F_i}(T)$ of F_i decomposes into $\prod_{\mathfrak{a}} P_{i,\mathfrak{a}}(T - z_i\partial_{z_i}\mathfrak{a})$, where $P_{i,\mathfrak{a}}(T)$ has holomorphic coefficients for all $i, \mathfrak{a}$, and $\chi_{G_j}(T) = \prod_{\mathfrak{a}} P_{j,\mathfrak{a}}(T - \partial_{z_i}\mathfrak{a})$, where $P_{j,\mathfrak{a}}$ have holomorphic coefficients. This last property is stronger than the meromorphy of the coefficients of χ_{F_i}, χ_{G_j}.

(iii) The conditions "moderate" or "wild" say nothing about the possible extension of the bundle E as a bundle on X nor, if necessary, about the extension of F_i, G_j as meromorphic or holomorphic endomorphisms of this extended bundle.

(iv) These conditions are preserved by inverse image by a morphism

$$f : (X', D') \longrightarrow (X, D)$$

of manifolds with a normal crossing divisor with $D' = f^{-1}(D)$. Conversely, given (E, θ) on X^o, *does there exist an proper modification $X' \to X$ which is an isomorphism over X^o, such that $X' \smallsetminus X^o$ is a normal crossing divisor, and (E, θ) admits a wild decomposition with ramification on X'?* Since such a modification is an isomorphism out of a 1-codimensional set in D, it is necessary that θ is wild in the neighborhood of any point of a dense Zariski open subset of D. Conversely, Mochizuki shows [Moc11a, Chap. 15] that this is indeed the case if the following properties are satisfied:

- D has normal crossings in X,
- θ admits a wild decomposition with ramification generically along D,
- in the neighborhood of any point of D, and in suitable local coordinates, one has, after eventual finite ramification, a decomposition $\chi_{F_i}(T) = \prod_{\mathfrak{a}\in A_i} P_{i,\mathfrak{a}}(T - z_i\partial_{z_i}\mathfrak{a})$ as above.

3.2. Flat bundles with irregular singularities

Let us stop for a moment to imagine the analogue of the two properties "moderate/wild" for a flat holomorphic bundle (V, ∇) on X^o when D has normal crossings. Considering as above the characteristic polynomials of the coefficients of the connection matrix does not make sense anymore, but there exists a unique meromorphic extension [(2)] of (V, ∇) in a basis of which the horizontal sections of the connection

2. i.e., an $\mathscr{O}_X(*D)$-locally free module of finite rank $\mathscr{V}$ equipped with an integrable connection ∇.

have coefficients with moderate growth (Deligne extension). Moreover, there exists a canonical holomorphic extension on which the connection has logarithmic poles, and the residues along the components of D are constant. This is the "moderate" situation (with regular singularities), which is therefore better behaved than the analog for Higgs bundles. The analog of the "wild" property requires on the other hand a reference meromorphic extension of the bundle V to be defined, unlike the Higgs case. Any meromorphic extension $(\mathscr{V},\nabla)$ distinct from the Deligne extension will be said to have *irregular singularities*. In the situation of the introduction, if Z is smooth, the algebraic condition of (V,∇) provides a well determined meromorphic extension of the associated holomorphic bundle $(V,\nabla)^{\mathrm{an}}$.

The decomposition property (after ramification) parametrized by polar parts $\mathfrak{a}$ analogous to (3.1 ∗∗) is not satisfied in general, even if $\dim X = 1$. In this case, it is only satisfied if we allow *formal* gauge changes for the connection.

For $\dim X = 2$, it was conjectured (and proved in a few special cases) in [Sab00, Conj. 2.5.1], that such a property is satisfied after blowing-ups of X. This property was proved by T. Mochizuki [Moc09a] when the connection is defined algebraically, by a method of reduction to characteristic p and a use of the p-curvature as an ersatz of a Higgs field, for which one can apply a statement of the type of that of Remark 3.2(iv). Some time later, Kedlaya proposed a completely different proof of the same statement, without any condition of algebraicity of the connection. It is based on techniques inspired by p-adic differential equations.

In [Moc11a, Th. 16.2.1], T. Mochizuki extended this result in any dimension (still for an algebraically defined connection), relying on the result in dimension 2 and using a harmonic metric to go back and forth between the flat bundle and the Higgs bundle by the wild Hitchin-Kobayashi correspondence explained below. Therefore, the proof contains a big part of analysis. On the other hand, Kedlaya [Ked11] has also been able to extend his own methods to any dimension, without any algebraic assumption.

THEOREM 3.3 ([Moc09a, Moc11a], [Ked10, Ked11]). — *Let X' be a smooth algebraic variety (resp. a germ of a complex analytic variety) and let $(\mathscr{V},\nabla)$ be a meromorphic bundle on X' with integrable connection, holomorphic on a Zariski open subset X'^o of X'. Then there exists a projective modification $\pi : X \to X'$ with X smooth, which is an isomorphism over X'^o, such that $X \smallsetminus X'^o$ is a divisor with normal crossings D, and that at any point $x \in D$, the formalized bundle $(\widehat{\mathscr{O}}_{X,x}\otimes_{\mathscr{O}_x}\mathscr{V},\widehat{\nabla})$ can be decomposed, after possible ramification around the local components of D, into the form*

$$(3.3\,*)\qquad (\widehat{\mathscr{O}}_{X,x}\otimes_{\mathscr{O}_x}\mathscr{V},\widehat{\nabla}) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}_x(\nabla)}(\widehat{\mathscr{V}}_{\mathfrak{a}},\widehat{\nabla}_{\mathfrak{a}}),$$

where $\widehat{\nabla}_{\mathfrak{a}}^{\mathrm{reg}} := \widehat{\nabla}_{\mathfrak{a}} - \mathrm{d}\mathfrak{a}\otimes \mathrm{Id}_{\widehat{\mathscr{V}}_{\mathfrak{a}}}$ has regular singularities ([Del70]*).*

This theorem, as proved by these authors in its more precise version with the property (Good) below, was the missing link to analyze irregular singularities of holonomic systems of partial differential equations.

Example 1.2, *continued.* — For $(V^\lambda, (1/\lambda)\nabla^\lambda)$, the regular/irregular singularity dichotomy is the same as the moderate/wild dichotomy of the Higgs case.

3.3. The condition "good wild"

The decomposition property (3.1 $**$) (after local ramification around the components of the divisor D) for a Higgs bundle or, taking formal coefficients, for a meromorphic bundle with a flat connection, is still insufficient when $n = \dim X \geqslant 2$ for the analysis of the asymptotic properties of a harmonic metric or of the Stokes phenomenon. For example, in the case of two variables x_1, x_2, we try to avoid the existence of horizontal sections of the connection which have a behavior like $\exp(x_1/x_2)$ because of the "indeterminacy" of the limit of x_1/x_2 when $x_1, x_2 \to 0$. On the other hand, we accept $\exp(1/x_2)$ or $\exp(1/x_1x_2)$.

In local coordinates adapted to the divisor as well as in Section 3.1, we associate to any polar part $\mathfrak{a} \in \mathscr{O}(*D)/\mathscr{O}$, written in the form $\sum_{\boldsymbol{m}\in\mathbb{Z}^\ell\times\mathbb{N}^{n-\ell}} \mathfrak{a}_{\boldsymbol{m}} z^{\boldsymbol{m}}$, the polyhedron of $\mathbb{R}^n$ obtained as the convex hull of the octants $\mathbb{R}^n_+$ (to neglect $\mathscr{O}$) and $\boldsymbol{m} + \mathbb{R}^n_+$ for which $\mathfrak{a}_{\boldsymbol{m}} \neq 0$.

A finite family S of polar parts, like $\mathrm{Irr}(\theta)$, is said to be *good* if the following property is satisfied:

(Good) the Newton polyhedra of polar parts $\mathfrak{a} - \mathfrak{b}$, for $\mathfrak{a}, \mathfrak{b} \in S \cup \{0\}$, are octants of vertices in $-\mathbb{N}^\ell \times \{0_{n-\ell}\}$ and are pairwise nested.

[For several questions, it is enough to consider the weaker condition that the polyhedra of $\mathfrak{a} - \mathfrak{b}$, for $\mathfrak{a}, \mathfrak{b} \in S$, are octants of vertices in $-\mathbb{N}^\ell \times \{0_{n-\ell}\}$, in which case it concerns only the bundles of rank $\geqslant 2$. Note also that either of these properties is always satisfied in dimension 1].

We say that a Higgs bundle on X^o is *wild and good* along D if the decomposition (3.1 $**$) takes place in the neighborhood of any point of D after local ramification, with a *good* local set $\mathrm{Irr}(\theta)$.

Similarly, a flat meromorphic bundle $(\mathscr{V}, \nabla)$ on X with poles along D is said to admit an *good formal structure along* D if, for any point $x \in D$, the bundle with connection tensored by $CC[\![z_1, \dots, z_n]\!][1/z_1 \dots z_\ell]$ admits, after ramification, a decomposition (3.3 $*$) parametrized by a *good* finite set $\mathrm{Irr}_x(\nabla) \subset \mathscr{O}(*D)/\mathscr{O}$. This property has been used when D is *smooth*, in the study of isomonodromic deformations of differential equations of one variable with irregular singularities. The first works in the case of normal crossings, after the first cases considered in [LvdE82], are those of Majima [Maj84], continued by [Sab00] in the case of two variables. The situation is now clear thanks to the very detailed analysis made by T. Mochizuki in [Moc11c, Moc11a].

As for the regular singularities with the canonical extension of Deligne [Del70], it is important to be able to work with an $\mathscr{O}_X$-coherent module $\mathcal{V}$ such that $\mathscr{V} = \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathcal{V}$, called a *lattice* of $\mathscr{V}$. The global existence of such a lattice is not obvious. Even in the local situation, when $(\mathscr{V}, \nabla)$ admits a good formal structure, it is not clear that there exists a lattice whose formalization is adapted to the decomposition (3.3 ∗). The existence of such a lattice is however essential to show the asymptotic properties of the horizontal sections of the connection in the neighborhood of the divisor: it is first in a local basis of such a lattice that we can express them.

Definition 3.4 (Good lattice [Moc11a, § 2.3]). — *A* good lattice *is a $\mathscr{O}_X$-torsion-free coherent submodule $\mathcal{V}$ of the meromorphic bundle $\mathscr{V}$, which generates it by tensorization by $\mathscr{O}_X(*D)$ and such that the formalized module $\widehat{\mathcal{V}} := \widehat{\mathscr{O}}_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathcal{V}$ at each point $x \in D$ is the invariant part by the action of the Galois group of a local ramification of a* good unramified lattice*, i.e., which decomposes in a way compatible with the decomposition* (3.3 ∗) *of $\widehat{\mathscr{O}}_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{V}$, such that on the component $(\widehat{\mathcal{V}}_{\mathfrak{a}}, \widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}})$ is logarithmic poles in the sense of* [Del70].

Malgrange [Mal96] has shown the existence of a "canonical lattice" that is *good on a dense Zariski open of D*. The construction is local and, as for the Deligne canonical lattice in the case of regular singularities, it is by controlling the residues along the components of D that Malgrange can globalize various local constructions. This lattice is called the *canonical Deligne-Malgrange lattice* by Mochizuki. From the point of view of asymptotic analysis in the neighborhood of singularities of D, this lattice is still not enough, but Mochizuki shows:

Theorem 3.5 ([Moc11c, Cor. 2.24]). — *If $(\mathscr{V}, \nabla)$ admits a good formal structure along D, then the canonical Deligne-Malgrange lattice is good at any point of D.*

Remarks 3.6

(i) The notion of a good lattice extends in an obvious way to the case of a λ-connection if $\lambda \neq 0$. In the Higgs case ($\lambda = 0$), it is even useless to consider the formalization.

(ii) In the results mentioned in Remark 3.2(iv) as well as in Theorem 3.3, it is the "wild and good" property which is obtained after blowing-ups, that is to say the sets $\mathrm{Irr}_x(\theta)$ and $\mathrm{Irr}_x(\nabla)$ are good for any $x \in D$.

(iii) In the following, we will admit Theorem 3.3, since it now has an independent proof by Kedlaya. In [Moc11a], Mochizuki could not afford this shortcut, as he did not have this theorem at his disposal at that time, and he proved it by the arguments given above, which we will not explain.

3.4. Good wild harmonic bundles

The property "moderate" or "wild" for a flat (or Higgs) bundle with a harmonic metric concerns the associated Higgs field. In the case of a curve, the "moderate" property was introduced by Simpson [Sim90]. It was extended by Biquard [Biq97] to the case where D is a smooth divisor, then by Mochizuki [Moc02, Moc07] to the case where D is a divisor with normal crossings. The "wild" condition, already mentioned by Simpson

[Sim90], was considered on curves in [Sab99, BB04, Sab09] and finally, in all generality, in [Moc11a].

Definition 3.7. — *A harmonic flat bundle* (V, ∇, h) *(resp. a harmonic Higgs bundle* (E, θ, h)*) on* X^o *is* wild *(resp.* good wild*) if the associated Higgs bundle* (E, θ) *is wild, see Section* 3.1 *(resp.* good wild*, see Section* 3.3*).*

Remarks 3.8

(i) For a harmonic Higgs bundle, the holomorphy of $f_{i,k}$ (see Section 3.1) implies the constancy of $f_{i,k|D_i}$ (see [Moc07, Lem. 8.2]). In the case of a harmonic Higgs bundle satisfying (3.1 ∗∗), it is not clear a priori that the components $(E_{\mathfrak{a}}, \theta_{\mathfrak{a}} - d\mathfrak{a} \otimes \mathrm{Id})$, equipped with the induced metric, are harmonic. We must therefore treat them as any Higgs bundles, and impose the constancy of $f_{i,k|D_i}$.

(ii) Mochizuki [Moc07, Chap. 8] gives, for a harmonic Higgs bundle, a moderation criterion by restriction to curves transverse to the smooth part of D, reminiscent of that given by Deligne [Del70] for flat connections with regular singularities.

4. WILD HITCHIN-KOBAYASHI CORRESPONDENCE

In the following, we will sketch a proof that any simple holonomic $\mathscr{D}_X$-module comes from a $\mathscr{D}$-holonomic module with polarizable twistor structure (point (3) of Section 2). We start with the end, namely the point (6).

In the global situation (see convention 3.0), let (V, ∇) be a flat algebraic bundle on X^o. It corresponds bijectively to a $\mathscr{O}_X(*D)$-coherent module $\mathscr{V}$ with flat connection ∇. Let us assume (V, ∇) is simple. Mochizuki shows the existence of a harmonic metric for (V, ∇) with good properties. Let us describe the steps by referring below for the precise definitions.

(a) Malgrange's construction of a canonical lattice [Mal96] allows to equip $(\mathscr{V}, \nabla)$ with a parabolic filtration ${}_\bullet\mathscr{V}^{\mathrm{DM}}$.

(b) Given an ample line bundle L on X, we associate to any parabolic flat $\mathscr{O}_X(*D)$-coherent module a slope μ_L, hence a notion of μ_L-stability. Then $(\mathscr{V}, \nabla)$ is simple if and only if $(\mathscr{V}, {}_\bullet\mathscr{V}^{\mathrm{DM}}, \nabla)$ is μ_L-stable.

(c) We also have a notion of parabolic characteristic numbers. According to Theorem 3.3 (using Kedlaya's version), even if we change X and D, we can even suppose that $(\mathscr{V}, \nabla)$ admits a good formal structure along D. In this case, the characteristic numbers of $(\mathscr{V}, {}_\bullet\mathscr{V}^{\mathrm{DM}})$ are zero.

(d) The Hitchin-Kobayashi correspondence then consists, in this framework, in the construction of a harmonic metric h for (V, ∇) adapted to the filtration $({}_\bullet\mathscr{V}^{\mathrm{DM}})$, hence a harmonic Higgs bundle (E, θ, h) and a variation of pure polarized twistor structure of weight 0 on X^o. This construction also relies on the fact that the harmonic Higgs bundle (E,θ,h) thus obtained is wild and good.

The remaining questions are whether (E,θ) extends (and how) to X, and whether the polarized twistor structure variation extends (and how) to X. We will discuss them at the end of this paper.

4.1. Parabolic filtrations and adapted metrics

We place ourselves in the local or global situation (convention 3.0). Mochizuki considered a property similar to the following one in [Moc07, § 4.2].

Parabolic filtration. — Let ${}_{\infty}E$ be a torsion-free coherent $\mathscr{O}_X(*D)$-module. A *parabolic filtration* of ${}_{\infty}E$ consists in giving an increasing filtration, indexed by $\mathbb{R}^{\mathfrak{I}}$ provided with its natural partial order, of ${}_{\infty}E$ by torsion-free coherent sub$\mathscr{O}_X$-modules ${}_{\boldsymbol{a}}E$, which satisfies the following properties:

(1) (translation) for any $\boldsymbol{a}\in\mathbb{R}^{\mathfrak{I}}$, we have ${}_{\infty}E=\mathscr{O}_X(*D)\otimes_{\mathscr{O}_X}{}_{\boldsymbol{a}}E$ and, for any $\boldsymbol{n}\in\mathbb{Z}^{\mathfrak{I}}$, ${}_{\boldsymbol{a}-\boldsymbol{n}}E=\mathscr{O}_X(-\sum_{i\in\mathfrak{I}}n_iD_i)\otimes_{\mathscr{O}_X}{}_{\boldsymbol{a}}E$;

(2) (finiteness) there exists for any $i\in\mathfrak{I}$ a finite subset $\mathcal{A}_i\subset\mathbb{R}$ modulo $\mathbb{Z}$ such that the filtration is determined by its restriction to $\mathcal{A}=\prod_{i\in\mathfrak{I}}\mathcal{A}_i$, i.e., for any $\boldsymbol{a}'\in\mathbb{R}^{\mathfrak{I}}$, we have ${}_{\boldsymbol{a}'}E=\bigcup_{\substack{\boldsymbol{a}'\in\mathcal{A}\\ \boldsymbol{a}'\leqslant\boldsymbol{a}'}}{}_{\boldsymbol{a}}E$.

If ${}_{\boldsymbol{a}}E$ are $\mathscr{O}_X$-locally free, we will also say that $({}_{\infty}E,{}_{\bullet}E)$ is a *parabolic meromorphic bundle on* (X,D). If ${}_{\infty}E$ is locally free of rank 1, then giving a parabolic filtration is equivalent to giving $\boldsymbol{b}\in\mathbb{R}^{\mathfrak{I}}$ modulo $\mathbb{Z}^{\mathfrak{I}}$. We then have locally ${}_{\boldsymbol{a}}E\simeq\mathscr{O}_X(\sum_i[a_i+b_i]D_i)$. The following proposition simplifies various notions introduced in [Moc06, Chap. 3], [Moc07, Chap. 4] and [IS07, § 2].

PROPOSITION 4.1 ([Bor09, Th. 2.4.20], [HS10, Th. 4.2]). — *Any parabolic meromorphic bundle* $({}_{\infty}E,{}_{\bullet}E)$ *on* (X,D) *is locally abelian, i.e., locally isomorphic to a direct sum of parabolic bundles of rank* 1.

DEFINITION 4.2. — *Let* $({}_{\infty}E,{}_{\bullet}E)$ *be a parabolic meromorphic bundle over* (X,D). *A local basis* $\boldsymbol{e}$ *of* ${}_{\infty}E$ *as* $\mathscr{O}_X(*D)$*-module is said to be* adapted *to the parabolic filtration if it defines a decomposition of* $({}_{\infty}E,{}_{\bullet}E)$ *into rank-*1 *parabolic bundles. Each element* e_k *has then a multi-order* $\boldsymbol{a}(k)$. *We associate to* $\boldsymbol{e}$ *a normalized basis* $\boldsymbol{e}'$ *defined by* $e'_k=\prod_{i=1}^{\ell}|z_i|^{a_i(k)}\cdot e_k$.

Parabolic characteristic numbers. — Let $({}_{\infty}E,{}_{\bullet}E)$ be a parabolic meromorphic bundle. Let us consider one of the bundles ${}_{\boldsymbol{a}}E$ (for $\boldsymbol{a}\in\mathcal{A}$, this is sufficient). On each component D_i of D we then have the $\mathscr{O}_{D_i}$-locally free module ${}_{\boldsymbol{a}}E/{}_{\boldsymbol{a}^{-i}}E$, if $\boldsymbol{a}^{-i}$ is the predecessor of $\boldsymbol{a}$ in the i direction only. Let us denote $\boldsymbol{a}E_{||D_i}$ this bundle and $\operatorname{rk}{}_{\boldsymbol{a}}E_{||D_i}$ its rank. We observe that, for $\boldsymbol{b}$, the usual restriction ${}_{\boldsymbol{b}}E_{|D_i}={}_{\boldsymbol{b}}E/{}_{\boldsymbol{b}-1_i}E$ has rank

$$\operatorname{rk}{}_{\boldsymbol{b}}E_{|D_i}=\sum_{\substack{a_i\in(b_i-1,b_i]\\ a_j=b_j\,\forall\,j\neq i}}\operatorname{rk}{}_{\boldsymbol{a}}E_{||D_i}$$

Definition 4.3 ([Moc06, § 3.1.2]). — *In the global situation, let L be an ample bundle on X. The parabolic degree* $\text{par-}\deg_L({}_{\infty}E, {}_{\bullet}E)$ *is defined by the formula, independent of the choice of* $\boldsymbol{b} \in \mathbb{R}^{\mathcal{I}}$,

$$\text{par-}\deg_L({}_{\infty}E, {}_{\bullet}E) = \deg_L {}_{\boldsymbol{b}}E - \sum_{i\in\mathcal{I}} \Big(\sum_{a_i \in (b_i-1, b_i]} a_i \operatorname{rk} {}_{\boldsymbol{a}}E_{||D_i} \Big) \deg_L D_i.$$

The slope $\mu_L({}_{\infty}E, {}_{\bullet}E)$ *is the quotient* $\text{par-}\deg_L({}_{\infty}E, {}_{\bullet}E)/\operatorname{rk} {}_{\infty}E$.

We can also define a class $\text{par-}c_1({}_{\infty}E, {}_{\bullet}E)$ by replacing in the above formula the number $\deg_L D_i$ by the class $[D_i]$ in $H^2(X, \mathbb{R})$ and $\deg_L {}_{\boldsymbol{b}}E$ by $c_1({}_{\boldsymbol{b}}E)$, and we can also define a number $\text{par-}\deg_L \operatorname{ch}_2({}_{\infty}E, {}_{\bullet}E)$ (see loc. cit.).

Hermitian metric adapted to a parabolic filtration. — In the local situation, let $({}_{\infty}E, {}_{\bullet}E)$ be a parabolic meromorphic bundle over (X, D). Let furthermore h be a Hermitian metric on $E = {}_{\infty}E_{|X^{\circ}}$. For each $\boldsymbol{a} \in \mathbb{R}^{\mathcal{I}}$, let us define the subsheaf of $\mathscr{O}_X$-modules ${}_{\boldsymbol{a}}\widetilde{E} \subset j_*E$ by

$$\forall\, U \subset X, \quad {}_{\boldsymbol{a}}\widetilde{E}(U) = \Big\{ e \in E(U \smallsetminus D) \mid \forall\, \varepsilon > 0,\ |e|_h = O\big(\textstyle\prod_{i\in\mathcal{I}} |z_i|^{-a_i-\varepsilon}\big) \text{ loc. on } U \Big\},$$

and ${}_{\infty}\widetilde{E} = \bigcup_{\boldsymbol{a}} {}_{\boldsymbol{a}}\widetilde{E}$, which is a $\mathscr{O}_X(*D)$-module, filtered by the torsion-free $\mathscr{O}_X$-submodules ${}_{\boldsymbol{a}}\widetilde{E}$. In general, these sheaves have no coherence properties.

Definition 4.4 ([Moc06, § 3.5]). — *The metric h is said to be* adapted to the parabolic meromorphic bundle $({}_{\infty}E, {}_{\bullet}E)$ *if* ${}_{\boldsymbol{a}}\widetilde{E} = {}_{\boldsymbol{a}}E$ *for all* $\boldsymbol{a} \in \mathbb{R}^{\mathcal{I}}$.

4.2. The Deligne-Malgrange filtration

Let us return to our problem, in the global situation. Let $(\mathscr{V}, \nabla)$ be an $\mathscr{O}_X(*D)$-coherent module with integrable connection. When $(\mathscr{V}, \nabla)$ has *regular singularities* along D, Deligne [Del70] has constructed a canonical vector bundle on which the connection has logarithmic poles and the eigenvalues α_i of the residue endomorphism on each component D_i of D have a real part in $[0, 1[$. For each $\boldsymbol{a} \in \mathbb{R}^{\mathcal{I}}$, we can define the bundle with logarithmic connection ${}_{\boldsymbol{a}}\mathscr{V}$ by imposing that $-\operatorname{Re}\alpha_i \in \,]a_i - 1, a_i]$ for any $i \in \mathcal{I}$. We thus obtain the canonical Deligne filtration and a flat parabolic meromorphic bundle $(\mathscr{V}, {}_{\bullet}\mathscr{V}, \nabla)$, all this compatible with the formation of the determinant. In the rank-1 case, the connection on the associated C^{∞} bundle has distributional coefficients, and the Chern-Weil formula (in the sense of currents) for $c_1({}_{\bullet}\mathscr{V})$ shows that $\text{par-}c_1(\mathscr{V}, {}_{\bullet}\mathscr{V}) = 0$, and this remains true in any rank by passing to the determinant, as does the equality $\text{par-}\deg_L(\mathscr{V}, {}_{\bullet}\mathscr{V}) = 0$.

We also notice that any coherent $\mathscr{O}_X(*D)$-submodule of $\mathscr{V}$ stable by the connection is still $\mathscr{O}_X(*D)$-locally free, and the connection there is still with regular singularities. Moreover, the Deligne filtration of $\mathscr{V}$ induces on this submodule its own Deligne filtration.

We will say that the flat parabolic meromorphic bundle $(\mathscr{V}, {}_{\bullet}\mathscr{V}, \nabla)$ is μ_L-stable if every flat meromorphic subbundle, equipped with the induced parabolic filtration, has strictly smaller slope.

We then check that $(\mathscr{V}, {}_{\bullet}\mathscr{V}, \nabla)$ is μ_L-stable if and only if $(\mathscr{V}, \nabla)$ is simple (indeed, the Deligne filtration induces on any nontrivial flat meromorphic subbundle the Deligne filtration of the latter, which is thus of zero slope, in contradiction with stability).

When $(\mathscr{V}, \nabla)$ has irregular singularities, Malgrange [Mal96] constructed a canonical lattice (cf. Section 3.3) by imposing analogous conditions on the residues of the connections $\widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}}$, which exist a priori on a Zariski dense open subset of each D_i. We deduce a canonical filtration $({}_{\bullet}\mathscr{V}^{\mathrm{DM}}, \nabla)$, called *the Deligne-Malgrange filtration.* Note that each $({}_{\boldsymbol{a}}\mathscr{V}^{\mathrm{DM}}, \nabla)$ is not necessarily logarithmic, and is not necessarily locally free. Nevertheless, every ${}_{\boldsymbol{a}}\mathscr{V}^{\mathrm{DM}}$ is $\mathscr{O}_X$-coherent and reflexive (see [Moc11a, Lem. 2.7.8]).

If moreover $(\mathscr{V}, \nabla)$ admits a good formal structure along D then, as indicated in Section 3.3, the canonical Deligne-Malgrange lattice is locally the invariant part in a ramification of a lattice which decomposes formally in the neighborhood of each point of D as $(\mathscr{V}, \nabla)$. It follows that it is a vector bundle, and $(\mathscr{V}, {}_{\bullet}\mathscr{V}^{\mathrm{DM}})$ is a parabolic flat meromorphic bundle. In the case of rank 1, worrying about smoothness and ramification is superfluous and, using the quasi-isomorphism $\Omega^{\bullet}(\log D) \simeq \Omega^{\bullet}(*D)$ (see [Del70, Prop. II.3 .13]), one shows that the connection $\nabla + \overline{\partial}$ on $\mathscr{C}^{\infty}_X \otimes \mathscr{V}$ is written as the sum of a C^{∞} logarithmic flat connection and an exact form $\mathrm{d}\varphi$ with $\varphi \in \Gamma(X, \mathscr{C}^{\infty}_X(*D))$. It follows that, as in the logarithmic case, we have, in any rank, $\text{par-}c_1(\mathscr{V}, {}_{\bullet}\mathscr{V}) = 0$ and $\text{par-}\deg_L(\mathscr{V}, {}_{\bullet}\mathscr{V}) = 0$.

One shows in the same way that any $\mathscr{O}_X(*D)$-coherent submodule of $\mathscr{V}$ stable by the connection is still $\mathscr{O}_X(*D)$-locally free, and admits a good formal structure along D (with local exponential factors contained in those of $(\mathscr{V}, \nabla)$). Finally, the Deligne-Malgrange filtration of $\mathscr{V}$ induces that of its submodules.

We deduce, as above, the equivalence μ_L-stability of $(\mathscr{V}, {}_{\bullet}\mathscr{V}^{\mathrm{DM}}, \nabla) \iff$ simplicity of $(\mathscr{V}, \nabla)$ (see [Moc11a, § 2.7.2.2]) We also have:

Proposition 4.5 ([Moc11a, Cor. 14.3.4]). — *One has* $\text{par-}\deg_L \mathrm{ch}_2({}_{\bullet}\mathscr{V}^{\mathrm{DM}}) = 0$.

4.3. Construction of an adapted harmonic metric

We assume that $(\mathscr{V}, \nabla)$ is simple and admits a good formal structure along D. We try to construct a harmonic metric adapted to the parabolic structure of Deligne-Malgrange.

The rank-1 case. — It is instructive to start by considering the case of rank 1 bundles. We first notice that, in rank 1, the pseudocurvature $G(\nabla, h)$ (cf. (1.1)) is equal to twice the curvature of h (see [Moc09b, Lem. 2.31]). So we have to construct a metric with zero curvature adapted to a parabolic filtration, itself determined by the data of $\boldsymbol{a} \in \mathbb{R}^{\mathfrak{I}}$. We construct a singular metric h_0 on ${}_{\boldsymbol{a}}\mathscr{V}$ by first imposing that, in any local situation, a local basis e of ${}_{\boldsymbol{a}}\mathscr{V}$ has norm $\|e\|_{|h_0} = |z|^{-\boldsymbol{a}} \cdot \|e\|_{h_{\mathrm{loc}}}$, where h_{loc} is a local C^{∞} metric on ${}_{\boldsymbol{a}}\mathscr{V}$, then using a partition of the unity to glue. The curvature $R(h_0)$ is the sum of a closed C^{∞} form $R'(h_0)$ of type $(1,1)$ on X and a closed current of type $(1,1)$ supported on D, and its class is $c_1({}_{\boldsymbol{a}}\mathscr{V})$. The parabolic correction is made so that the class of $R'(h_0)$ is equal to $c_1({}_{\bullet}\mathscr{V},{}_{\bullet}\mathscr{V})$, which we have seen above is zero. Hodge theory then implies that $R'(h_0)=\overline{\partial}\partial g$ for some function g of class C^∞, and the metric $h=e^{-g}h_0$ is still adapted to ${}_{\boldsymbol{a}}\mathscr{V}$ and has zero curvature.

The case of curves. — In the global situation, let X be a smooth algebraic curve and $(\mathscr{V},\nabla)$ a meromorphic bundle with connection on (X,D). The Deligne-Malgrange filtration is classically defined in this case (see [Mal96]) and the goodness condition is trivially satisfied. Let us assume $(\mathscr{V},\nabla)$ is simple. It follows essentially from Simpson's work (see [Sab99]) that there exists a harmonic metric h for $(\mathscr{V}_{|X^\circ},\nabla)$ adapted to the Deligne-Malgrange filtration.

In [BB04] Biquard and Boalch extend this result to a more general parabolic situation, and specify it in a Hitchin-Kobayashi correspondence between flat bundles and Higgs bundles.

In [Moc11a, § 13.4], Mochizuki gives another proof of this result, in the spirit of that of [Sim90], and he specifies a uniqueness property of the harmonic metric.

A Metha-Ramanathan type theorem. — This reduction result to general curves when $\dim X\geqslant 2$ is important in several places in the following proof. Simpson's argument [Sim92] has already been generalized by Mochizuki in [Moc06] to the case of regular singularities, and the proof is extended to the case of irregular singularities:

Proposition 4.6 ([Moc11a, Cor. 13.2.3]). — *In the global situation, let $(\mathscr{V},\nabla)$ be a flat meromorphic bundle with poles along D and L be an ample line bundle on X. Then $(\mathscr{V},\nabla)$ is simple if and only if its restriction to any general enough curve which is a complete intersection of sections of $L^{\otimes m_\nu}$, for a sequence $m_\nu\to+\infty$, is simple.*

Constructing an adapted harmonic metric. — Let $(\mathscr{V},\nabla)$ be a flat meromorphic bundle on (X,D). Suppose that $(\mathscr{V},\nabla)$ admits a *good formal structure along* D, and thus gives rise to a flat parabolic meromorphic bundle $(\mathscr{V},{}_{\bullet}\mathscr{V}^{\mathrm{DM}},\nabla)$. Let us also assume that $(\mathscr{V},\nabla)$ is *simple*, so that $(\mathscr{V},{}_{\bullet}\mathscr{V}^{\mathrm{DM}},\nabla)$ is μ_L-stable, with zero characteristic numbers. Moreover, as indicated above, the first parabolic Chern class is zero. We can define a flat parabolic meromorphic bundle of rank 1, namely the determinant $\det({}_{\bullet}\mathscr{V}^{\mathrm{DM}},\nabla)$, which also has zero par-c_1 and which, according to what we have seen above, admits a harmonic metric $h_{\det}$.

Theorem 4.7 ([Moc07, Th. 25.28], [Moc09b, Th. 5.16], [Moc11a, Th. 16.1.1])
Under these conditions, there exists a unique harmonic metric adapted to $(\mathscr{V},{}_{\bullet}\mathscr{V}^{\mathrm{DM}},\nabla)$ normalized by $\det(h)=h_{\det}$. Moreover this metric makes the associated harmonic Higgs bundle (E,θ,h) a good wild Higgs bundle.

Let us first consider the last property, which will be useful to show uniqueness, and ultimately existence in dimension $\geqslant 3$. This is a more general result.

PROPOSITION 4.8 (Adaptation implies good wildness, [Moc11a, Prop. 13.5.2])
Let $(\mathscr{V},\nabla)$ be a flat meromorphic bundle on (X,D) admitting a good formal structure along D. Let us also assume $(\mathscr{V},\nabla)_{|X^o}$ equipped with a harmonic metric h adapted to ${}_\bullet\mathscr{V}^{\mathrm{DM}}$. Then the associated harmonic Higgs bundle (E,θ,h) is good wild.

Uniqueness in 4.7. — Let us take two such metrics h_1 and h_2. In restriction to a general curve C as in 4.6, $(\mathscr{V},\nabla)$ is simple and wild (automatically good in dimension 1). Anticipating the next section (Theorems 5.2 and 5.3), the estimate of Theorem 5.2(ii) for $({}_\bullet\mathscr{V}^{\mathrm{DM}},h)$ is restricted to the curve C, which allows us to see that the restrictions of h_1,h_2 to C^o are adapted to $({}_\bullet\mathscr{V}^{\mathrm{DM}})_{|C}$. The uniqueness seen above in the case of curves shows that $h_{1|C^o}=h_{2|C^o}$. Since we can pass such a general curve through every point of X^o, we deduce uniqueness.

Existence in 4.7. — The technique was developed by Mochizuki in the moderate case in [Moc07, Moc09b] and is adapted to the good wild case using the results already obtained and those in 5.2 and 5.3 below. Let us summarize it very quickly.

On the one hand Mochizuki extends results of Donaldson and Simpson (see [Sim88]), and on the other hand he starts with the case of surfaces. In this case, he constructs a family, parametrized by $\varepsilon>0$, of perturbations of the parabolic structure to remove the possible nilpotent part of the parabolic gradings of the residues of $\widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}}$ along the components of D. Using a generalization of the results of Donaldson and Simpson, and starting from a suitable metric on the different flat parabolic graded bundles on the components of D_i, he obtains for each ε a harmonic metric [3] adapted to the perturbed parabolic filtration. He then shows the convergence of these metrics for $\varepsilon\to 0$, in a suitable sense, to a harmonic metric [4] adapted to the parabolic filtration $({}_\bullet\mathscr{V}^{\mathrm{DM}})$.

The case of dimension $\geqslant 3$ can be treated thanks to the uniqueness argument: the metric defined on each rather general surface, thanks to 4.6, is indeed the restriction of a metric existing on X^o, and it satisfies the required properties.

5. EXTENSION OF WILD HARMONIC BUNDLES

5.1. The problems of extension

We place ourselves in the local situation. Let (E,θ,h) be a good wild harmonic Higgs bundle on X^o. In particular, we try to solve the following problems:

(a) extend the bundle E into a meromorphic bundle $\mathscr{E}$ (i.e., a $\mathscr{O}_X(*D)$-locally free module of finite rank) on X,

(b) show that θ extends meromorphically to the extension $\mathscr{E}$ (i.e., the coefficients of θ in a local basis of $\mathscr{E}$ are meromorphic),

3. in the first sense of the footnote 5.
4. in the sense "pluri-harmonic" of the footnote 5.

(c) for each λ, also extend the flat bundles (V^λ, ∇) into flat meromorphic bundles $(\mathscr{V}^\lambda, \nabla)$.

(d) perform this last extension holomorphically with respect to λ.

We explain in this section how Mochizuki proceeds in the wild and good case [Moc11a], after having solved these questions in the moderate case [Moc07].

Insofar as the local extension problem is solved canonically using the metric (see below), it leads to results applicable in the global situation.

5.2. Acceptable Hermitian holomorphic bundles

Let us consider the local situation. Let (E, h) be a Hermitian holomorphic bundle on X^o. The condition of *acceptability* goes back to the paper of Cornalba and Griffiths [CG75], and was used in this framework by Simpson [Sim88, Sim90]:

Definition 5.1. — *The Hermitian bundle (E, h) is* acceptable *if the norm of the curvature of h, computed with respect to h itself (on the bundle of endomorphisms) and to the Poincaré metric on $U \setminus D$, is* bounded.

Mochizuki refines the results on acceptable bundles in the following way, with the notation used in Definition 4.4:

Theorem 5.2 ([Moc07] and [Moc11a, 21.3.1–3])

(i) If (E, h) is acceptable, then every ${}_{\boldsymbol{a}}\widetilde{E}$ is $\mathscr{O}_X$-locally free and $({}_{\boldsymbol{a}}\widetilde{E})$ makes ${}_{\boldsymbol{\infty}}\widetilde{E}$ a parabolic meromorphic bundle $({}_{\boldsymbol{\infty}}\widetilde{E}, {}_{\bullet}\widetilde{E})$.

(ii) Moreover, if $\boldsymbol{e}$ is a local basis of ${}_{\boldsymbol{\infty}}\widetilde{E}$ adapted to the decomposition given by Proposition 4.1 and $\boldsymbol{e}'$ the associated normalized basis (see Definition 4.2), the eigenvalues eta(z) of the matrix $h(\boldsymbol{e}', \boldsymbol{e}')$ of the metric in this basis satisfy the inequalities

$$C\Big(\sum_{i=1}^{\ell} \mathrm{L}(z_i)\Big)^{-N} \leqslant \eta(z) \leqslant C'\Big(\sum_{i=1}^{\ell} \mathrm{L}(z_i)\Big)^{N}$$

for C, C', N positive suitable, posing $\mathrm{L}(z) = |\log |z||$ ($|z| < 1$).

(iii) Finally, the bundle $(\mathrm{End}({}_{\boldsymbol{\infty}}\widetilde{E}), h)$ is also acceptable, and ${}_{\boldsymbol{0}}\mathrm{End}({}_{\boldsymbol{\infty}}\widetilde{E})$ is the sheaf of endomorphisms of ${}_{\boldsymbol{\infty}}\widetilde{E}$ which preserve the filtration ${}_{\boldsymbol{a}}\widetilde{E}$ and whose restriction to each component D_i preserves the natural filtration of ${}_{\boldsymbol{a}}\widetilde{E}_{|D_i}$.

5.3. Acceptability of good wild harmonic bundles

Let us stay in a local setting as above. Let (E, θ, h) be a harmonic Higgs bundle on X^o. The curvature of the Chern connection is calculated, because of the harmonicity, by the formula $R(h, \overline{\partial}_E) = -[\theta, \theta^\dagger]$. More generally, for any $\lambda \in \mathbb{C}$, $R(h, \overline{\partial}_E + \lambda\theta^\dagger) = -(1 + |\lambda|^2)[\theta, \theta^\dagger]$. When (E, θ, h) is good wild, Mochizuki shows the acceptability of (E, h), which therefore entails the acceptability of all (V^λ, h). More precisely, he gives a geometric interpretation of this property, which we now describe, generalizing the one given by Simpson [Sim90] in dimension 1 and in the moderate situation.

The decomposition (3.1 ∗∗) can be refined: $(E,\theta)\simeq\bigoplus_{(\mathfrak{a},\boldsymbol{\alpha})}(E_{(\mathfrak{a},\boldsymbol{\alpha})},\theta_{(\mathfrak{a},\boldsymbol{\alpha})})$, using (3.1 ∗). This decomposition depends only on the Higgs field, and we will analyze it with respect to the metric h. We will say that $E_{(\mathfrak{a},\boldsymbol{\alpha})}$ is $g_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})}$*-asymptotically h-orthogonal to* $E_{(\mathfrak{b},\boldsymbol{\beta})}$ if the norm of the projection of $E_{(\mathfrak{b},\boldsymbol{\beta})}$ onto $E_{(\mathfrak{a},\boldsymbol{\alpha})}$ parallelly to $E^{\perp_h}_{(\mathfrak{a},\boldsymbol{\alpha})}$ and that of $E_{(\mathfrak{a},\boldsymbol{\alpha})}$ on $E_{(\mathfrak{b},\boldsymbol{\beta})}$ parallelly to $E^{\perp_h}_{(\mathfrak{b},\boldsymbol{\beta})}$ are locally bounded by a function $g_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})}(z)$.

In the following, we will take $g_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})}(z)=\exp(-\varepsilon|z^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}|)\prod_{j|\alpha_i\neq\beta_i}|z_i|^{\varepsilon}$ $(\varepsilon>0)$ The condition (Good) implies indeed that for all $\mathfrak{a}-\neq\mathfrak{b}$, $\mathfrak{a}-\mathfrak{b}=z^{-\boldsymbol{m}}\mathfrak{c}(z)$ with $\mathfrak{c}$ holomorphic and $\mathfrak{c}(0)\neq0$, for a certain multi-index $\boldsymbol{m}\in\mathbb{Z}^\ell\smallsetminus\mathbb{N}^\ell$, which we denote by $\operatorname{ord}(\mathfrak{a}-\mathfrak{b})$. We thus see that if $\mathfrak{a}\neq\mathfrak{b}$, the function $g_{\varepsilon,(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})}$ is exponentially decaying near the origin, while if $\mathfrak{a}=\mathfrak{b}$, we have only a moderate decay.

Theorem 5.3 ([Moc07, Chap. 8] and [Moc11a, Chap. 7]). — *Let (E,θ,h) be a good wild harmonic Higgs bundle. Then (E,h) is acceptable, as is any (V^λ,h), and* $(\operatorname{End}V^\lambda,h)$. *More precisely, for all* $(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})$,

- *(i)* $E_{(\mathfrak{a},\boldsymbol{\alpha})}$ *is* $g_{\varepsilon,(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})}$*-asymptotically h-orthogonal to* $E_{(\mathfrak{b},\boldsymbol{\beta})}$ *for epsilon* >0 *small enough;*
- *(ii) the norm of the component* $[\theta,\theta^\dagger]_{(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})}$ *relative to h and to the Poincaré metric is* $O(g_{\varepsilon,(\mathfrak{a},\boldsymbol{\alpha}),(\mathfrak{b},\boldsymbol{\beta})})$.

Remark 5.4. — This type of estimation goes back to Simpson's work [Sim90, § 2] and is for this reason named "Simpson's main estimate" by Mochizuki. It is remarkable that the presence of non-zero polar parts clearly improves the estimates of asymptotic orthogonality compared to the moderate case, since they are exponentially small. Such a phenomenon had already been observed in dimension 1 by Biquard and Boalch [BB04, Lem. 4.6]. Nevertheless, let us not be too quick to rejoice...

5.4. Extension with λ fixed

Let us always assume (E,θ,h) harmonic and good wild in the local situation of the § 3.1. The general results on acceptable Hermitian bundles (Theorem 5.2) and the acceptability theorem for good wild harmonic bundles (Theorem 5.3) allow us to answer the questions (a) and (b) at the beginning of this section. More precisely, for each λ, we obtain a parabolic meromorphic bundle on X, denoted $(\mathcal{P}\mathscr{E}^\lambda,\mathcal{P}_\bullet\mathscr{E}^\lambda)$.

Theorem 5.5 ([Moc07, Cor. 8.89], [Moc11a, Th. 7.4.5]). — *For any λ, the λ-connection ∇^λ is meromorphic on $\mathcal{P}_{\boldsymbol{a}}\mathscr{E}^\lambda$ for any $\boldsymbol{a}\in\mathbb{R}^\ell$ (and logarithmic in the moderate case), and makes it a good lattice, which is unramified if θ is.*

The proof of the moderate case can be adapted and extended to the wild case, thanks in particular to the asymptotic orthogonality seen above. Suppose further that (E,θ,h) is wild without ramification and good. The decomposition (3.1 ∗∗) involves a finite set $\operatorname{Irr}(\theta)$ of polar parts. Mochizuki furthermore obtains:

(5.5 $*$) *For any* $\lambda \neq 0$, *the set* $\mathrm{Irr}(\nabla^\lambda)$ *parametrizing the decomposition* (3.3 $*$) *for* $(\mathcal{P}\mathscr{E}^\lambda, \nabla^\lambda)$ *is equal to* $(1+|\lambda|^2)\,\mathrm{Irr}(\theta)$.

This behavior completes the behavior of the eigenvalues of the residue of $(\widehat{\nabla}^\lambda)^{\mathrm{reg}}_{(1+|\lambda|^2)\mathfrak{a}}$, which is governed by the function $\mathfrak{e}$ ("eigenvalue") introduced in (1.2 $*$), as shown by Simpson in dimension 1 and Mochizuki in any dimension, in the moderate case. The function $\mathfrak{p}$ ("parabolic") governs the behavior with respect to λ of the parabolic structure, which we have not detailed here. The bad news, anticipated in Example 1.2, is that, contrary to $\mathfrak{e}$, the behavior of $\mathrm{Irr}(\nabla^\lambda)$ is not holomorphic in λ.

5.5. Extension when λ varies

In the moderate case, the passage from fixed λ to variable λ does not cause any important complication. One just has to be careful about the local uniformity with respect to λ of the asymptotic estimates. In particular, in the estimate of Theorem 5.2(ii), one replaces the $\big(\sum \mathrm{L}(z_i)\big)^{\pm N}$ by $\prod |z_i|^{\mp\varepsilon}$. So I will not insist on this point, which is treated in detail in [Moc07].

On the other hand, in the wild case, the behavior of Example 1.2 seen at the end of Section 1 is not at all trivial, and requires the setting up of a faithful description, in the good wild multivariable case, of the meromorphic bundles equipped with an λ-integrable connection in terms of the associated formal object and a *Stokes structure.* This is done in Chapters 2 to 4, and 20 (Appendix) of [Moc11a], and represents a major contribution to the theory of holonomic $\mathscr{D}$-modules, independently of the other points considered here. The question which is important for us is then treated in Chapters 9 to 11. Since the whole thing takes more than 200 pages, we will not go into details.

Let us consider the simplified situation, with $\lambda = 1$ fixed, of a meromorphic bundle $(\mathscr{V}, \nabla)$ with connection on a disk Δ of coordinate z, with a unique pole at $z = 0$. The Levelt-Turrittin theorem gives, after a suitable ramification $z' \mapsto z = z'^q$, a decomposition of the formalized bundle $(\mathscr{V}, \nabla)$ at 0. Let us assume for simplicity of explanation that $q = 1$ (unramified case). Then we have a decomposition (3.3 $*$).

Proposition 5.6. — *Let t be a positive real number. There exists, in a canonical and functorial way, a meromorphic bundle with connection $(\mathscr{V}_t, \nabla_t)$ which has the formal decomposition*

$$(\widehat{\mathscr{V}}_t, \widehat{\nabla}_t) \simeq \bigoplus_{\mathfrak{a}} (\widehat{\mathscr{V}}_{\mathfrak{a}}, \widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}} + t \cdot \mathrm{d}\mathfrak{a} \otimes \mathrm{Id}_{\widehat{\mathscr{V}}_{\mathfrak{a}}}).$$

In particular, for $t = 1$ we find $(\mathscr{V}, \nabla)$, and if $t = |\tau|$ with $\tau \in \mathbb{C}^*$, then the family $(\mathscr{V}_t, \nabla_t)$ is *isomonodromic* with respect to τ.

The proof of the proposition relies on the irregular Riemann-Hilbert correspondence, as explained for example in [Del07a] (see also [BV89], [Mal91, Chap. IV]). The formal structure of the supposed $(\mathscr{V}_t, \nabla_t)$ being fixed by the decomposition, it suffices, in order to show its existence, to enrich it by a Stokes structure (passing from a I-graded local system to a I-filtered local system, in the language of loc. cit.). Once the formal decomposition is fixed, the Stokes structure depends on the combinatorial structure on the unit circle of the sets defined by the inequalities $\arg(t\mathfrak{a} - t\mathfrak{b}) \in [-\pi/2, \pi/2]$ for $\mathfrak{a} \neq \mathfrak{b}$ intervening in the decomposition of $(\widehat{\mathscr{V}}, \widehat{\nabla})$. These intervals being independent of $t > 0$, the Stokes structure given by $(\mathscr{V}, \nabla)$ for $t = 1$ extends uniquely for all $t > 0$.

In order to adapt this proof to the local situation considered above, it is appropriate

(a) to extend, for a meromorphically connected bundle on X with poles along D, which is wild and good, the asymptotic theory of Sibuya-Majima; for this we need the existence of a good lattice (see § 3.3) to argue by induction on the dimension;

(b) to extend to this case the irregular Riemann-Hilbert correspondence; Mochizuki thus rediscovered the notion of local I-filtered system of [Del07a], adapted it to any dimension by proving also the efficiency of this approach, from the metric point of view in particular;

(c) to apply, for λ fixed, the analog of Proposition 5.6 to $\mathcal{P}\mathscr{E}^\lambda$ with the multiplier $1/(1 + |\lambda|^2)$ to obtain a prolongation noted $\mathcal{Q}\mathscr{E}^\lambda$;

(d) to extend also the irregular Riemann-Hilbert correspondence to families of connections parametrized by λ, like those induced by a λ-connection if $\lambda \neq 0$;

(e) to correct the non-holomorphic dependence of the exponential factors by an argument analogous to that of Proposition 5.6 by suitably choosing the multiplier (corresponding to t) so that, by fixing λ we find $\mathcal{Q}\mathscr{E}^\lambda$; characterizing the extension $\mathcal{Q}\mathscr{E}$ thus obtained by a condition of moderate growth requires to modify the metric h and make it depend on λ; [such a correction already appears in [Sza07] for a Nahm transform calculation of a Higgs bundle, and in [Sab04] for a Fourier transform calculation];

(f) to glue the previous construction, made for $\lambda \neq 0$, to $\mathcal{P}\mathscr{E}^0$ (in the Higgs case, there is no Stokes structure and the decomposition (3.1 ∗∗) is already holomorphic).

We deduce:

Theorem 5.7 ([Moc11a, Th. 11.12]). — *If (E, θ, h) is a good wild harmonic Higgs bundle, there exists a unique $\mathscr{O}_{X\times\mathbb{C}_\lambda}(*(D \times \mathbb{C}_\lambda))$-locally free $\mathcal{Q}\mathscr{E}$-module with λ-meromorphic connection whose restriction to each λ is equal to $(\mathcal{Q}\mathscr{E}^\lambda, \nabla^\lambda)$.*

Remark 5.8. — The fact that the Stokes phenomenon does not appear for some questions related to the wild case (it does not appear in [BB04] for example) comes from the fact that this phenomenon does not exist for Higgs bundles and that, for the case of flat meromorphic bundles, several points can be treated with an approximation to a sufficiently large order of the formal structure.

6. $\mathscr{D}$-MODULES WITH A WILD TWISTOR STRUCTURE

We saw in Step (1) of Section 2 that the extension of a variation of polarized twistor structure on X^o is an object of a category of quadruplets $(\mathscr{M}', \mathscr{M}'', C, \mathscr{S})$, where $\mathscr{S}$ is the polarization. The $\mathcal{Q}\mathscr{E}$ construction of Theorem 5.7 is the main step to construct $\mathscr{M}'$, and we will set $\mathscr{M}'' = \mathscr{M}'$ and $\mathscr{S} = \mathrm{Id}$ to have a simple model when the weight w is zero. To go from $\mathscr{Q}\mathscr{E}$ to $\mathscr{M}'$, the analog of the intermediate extension $j_{!*}$ is missing. It is also important to extend C obtained from the harmonic metric as in the dictionary of Section 1.

An essential point to ensure, for this construction, and for the constructions that will follow, is that the $\mathscr{R}_{X\times\mathbb{C}_\lambda}$-modules are *strict*, i.e., without $\mathscr{O}_{\mathbb{C}_\lambda}$-torsion. However, this property will be implicitly considered in the following. For example, for the direct image by a proper morphism, the preservation of this condition is analogous to the degeneracy in E_1 of the Hodge $\Rightarrow$ de Rham spectral sequence.

6.1. Nearby cycles

In this paragraph we are interested in local properties for a $\mathscr{R}_{X\times\mathbb{C}_\lambda}$-holonomic module.

The construction of the functor of moderated nearby cycles ψ_f^{mod} with respect to a holomorphic function $f : X \to \mathbb{C}$ for a holonomic $\mathscr{D}_X$-module M relies on the existence theorem of a Bernstein-Sato polynomial, and is expressed as the grading with respect to the Kashiwara-Malgrange filtration of M. Then $\psi_f^{\mathrm{mod}}M$ is a holonomic $\mathscr{D}_X$-module with a semisimple endomorphism and a nilpotent endomorphism N which commute. For a holonomic $\mathscr{R}_X$-module $\mathscr{M}$, such a Bernstein-Sato polynomial does not necessarily exist, but one can define the subcategory of $\mathscr{R}_X$-modules which are moderately specializable along any germ of holomorphic function on X by imposing the existence of a Bernstein-Sato type functional equation. For $\mathscr{M}$ specializable, $\psi_f^{\mathrm{mod}}\mathscr{M}$ is defined, supported in $\{f = 0\} \times \mathbb{C}_\lambda$ and equipped with two endomorphisms as above.

For holonomic $\mathscr{D}_X$-modules with irregular singularities, this functor ψ_f^{mod} may be of little use. For example, in the situation of Example 1.2, for any holomorphic function $f(z)$ such that $f(0) = 0$, the functor ψ_f^{mod} applied to the $\mathscr{D}_\Delta$-module $\mathscr{D}_\Delta/\mathscr{D}_\Delta \cdot (z^2\partial_z + 1)$ gives 0 as a result. In [Del07b], Deligne defined from ψ_f^{mod} a functor which we note ψ_f^{mod}, and which allows to avoid this trivial behavior: we have $\psi_f^{\mathrm{Del}}(\mathscr{M}, \nabla) = \bigoplus_{\mathfrak{a}} \psi_f^{\mathrm{mod}}(\mathscr{M}, \nabla + \mathrm{d}\mathfrak{a})$, where $\mathfrak{a}$ runs in the set of polar parts of the variable $z^{1/q}$ and q is any integer $\geqslant 1$. We can then define the subcategory of $\mathscr{R}_{X\times\mathbb{C}_\lambda}$-Deligne-specializable modules along any germ of holomorphic function on X, and for $\mathscr{M}$ which is Deligne-specializable, $\psi_f^{\mathrm{Del}}\mathscr{M}$ is supported in $\{f = 0\} \times \mathbb{C}_\lambda$ and has two endomorphisms as above. In this framework, the "monodromic filtration" $\mathrm{M}_\bullet\psi_f^{\mathrm{Del}}\mathscr{M}$ associated to the nilpotent endomorphism N is well-defined ([Del80, Prop. 1.6.1]), and we will consider below the functors $\mathrm{gr}_\ell^{\mathrm{M}}\psi_f^{\mathrm{Del}}$. These functors extend in a natural way to pairings C.

Finally, the property of decomposability according to the support, introduced by Saito, will also be important. The strict holonomic $\mathscr{M}$-$\mathscr{R}_{X\times\mathbb{C}_\lambda}$-modules that we consider have as support an analytic (or algebraic) closed subset $Z \times \mathbb{C}_\lambda$ of $X \times \mathbb{C}_\lambda$. In the neighborhood of any point x of Z, there are submodules $\mathscr{M}_i$ supported in a germ of an irreducible closed analytic subset Z_i of Z in x. The condition of *S-decomposability* consists in requiring a local decomposability $\mathscr{M} = \bigoplus_i \mathscr{M}_i$ at any point of Z. If we have a pairing C, we also impose the diagonality of C with respect to this decomposition.

6.2. The category of holonomic $\mathscr{D}$-modules with a polarizable wild twistor structure

This category is defined (see [Sab09]) following Saito's procedure for polarizable Hodge modules, by induction on the dimension of the support.

Definition 6.1. — *The category* $\mathrm{MT}^{(\mathrm{wild})}_{\leqslant d}(X, w)$ *of holonomic* $\mathscr{D}$*-modules with pure wild twistor structure of weight* $w \in \mathbb{Z}$ *is the full subcategory of the category of triples* $\mathscr{T} = (\mathscr{M}', \mathscr{M}'', C)$ *whose objects satisfy the following properties:*

(HSD) $\mathscr{T}$ *is holonomic, S-decomposable and has a support of dimension* $\leqslant d$ *in* X.

($\mathrm{MT}^{(\mathrm{wild})}_{>0}$) *For any open* $U \subset X$ *and any holomorphic function* $f : U \to \mathbb{C}$, $\mathscr{T}$ *is Deligne-specializable along* $\{f = 0\}$ *and, for any integer* $\ell \geqslant 0$, *the triple* $\mathrm{gr}^{\mathrm{M}}_{\ell} \Psi^{\mathrm{Del}}_{f} \mathscr{T}$ *is an object of* $\mathrm{MT}^{(\mathrm{wild})}_{\leqslant d-1}(X, w + \ell)$.

(MT_0) *for any* x_o, *the S-component* $(\mathscr{M}'_{x_o\}}, \mathscr{M}''_{x_o\}}, C_{x_o\}})$ *is a Dirac mass in* x_o *carrying a pure twistor structure of weight* w.

We can also define polarizable objects by analogous constraints on the polarization $\mathscr{S}$.

We indicated in Section 2(4) that Theorem 0.1 applies to these polarizable objects. We refer to [Moc11a, Chap. 18] for the proof, which is inspired by Saito's proof for polarizable Hodge modules.

6.3. End of the proof of the hard Lefschetz theorem

Let Z^o be an irreducible smooth quasi-projective variety. We have seen (Section 1) that a variation of pure polarizable twistor structure of weight 0 on Z^o corresponds to a harmonic Higgs bundle. We will say that this variation is *wild* if there exists a projective compactification Z' of Z^o such that $Z' \smallsetminus Z^o$ is a normal crossing divisor whose components are all smooth and the harmonic Higgs bundle is good wild along this divisor. Let $\mathrm{VTP}^{(\mathrm{wild})}(Z^o, 0)$ be the corresponding category.

Let on the other hand Z be any projective compactification of Z^o contained in a smooth projective variety X and $\mathrm{MTP}^{(\mathrm{wild})}(Z, Z^o, 0)$ be the category of holonomic $\mathscr{D}_X$-modules with pure wild twistor structure of weight 0 and polarizable, smooth on Z^o and with no supported component in a strict closed subset of Z.

Theorem 6.2 ([Moc07, Th. 19.2], [Moc11a, Cor. 19.1.4]). — *The restriction to* Z^o *defines a functor* $\mathrm{MTP}^{(\mathrm{wild})}(Z, Z^o, 0) \to \mathrm{VTP}^{(\mathrm{wild})}(Z^o, 0)$. *This functor is a category equivalence.*

Remark 6.3 (How to change weight). — The above objects have a Tate twist (k) for any $k \in \frac{1}{2}\mathbb{Z}$, analogous to complex Hodge structures. This twist goes from weight 0 to weight $-2k$.

This theorem answers the point (6) of Section 2. One of the difficult points is the essential surjectivity of the functor. The construction of the $\mathscr{R}_{X\times\boldsymbol{S}}$-module $\mathfrak{Q}\mathscr{E}$, outcome of Section 5, is the essential ingredient. The construction of the extension of the pairing C, as well as the proof of the $\mathrm{MTP}^{(\mathrm{wild})}$-properties of the object $\mathscr{T}$ thus obtained, in the case where Z is smooth, $Z \smallsetminus Z^o$ is a divisor with normal crossings and the variation is good wild along $Z \smallsetminus Z^o$, are explained in [Moc11a, Chap. 12] (and in [Moc07, Chap. 18] for the moderate case). Finally, the general case is treated in Chapter 19 of loc. cit., using, as Saito did in [Sai88], a relative version of Theorem 0.1 to go from a good compactification of Z^o to a less good one.

7. WILD HODGE THEORY

Variations of complex Hodge structure and integrable variations of twistor structure

The preceding theory provides few new numerical invariants for simple holonomic $\mathscr{D}$-modules, in contrast to classical Hodge theory which provides the Hodge numbers $h^{p,q}$. It is known ([Sim97]) that the polarizable variations of twistor structure of weight w which arise from a polarizable variation of Hodge structure of weight w are those which are equipped with a natural $\mathbb{C}^*$-action (on the $\mathbb{C}_\lambda$-factor), an action which recovers the Hodge grading. We can also characterize them as those admitting an infinitesimal action of $\mathbb{C}^*$, when the base X^o is quasi-projective, if we impose the moderate condition at infinity (Theorem 7.2 below).

Definition 7.1. — *A variation of twistor structure* $(\mathscr{H}', \mathscr{H}'', C)$ *(see Section* 1*) is* integrable *if the λ-connections on $\mathscr{H}'$ and $\mathscr{H}''$ come from a flat (absolute) connection with a pole of Poincaré rank equal to* 1 *along* $\lambda = 0$*, and if C is compatible (in a natural sense) to these connections.*

In other words, there exists $\nabla : \mathscr{H}' \to \frac{1}{\lambda}\Omega^1_{X\times\mathbb{C}_\lambda}(\log\{\lambda=0\}) \otimes \mathscr{H}'$ (ditto for $\mathscr{H}''$), such that $\nabla^2 = 0$, that the component on $\frac{1}{\lambda}\Omega^1_{X\times\mathbb{C}_\lambda/\mathbb{C}_\lambda}$ is the λ-connection, and finally

$$\lambda\frac{\partial}{\partial\lambda} C(m', \sigma^*\overline{m''}) = C(\lambda\nabla_{\partial_\lambda} m', \sigma^*\overline{m''}) - C(m', \sigma^*\overline{\lambda\nabla_{\partial_\lambda} m''}).$$

Theorem 7.2 (see [HS10, Th. 6.2]). — *Integrable variations of polarized twistor structure of weight w on a Zariski open X^o of X (projective smooth) which are moderate at infinity correspond bijectively to variations of polarized complex Hodge structure equipped with a self-adjoint semisimple automorphism for h.*

More generally, one can then consider the integrable and wild variations of polarized pure twistor structure on a quasi-projective variety as irregular analogues of the variations of polarized complex Hodge structure.

Hertling observed that a new invariant appears in this framework, already considered by the physicists Cecotti and Vafa [CV91, CFIV92], called "new super-symmetric index". A pure polarized twistor structure of weight 0 corresponds to a complex vector space with a positive definite Hermitian form. It is integrable if it has two endomorphisms $\mathscr{U}$ and $\mathscr{Q}$, with $\mathscr{Q}$ self-adjoint with respect to the Hermitian form. The vector space is decomposed according to the eigenvalues $p \in \mathbb{R}$ of $\mathscr{Q}$, and the components have dimension $h^{p,-p}$. For a pure polarized complex Hodge structure of weight 0, this is nothing other than the Hodge decomposition and the Hodge numbers, with $p \in \mathbb{Z}$ in this case, and we also have $\mathscr{U} = 0$.

In a variation parametrized by $x \in X$, the exponent p can vary with x in a real analytic way, while it is constant (integer) for variations of Hodge structure. Also, grouping the eigenspaces of $\mathscr{Q}$ according to the p having the same integral part, in order to have a Hodge decomposition in the usual sense, can cause dimension jumps according to the values of x.

The behavior of this index at infinity of such a variation (moderate or wild) is analyzed in [Sab10] in dimension 1 and in [Moc11b] in any dimension.

Real and rational structures. — Hertling [Her03] also considered such variations with a real structure (structure he calls TERP). The previous theorem is in fact shown with real structure.

More recently, Katzarkov, Kontsevich and Pantev [KKP08] have proposed the notion of rational structure (see also [Sab11]), and in this framework the name of "variation of noncommutative Hodge structure".

Wild mixed Hodge theory. — In a similar way to the theory of mixed Hodge modules of M. Saito [Sai90], T. Mochizuki [Moc15] has developed the theory of holonomic $\mathscr{D}$-modules with wild, possibly integrable, mixed twistor structure. In particular, he explains a duality functor, which was not defined in the previous framework, and which allows to define the notion of real structure. Only the rational structure is still missing, but the arguments of [Moc14] should be able to be applied to this framework too.

Application to quantum cohomology. — The results of Iritani on mirror symmetry for toric Fano varieties [Iri09a, Iri09b] (see also [RS15]), together with the results on the Fourier transformation of [Sab08], allow to show that the quantum $\mathscr{D}$-module of a toric Fano variety underlies a variation of noncommutative Hodge structure.

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