# SOME PROPERTIES AND APPLICATIONS OF BRIESKORN LATTICES

*by*

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***Abstract.*** After reviewing the main properties of the Brieskorn lattice in the framework of tame regular functions on smooth affine complex varieties, we prove a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

**Contents**



## 1. Introduction

The *Brieskorn lattice*, introduced by Brieskorn in [**Bri70**] in order to provide an algebraic computation of the Milnor monodromy of a germ of complex hypersurface with an isolated singularity, has also proved central in the Hodge theory for vanishing cycles of such a singularity, as emphasized by Pham [**Pha80, Pha83**]. Hodge theory for vanishing cycles, as developed by Steenbrink [**Ste76, Ste77, SS85**] and Varchenko [**Var82**], makes it an analogue of the Hodge filtration in this context, and fundamental results have been obtained by M. Saito [**Sai89**] in order to characterize it among other lattices in the Gauss-Manin system of an isolated singularity of complex hypersurface. As such, it leads to the definition of a period mapping, as introduced and studied with much detail by K. Saito for some singularities [**Sai83**]. It is also a basic constituent of the period mapping restricted to the $\mu$-constant stratum [**Sai91**], where a natural Torelli problem occurs (see [**Sai91**], [**Her99**]).

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For a holomorphic germ $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ with an isolated singularity, denoting by $t$ the coordinate on the target space $\mathbb{C}$, the space

$$\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0} \tag{1.1}$$

is naturally endowed with a $\mathbb{C}\{t\}$-module structure (where $t$ acts as the multiplication by $f$), and the *Brieskorn lattice* is the $\mathbb{C}\{t\}$-module (see [**Bri70**, p. 125])

$$''H^n_{f,0}=\Big(\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{\mathbb{C}^{n+1},0}\Big)\Big/\mathbb{C}\{t\}\text{-torsion}. \tag{1.2}$$

Brieskorn shows that (1.2) is free of finite rank equal to the Milnor number $\mu(f,0)$, and Sebastiani [**Seb70**] shows the torsion freeness of (1.1), which can thus also serve as an expression for $''H^n_{f,0}$. It is also endowed with a meromorphic connection $\nabla$ having a pole of order at most two at $t=0$, and the $\mathbb{C}(\{t\})$-vector space with connection generated by $''H^n_{f,0}$ is isomorphic to the Gauss-Manin connection, which has a regular singularity there. $''H^n_{f,0}$ is thus a $\mathbb{C}\{t\}$-lattice of this $\mathbb{C}(\{t\})$-vector space. While the action of $\nabla_{\partial_t}$, simply written as $\partial_t$, introduces a pole, there is a well-defined action of its inverse $\partial_t^{-1}$ that makes $''H^n_{f,0}$ a module over the ring of $\mathbb{C}\{\{\partial_t^{-1}\}\}$ of 1-Gevrey series (i.e., formal power series $\sum_{n\geqslant 0}a_n\partial_t^{-n}$ such that the series $\sum_n a_nu^n/n!$ converges). It happens to be also free of rank $\mu$ over this ring ([**Mal74, Mal75**]). The relation between the rings $\mathbb{C}\{t\}$ and $\mathbb{C}\{\{\partial_t^{-1}\}\}$ is called *microlocalization*. In the global case below, we will use instead the Laplace transformation. The mathematical richness of this object leads to various generalizations.

For non-isolated hypersurface singularities, the objects with definition as in (1.2) (but in various degrees) have been introduced by Hamm in his Habilitationsschrift (see [**Ham75**, §II.5]), who proved that they are $\mathbb{C}\{t\}$-free of finite rank, but do not coincide with (1.1) in general. A natural $\mathbb{C}\{\{\partial_t^{-1}\}\}$-structure still exists on (1.1), and Barlet and Saito [**BS07**] have shown that the $\mathbb{C}\{t\}$-torsion and the $\mathbb{C}\{\{\partial_t^{-1}\}\}$-torsion coincide, so that $''H^k_{f,0}$ remains $\mathbb{C}\{\{\partial_t^{-1}\}\}$-free of finite rank.

The Brieskorn lattice has also a global variant. On the one hand, the Brieskorn lattice for tame regular functions on smooth affine complex varieties (see Section 2) is a direct analogue of the case of an isolated singularity, but the double pole of the action of $t$ with respect to the variable $\partial_t^{-1}$ cannot in general be reduced to a simple one by a meromorphic (even formal) gauge transformation i.e., the Gauss-Manin system with respect to the variable $\partial_t^{-1}$ has in general an irregular singularity. The properties of the Brieskorn module for regular functions on affine manifolds which are not tame have been considered by Dimca and M. Saito [**DS01**].

On the other hand, given a *projective* morphism $f:X\to\mathbb{A}^1$ on a smooth quasi-projective variety $X$, the Brieskorn modules, defined as the hypercohomology $\mathbb{C}[\partial_t^{-1}]$-modules of the twisted de Rham complex $(\Omega^\bullet_X[\partial_t^{-1}],\mathrm{d}-\partial_t^{-1}\mathrm{d}f)$, have been shown to be $\mathbb{C}[\partial_t^{-1}]$-free (Barannikov-Kontsevich, see [**Sab99b**]), and a similar result holds when one replaces $\Omega^\bullet_X$ with $\Omega^\bullet_X(\log D)$ for some divisor with normal crossings. More generally, one can adapt the definition of the Brieskorn modules for the twisted de Rham complex attached to a mixed Hodge module, and the $\mathbb{C}[\partial_t^{-1}]$-freeness still holds, so that they can be called Brieskorn lattices (see loc. cit.). This enables one to use the push-forward operation by the map $f$ and reduce the study to that of Brieskorn lattices attached to mixed Hodge modules on the affine line, as for example the mixed Hodge modules that the Gauss-Manin systems of $f$ underlie. In such a way, the Brieskorn lattice has a *purely Hodge-theoretic definition*, which does not refer to the underlying geometry, and can thus be attached, for example, to any polarizable variation of Hodge structure on a punctured affine line (see [**Sab08**, §1.d]).

The Brieskorn lattice of tame functions is of particular interest and has been considered in [**Sab06**] for example. The Brieskorn lattice for families of such functions, considered in [**DS03**], has been investigated with much care for families of Laurent polynomials in relation with mirror symmetry by Reichelt and Reichelt-Sevenheck [**RS15, Rei14, Rei15, RS17**].

Lastly, in the global setting as above, the pole of order two of the action of $t$ with respect to the variable $\partial_t^{-1}$ produces in general a truly irregular singularity, and the Brieskorn lattice is an essential tool to produce the *irregular Hodge filtration* attached to such a singularity (see [**SY15, Sab17**]).

The contents of this article is as follows. In Section 2, we review known results on the Brieskorn lattice for a tame function. We show in Section 3 how these results enables one to obtain a simple proof of a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

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## 2. The Brieskorn lattice of a tame function

In this section, we review the main properties of the Brieskorn lattice attached to a tame function on an affine manifold, following [**Sab99a, Sab06, DS03**].

Let $U$ be a smooth complex affine variety of dimension $n$ and let $f \in \mathscr{O}(U)$ be a regular function on $U$. There are various notions of tameness for such a function, which are not known to be equivalent, but for what follows they have the same consequences. One of the definitions, given by Katz in [**Kat90**, Th. 14.13.3], is that the cone of $f_!\mathbb{C}_U \to \boldsymbol{R}f_*\mathbb{C}_U$ should have constant cohomology on $\mathbb{A}^1$. We will use the notion of a weakly tame function, as defined in [**NS99**], that is, either cohomologically tame or M-tame.

We assume that $f$ is weakly tame. Let $\theta$ be a new variable. The *Brieskorn lattice* attached to $f$ is the $\mathbb{C}[\theta]$-module

$$G_0 := \Omega^n(U)[\theta]\big/(\theta \mathrm{d} - \mathrm{d}f)\Omega^{n-1}(U)[\theta].$$

An expression like (1.1) also exists if $U$ is the affine space $\mathbb{A}^{n+1}$, but the above one is valid for any smooth affine variety $U$. The variable $\theta$ is for $\partial_t^{-1}$. We already notice that

$$G_0/\theta G_0 \simeq \Omega^n(U)/\mathrm{d}f \wedge \Omega^{n-1}(U) \tag{2.1}$$

has dimension equal to the sum $\mu = \mu(f)$ of the Milnor numbers of $f$ at all its critical points in $U$. The following properties are known in this setting.

(1) The algebraic Gauss-Manin systems $\mathscr{H}^k f_+\mathscr{O}_U$ are isomorphic to powers of the $\mathbb{C}[t]\langle\partial_t\rangle$-module $(\mathbb{C}[t], \partial_t)$, except for $k = 0$, so their localized Laplace transforms vanish except that for $k = 0$. If we regard the Laplace transform of $\mathscr{H}^0 f_+\mathscr{O}_U$ as a

$\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, we know that it has finite type as such, and its localized Laplace transform $G$, that is, the $\mathbb{C}[\tau,\tau^{-1}]$-module obtained by localization, is free of rank $\mu$. We have

$$G = \Omega^n(U)[\tau,\tau^{-1}]/(\mathrm{d}-\tau\mathrm{d}f)\Omega^{n-1}(U)[\tau,\tau^{-1}].$$

(2) Setting $\theta=\tau^{-1}$, we write

$$G = \Omega^n(U)[\theta,\theta^{-1}]/(\theta\mathrm{d}-\mathrm{d}f)\Omega^{n-1}(U)[\theta,\theta^{-1}],$$

and there is therefore a natural morphism $G_0\to G$. This morphism is *injective*, so that $G_0$ is a *free* $\mathbb{C}[\theta]$-module of rank $\mu$ such that $\mathbb{C}[\theta,\theta^{-1}]\otimes_{\mathbb{C}[\theta]}G_0=G$, i.e., $G_0$ is a $\mathbb{C}[\theta]$-lattice of $G$, on which the restriction of the Gauss-Manin connection has a pole of order at most two. Moreover, the action of $\theta^2\partial_\theta$ on the class $[\omega]$ of $\omega\in\Omega^n(U)$ in $G_0$ is given by

$$\theta^2\partial_\theta[\omega]=[f\omega],$$

and the action of $\theta^2\partial_\theta$ on a polynomial $\sum_{k\geqslant 0}[\omega_k\theta^k]$ is obtained by the usual formulas.

(3) Let $V_\bullet G$ be the (increasing) $V$-filtration of $G$ with respect to the function $\tau$ (recall that $G$ has a regular singularity at $\tau=0$, while that at infinity is usually irregular). It is a filtration by free $\mathbb{C}[\tau]$-modules of rank $\mu$ indexed by $\mathbb{Q}$. The jumping indices of the induced filtration $V_\bullet(G_0/\theta G_0)$, together with their multiplicities (the dimension of $\mathrm{gr}^V_\beta(G_0/\theta G_0)$) form the *spectrum of $f$ at* $\infty$. The jumping indices are contained in the interval $[0,n]\cap\mathbb{Q}$ and the spectrum is symmetric with respect to $n/2$.

(4) On the other hand, for $\alpha\in[0,1)\cap\mathbb{Q}$, the vector space $\mathrm{gr}^V_\alpha G$ is endowed with the nilpotent endomorphism $\mathrm{N}$ induced by the action of $-(\tau\partial_\tau+\alpha)$ and with the increasing filtration $G_\bullet\,\mathrm{gr}^V_\alpha G$ naturally induced by the filtration $G_p=\theta^{-p}G_0$, i.e.,

$$G_p\,\mathrm{gr}^V_\alpha G=(G_p\cap V_\alpha G)/(G_p\cap V_{<\alpha}G),$$

where the intersections are taken in $G$. As a consequence, we have isomorphisms ($p\in\mathbb{Z}$, $\alpha\in[0,1)$)

$$\mathrm{gr}^G_p\,\mathrm{gr}^V_\alpha G\xrightarrow[\sim]{\theta^p}\mathrm{gr}^V_{\alpha+p}(G_0/\theta G_0).$$

(5) The $\mathbb{C}$-vector space $H_{\neq 1}:=\bigoplus_{\alpha\in(0,1)\cap\mathbb{Q}}\mathrm{gr}^V_\alpha G$, resp. $H_1:=\mathrm{gr}^V_0 G$, endowed with

- the filtration

$$F^pH_{\neq 1}:=\bigoplus_{\alpha\in(0,1)\cap\mathbb{Q}}G_{n-1-p}\,\mathrm{gr}^V_\alpha G\quad\text{resp. } F^pH_1=G_{n-p}\,\mathrm{gr}^V_0 G,$$

- and the weight filtration $W_\bullet=\mathrm{M}(\mathrm{N})[n-1]$ (resp. $\mathrm{M}(\mathrm{N})[n]$), i.e., the monodromy filtration of $\mathrm{N}$ centered at $n-1$ (resp. $n$),

is part of a mixed Hodge structure. In particular, $\mathrm{N}$ strictly shifts by one the filtration $G_\bullet\,\mathrm{gr}^V_\alpha G$ and acts on the graded space $\mathrm{gr}^G_\bullet\,\mathrm{gr}^V_\alpha G$ as the degree-one morphism induced by $-\tau\partial_\tau$. We therefore have a commutative diagram, for any $\alpha\in[0,1)$ and $p\in\mathbb{Z}$, (see **[Var81]** and **[SS85**, §7] in the singularity case):

$$\begin{array}{ccc}\mathrm{gr}^G_p\,\mathrm{gr}^V_\alpha G & \xrightarrow[\sim]{\theta^p} & \mathrm{gr}^V_{\alpha+p}(\Omega^n(U)/\mathrm{d}f\wedge\Omega^{n-1}(U))\\ {\scriptstyle[\mathrm{N}]}\downarrow & & \downarrow{\scriptstyle[f]}\\ \mathrm{gr}^G_{p+1}\,\mathrm{gr}^V_\alpha G & \xrightarrow[\sim]{\theta^{p+1}} & \mathrm{gr}^V_{\alpha+p+1}(\Omega^n(U)/\mathrm{d}f\wedge\Omega^{n-1}(U)),\end{array}\tag{2.2}$$

by using the relation $-\tau\partial_\tau=\theta\partial_\theta$.

To see this, write the commutative diagram

$$\begin{array}{ccccc}
\operatorname{gr}^G_p \operatorname{gr}^V_\alpha G & \xrightarrow[\sim]{\theta^p} & \operatorname{gr}^V_{\alpha+p} \operatorname{gr}^G_0 G & & \\
{\scriptstyle \theta\partial_\theta - \alpha}\big\downarrow & & {\scriptstyle \theta\partial_\theta - (\alpha+p)}\big\downarrow & \searrow & \\
\operatorname{gr}^G_{p+1} \operatorname{gr}^V_\alpha G & \xrightarrow[\sim]{\theta^p} & \operatorname{gr}^V_{\alpha+p} \operatorname{gr}^G_1 G & \xrightarrow{\theta} & \operatorname{gr}^V_{\alpha+p+1} \operatorname{gr}^G_0 G
\end{array}$$

and use that in the vertical morphisms, the constant part $\alpha$ or $\alpha + p$ induces the morphism 0.

(6) Recall that a mixed Hodge structure $(H_{\mathbb{Q}}, F^\bullet H_{\mathbb{C}}, W_\bullet H_{\mathbb{Q}})$ is said to be *of Hodge-Tate type* if

(a) the filtration $W_\bullet$ has only even jumping indices

(b) and $W_{2\bullet} H_{\mathbb{C}}$ is opposite to $F^\bullet H_{\mathbb{C}}$.

The description of the mixed Hodge structure given in (5) implies the following criterion. We will set $\nu = n - 1$ when considering $H_{\neq 1}$ and $\nu = n$ when considering $H_1$. We will then denote by $H$ either $H_{\neq 1}$ or $H_1$.

***Corollary 2.3.*** *The mixed Hodge structure that the triple $(H, F^\bullet H, W_\bullet H)$ underlies is of Hodge-Tate type if and only if, for any integer $k$ such that $0 \leqslant k \leqslant [\nu/2]$, the $(\nu - 2k)$th power of* N *induces an isomorphism*

$$[\mathrm{N}]^{\nu-2k} : \operatorname{gr}^G_k H \xrightarrow{\sim} \operatorname{gr}^G_{\nu-k} H.$$

*Proof.* We define the filtration $W'_\bullet H$ indexed by $2\mathbb{Z}$ by the formula $W'_{2k} H = G_{\nu-k} H$, so that $G_k H = W'_{2(\nu-k)} H$. If we set $\ell = \nu - 2k$ for $0 \leqslant k \leqslant \nu/2$, we have $0 \leqslant \ell \leqslant \nu$ and the isomorphism in the corollary is written

$$[\mathrm{N}]^\ell : \operatorname{gr}^{W'}_{\nu+\ell} H \xrightarrow{\sim} \operatorname{gr}^{W'}_{\nu-\ell} H.$$

We can conclude that $W'_\bullet H = W_\bullet H$ if we know that $\mathrm{N}^{\nu+1} = 0$, that is, $\operatorname{gr}^G_{\nu+1} H = 0$. This is a consequence of the positivity of the spectrum [**Sab06**, Cor. 13.2], which says that, if $\alpha \in [0, 1)$, we have $\operatorname{gr}^G_k \operatorname{gr}^V_\alpha G = 0$ for $k \notin [0, \nu] \cap \mathbb{N}$. □

The following lemma was pointed out to me by Claus Hertling.

***Lemma 2.4.*** *A mixed Hodge structure $(H_{\mathbb{Q}}, F^\bullet H_{\mathbb{C}}, W_\bullet H_{\mathbb{Q}})$ is Hodge-Tate if and only if we have, for all $p \in \frac{1}{2}\mathbb{Z}$,*

$$\dim \operatorname{gr}^p_F H_{\mathbb{C}} = \dim \operatorname{gr}^W_{2p} H_{\mathbb{Q}}.$$

*Proof.* Indeed, one direction is clear. Conversely, if the equality of dimensions holds, then (6a) holds since $F^\bullet H$ has only integral jumps; moreover, up to a Tate twist, one can assume that $W_{<0} H = 0$, so $\operatorname{gr}^k_F H = 0$ for $k < 0$. It is enough to prove that $\operatorname{gr}^p_F \operatorname{gr}^W_{2\ell} H = 0$ for all $p \neq \ell$. We prove this by induction on $\ell$. If $\ell = 0$, the result follows from the property that $F^p H = 0$ for $p < 0$ and Hodge symmetry. Assume the result up to $\ell$. For $j \leqslant \ell$ we thus have $\dim \operatorname{gr}^j_F \operatorname{gr}^W_{2j} H = \dim \operatorname{gr}^W_{2j} H = \dim \operatorname{gr}^j_F H$ (the latter equality by the assumption), and therefore $\operatorname{gr}^W_{2i} \operatorname{gr}^j_F H = 0$ for $i \neq j$. In particular, taking $i = \ell + 1$, we have $\operatorname{gr}^p_F \operatorname{gr}^W_{2(\ell+1)} H = 0$ for all $p \leqslant \ell$. By Hodge symmetry, we obtain $\operatorname{gr}^p_F \operatorname{gr}^W_{2(\ell+1)} H = 0$ for all $p \neq \ell + 1$, as wanted. □

(7) We now consider the case where $U = (\mathbb{C}^*)^n$, endowed with coordinates $x = (x_1, \dots, x_n)$. Let $f \in \mathbb{C}[x, x^{-1}]$ be a Laurent polynomial in $n$ variables, with Newton polyhedron $\Delta(f)$. We assume that $f$ is *nondegenerate with respect to its Newton polyhedron* and *convenient* (see [**Kou76**]). In particular, 0 belongs to the interior of its Newton polyhedron. It is known that such a function is M-tame.

For any face $\sigma$ of dimension $n-1$ of the boundary $\partial\Delta(f)$, we denote by $L_\sigma$ the linear form with coefficients in $\mathbb{Q}$ such that $L_\sigma \equiv 1$ on $\sigma$. For $g \in \mathbb{C}[x, x^{-1}]$, we set $\deg_\sigma(g) = \max_m L_\sigma(m)$, where the max is taken on the exponents of monomials $x^m$ appearing in $g$, and $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$. We denote the volume form $\mathrm{d}x_1/x_1 \wedge \cdots \wedge \mathrm{d}x_n/x_n$ by $\omega$, giving rise to an identification $\mathbb{C}[x, x^{-1}] \xrightarrow{\sim} \Omega^n(U)$ and $\mathbb{C}[x, x^{-1}]/(\partial f) \xrightarrow{\sim} G_0/\theta G_0$ (see (2.1)).

The Newton increasing filtration $\mathcal{N}_\bullet\Omega^n(U)$ indexed by $\mathbb{Q}$ is defined by

$$\mathcal{N}_\beta\Omega^n(U) := \{g\omega \in \Omega^n(U) \mid \deg_{\Delta^*}(g) \leqslant \beta\}.$$

We have $\mathcal{N}_\beta\Omega^n(U) = 0$ for $\beta < 0$ and $\mathcal{N}_0\Omega^n(U) = \mathbb{C}\cdot\omega$. We can extend this filtration to $\Omega^n(U)[\theta]$ by setting

$$\mathcal{N}_\beta\Omega^n(U)[\theta] := \mathcal{N}_\beta\Omega^n(U) + \theta\mathcal{N}_{\beta-1}\Omega^n(U) + \cdots + \theta^k\mathcal{N}_{\beta-k}\Omega^n(U) + \cdots$$

and then naturally induce this filtration on $G_0$, to obtain a filtration $\mathcal{N}_\bullet G_0$ and then on $G_0/\theta G_0$. We have

$$\mathcal{N}_\bullet G_0 = V_\bullet G \cap G_0 \quad \text{and} \quad \mathcal{N}_\bullet(G_0/\theta G_0) = V_\bullet(G_0/\theta G_0). \tag{2.5}$$

Corollary 2.3 now reads, according to (2.2) and by using the above identification through multiplication by $\omega$:

***Corollary 2.6.*** *The mixed Hodge structure that the triple $(H, F^\bullet H, W_\bullet H)$ underlies is of Hodge-Tate type if and only if, for any integer $k$ such that $0 \leqslant k \leqslant [\nu/2]$ ($\nu = n-1$, resp. $n$), we have isomorphisms*

$$\mathrm{gr}^{\mathcal{N}}_{\alpha+k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow[\sim]{[f]^{n-1-2k}} \mathrm{gr}^{\mathcal{N}}_{\alpha+n-1-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \quad \forall\alpha \in (0,1),$$

$$\text{resp.}\quad \mathrm{gr}^{\mathcal{N}}_{k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow[\sim]{[f]^{n-2k}} \mathrm{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big).$$

## 3. On a conjecture of Katzarkov-Kontsevich-Pantev

In this section we use the algebraic Brieskorn lattice of a convenient nondegenerate Laurent polynomial to solve the toric case of the part "$f^{p,q} = h^{p,q}$" of Conjecture 3.6 in [**KKP17**] (the other equality "$h^{p,q} = i^{p,q}$" is obviously not true by simply considering the case of the standard Laurent polynomial mirror to the projective space $\mathbb{P}^n$, see also another counter-example in [**LP18**]). We refer to [**LP18, Har17, Sha17**] for further discussion and positive results on this conjecture.

### 3.a. The Brieskorn lattice and the conjecture of Katzarkov-Kontsevich-Pantev

Given a smooth quasi-projective variety $U$ and a morphism $f : U \to \mathbb{A}^1$, every twisted de Rham cohomology $H^k_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f)$, i.e., the $k$th hypercohomology of the twisted de Rham complex $(\Omega_U^\bullet, \mathrm{d}+\mathrm{d}f)$, is endowed with a decreasing filtration $F_{\mathrm{Yu}}^\bullet H^k_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)$ indexed by $\mathbb{Q}$ (see [**Yu14**]). For $\alpha\in[0,1)$, the filtration indexed by $\mathbb{Z}$ defined by $F^p_{\mathrm{Yu},\alpha}=F^{p-\alpha}_{\mathrm{Yu}}$ can also be computed in terms of the Kontsevich complex $\Omega_f^\bullet(\alpha)$ together with its stupid filtration (see [**ESY17**, Cor. 1.4.5]). The irregular Hodge numbers $h^{p,q}_\alpha(f)$ are defined as

$$h^{p,q}_\alpha(f):=\dim\mathrm{gr}^{p-\alpha}_{F_{\mathrm{Yu}}}H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f). \tag{3.1}$$

It is well-known that $\dim H^k_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)=\dim H^k(U,f^{-1}(t))$ for $|t|\gg 0$. This space is endowed with a monodromy operator (around $t=\infty$), and we will consider the case where this monodromy operator is *unipotent*. In such a case, the filtration $F^\bullet_{\mathrm{Yu}}H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)$ is known to jump at integers only, and in (3.1) only $\alpha=0$ occurs. We then simply denote this number by $h^{p,q}(f)$, so that, in such a case,

$$h^{p,q}(f):=\dim\mathrm{gr}^p_{F_{\mathrm{Yu}}}H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f).$$

Let $W_\bullet$ be the monodromy filtration on $H^k(U,f^{-1}(t))$ centered at $k$. The conjecture of [**KKP17**] that we consider is the possible equality (see [**LP18, Har17, Sha17**])

$$h^{p,q}(f)=\dim\mathrm{gr}^W_{2p}H^{p+q}(U,f^{-1}(t)). \tag{3.2}$$

If moreover $U$ is affine and $f$ is weakly tame, so that $H^{p+q}_{\mathrm{DR}}(U,\mathrm{d}+\mathrm{d}f)=0$ unless $p+q=n$, [**SY15**, Cor. 8.19] gives, using the notation of Section 2:[(1)]

$$h^{p,q}(f)=\begin{cases}\dim\mathrm{gr}^V_{n-p}(G_0(f)/\theta G_0(f))=\dim\mathrm{gr}^p_F\mathrm{gr}^V_0 G & \text{if } p+q=n,\\ 0 & \text{if } p+q\neq n,\end{cases}$$

and this is the number denoted by $f^{p,q}$ in [**KKP17**]. In such a case, we have $H=H_1$ in the notation of Section 2(5).

The following criterion has been obtained, with a different approach of the irregular Hodge filtration, by Y. Shamoto.

***Proposition 3.3*** **([Sha17]).** *Assume $U$ affine and $f$ weakly tame with unipotent monodromy operator at infinty. Then* (3.2) *holds true if and only if the mixed Hodge structure of Section* 2(5) *on $H=H_1$ is of Hodge-Tate type.*

*Proof.* According to Lemma 2.4, proving the result amounts to identifying the space $\mathrm{gr}^V_0 G$ endowed with its nilpotent operator N with the space $H^n(U,f^{-1}(t))$ endowed with the nilpotent part of the (unipotent) monodromy (up to a nonzero constant). Choosing an extension $F:\mathcal{X}\to\mathbb{P}^1$ of $f$ as a projective morphism on a smooth variety $\mathcal{X}$ such that $\mathcal{X}\smallsetminus U$ is a divisor, and setting $\mathcal{F}=\boldsymbol{R}j_*\mathbb{C}_U$ $(j:U\hookrightarrow\mathcal{X})$, we identify the dimension of $H^k(U,f^{-1}(t))$ with that of the $k$th-hypercohomology on $\mathcal{X}$ of the Beilinson extension $\Xi_F\mathcal{F}$. Then the desired identification is given by [**Sab97**, Cor. 1.13]. ☐

(1) The definition of $\delta_\gamma$ in [**SY15**] should read $\dim\mathrm{gr}^V_\gamma(G_0(f)/uG_0(f))$.

### 3.b. The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, first part

As usual in toric geometry, we denote by $M$ the lattice $\mathbb{Z}^n$ in $\mathbb{C}^n$ and by $N$ its dual lattice. We fix a reflexive simplicial polyhedron $\Delta \subset \mathbb{R} \otimes M$ with vertices in $M$ and having 0 in its interior (it is then the unique integral point in its interior), see [**Bat94**, §4.1]. We denote by $\Delta^*$ the dual polyhedron with vertices in $N$, which is also simplicial reflexive and has 0 in its only interior point, and by $\Sigma \subset N$ the fan dual to $\Delta$, which is also the cone on $\Delta^*$ with apex 0. We assume that $\Sigma$ is the fan of nonsingular toric variety $X$ of dimension $n$, that is, each set of vertices of the same $(n-1)$-dimensional face of $\partial\Delta^*$ is a $\mathbb{Z}$-basis of $N$. We know that

- $X$ is Fano ([**Bat94**, Th. 4.1.9]),
- the Chow ring $A^*(X) \simeq H^{2*}(X, \mathbb{Z})$ is generated by the divisor classes $D_v$ corresponding to vertices $v \in V(\Delta^*)$ of $\Delta^*$, i.e., primitive elements on the rays of $\Sigma$ (see [**Ful93**, p. 101]),
- we have $c_1(TX) = c_1(K_X^\vee) = \sum_{v\in V(\Delta^*)} D_v$ in $H^{2*}(X, \mathbb{Z})$ (see [**Ful93**, p. 109]), which satisfies Hard Lefschetz on $H^{2*}(X, \mathbb{Q})$, by ampleness of $K_X^\vee$.

Let us fix coordinates $x = (x_1, \dots, x_n)$ such that $\mathbb{Q}[N] = \mathbb{Q}[x, x^{-1}]$. We use the notation of Section 2(7). Due to the reflexivity of $\Delta^*$, $L_\sigma$ has coefficients in $\mathbb{Z}$ (it corresponds to a vertex of $\Delta$). For $g \in \mathbb{C}[x, x^{-1}]$, the $\sigma$-degree $\deg_\sigma(g) = \max_m L_\sigma(m)$ and the $\Delta^*$-degree $\deg_{\Delta^*}(g) = \max_\sigma \deg_\sigma(g)$ are thus nonnegative integers.

***Proposition 3.4.*** *The case "$f^{p,q} = h^{p,q}$" of* [**KKP17**, Conj. 3.6] *holds true if $f$ is the Laurent polynomial*

$$f(x) = \sum_{v\in V(\Delta^*)} x^v \in \mathbb{Q}[x, x^{-1}].$$

The idea of the proof is to notice that the property for the second morphism in Corollary 2.6 to be an isomorphism is exactly the property that $c_1(TX)$ satisfies the Hard Lefschetz property, and thus to identify its source and target as the cohomology of $X$ in suitable degree.

***Lemma 3.5.*** *For $\Delta$ as above, any Laurent polynomial*

$$f_{\boldsymbol{a}}(x) = \sum_{v\in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \boldsymbol{a} = (a_{v\in V}) \in (\mathbb{C}^*)^{V(\Delta^*)}.$$

*is convenient and non-degenerate in the sense of Kouchnirenko.*

*Proof.* The Newton polyhedron of $f_{\boldsymbol{a}}$ is equal to $\Delta^*$, and 0 belongs to its interior. In order to prove the non-degeneracy, we note that the vertices of any $(n-1)$-dimensional face $\sigma$ of $\partial\Delta^*$ form a $\mathbb{Z}$-basis. It follows that, in suitable toric coordinates $y_1, \dots, y_n$, the restriction $f_{\boldsymbol{a}|\sigma}$ can be written as $y_1 + \cdots + y_n$, and the non-degeneracy is then obvious. □

*Proof of Proposition* 3.4. Note that $\deg_{\Delta^*}(f) = 1$, as well as $\deg_{\Delta^*}(x_i\partial f/\partial x_i) = 1$. The Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$ is endowed with the Newton filtration $\mathcal{N}_\bullet$ induced by the $\Delta^*$-degree $\deg_{\Delta^*}$, and corresponds to $\mathcal{N}_\bullet(G_0/\theta G_0)$ by multiplication by $\omega$. In the present setting, [**BCS05**, Th. 1.1] identifies the graded ring $A^*(X)_{\mathbb{Q}}$ with the graded ring

$$\operatorname{gr}^{\mathcal{N}}_{\bullet}\big(\mathbb{Q}[x,x^{-1}]/(\partial f)\big).$$

By applying Hard Lefschetz to $c_1(TX)$, we deduce that, for every $k\in\mathbb{N}$ such that $0\leqslant k\leqslant [n/2]$, multiplication by the $(n-2k)$th power of the $\mathcal{N}$-class $[f]$ of $f$ induces an isomorphism

$$[f]^{n-2k}:\operatorname{gr}^{\mathcal{N}}_{k}\big(\mathbb{Q}[x,x^{-1}]/(\partial f)\big)\xrightarrow{\ \sim\ }\operatorname{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{Q}[x,x^{-1}]/(\partial f)\big).$$

By Corollary 2.6 for $H=H_1$, we deduce the assertion of the proposition from Proposition 3.3. ☐

### 3.c. The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, second part

We now prove the main result of this note.

***Theorem 3.6.*** *The case "$f^{p,q}=h^{p,q}$" of* [**KKP17**, Conj. 3.6] *holds true for any Laurent polynomial*

$$f_{\boldsymbol{a}}(x)=\sum_{v\in V(\Delta^*)}a_vx^v\in\mathbb{C}[x,x^{-1}],\quad \boldsymbol{a}=(a_{v\in V})\in(\mathbb{C}^*)^{V(\Delta^*)}.$$

***Remark 3.7.*** The case where $n=3$ was already proved differently by Y. Shamoto [**Sha17**, §4.2].

*Proof.* Let us set $H(f_{\boldsymbol{a}})=H_1(f_{\boldsymbol{a}})=\operatorname{gr}^V_0G(f_{\boldsymbol{a}})$, where $G(f_{\boldsymbol{a}})$ is the localized Laplace transform of the Gauss-Manin system for $f_{\boldsymbol{a}}$ as in Section 2(2). By Lemma 3.5, we can apply the results of Section 2 to $f_{\boldsymbol{a}}$ for any $\boldsymbol{a}\in(\mathbb{C}^*)^{V(\Delta^*)}$. We will prove that, for fixed $p$, both terms $\dim\operatorname{gr}^G_{n-p}H(f_{\boldsymbol{a}})$ and $\dim\operatorname{gr}^W_{2p}H(f_{\boldsymbol{a}})$ in Lemma 2.4 are independent of $\boldsymbol{a}$. Since they are equal if $\boldsymbol{a}=(1,\dots,1)$, after Proposition 3.4, they are equal for any $\boldsymbol{a}\in(\mathbb{C}^*)^{V(\Delta^*)}$, as wanted.

(1) For the first term, we will use [**NS99**]. We have denoted there $\dim\operatorname{gr}^G_pH(f_{\boldsymbol{a}})$ by $\nu_p(f_{\boldsymbol{a}})$ and, since $\operatorname{gr}^V_\alpha G=0$ for $\alpha\notin\mathbb{Z}$, it is also equal to the number denoted there by $\Sigma_{p-1}(f_{\boldsymbol{a}})$. By the theorem in [**NS99**] and Lemma 3.5, $\Sigma_{p-1}(f_{\boldsymbol{a}})$ depends semi-continuously on $\boldsymbol{a}$. On the other hand, according to [**Kou76**], $\dim H(f_{\boldsymbol{a}})$ is independent of $\boldsymbol{a}$ and is computed only in terms of $\Delta^*$. Since $\dim H(f_{\boldsymbol{a}})=\sum_p\Sigma_{p-1}(f_{\boldsymbol{a}})$, each term in this sum is also constant with respect to $\boldsymbol{a}$.

(2) We will prove the local constancy of $\dim\operatorname{gr}^W_{2p}H(f_{\boldsymbol{a}})$ near any $\boldsymbol{a}_o\in(\mathbb{C}^*)^{V(\Delta^*)}$. As noticed in [**DS03**, §4], we can apply the results of Section 2 of loc. cit. to $f_{\boldsymbol{a}_o}$. We fix a Stein open set $\mathcal{B}^o$ adapted to $f_{\boldsymbol{a}_o}$ as in [**DS03**, §2a], and fix a neighbourhood $X$ of $\boldsymbol{a}_o$ so that it is also adapted to any $f_{\boldsymbol{a}}$ for $\boldsymbol{a}$ in this neighbourhood. By construction, all the critical points of $f_{\boldsymbol{a}_o}$ are contained in the interior of $\mathcal{B}^o$ if $X$ is chosen small enough, and since $\mu(f_{\boldsymbol{a}})$ is constant, the same property holds for $\boldsymbol{a}\in X$. By using successively Theorem 2.9, Remark 2.11 and Proposition 1.20(1) in [**DS03**], we deduce that, when $\boldsymbol{a}$ varies in $X$, the localized partial Laplace transformed Gauss-Manin systems $G(f_{\boldsymbol{a}})$ form an $\mathscr{O}_X[\tau,\tau^{-1}]$-free module with integrable connection and regular singularity along $\tau=0$, which is compatible with base change with respect to $X$.

As a consequence, the monodromy of each $G(f_{\boldsymbol{a}})$ around $\tau=0$ is constant, and the assertion follows. □

***Remark 3.8* (suggested by the referee).** If we relax the condition in Section 3.b that the toric Fano variety $X$ is *nonsingular*, then we have to consider the orbifold Chow ring of $X$ as in [**BCS05**], or the Chen-Ruan orbifold cohomology of $X$. For the cohomology of the untwisted sector (i.e., the usual cohomology), the Hard Lefschetz theorem is still valid (see [**Ste77**]) and Proposition 3.4 still holds, i.e., (3.2) holds for $f$. Moreover, Part (2) of the proof of Theorem 3.6 also extends to this setting. However, the semicontinuity result of [**NS99**] used in Part (1) of the proof is not enough to imply the constancy (with respect to $\boldsymbol{a}$) of $\nu_p(f_{\boldsymbol{a}})$.

On the other hand, one can also consider the various $h^{p,q}_\alpha(f)$ for $\alpha\in(0,1)\cap\mathbb{Q}$ and, correspondingly, the twisted sectors of the orbifold $X$. In such a case, Hard Lefschetz for $f$ may already give trouble (see [**Fer06**]).

## References

[BS07] D. Barlet & M. Saito – Brieskorn modules and Gauss-Manin systems for non-isolated hypersurface singularities, *J. London Math. Soc. (2)* **76** (2007), no. 1, p. 211–224.

[Bat94] V.V. Batyrev – Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), p. 493–535.

[BCS05] L.A. Borisov, L. Chen & G.G. Smith – The orbifold Chow ring of toric Deligne-Mumford stacks, *J. Amer. Math. Soc.* **18** (2005), no. 1, p. 193–215.

[Bri70] E. Brieskorn – Die Monodromie der isolierten Singularitäten von Hyperflächen, *Manuscripta Math.* **2** (1970), p. 103–161.

[DS01] A. Dimca & M. Saito – Algebraic Gauss-Manin systems and Brieskorn modules, *Amer. J. Math.* **123** (2001), no. 1, p. 163–184.

[DS03] A. Douai & C. Sabbah – Gauss-Manin systems, Brieskorn lattices and Frobenius structures (I), *Ann. Inst. Fourier (Grenoble)* **53** (2003), no. 4, p. 1055–1116.

[ESY17] H. Esnault, C. Sabbah & J.-D. Yu – $E_1$-degeneration of the irregular Hodge filtration (with an appendix by M. Saito), *J. reine angew. Math.* **729** (2017), p. 171–227.

[Fer06] J. Fernandez – Hodge structures for orbifold cohomology, *Proc. Amer. Math. Soc.* **134** (2006), no. 9, p. 2511–2520.

[Ful93] W. Fulton – *Introduction to toric varieties*, Ann. of Math. Studies, vol. 131, Princeton University Press, Princeton, N.J., 1993.

[Ham75] H. Hamm – Zur analytischen und algebraischen Beschreibung der Picard-Lefschetz-Monodromie, Habilitationsschrift, Göttingen, 1975.

[Har17] A. Harder – Hodge numbers of Landau-Ginzburg models, `arXiv:1708.01174`, 2017.

[Her99] C. Hertling – Classifying spaces for polarized mixed Hodge structures and for Brieskorn lattices, *Compositio Math.* **116** (1999), no. 1, p. 1–37.

[Kat90] N. Katz – *Exponential sums and differential equations*, Ann. of Math. studies, vol. 124, Princeton University Press, Princeton, N.J., 1990.

[KKP17] L. Katzarkov, M. Kontsevich & T. Pantev – Bogomolov-Tian-Todorov theorems for Landau-Ginzburg models, *J. Differential Geometry* **105** (2017), no. 1, p. 55–117.

[Kou76] A.G. Kouchnirenko – Polyèdres de Newton et nombres de Milnor, *Invent. Math.* **32** (1976), p. 1–31.

[LP18] V. Lunts & V. Przyjalkowski – Landau-Ginzburg Hodge numbers for mirrors of Del Pezzo surfaces, *Adv. in Math.* **329** (2018), p. 189–216.

[Mal74] B. Malgrange – Intégrales asymptotiques et monodromie, *Ann. Sci. École Norm. Sup. (4)* **7** (1974), p. 405–430.

[Mal75] ______, Le polynôme de Bernstein d'une singularité isolée, in *Fourier integral operators and partial differential equations (Nice, 1974)*, Lect. Notes in Math., vol. 459, Springer-Verlag, 1975, p. 98–119.

[NS99] A. Némethi & C. Sabbah – Semicontinuity of the spectrum at infinity, *Abh. Math. Sem. Univ. Hamburg* **69** (1999), p. 25–35.

[Pha80] F. Pham – *Singularités des systèmes de Gauss-Manin*, Progress in Math., vol. 2, Birkhäuser, Basel, Boston, 1980.

[Pha83] ______, Structure de Hodge mixte associée à un germe de fonction à point critique isolé, in *Analyse et topologie sur les espaces singuliers (Luminy, 1981)* (B. Teissier & J.-L. Verdier, eds.), Astérisque, vol. 101-102, Société Mathématique de France, 1983, p. 268–285.

[Rei14] T. Reichelt – Laurent polynomials, GKZ-hypergeometric systems and mixed Hodge modules, *Compositio Math.* **150** (2014), no. 6, p. 911–941.

[Rei15] ______, A comparison theorem between Radon and Fourier-Laplace transforms for $\mathscr{D}$-modules, *Ann. Inst. Fourier (Grenoble)* **65** (2015), no. 4, p. 1577–1616.

[RS15] T. Reichelt & C. Sevenheck – Logarithmic Frobenius manifolds, hypergeometric systems and quantum $\mathscr{D}$-modules, *J. Algebraic Geom.* **24** (2015), no. 2, p. 201–281.

[RS17] ______, Non-affine Landau-Ginzburg models and intersection cohomology, *Ann. Sci. École Norm. Sup. (4)* **50** (2017), no. 3, p. 665–753.

[Sab97] C. Sabbah – Monodromy at infinity and Fourier transform, *Publ. RIMS, Kyoto Univ.* **33** (1997), no. 4, p. 643–685.

[Sab99a] ______, Hypergeometric period for a tame polynomial, *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999), p. 603–608.

[Sab99b] ______, On a twisted de Rham complex, *Tôhoku Math. J.* **51** (1999), p. 125–140.

[Sab06] ______, Hypergeometric periods for a tame polynomial, *Portugal. Math.* **63** (2006), no. 2, p. 173–226, `arXiv:math/9805077` (1998).

[Sab08] ______, Fourier-Laplace transform of a variation of polarized complex Hodge structure, *J. reine angew. Math.* **621** (2008), p. 123–158.

[Sab17] ______, Irregular Hodge theory, Chap. 3 in collaboration with Jeng-Daw Yu, `arXiv:1511.00176v4`, 2017.

[SY15] C. Sabbah & J.-D. Yu – On the irregular Hodge filtration of exponentially twisted mixed Hodge modules, *Forum Math. Sigma* **3** (2015), `doi:10.1017/fms.2015.8`.

[Sai83] K. Saito – Period mapping associated to a primitive form, *Publ. RIMS, Kyoto Univ.* **19** (1983), p. 1231–1264.

[Sai89] M. Saito – On the structure of Brieskorn lattices, *Ann. Inst. Fourier (Grenoble)* **39** (1989), p. 27–72.

[Sai91] ______, Period mapping via Brieskorn modules, *Bull. Soc. math. France* **119** (1991), p. 141–171.

[SS85] J. Scherk & J.H.M. Steenbrink – On the mixed Hodge structure on the cohomology of the Milnor fiber, *Math. Ann.* **271** (1985), p. 641–655.

[Seb70] M. Sebastiani – Preuve d'une conjecture de Brieskorn, *Manuscripta Math.* **2** (1970), p. 301–308.

[Sha17] Y. Shamoto – Hodge-Tate conditions for Landau-Ginzburg models, `arXiv:1709.03244`, 2017.

[Ste76] J.H.M. Steenbrink – Limits of Hodge structures, *Invent. Math.* **31** (1976), p. 229–257.

[Ste77] ______, Mixed Hodge structure on the vanishing cohomology, in *Real and Complex Singularities (Oslo, 1976)* (P. Holm, ed.), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, p. 525–563.

[Var81] A.N. Varchenko – On the monodromy operator in vanishing cohomology and the operator of multiplication by $f$ in the local ring, *Soviet Math. Dokl.* **24** (1981), p. 248–252.

[Var82] ______, Asymptotic Hodge structure on the cohomology of the Milnor fiber, *Izv. Akad. Nauk SSSR Ser. Mat.* **18** (1982), p. 469–512.

[Yu14] J.-D. Yu – Irregular Hodge filtration on twisted de Rham cohomology, *Manuscripta Math.* **144** (2014), no. 1–2, p. 99–133.

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