INTEGRABLE DEFORMATIONS AND DEGENERATIONS OF SOME IRREGULAR SINGULARITIES

by

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Abstract. We extend to a degenerate case a result of B. Malgrange on integrable deformations of irregular singularities, inspired by an article of G. Cotti, B. Dubrovin and D. Guzzetti [**CDG19**]. We give an application to integrable deformations of some meromorphic connections in Birkhoff normal form and to the construction of Frobenius manifolds.

Contents

1. Introduction

Motivations. Let us consider a family of differential systems of a complex variable z parametrized by t with matrix $A(t,z)\,\mathrm{d}z/z$ given by

$$A(t,z) = \frac{A_0(t)}{z} + A_1(t). \tag{1.1}$$

We assume that $A_0(t), A_1(t)$ are square matrices depending holomorphically on the parameter $t \in T$. Assume for example that $A_0(t)$ is non-resonant for generic values of t, that is, its eigenvalues are pairwise distinct. Then one can locally take these eigenvalues as parameters, and a suitable base change, formal with respect to z and locally holomorphic with respect to t, reduces the system into a diagonal form, so that its z-formal solutions are easy to obtain.

2020 ***Mathematics Subject Classification.*** 34M40, 32C38, 35A27.
Key words and phrases. Irregular singularity, formal decomposition, real blow-up, isomonodromic deformation, Birkhoff problem, Frobenius manifold.

If on the other hand some eigenvalues of $A_0(t)$ coincide at some $t_o \in T$, the z-formal behaviour of solutions in the neighbourhood of t_o becomes much harder to understand. The holomorphic behaviour of solutions is the subject of singular perturbation theory, and is addressed in an extensive literature, and the question is: how much do the Stokes matrices of the one-variable system at $t = t_o$ determine the Stokes matrices at neighbouring points?

Adding an integrability assumption to (1.1) makes the complexity of the problem bounded, in the following sense. If we assume that there exists an integrable meromorphic connection (on the trivial vector bundle) with respect to all variables z, t so that the original system (1.1) is the z-part of this connection, then it is worthwhile treating all variables on the same footing, allowing meromorphic changes of all the variables, also called complex blowing-ups. The system (1.1) can then be reduced to a simpler one after a suitable meromorphic change of variables: this is the content of a fundamental theorem of K. Kedlaya [**Ked10, Ked11**] in the present setting, while one can also refer to the work of T. Mochizuki [**Moc09, Moc11a**] in an algebraic setting (with respect to z and t). The price to pay is the introduction of singularities in the set of poles of the system, and the loss of the distinction between the notion of a variable and that of a parameter.

The integrability property can arise in at least two ways on a given system (1.1).

- An integrability property at a generic point of t may propagate all along the set of parameters where the system is defined.
- The system (1.1) is obtained as the solution to an isomonodromy deformation problem.

In the very interesting and inspiring article [**CDG19**] (see also [**CG17, CG18**]), G. Cotti, B. Dubrovin and D. Guzzetti have analyzed with much care the case where T parametrizes the eigenvalues of the matrix A_0, which is then assumed to be the matrix $\operatorname{diag}(t_1, \dots, t_n)$, and t_o belongs to the union Δ of diagonal hyperplanes (coalescing eigenvalues). In particular, they have shown how Stokes data at t_o can be extended in some neighbourhood of t_o, and they gave applications to Frobenius manifolds.

Our aim is to revisit some of their results from a different point of view, namely that of isomonodromic deformations of irregular differential equations, with the geometric perspective of [**Mal83c**]. We will take advantage of the results of K. Kedlaya and T. Mochizuki mentioned above on meromorphic connections in dimension $\geqslant 2$. In this setting, the behaviour of Stokes matrices is sufficiently well understood after a suitable meromorphic base change of variables, under the name of sheaf of Stokes torsors, which we will explain in Section 2.f.

The setting. Let $T = \mathbb{C}^n$ with coordinates $t_1, \dots, t_n$ and let X be a neighbourhood of $T \times \{0\}$ in $T \times \mathbb{C}$. We equip the extra factor $\mathbb{C}$ with the coordinate z, and we regard $T \subset X$ as the smooth hypersurface defined by the equation $z = 0$.

We consider as a *model system* the system (1.1) where the matrix $A_0(t)$ is the block-diagonal matrix $\Lambda(t)$ with diagonal blocks $t_i \operatorname{Id}$, and the matrix $A_1(t)$ is constant and block diagonal correspondingly. This system is integrable, and the T-component of the connection has a block-diagonal matrix with blocks $(-1/z)\mathrm{d}t_i \cdot \mathrm{Id}$.

We aim at analyzing integrable systems (1.1) that are formally isomorphic, as integrable systems, to the model system above. We will keep the formal isomorphism as part of the data, in other words, we are mainly interested in all possible Stokes data that can occur on such a model system.

We will use the language of meromorphic flat bundles, which happens to be more flexible when considering the meromorphic base changes like complex blowing-ups.

For each $i \in \{1, \dots, n\}$, let us be given a locally free $\mathscr{O}_X(*T)$-module $\mathscr{R}_i$ of finite rank, endowed with an integrable meromorphic connection ∇ (an object we call a *T-meromorphic flat bundle*). We assume that ∇ has *regular singularities* along T. We then set $\mathscr{E}^{-t_i/z} \otimes \mathscr{R}_i := (\mathscr{R}_i, \nabla - \mathrm{d}(t_i/z))$, and we consider the model T-meromorphic flat bundle

$$\mathscr{N} := \bigoplus_{i=1}^{n} (\mathscr{E}^{-t_i/z} \otimes \mathscr{R}_i). \tag{1.2}$$

Let $\mathscr{O}_{\widehat{T}}$ be the formal completion of $\mathscr{O}_X$ along T and let us set $\mathscr{M}_{\widehat{T}} := \mathscr{O}_{\widehat{T}} \otimes_{\mathscr{O}_X} \mathscr{M}$. We will be concerned with T-meromorphic flat bundles $\mathscr{M}$ endowed with an isomorphism $\mathrm{iso}_{\widehat{T}} : \mathscr{M}_{\widehat{T}} \xrightarrow{\sim} \mathscr{N}_{\widehat{T}}$. We note that a morphism $(\mathscr{M}_1, \mathrm{iso}_{\widehat{T}}) \to (\mathscr{M}_2, \mathrm{iso}_{\widehat{T}})$ between such objects (with the obvious definition) is uniquely determined by the associated formal morphism $\mathscr{M}_{1\widehat{T}} \to \mathscr{M}_{2\widehat{T}}$, and there is at most one isomorphism between two objects.

For $t_o \in T$, we denote by $\gamma_{t_o} : \{t_o\} \times (\mathbb{C}_z, 0) \hookrightarrow X$ the inclusion. Then $\gamma_{t_o}^* \mathscr{N}$ is a free $\mathscr{O}_{\mathbb{C}_z,0}(*0)$-module that can be endowed with the pullback connection, which we denote by $\gamma_{t_o}^+ \mathscr{N}$, and it has a form similar to (1.2) by replacing t_i with the constant $t_{o,i}$ and $\mathscr{R}_i$ with its restriction $\gamma_{t_o}^+ \mathscr{R}_i$. However some of the $t_{o,i}$ may coincide, which leads to grouping the corresponding $\gamma_{t_o}^+ \mathscr{R}_i$. The most degenerate case is when $t_{o,i} = 0$ for all i or, similarly, when all $t_{o,i}$ coincide.

The nondegenerate case is when t_o does not belong to the union Δ of the diagonal hyperplanes in T, i.e., when the $t_{o,i}$ are pairwise distinct, which is the case considered in [**JMU81**] as well as in [**Mal83c**]. In such a case, for any simply connected open subset U of $T \smallsetminus \Delta$ and for any $t_o \in U$, the restriction $\gamma_{t_o}^+$ induces a bijection between the set of isomorphism classes of pairs $(\mathscr{M}_U, \mathrm{iso}_{\widehat{U}})$ and pairs $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$ on $\mathbb{C}_z$. This is more classically expressed in terms of Stokes matrices. From the point of view developed here, we interpret this result by saying that, on a simply connected set U, giving a global section of a local system (the sheaf of Stokes torsors) is equivalent to giving a germ of section at one point of U. Our aim is to show that, for $\mathscr{N}$ as in (1.2), a similar result holds for any $t_o \in \Delta$.

The set Δ is naturally stratified: the stratum of a point is defined by specifying the precise sets of coordinates that coincide at this point. Given $t_o \in \Delta$, we denote by $S(t_o)$ its stratum. Its closure is a linear subspace of T.

The results.

Theorem 1.3. *Let S_o be a stratum of Δ and let U be an open subset of T such that*

(a) *$U\cap S_o$ is simply connected,*
(b) *U is star shaped with respect to $U\cap S_o$ (see Definition* 3.6*),*
(c) *for every stratum S such that $\overline{S}\cap S_o=\varnothing$, we also have $U\cap\overline{S}=\varnothing$.*

Then for any $t_o\in S_o$, the restriction $\gamma_{t_o}^+$ induces a bijection between the set of isomorphism classes of pairs of $(\mathscr{M},\mathrm{iso}_{\widehat{T}})$ defined on U and that of pairs $(\gamma_{t_o}^+\mathscr{M},\mathrm{iso}_{\widehat{0}})$.

We can interpret Theorem 1.3 in two ways.

Corollary 1.4.

(1) *Any pair $(\mathscr{M}^{t_o},\mathrm{iso}_{\widehat{0}})$ with formal model $\gamma_{t_o}^+\mathscr{N}$ on $(\mathbb{C}_z,0)$ can be extended as a pair $(\mathscr{M},\mathrm{iso}_{\widehat{T}})$ on any open set U satisfying* 1.3(a)–(c)*.*

(2) *Any $(\mathscr{M},\mathrm{iso}_{\widehat{T}})_V$ defined on a connected open set V of T is uniquely determined by any of its restrictions $\gamma_{t_o}^+(\mathscr{M},\mathrm{iso}_{\widehat{T}})$ at $t_o\in V$.*

We will see (Corollary 2.36) that the second part of the corollary holds in a much more general setting, according to recent results of J.-B. Teyssier [**Tey18**].

As a special case, let us assume that t_o belongs to the smallest stratum, where all coordinates coincide. Then $\gamma_{t_o}^+\mathscr{N}$ has a regular singularity at 0 up to an exponential twist, and any $(\mathscr{M}^{t_o},\mathrm{iso}_{\widehat{0}})$ is isomorphic to $(\gamma_{t_o}^+\mathscr{N},\mathrm{Id})$. We deduce that any $(\mathscr{M},\mathrm{iso}_{\widehat{T}})$ defined everywhere on T is isomorphic to $(\mathscr{N},\mathrm{Id})$, since $S(t_o)\simeq\mathbb{C}$ is simply connected and we can apply the theorem with $U=T$.

The theorem has a generalization as follows. Let Y be a 1-connected complex manifold and let $f=(f_1,\dots,f_n):Y\to T$ be a holomorphic mapping. We still denote by f the map $f\times\mathrm{Id}:Z:=Y\times\mathbb{C}\to T\times\mathbb{C}=X$ when the context is clear. There is no loss of generality in assuming that $f^{-1}(\Delta)$ is a hypersurface in Y (otherwise, one can replace n with a smaller n'). Then a Y-meromorphic flat bundle on Z with regular singularities is nothing but the pullback of a T-meromorphic flat bundle on X with regular singularities, and a Y-meromorphic flat bundle of the form

$$\mathscr{N}_Y=\bigoplus_{i=1}^n(\mathscr{E}^{-f_i(y)/z}\otimes\mathscr{R}_{Y,i})$$

is isomorphic to $f^+\mathscr{N}$, for $\mathscr{N}$ as in (1.2) with $\mathscr{R}_i$ such that $\mathscr{R}_{Y,i}=f^+\mathscr{R}_i$ (where f^+ is the pullback f^* of $\mathscr{O}$-modules together with the pullback connection). How much do this property and Theorem 1.3 extend to Y-meromorphic flat bundles formally modelled on $\mathscr{N}_Y$?

Notation 1.5. Let $y_o\in Y$ and set $t_o=f(y_o)$, contained in the stratum S_o of Δ. Given an open connected neighbourhood U of t_o in T, we denote by $f^{-1}(U)^{y_o}$ the connected component of $f^{-1}(U)$ containing y_o.

Corollary 1.6. *With the previous notation, if $U \subset T$ satisfies the assumptions as in Theorem* 1.3*, the restriction $\gamma_{y_o}^+$ (resp. the pullback f^+) induces a bijection between the set of isomorphism classes of pairs $(\mathscr{M}_{f^{-1}(U)^{y_o}}, \mathrm{iso}_{\widehat{f^{-1}(U)^{y_o}}})$ (i.e., defined on $f^{-1}(U)^{y_o}$) and that of pairs $(\mathscr{M}^{y_o}, \mathrm{iso}_{\widehat{0}})$ (resp. with that of pairs $(\mathscr{M}_U, \mathrm{iso}_{\widehat{U}})$).*

The proofs of Theorem 1.3 and Corollaries 1.4 and 1.6 are given in Section 3, where we also interpret them in terms of the sheaf of Stokes torsors (Corollary 3.8). The notion of very good formal decomposition, as well as the main properties we use, are developed in Section 2. In Section 4, we give an application to deformations of a connection in the Birkhoff normal form and to the construction of Frobenius manifolds, in the spirit of [**CDG19, CDG20, CG17, CG18**].

Acknowledgements. This work has benefitted from fruitful discussions with Giordano Cotti and Jean-Baptiste Teyssier. In particular, the uniqueness results of Section 2.i, originally obtained in some special cases, are due in this generality to Jean-Baptiste Teyssier in [**Tey18**], in an even stronger form. We thank the anonymous referee for useful suggestions for improving the presentation of the article.

2. General results on the notion of very good formal decomposition

The objects considered in this section are meromorphic bundles on a complex manifold, with poles along a divisor having normal crossing, and endowed with a flat meromorphic connection with poles at most along the same divisor. Moreover, we assume that they are isomorphic, in the formal neighbourhood of the divisor, to a simpler model, which we fix, and we wish to classify pairs consisting of a meromorphic flat bundle and a formal isomorphism with this model, up to isomorphism. When the model is *good*, then in the neighbourhood of a point on the divisor, such pairs are uniquely determined by their restriction to a slice that is transversal to the natural stratum of the divisor passing through this point (Proposition 2.32). We also prove a global uniqueness result (Corollary 2.35) obtained differently by J.-B. Teyssier: if the divisor is connected, then a pair is uniquely determined by its restriction to a generic smooth curve transverse to the divisor at a smooth point. This result is useful when combined with the Kedlaya-Mochizuki theorem, as it has consequences when the model is not good (Corollary 2.36). The techniques that we use mix a variant of the Malgrange-Sibuya theorem and recent results by T. Mochizuki on Stokes-filtered local systems on the oriented real blow-up along the divisor.

2.a. Setting and notation. Throughout this section, X denotes a germ of complex analytic manifold of dimension m along a *connected* reduced divisor $D \subset X$ with normal crossings, whose components are denoted by $D_{i\in I}$. We will assume, for the sake of simplicity, that these components are *smooth*. We denote by $\mathscr{O}_X(*D)$ the sheaf of meromorphic functions with poles on D at most. Let U be an open set in X

and $x_o \in U$. We denote by $D_1, \dots, D_\ell$ the components of D passing through x_o, and we choose local coordinates $(x_1, \dots, x_m)$ such that $D_i = \{x_i = 0\}$. The divisor D has a natural stratification by locally closed smooth connected complex submanifolds and, for $x_o \in D$, we denote by $D^{(x_o)}$ the stratum passing through x_o and by D_{x_o} the germ of D at x_o. With the above notation, it is equal to the connected component of $(\bigcap_{i=1,\dots,\ell} D_i) \smallsetminus (\bigcup_{j\neq 1,\dots,\ell} D_j)$ containing x_o.

2.b. Good sets of polar parts. The exponential behaviour of horizontal holomorphic sections of meromorphic connections is governed by exponential factors that are polar parts of meromorphic functions along D, that is, sections of the sheaf $\mathscr{O}_X(*D)/\mathscr{O}_X$. In higher dimensions, the asymptotic behaviour of such exponential factors can be complicated (the geometry of the sectors on which they have rapid decay or exponential growth can be complicated, even if D has normal crossings), and it is useful to select a class of such polar parts for which the sectors have a simple geometry. This leads to the goodness property.

***Definition 2.1* (Goodness).**

(1) A nonzero germ $\varphi \in \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ is *purely monomial* if some (or any) local representative in $\mathscr{O}_{X,x_o}(*D)$ can be written as $u(x)/x^{\boldsymbol{m}}$, where $\boldsymbol{m} \in \mathbb{N}^\ell \smallsetminus \{0\}$, and $u(x)$ is holomorphic and nonvanishing at x_o.

(2) For $\varphi \in \Gamma(U, \mathscr{O}_X(*D)/\mathscr{O}_X)$, we say that φ is *good* if its germ at any $x_o \in U$ is purely monomial.

(3) A finite set Φ of polar parts $\varphi \in \Gamma(U, \mathscr{O}_X(*D)/\mathscr{O}_X)$ is *good* if, for every pair $(\varphi, \psi) \in \Phi^2$ with $\varphi \neq \psi$, the difference $\varphi - \psi$ is good.

Convention 2.2. For a finite subset $\Phi_{x_o} \subset \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$, there exists a fundamental basis of Stein open neighbourhoods V of x_o such that each element of Φ_{x_o} is the germ at x_o of a unique element of $\Gamma(V, \mathscr{O}_X(*D))/\Gamma(V, \mathscr{O}_X)$. We abuse the notation by considering Φ_{x_o} as a subset of the latter quotient, and also by denoting with the same letter an element of this subset and any of its lifts in $\Gamma(V, \mathscr{O}_X(*D))$. For $\varphi \in \Phi_{x_o}$, it is then meaningful to say φ vanishes along some connected component of the smooth part of D, or that φ has no pole along some component of D. We can also consider the germ of φ at any $x \in D$ in some neighbourhood of x_o.

Let $x_o \in D$ and let Φ_{x_o} be a good (finite) set in $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$. We denote by $D(\Phi_{x_o})$ the union of (germs of) components of D at x_o along which at least some nonzero difference $\varphi - \psi$, with $\varphi, \psi \in \Phi_{x_o}$, has a pole.

Note that if Φ_{x_o} is good then for any $\varphi_o \in \Phi_{x_o}$, the set $\Phi_{x_o} - \varphi_o$ is also good, and, moreover, contains 0.

Let us fix $\varphi_o \in \Phi_{x_o}$. Recall that goodness then implies that the pole divisors of the elements $\varphi - \varphi_o$ in $\Phi_{x_o} - \varphi_o$ are totally ordered. Let us make this explicit in coordinates. By goodness, any $\varphi-\varphi_o\in(\Phi_{x_o}-\varphi_o)\smallsetminus\{0\}$ can be written as

$$\varphi-\varphi_o=u_\varphi\cdot\boldsymbol{x}^{-\boldsymbol{m}_\varphi}:=u_\varphi x_1^{-m_1}\cdots x_\ell^{-m_\ell}, \tag{2.3}$$

with $\boldsymbol{m}_\varphi\in\mathbb{N}^\ell\smallsetminus\{0\}$ and u_φ is an invertible holomorphic function. The goodness condition implies that the set $\{\boldsymbol{m}_\varphi\mid\varphi\in\Phi_{x_o}\smallsetminus\{\varphi_o\}\}$ is totally ordered with respect to the partial order on $\mathbb{N}^\ell$.

If there exist $\varphi\neq\psi$ in Φ_{x_o} such that $\varphi-\psi$ has no pole along some component of D at x_o, it will be convenient to consider the *level structure* of Φ_{x_o} as defined now. Although the family $(\boldsymbol{m}_\varphi)_{\varphi\in\Phi_{x_o}}$ depends on the choice of the base point φ_o, its maximum $\boldsymbol{m}_o$ does not. For any $\varphi\in\Phi_{x_o}$, using the notation of (2.3) we set

$$c(\varphi,\varphi_o)=\begin{cases}u_\varphi(x_o)&\text{if }\boldsymbol{m}_\varphi=\boldsymbol{m}_o,\\0&\text{if }\boldsymbol{m}_\varphi<\boldsymbol{m}_o,\end{cases}\quad\text{and}\quad C(\varphi_o)=\{c(\varphi,\varphi_o)\mid\varphi\in\Phi_{x_o}\}\subset\mathbb{C}. \tag{2.4}$$

Lemma 2.5. *If $\#\Phi_{x_o}\geqslant 2$, there exists $\varphi_o\in\Phi_{x_o}$ such that $\#C(\varphi_o)\geqslant 2$.*

Proof. If for the chosen φ_o we have $\#C(\varphi_o)=1$, then for any $\varphi'_o\in\Phi_{x_o}$ such that $\boldsymbol{m}_{\varphi'_o}=\boldsymbol{m}_o$, we have $\#C(\varphi'_o)=2$. Indeed, we then have $C(\varphi'_o)=-C(\varphi_o)\cup\{0\}$. ☐

Assume $\#C(\varphi_o)\geqslant 2$. We then obtain a nontrivial decomposition

$$\Phi_{x_o}=\bigsqcup_{c\in C(\varphi_o)}\Phi_{x_o}(\varphi_o,c),\quad \Phi_{x_o}(\varphi_o,c)=\{\varphi\in\Phi_{x_o}\mid c(\varphi,\varphi_o)=c\}. \tag{2.6}$$

Let $\boldsymbol{m}'_o$ denote the submaximal value of the sequence $(\boldsymbol{m}_\varphi)_{\varphi\in\Phi_{x_o}}$ (it may depend on the choice of φ_o). Then $\Phi_{x_o}(\varphi_o,c)$ is the inverse image in $\Phi_{x_o}-\varphi_o$ of $c/x^{\boldsymbol{m}_o}$ by the map induced by $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}\to\mathscr{O}_{X,x_o}(*D)/x^{-\boldsymbol{m}'_o}\mathscr{O}_{X,x_o}$.

***Definition 2.7* (Level decomposition (first step)).** The decomposition (2.6) is called the first step of the level decomposition of Φ_{x_o} with base point φ_o.

Every $\Phi_{x_o}(\varphi_o,c)$ is good, so that we can perform the same construction to it and get the complete level decomposition, which we will not define, since we will argue by induction only step by step.

Remark 2.8. The above notions can be defined similarly along the stratum $D^{(x_o)}$, and then they restrict to the previous ones at x_o (or at any point of the stratum). We then use the notation $C(x_o,\varphi_o)$ and $\Phi(x_o,\varphi_o,c)$.

2.c. Classes of D-meromorphic flat bundles. Let $\mathscr{I}_D$ be the reduced ideal of D. We will set $X^*=X\smallsetminus D$ and, for any subset $J\subset I$, $D_J:=\bigcap_{i\in J}D_i$ and $D_J^\circ:=D_J\smallsetminus\bigcup_{i\notin J}D_i$. We denote by $\mathscr{O}_{\widehat{D}}$ the formal completion $\varprojlim_k\mathscr{O}_X/\mathscr{I}_D^k$, which we regard in an obvious way as a sheaf on D. Recall (see e.g. [**Sab00**, Lem. I.1.1.13]) that a section f of $\mathscr{O}_{\widehat{D}}$ at a point $x_o\in D$ where the components of D are $D_1,\dots,D_\ell$ consists of the data $(f_i)_{i\in\{1,\dots,\ell\}}$ of sections f_i of $\mathscr{O}_{\widehat{D}_i}$ such that f_i and f_j coincide on $\mathscr{O}_{\widehat{D_i\cap D_j}}$ for all pairs $i,j=1,\dots,\ell$. In particular, $\mathscr{O}_{\widehat{D}}$ is naturally endowed with a differential d, extending d on $\mathscr{O}_{X|D}$.

For an $\mathscr{O}_X$-module $\mathscr{M}$, we denote by $\mathscr{M}_{|D}$ its sheaf-theoretic restriction to D and we set $\mathscr{M}_{\widehat{D}} := \mathscr{O}_{\widehat{D}} \otimes_{\mathscr{O}_{X|D}} \mathscr{M}_{|D}$. By a *D-meromorphic flat bundle* we mean a locally free $\mathscr{O}_X(*D)$-module of finite rank endowed with an integrable connection. For such an $\mathscr{M}$, we say that $\mathscr{M}_{\widehat{D}}$ is a D-meromorphic formal flat bundle: it is $\mathscr{O}_{\widehat{D}}(*D)$-locally free of finite rank with an integrable connection.

Let us fix a D-meromorphic flat bundle $\mathscr{N}$ on X.

Definition 2.9. Let $\mathscr{M}$ be a coherent $\mathscr{O}_X(*D)$-module an with integrable connection. We say that *$\mathscr{M}$ has $\mathscr{N}$ as a D-formal model* if there exists an isomorphism $\mathrm{iso}_{\widehat{D}} : \mathscr{M}_{\widehat{D}} \xrightarrow{\sim} \mathscr{N}_{\widehat{D}}$.

Note that $\mathscr{N}$ has $\mathscr{N}$ as a D-formal model when equipped with $\mathrm{Id} : \mathscr{N}_{\widehat{D}} \xrightarrow{\sim} \mathscr{N}_{\widehat{D}}$. If $\mathscr{M}$ has $\mathscr{N}$ as a D-formal model, then $\mathscr{M}$ is also a D-meromorphic flat bundle, i.e., it is $\mathscr{O}_X(*D)$-locally free of finite rank. This justifies the terminology of [**Tey19**] that $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})$ is an *$\mathscr{N}$-marked D-meromorphic flat bundle*.

We define the category of $\mathscr{N}$-marked D-meromorphic flat bundles in an obvious way: a morphism $\lambda : (\mathscr{M}, \mathrm{iso}_{\widehat{D}}) \to (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ is a morphism $\mathscr{M} \to \mathscr{M}'$ such that $\mathrm{iso}'_{\widehat{D}} \circ \lambda_{\widehat{D}} = \mathrm{iso}_{\widehat{D}}$. If $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})$ and $(\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ are isomorphic by an isomorphism ι, then such an isomorphism ι is unique, since $\iota_{\widehat{D}}$ is uniquely determined. When $(X, D) = (\mathbb{C}, 0)$, the interest of considering such pairs has been emphasized by B. Malgrange in [**Mal83b**].

For any open set $U \subset D$, we denote by $\mathscr{H}(U, \mathscr{N})$ the set of isomorphism classes of pairs $(\mathscr{M}_U, \mathrm{iso}_{\widehat{D}})$ consisting of a (germ of) D-meromorphic flat bundle $\mathscr{M}_U$ on some open neighbourhood of U in X and a formal isomorphism $\mathrm{iso}_{\widehat{D}}$ with $\mathscr{N}_{\widehat{D}}$ on U. Owing to the uniqueness of isomorphisms between such pairs, we deduce that the presheaf $U \mapsto \mathscr{H}(U, \mathscr{N})$ is a sheaf of sets $\mathscr{H}_D(\mathscr{N})$ with a given section $(\mathscr{N}, \mathrm{Id})$, and sections on D of this sheaf consist of pairs $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})$ as above (up to unique isomorphism).

The basic operations we will use are the complex blowing-ups, or sequences of such, and more generally proper modifications. A proper modification of X is a proper morphism $X' \to X$ of complex manifolds inducing an isomorphism between open dense subsets of these manifolds. Since we consider pairs (X, D) of manifolds endowed with a normal crossing divisor, we extend this notion as follows: a proper modification $e : (X', D') \to (X, D)$ is a proper modification $e : X' \to X$ such that D' is contained $e^{-1}(D)$ and which induces an isomorphism $X \smallsetminus e^{-1}(D) \to X \smallsetminus D$. When $D' = e^{-1}(D)$, we can compare the sheaves $\mathscr{H}_D(\mathscr{N})$ and $\mathscr{H}_{D'}(e^+\mathscr{N})$.

Lemma 2.10. *Let $e : (X', D') \to (X, D)$ be a proper modification such that $D' = e^{-1}(D)$. Then $e^+ : \mathscr{H}_D(\mathscr{N}) \to e_*\mathscr{H}_{D'}(e^+\mathscr{N})$ is an isomorphism, having e_+ as its inverse.*

Proof. It is known that (e^+, e_+) (where e_+ is the pushforward e_* of $\mathscr{O}_{X'}(*D')$-modules with the pushforward connection) forms a pair of quasi-inverse functors between the categories of D- (resp. D'-) meromorphic flat bundles on X (resp. X'). On the other hand, by right exactness of the tensor product, we have $\mathscr{O}_{\widehat{D}'} = e^*\mathscr{O}_{\widehat{D}}$. We thus have a functorial isomorphism of $\mathscr{O}_{\widehat{D}}$-modules, compatible with connections

$$e_*(\mathscr{O}_{\widehat{D}'} \otimes \mathscr{M}') \simeq R^0 e_*(e^*\mathscr{O}_{\widehat{D}} \otimes e^*(e_*\mathscr{M}')) \simeq \mathscr{H}^0(\mathscr{O}_{\widehat{D}} \otimes \boldsymbol{R}e_*e^*(e_*\mathscr{M}') \simeq \mathscr{O}_{\widehat{D}} \otimes (e_*\mathscr{M}').$$

The result follows. □

Recall (see [**Mal04**, Th. 5.5]) that, if a closed analytic subset $S \subset D$ has codimension $\geqslant 2$ in D, then the restriction functor, from the category of D-meromorphic flat bundles on X to that of $(D \smallsetminus S)$-meromorphic flat bundles on $X \smallsetminus S$, is an equivalence of categories. The next proposition shows that a similar result holds for $\mathscr{N}$-marked D-meromorphic flat bundles.

Proposition 2.11. *The restriction functor, from the category of $\mathscr{N}$-marked D-meromorphic flat bundles on X to that of $\mathscr{N}$-marked $(D \smallsetminus S)$-meromorphic flat bundles on $X \smallsetminus S$, is an equivalence of categories.*

Proof. By the result of B. Malgrange aforementioned, we are reduced to proving essential surjectivity, that is, given D-meromorphic flat bundles $\mathscr{M}, \mathscr{N}$, any formal isomorphism $\mathrm{iso}_{\widehat{D \smallsetminus S}} : \mathscr{M}_{\widehat{D \smallsetminus S}} \simeq \mathscr{N}_{\widehat{D \smallsetminus S}}$ extends as an isomorphism $\mathrm{iso}_{\widehat{D}} : \mathscr{M}_{\widehat{D}} \simeq \mathscr{N}_{\widehat{D}}$. This is a local question on S, and we can assume that $\mathscr{M}$ and $\mathscr{N}$ are $\mathscr{O}_X(*D)$-free. The components of $\mathrm{iso}_{\widehat{D \smallsetminus S}}$ on bases of $\mathscr{M}, \mathscr{N}$ are sections of $\mathscr{O}_{\widehat{D \smallsetminus S}}(*(D \smallsetminus S))$. By considering the polar coefficients, we are led to showing that, if $j : D \smallsetminus S \hookrightarrow D$ denotes the inclusion, the natural monomorphism $\mathscr{O}_{\widehat{D}} \to j_*\mathscr{O}_{\widehat{D \smallsetminus S}}$ is an isomorphism.

Let $f = (f_i)$ be a section of $\mathscr{O}_{\widehat{D \smallsetminus S}}$ on $\mathrm{nb}_D(x_o) \smallsetminus S$ for some $x_o \in S$. We simply write $D = \mathrm{nb}(x_o)$. Since S has codimension $\geqslant 2$ in each D_i, any section f_i of $\mathscr{O}_{\widehat{D_i \smallsetminus S}}$ extends in a unique way as a section of $\mathscr{O}_{\widehat{D_i}}$, as seen by applying Hartogs's theorem to the coefficients $f_{i,k} \in \mathscr{O}(D_i \smallsetminus S)$ of the formal series $f_i = \sum_{k \geqslant 0} f_{i,k} x_i^k$. At a point of $D_i \cap D_j \cap S$, the coefficients $f_{i,k,\ell}, f_{j,k,\ell} \in \mathscr{O}((D_i \cap D_j) \smallsetminus S)$ of f_i, f_j on the monomial $x_i^k x_j^\ell$ coincide on their domains, and since $D_i \cap D_j \cap S$ has codimension $\geqslant 1$ in $D_i \cap D_j$, they coincide everywhere on $D_i \cap D_j$. As a consequence, the section f of $\mathscr{O}_{\widehat{D \smallsetminus S}}$ extends (in a unique way) as a section of $\mathscr{O}_{\widehat{D}}$ on $\mathrm{nb}_D(x_o)$, as wanted. □

On the other hand, extension of morphisms can be done in codimension one with respect to D, according to Hartogs's theorem. Assume for example that D is smooth, and let $H \subset D$ be a codimension-one closed analytic subset. Let us denote by $j : U = D \smallsetminus H \hookrightarrow D$ the open inclusion. For a D-meromorphic flat bundle $\mathscr{M}$ on X, we denote by $\mathscr{M}_{|U}$ its sheaf-theoretic restriction to $U \subset X$.

Lemma 2.12. *Let $\mathscr{M}, \mathscr{M}'$ be D-meromorphic flat bundles on X. Then, under the above assumptions, any morphism $f_U : \mathscr{M}_{|U} \to \mathscr{M}'_{|U}$ extends in a unique way as a morphism $f : \mathscr{M} \to \mathscr{M}'$. If $(\mathscr{M}, \mathrm{iso}_{\widehat{D}}), (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ are $\mathscr{N}$-marked D-meromorphic flat bundles, any morphism $f_U : (\mathscr{M}, \mathrm{iso}_{\widehat{D}})_{|U} \to (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})_{|U}$ extends in a unique way as* *a morphism f between these pairs. In both cases, if f_U is an isomorphism, then so is f.*

Proof. By the uniqueness assertion, the question is local on D and since D is smooth, we can assume that $X = D \times (\mathbb{C}, 0)$ with D simply connected. Since $\mathscr{M}, \mathscr{M}'$ are flat bundles on $X \smallsetminus D$ and $\pi_1(X \smallsetminus D) = \pi_1(\mathbb{C} \smallsetminus \{0\})$, we can assume that $X = D \times B_\varepsilon$, where B_ε is the open disc of radius $\varepsilon > 0$ centered at the origin in $\mathbb{C}$. By horizontality, f_U is defined on $U \times B_\varepsilon$.

For the same reason, the sheaf $\mathscr{H}om^\nabla(\mathscr{M}, \mathscr{M}')_{X \smallsetminus D}$ is a locally constant sheaf on $X \smallsetminus D = D \times B_\varepsilon^*$ ($B_\varepsilon^* = B_\varepsilon \smallsetminus \{0\}$). It follows that any section of this sheaf on $\{x\} \times B_\varepsilon^*$ uniquely extends as a global section. In particular f_U uniquely extends to $X \smallsetminus D$.

We now apply Hartogs theorem. Let us fix bases of $\mathscr{M}, \mathscr{M}'$ as $\mathscr{O}_X(*D)$-modules (recall that we work locally on D). We have obtained a morphism $\mathscr{M}_{X \smallsetminus H} \to \mathscr{M}'_{X \smallsetminus H}$ extending f_U. In the chosen bases, the entries of the matrix of this morphism are holomorphic functions on $X \smallsetminus H$. Since H has codimension two in X, they extend (in a unique way) as holomorphic functions on X, hence the first statement of the lemma. The other statements are then straightforward. □

Corollary 2.13. *Under the above assumptions, the natural morphism $\mathscr{H}_D(\mathscr{N}) \to j_* j^{-1} \mathscr{H}_D(\mathscr{N})$ is injective.*

Proof. The question is local at points of H and we can assume that D is a small open neighbourhood of such a point. We are thus reduced to proving that if two pairs $(\mathscr{M}, \mathrm{iso}_{\widehat{D}}), (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ are isomorphic on $D \smallsetminus H$, they are isomorphic. But by the previous lemma, the isomorphism on $D \smallsetminus H$ lifts in a unique way as an isomorphism on D. □

2.d. Very good formal decomposition. By a *good decomposable D-meromorphic flat bundle* we mean a locally free $\mathscr{O}_X(*D)$-module $\mathscr{M}^{\mathrm{good}}$ with integrable connection ∇^{good} (that we usually omit to mention), which is globally (on X) isomorphic as such to a direct sum

$$\mathscr{M}^{\mathrm{good}} \simeq \bigoplus_{\varphi \in \Phi} (\mathscr{E}^\varphi \otimes \mathscr{R}_\varphi), \tag{2.14}$$

where

- Φ is a finite subset of $\Gamma(X, \mathscr{O}_X(*D)/\mathscr{O}_X)$ which is *good* at every point $x_o \in D$,
- $\mathscr{R}_\varphi$ is a D-meromorphic flat bundle having regular singularities along D,
- $\mathscr{E}^\varphi := (\mathscr{O}_X(*D), \mathrm{d} + \mathrm{d}\varphi)$.[1]

[1] This definition is local on X, since φ lifts to $\widetilde{\varphi} \in \Gamma(X, \mathscr{O}_X(*D))$ if X is Stein. Otherwise, on a Stein open covering (U_i) of X, various liftings $\widetilde{\varphi}_i$ give rise to the cocycle $(\exp(\widetilde{\varphi}_i - \widetilde{\varphi}_j))_{ij} \in H^1(X, \mathscr{O}_X^*)$ defining a rank-one vector bundle $\mathscr{L}_\varphi$, and $\mathrm{d} + \mathrm{d}\varphi$ is a well-defined connection on $\mathscr{L}_\varphi(*D)$.

***Definition 2.15* (Very good formal decomposition, [Sab93]).** Let $\mathscr{M}$ be a D-meromorphic flat bundle and let $\mathscr{M}^{\rm good}$ be a good decomposable D-meromorphic flat bundle. If $\mathscr{M}$ has $\mathscr{M}^{\rm good}$ as a D-formal model, we say that $\mathscr{M}$ has a *very good formal decomposition* along D at each point of D.

If $\mathscr{M}$ has $\mathscr{M}^{\rm good}$ D-formal model, we can fix a D-formal isomorphism $\mathrm{iso}_{\widehat{D}}:\mathscr{M}_{\widehat{D}}\xrightarrow{\sim}\mathscr{M}^{\rm good}_{\widehat{D}}$. We thus have an $\mathscr{M}^{\rm good}$-marked D-meromorphic flat bundle $(\mathscr{M},\mathrm{iso}_{\widehat{D}})$, which defines a section on D of the sheaf $\mathscr{H}_D(\mathscr{M}^{\rm good})$ (see Section 2.c).

Let $\gamma:(\mathbb{C}^\ell,0)\to(X,D)$ be a germ of holomorphic map such that $\gamma^{-1}(D)$ is a normal crossing divisor. Then $\gamma^+\mathscr{M}^{\rm good}$ is a good model (but various $\gamma^*\varphi$ may coincide, leading to grouping the corresponding $\gamma^+\mathscr{R}_\varphi$) and $\gamma^+(\mathscr{M},\mathrm{iso}_{\widehat{D}}):=(\gamma^+\mathscr{M},\gamma^*\mathrm{iso}_{\widehat{D}})$ is a $\gamma^+\mathscr{M}^{\rm good}$-marked $\gamma^{-1}(D)$-meromorphic flat bundle on $(\mathbb{C}^\ell,\gamma^{-1}(D))$ near $0\in\mathbb{C}^\ell$.

By definition, $\mathscr{M}^{\rm good}$ is endowed with the marking Id. Any other marking is an automorphism $\mathrm{iso}_{\widehat{D}}:\mathscr{M}^{\rm good}_{\widehat{D}}\xrightarrow{\sim}\mathscr{M}^{\rm good}_{\widehat{D}}$.

Proposition 2.16. *Any such automorphism is block diagonal with respect to the decomposition* (2.14)*, with the* (φ,φ)*-block being induced by an automorphism of* $\mathscr{R}_\varphi$*.*

Sketch of proof. Let $x_o\in D$. One computes that, if $\varphi_{x_o}\neq\psi_{x_o}$ in Φ_{x_o}, then

$$\mathscr{H}om^\nabla((\mathscr{E}^\varphi\otimes\mathscr{R}_\varphi)_{\widehat{D},x_o},(\mathscr{E}^\psi\otimes\mathscr{R}_\psi)_{\widehat{D},x_o})=0,$$

being formed of ∇-horizontal sections of $\mathscr{E}^{\psi-\varphi}\otimes\mathscr{H}om(\mathscr{R}_{\varphi,\widehat{D},x_o},\mathscr{R}_{\psi,\widehat{D},x_o})$. On the other hand, if $\varphi_{x_o}=\psi_{x_o}$ in Φ_{x_o}, we have by regularity,

$$\begin{aligned}\mathscr{H}om^\nabla((\mathscr{E}^\varphi\otimes\mathscr{R}_\varphi)_{\widehat{D},x_o},(\mathscr{E}^\psi\otimes\mathscr{R}_\psi)_{\widehat{D},x_o})&=\mathscr{H}om^\nabla(\mathscr{R}_{\varphi,\widehat{D},x_o},\mathscr{R}_{\psi,\widehat{D},x_o})\\&\simeq\mathscr{H}om^\nabla(\mathscr{R}_\varphi,\mathscr{R}_\psi)_{|D,x_o}.\end{aligned}$$

Let $\varphi\neq\psi$ in Φ and let $D(\varphi,\psi)$ be the (nonempty) union of components of D where $\varphi\neq\psi$. Let D' be the union of the other components. Since D is assumed to be connected, each connected component of D' cuts $D(\varphi,\psi)$. The above computation shows that the block $\mathrm{iso}_{\varphi,\psi,\widehat{D}}$ is a section of a locally constant sheaf on $W'\smallsetminus D'$, where W' is a neighbourhood of D' in $X\smallsetminus D(\varphi,\psi)$. Since each connected component of this neighbourhood has a limit point in $D(\varphi,\psi)$, the section must vanish near this limit point. Hence it is zero. □

Corollary 2.17. *Let* $\mathrm{iso}_{\widehat{D}}$ *and* $\mathrm{iso}'_{\widehat{D}}$ *be two markings of* $\mathscr{M}$ *with model* $\mathscr{M}^{\rm good}$*. Let* $x_o\in D$*, let* $\ell=\mathrm{codim}_X D^{(x_o)}$ *and let* $\gamma:(\mathbb{C}^\ell,0)\to(X,D)$ *be a transversal slice of* $D^{(x_o)}$ *at* $x_o=\gamma(0)$*. If* $\gamma^*\mathrm{iso}_{\widehat{D}}=\gamma^*\mathrm{iso}'_{\widehat{D}}$*, then* $\mathrm{iso}_{\widehat{D}}=\mathrm{iso}'_{\widehat{D}}$*.*

Proof. It is enough to prove the assertion for markings Id and $\mathrm{iso}_{\widehat{D}}$ of $\mathscr{M}^{\rm good}$. We can work on a connected open neighbourhood of D in X, which we still denote by X, so that we can assume that $X\smallsetminus D$ is connected. Since, by Proposition 2.16, $\mathrm{iso}_{\widehat{D}}$ is block diagonal with blocks $\mathrm{iso}_{\widehat{D},\varphi,\varphi}$, we are reduced to proving that if $\gamma^*\mathrm{iso}_{\widehat{D},\varphi,\varphi}=\mathrm{Id}$, then

$\mathrm{iso}_{\widehat{D},\varphi,\varphi}=\mathrm{Id}$. This is a consequence of the fact that $\mathrm{iso}_{\widehat{D},\varphi,\varphi}$ is a global section of a locally constant sheaf on a connected set, as shown in the same proposition. □

2.e. Good versus very good formal decomposition. More general is the notion of *good formal decomposition* (see [**Sab00, Moc09, Ked10, Moc11a, Ked11**]). We will make explicit the differences between the two notions. We denote by $\mathscr{O}_{\widehat{D^{(x_o)}}}$ the sheaf locally defined as $\varprojlim_k \mathscr{O}_X/(x_1,\dots,x_\ell)^k$ (i.e., the formalization of $\mathscr{O}_X$ along the stratum of x_o) and we say that $\mathscr{M}$ has a good decomposition at x_o with formal model $\mathscr{M}^{\mathrm{good}}$ if there exists an isomorphism in the neighbourhood of x_o:

$$\mathscr{M}_{\widehat{D^{(x_o)}}}:=\mathscr{O}_{\widehat{D^{(x_o)}}}\otimes\mathscr{M}\xrightarrow{\sim}\mathscr{O}_{\widehat{D^{(x_o)}}}\otimes\mathscr{M}^{\mathrm{good}}=:\mathscr{M}^{\mathrm{good}}_{\widehat{D^{(x_o)}}}. \tag{2.18}$$

Remark 2.19. Let us make clear that, starting from any D-meromorphic flat bundle, one can find a sequence of blowing-ups (locally on X in the complex analytic setting, see [**Ked10, Ked11**], and globally in the projective setting, see [**Moc09, Moc11a**]) so that, after local ramifications on the blown-up space giving rise to a space denoted by (X',D'), the pullback of the D-meromorphic flat bundle, which is now a D'-meromorphic flat bundle, admits a good formal decomposition, i.e., for each stratum D'_I of the natural stratification of D', when tensored with the formal completion $\mathscr{O}_{\widehat{D'_I}}$ along this stratum, it admits an isomorphism with a good model. In general, such an isomorphism cannot be lifted as an isomorphism formally along the divisor D', i.e., by tensoring instead with $\mathscr{O}_{\widehat{D'}}$.

Example 2.20. However (see [**Sab00**, Th. I.2.2.4] in dimension two, and in general [**Sab13**, Cor. 11.28] which is a consequence of results by T. Mochizuki [**Moc11a**, §2.4.3] on good lattices), if we assume moreover that, given $x_o\in D$, for any pair $\varphi\neq\psi\in\Phi_{x_o}$ the difference $\varphi-\psi$ has poles along *all* components of D_{x_o}, then any good formal decomposition along $D^{(x_o)}$ is very good.

***Example 2.21* (See [Sab00, Lem. I.2.2.3]).** On the other hand, let $x_o\in D$ and assume that there exists a component D_i of D_{x_o} along which all nonzero differences $\varphi-\psi$ $(\varphi,\psi\in\Phi_{x_o})$ vanish (i.e., their representatives do not have a pole, in other words, $D(\Phi_{x_o})\neq D_{x_o}$. Then the germ $\mathscr{H}_D(\mathscr{M}^{\mathrm{good}})_{x_o}$ reduces to $(\mathscr{M}^{\mathrm{good}},\mathrm{Id})$, that is, for any germ $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_{x_o}$, there exists a (unique) lifting $\mathrm{iso}_{x_o}:\mathscr{M}_{x_o}\xrightarrow{\sim}\mathscr{M}^{\mathrm{good}}_{x_o}$. Indeed (see below), the sheaf $\mathscr{A}ut^{\mathrm{rd}\,D}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})_{|\varpi^{-1}(x_o)}$ is equal to Id, so $\mathrm{StT}_D(\mathscr{M}^{\mathrm{good}})_{x_o}$ is also reduced to Id, and we can apply Theorem 2.26 below.

Checking whether a good decomposition is very good can be done inductively with respect to the level decomposition of $\mathscr{M}_{\widehat{D}}$, which we define now, in a way parallel to the level decomposition of Φ_{x_o}.

***Proposition 2.22* (First step of the level decomposition).** *Assume that $\mathscr{M}$ is a (germ at x_o of a) good D-meromorphic flat bundle with good formal model* (2.14) *(i.e.,* (2.18) *holds) and that $D(\Phi_{x_o})=D_{x_o}$. Then, for every $c\in C(x_o,\varphi_o)$ (see* (2.4)*) there* *exists a D-meromorphic flat bundle $\mathscr{M}_c$ in the neighbourhood of $D^{(x_o)}$ satisfying the following properties with respect to* (2.14)*:*

$$\mathscr{M}_{\widehat{D}|D^{(x_o)}} \simeq \bigoplus_{c\in C(x_o,\varphi_o)} \mathscr{M}_{c,\widehat{D}|D^{(x_o)}}, \tag{2.22 $*$}$$

$$\mathscr{M}_{c,\widehat{D^{(x_o)}}} \simeq \bigoplus_{\eta\in\Phi(x_o,\varphi_o,c)} (\mathscr{E}^{\eta}\otimes\mathscr{R}_{\eta})_{\widehat{D^{(x_o)}}} \quad \forall\, c\in C(x_o,\varphi_o). \tag{2.22 $**$}$$

Moreover, we have $\mathscr{M}_{\widehat{D}|D^{(x_o)}} \simeq \mathscr{M}^{\mathrm{good}}_{\widehat{D}|D^{(x_o)}}$ if and only if the same property holds for each $\mathscr{M}_c$.

Proof. See [**Sab13**, p. 189–190] for the first part. For the second assertion, the "if" part follows from the first part. Conversely, assume $\mathscr{M}_{\widehat{D}|D^{(x_o)}} \simeq \mathscr{M}^{\mathrm{good}}_{\widehat{D}|D^{(x_o)}}$. It is enough to prove that this isomorphism is block diagonal with respect to the decomposition (2.22 $*$). This amounts to showing that there is no nonzero morphism $\mathscr{M}_{c,\widehat{D}|D^{(x_o)}} \to \mathscr{M}^{\mathrm{good}}_{c',\widehat{D}|D^{(x_o)}}$ if $c\neq c'$ and, by (2.22 $**$) and faithful flatness of $\mathscr{O}_{\widehat{D^{(x_o)}}}$ over $\mathscr{O}_{\widehat{D}|D^{(x_o)}}$, no nonzero morphism $(\mathscr{E}^{\varphi}\otimes\mathscr{R}_{\varphi})_{\widehat{D^{(x_o)}}} \to (\mathscr{E}^{\psi}\otimes\mathscr{R}_{\psi})_{\widehat{D^{(x_o)}}}$ if $c(\varphi,\varphi_o)\neq c(\psi,\varphi_o)$. This is implied by the vanishing of any horizontal section of $\mathscr{H}om(\mathscr{R}_{\varphi,\widehat{D^{(x_o)}}}, \mathscr{E}^{\psi-\varphi}\otimes\mathscr{R}_{\psi,\widehat{D^{(x_o)}}})$ if $\varphi\neq\psi$ in Φ in the neighbourhood of $D^{(x_o)}$, a property that is standard. $\square$

2.f. The sheaf of Stokes torsors.

Reminder of the theory in dimension one. The approach followed here intends to generalize in higher dimensions that of B. Malgrange [**Mal83b**] in dimension one, relying on the so-called Malgrange-Sibuya theorem (see also [**BV89**]), approach that we quickly recall here. We thus assume that $(X,D)=(\mathbb{C},0)$. Let $\widetilde{\mathbb{C}} = S^1\times\mathbb{R}_+$ be the real oriented blow-up of $\mathbb{C}$ at the origin, that is, the space of polar coordinates. It is endowed with the sheaf $\mathscr{A}_{\widetilde{\mathbb{C}}}$ (C^∞ functions on $\widetilde{\mathbb{C}}$ satisfying the Cauchy-Riemann equation on $\mathbb{C}^*$, hence $\mathscr{A}_{\widetilde{\mathbb{C}}|\mathbb{C}^*} = \mathscr{O}_{\mathbb{C}^*}$) and its subsheaf $\mathscr{A}^{\mathrm{rd}}_{\widetilde{\mathbb{C}}}$ consisting of functions having rapid decay along $S^1\times\{0\}$. We now only consider the sheaf-theoretic restrictions $\mathscr{A}_{S^1}, \mathscr{A}^{\mathrm{rd}}_{S^1}$ of these sheaves to the boundary $S^1\times\{0\}=S^1$. For a model meromorphic flat bundle $\mathscr{M}^{\mathrm{good}}$ on $(\mathbb{C},0)$, the sheaf $\mathscr{A}ut^{\mathrm{rd}}_{S^1}(\mathscr{M}^{\mathrm{good}})$ consists of local automorphisms of $\mathscr{A}_{S^1}\otimes\mathscr{M}^{\mathrm{good}}$ compatible with the connection that are asymptotic to Id on the open set of S^1 where they are defined. The set $H^1(S^1,\mathscr{A}ut^{\mathrm{rd}}_{S^1}(\mathscr{M}^{\mathrm{good}}))$, that is, the set of *Stokes torsors*, classifies meromorphic flat bundles $\mathscr{M}$ endowed with a formal isomorphism $\mathrm{iso}_{\widehat{0}}:\mathscr{M}_{\widehat{0}} \xrightarrow{\sim} \mathscr{M}^{\mathrm{good}}_{\widehat{0}}$, according to [**Mal83b**, Th. 3.4]. It can be endowed with a richer structure (see [**BV89**]) that we will not consider here.

Example 2.23. Let $t_{o,1},\dots,t_{o,n}$ be pairwise distinct complex numbers and let $\mathscr{R}_i$ $(i=1,\dots,n)$ be a free $\mathscr{O}_{\mathbb{C},0}(*0)$-module with a regular connection. Set

$$\mathscr{M}^{\mathrm{good}} = \bigoplus_{i=1}^{n} \mathscr{E}^{-t_{o,i}/z}\otimes\mathscr{R}_i.$$

If $\theta_o \in S^1$ is general, we can reindex the numbers $t_{o,i}$ so that, for $i, j \in \{1, \dots, n\}$,

$$i < j \iff \operatorname{Re}((t_{o,i} - t_{o,j})e^{-\mathrm{i}\theta_o}) < 0,$$

and an element of $H^1(S^1, \mathscr{A}ut^{\mathrm{rd}}_{S^1}(\mathscr{M}^{\mathrm{good}}))$ is a pair (S^+, S^-) of matrices (the Stokes matrices), one being upper triangular, the other one being lower triangular, both having Id as their diagonal part. Let us recall this correspondence. It also depends on a choice of a horizontal basis of $\bigoplus_i \mathscr{R}_i$ in a small open sector centered at θ_o and another one in the opposite sector. The set H^1 is computed as Čech cohomology via the Leray covering consisting of the two intervals with boundary points $\theta_o, \theta_o + \pi$, slightly extended, so that it is identified with

$$\Gamma\big((\theta_o - \varepsilon, \theta_o + \varepsilon), \mathscr{A}ut^{\mathrm{rd}}_{S^1}(\mathscr{M}^{\mathrm{good}})\big) \times \Gamma\big((\theta_o + \pi - \varepsilon, \theta_o + \pi + \varepsilon), \mathscr{A}ut^{\mathrm{rd}}_{S^1}(\mathscr{M}^{\mathrm{good}})\big),$$

and we identify the first (resp. second) term with upper (resp. lower) triangular constant matrices S^+ (resp. S^-) with Id on the diagonal since, for a constant matrix (S_{ij}), the matrix $S_{ij} \exp(t_{o,i} - t_{o,j})/z$ has rapid decay in a small open sector centered at θ_o (resp. $\theta_o + \pi$) if and only if $i < j$ (resp. $i > j$).

If we make the complex numbers $t_{o,i}$ vary as t_i, but remaining pairwise distinct, the space $H^1(S^1, \mathscr{A}ut^{\mathrm{rd}}_{S^1}(\mathscr{M}^{\mathrm{good}}))$ varies in a locally constant way, as we will recall below. Understanding the behaviour of the Stokes matrices in such a variation needs more care, since they depend on the choice of the generic $\theta_o \in S^1$, which can become non generic for some values of the parameters t_i. Apparent real singularities may thus appear in the parameter space. This explains why we will use the language of sheaves of Stokes torsors instead of that of Stokes matrices: we wish to avoid these apparent singularities. However, if we vary t_i along a real parameter in such a way that θ_o and the corresponding order can be chosen constant all along the deformation, then the representation in terms of Stokes matrices holds all along the deformation. This includes limit cases where some t_i's may coincide: coalescence of eigenvalues occurs in the sense of [**CDG19**].

Real oriented blow-up. In higher dimensions, the Stokes sectors are multi-sectors, which are conveniently defined on the real oriented blown-up space of X along the components of D, a space that has local coordinates given by the polar coordinates around each component of D.

Let us now consider the higher-dimensional situation of (X, D) as above. We denote by $\varpi : \widetilde{X} = \widetilde{X}(D_{i\in I}) \to X$ the real-oriented blowing-up of the components $D_{i\in I}$ in X (see e.g. [**Sab13**, §8.2] for the global setting). In a local coordinate system $(x_1, \dots, x_m)$ as above, we identify $\widetilde{X}$ with $(S^1)^\ell \times (\mathbb{R}_+)^\ell \times \mathbb{C}^{m-\ell}$ (polar coordinates with respect to $x_1, \dots, x_\ell$). We set $\partial\widetilde{X} := \varpi^{-1}(D)$, locally isomorphic to the product $(S^1)^\ell \times \partial(\mathbb{R}_+)^\ell \times \mathbb{C}^{m-\ell}$. On $\widetilde{X}$ we consider the sheaves $\mathscr{A}_{\widetilde{X}}$ (C^∞ functions on $\widetilde{X}$ satisfying the Cauchy-Riemann equation on X^*) and $\mathscr{A}^{\mathrm{rd}\,D}_{\widetilde{X}}$ (holomorphic functions on X^* having rapid decay along $\partial\widetilde{X}$). These sheaves coincide with $\mathscr{O}_{X^*}$ on X^*, so we will only consider their sheaf-theoretic restrictions to $\partial\widetilde{X}$, where we have a strict inclusion

$$\mathscr{A}^{\mathrm{rd}\,D}_{\partial\widetilde{X}} \subset \mathscr{A}_{\partial\widetilde{X}}.$$

On the other hand, setting

$$\mathscr{A}_{\widehat{\widetilde{X}|D}} := \varprojlim_k \bigl(\mathscr{A}_{\partial\widetilde{X}} \,/\, \varpi^{-1}\mathscr{I}_D^k \cdot \mathscr{A}_{\partial\widetilde{X}}\bigr),$$

we have an exact sequence

$$0 \longrightarrow \mathscr{A}^{\mathrm{rd}\,D}_{\partial\widetilde{X}} \longrightarrow \mathscr{A}_{\partial\widetilde{X}} \longrightarrow \mathscr{A}_{\widehat{\widetilde{X}|D}} \longrightarrow 0.$$

(See [**Sab00**, §II.1.1] and [**Moc14**], and the references therein for details). For an $\mathscr{O}_X(*D)$-module $\mathscr{M}$, we denote by $\mathscr{M}_{\partial\widetilde{X}}$ the $\mathscr{A}_{\partial\widetilde{X}}$-module $\mathscr{A}_{\partial\widetilde{X}} \otimes_{\varpi^{-1}\mathscr{O}_{X|D}} \varpi^{-1}\mathscr{M}_{|D}$, and similarly for $\mathscr{M}^{\mathrm{rd}\,D}_{\partial\widetilde{X}}$. If, moreover, $\mathscr{M}$ is endowed with an integrable connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$, then ∇ lifts as an operator $\nabla : \mathscr{M}_{\partial\widetilde{X}} \to \varpi^{-1}\Omega^1_X \otimes \mathscr{M}_{\partial\widetilde{X}}$ that satisfies $\nabla^2 = 0$, and similarly for $\mathscr{M}^{\mathrm{rd}\,D}_{\partial\widetilde{X}}$.

The sheaf of Stokes torsors. Let $\mathscr{E}nd^{\nabla}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})$ be the sheaf of endomorphisms of $\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}}$ compatible with ∇^{good}. We denote by $\mathscr{A}ut^{\mathrm{rd}\,D}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})$ its subsheaf consisting of sections whose image in $\mathscr{E}nd(\mathscr{M}^{\mathrm{good}}_{\widehat{\widetilde{X}D}})$ is equal to Id. It is a sheaf of groups on $\partial\widetilde{X}$. We consider the presheaf on X defined by

$$U \longmapsto H^1\bigl(\varpi^{-1}(U), \mathscr{A}ut^{\mathrm{rd}\,D}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})\bigr),$$

and we denote by $\mathcal{S}\mathrm{tT}_D(\mathscr{M}^{\mathrm{good}})$ the associated sheaf, which we call *the sheaf of Stokes torsors.*(2) This is a sheaf of pointed sets (pointed by the class of Id).(3)

We define below a morphism of presheaves

$$\mathscr{H}_D(U, \mathscr{M}^{\mathrm{good}}) \longrightarrow H^1\bigl(\varpi^{-1}(U), \mathscr{A}ut^{\mathrm{rd}\,D}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})\bigr) \tag{2.24}$$

and we then consider the associated morphism of sheaves of pointed sets

$$\mathscr{H}_D(\mathscr{M}^{\mathrm{good}}) \longrightarrow \mathcal{S}\mathrm{tT}_D(\mathscr{M}^{\mathrm{good}}). \tag{2.25}$$

By a theorem of H. Majima [**Maj84**, Th. III.2.1, p. 121] (see also [**Sab93**, Th. (3.1)] that is stated in dimension two, but the proof can be adapted to arbitrary dimensions, and [**Moc11a**, Prop. 20.1.1]), for any $x_o \in D$ and $\theta_o \in \varpi^{-1}(x_o)$, the germ of $\mathrm{iso}_{\widehat{D}}$ at x_o can be lifted as a germ of isomorphism

$$\mathscr{M}_{\partial\widetilde{X},\theta_o} \xrightarrow{\sim} \mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X},\theta_o}.$$

We can thus find a covering $\mathscr{U}$ of $\varpi^{-1}(U)$ by open subsets where such a lifting exists, and by comparing the liftings on the intersection of two open subsets we obtain a cocycle in $Z^1\bigl(\mathscr{U}, \mathscr{A}ut^{\mathrm{rd}\,D}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})\bigr)$. Two families of liftings on $\mathscr{U}$ define

(2) As pointed out to me by J.-B. Teyssier, this presheaf is already a sheaf, due to the theorem of Malgrange-Sibuya mentioned below.

(3) It could be given a richer structure as in [**Mal83c**, Th. 2.2], see [**Tey19**], but we will not need it.

two cocycles related by the action of a coboundary, so the corresponding class in $H^1\big(\mathscr{U},\mathscr{A}ut^{\mathrm{rd}\,D}(\mathscr{M}^{\mathrm{good}}_{\partial\widetilde{X}})\big)$ is independent of the choice of local liftings. Passing to the limit with respect to the coverings $\mathscr{U}$ leads to the definition of the morphism (2.24) and hence to that of (2.25).

Theorem 2.26. *The morphism of sheaves* (2.25) *is an isomorphism.*

Proof. It is completely similar to that of [**Mal83b**, Th. 3.4], as extended to the case where D is smooth (see [**Mal83c**, §2], see also [**Sab00**, §II.6.d]), where one has to replace S^1 with $\varpi^{-1}(x_o)\simeq(S^1)^\ell$ for some ℓ. The proof of injectivity needs no change, and the proof of surjectivity needs the reference to the (generalized) Malgrange-Sibuya theorem [**Sab13**, Th. 12.2]. □

Corollary 2.27. *Let $U\subset D$ be an open subset and $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_{|U}\in\Gamma(U,\mathscr{H}_D(\mathscr{M}^{\mathrm{good}}))$. The D-formal isomorphism $\mathrm{iso}_{\widehat{D}}$ can be lifted (in a unique way) as an isomorphism $\mathrm{iso}_U:\mathscr{M}_{|U}\xrightarrow{\sim}\mathscr{M}^{\mathrm{good}}_{|U}$ if and only if the image of $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_{|U}$ in $\Gamma(U,\mathcal{S}\mathrm{tT}_D(\mathscr{M}^{\mathrm{good}}))$ is equal to* Id. □

2.g. Generic local constancy. We keep the setting of Section 2.f.

Proposition 2.28. *The sheaf $\mathscr{H}_D(\mathscr{M}^{\mathrm{good}})$ is a locally constant sheaf of pointed sets when restricted to the smooth open subset of D, and its fiber at a smooth point $x_o\in D$ is in bijection with $\mathscr{H}_0(\gamma^+\mathscr{M}^{\mathrm{good}})$ for any germ $\gamma:(\mathbb{C},0)\to(X,x_o)$ transverse to D.*

First proof. It is known ([**Mal83c**, Th. 2.2], see also [**Sab07**, Th. II.6.1 & Cor. II.6.7]) that the sheaf $\mathcal{S}\mathrm{tT}_D(\mathscr{M}^{\mathrm{good}})$ is a *locally constant sheaf* of pointed sets when restricted to the smooth part of D and that its sheaf-theoretic restriction to a germ of a smooth curve transversal to D at a smooth point of D is equal to the sheaf of Stokes torsors of the meromorphic flat bundle that is the restriction to $\mathscr{M}^{\mathrm{good}}$ to this curve. We conclude with Theorem 2.26. □

Second proof. We use the notion of Stokes structure and Stokes filtration as in [**Moc11b**] and [**Sab13**]. The Riemann-Hilbert correspondence in this setting induces bijective correspondence between the isomorphism classes (on $U\subset D^{\mathrm{smooth}}$) of meromorphic flat bundles formally isomorphic to $\mathscr{M}^{\mathrm{good}}_{|U}$ and isomorphism classes of good Stokes-filtered local systems $(\mathscr{L},\mathscr{L}_\bullet)$ on $\varpi^{-1}(U)\subset\partial\widetilde{X}$ whose associated graded object is isomorphic to the Stokes-filtered local system $(\mathscr{L}^{\mathrm{good}},\mathscr{L}^{\mathrm{good}}_\bullet)$ attached to $\mathscr{M}^{\mathrm{good}}_{|U}$.

On the one hand, by applying the general result of [**Moc11b**, Th. 4.13], which we will also use in a more general setting below (see also [**Sab17**, Appendix]), the sheaf on D^{smooth} classifying the Stokes-filtered local systems is a locally constant sheaf of sets compatible with the restriction to the curve γ. The supplementary choice of a D-formal isomorphism $\mathrm{iso}_{\widehat{D}}$ is equivalent to the choice of an isomorphism $\mathrm{iso}_{\mathrm{gr}\,D}:(\mathrm{gr}\,\mathscr{L},\mathrm{gr}\,\mathscr{L}_\bullet)\xrightarrow{\sim}(\mathscr{L}^{\mathrm{good}},\mathscr{L}^{\mathrm{good}}_\bullet)$, which is a thus section of a local system.

It follows that the sets of isomorphism classes of pairs $\big((\mathscr{L},\mathscr{L}_\bullet),\mathrm{iso}_{\mathrm{gr}\,D}\big)_{|U}$ also define, when U varies among open subsets of D, a locally constant sheaf of pointed sets on D^{smooth}. $\square$

Corollary 2.29. *Let $U\subset D$ be a smooth connected open subset and let $(\mathscr{M},\mathrm{iso}_{\widehat{U}})$ and $(\mathscr{M}',\mathrm{iso}'_{\widehat{U}})$ be two elements of $\Gamma(U,\mathscr{H}_D(\mathscr{M}^{\mathrm{good}}))$. Then $(\mathscr{M},\mathrm{iso}_{\widehat{U}})\simeq(\mathscr{M}',\mathrm{iso}'_{\widehat{U}})$ if and only if, for some germ $\gamma:(\mathbb{C},0)\to(X,D)$ transverse to D at a point $\gamma(0)\in U$, we have $\gamma^+(\mathscr{M},\mathrm{iso}_{\widehat{U}})\simeq\gamma^+(\mathscr{M}',\mathrm{iso}'_{\widehat{U}})$.*

Proof. According to Proposition 2.28, this follows from the property that two sections on the connected set U of a locally constant sheaf coincide if and only if they coincide at one point. $\square$

Corollary 2.30. *Let $U\subset D$ be a smooth connected open subset and $(\mathscr{M},\mathrm{iso}_{\widehat{U}})\in\Gamma(U,\mathscr{H}_D(\mathscr{M}^{\mathrm{good}}))$. The D-formal isomorphism $\mathrm{iso}_{\widehat{U}}$ can be lifted (in a unique way) as an isomorphism $\mathrm{iso}_U:\mathscr{M}\xrightarrow{\sim}\mathscr{M}^{\mathrm{good}}_{|U}$ if and only if for some germ $\gamma:(\mathbb{C},0)\to(X,D)$ transverse to D at a point $\gamma(0)\in U$, $\gamma^*\mathrm{iso}_{\widehat{U}}$ lifts as an isomorphism $\gamma^+\mathscr{M}\xrightarrow{\sim}\gamma^+\mathscr{M}^{\mathrm{good}}$.*

Proof. Apply Corollary 2.29 with $(\mathscr{M}',\mathrm{iso}'_{\widehat{U}})=(\mathscr{M}^{\mathrm{good}}_{|U},\mathrm{Id})$. $\square$

Corollary 2.31. *Let $V\subset U$ be two connected nested open subsets contained in the smooth part of D. If the inclusion induces an isomorphism $\pi_1(V,x_o)\xrightarrow{\sim}\pi_1(U,x_o)$ for some $x_o\in V$, then any $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_V\in\Gamma(V,\mathscr{H}_D(\mathscr{M}^{\mathrm{good}}))$ extends in a unique way as $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_U\in\Gamma(U,\mathscr{H}_D(\mathscr{M}^{\mathrm{good}}))$.*

Proof. The data $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_V$ correspond to a section σ on V of the locally constant sheaf of sets $\mathscr{H}_D(\mathscr{M}^{\mathrm{good}})_{|D^{\mathrm{smooth}}}$. Under the assumption in the corollary, such a section extends in a unique way as a section on U. $\square$

2.h. Local constancy in arbitrary codimension. Our aim is to prove the analogue of Proposition 2.28 along higher-codimensional strata of D. Let $x_o\in D$ with stratum $D^{(x_o)}$ of codimension $\ell\geqslant 2$.

Proposition 2.32. *The sheaf $\mathscr{H}_D(\mathscr{M}^{\mathrm{good}})_{|D^{(x_o)}}$ is a locally constant sheaf of pointed sets, and its fiber at a smooth point x_o is in bijection with $\mathscr{H}_{\gamma^{-1}(D)}(\gamma^+\mathscr{M}^{\mathrm{good}})$ for any germ $\gamma:(\mathbb{C}^\ell,0)\to(X,x_o)$ transverse to $D^{(x_o)}$.*

Proof. We will use the same strategy as in the second proof of Proposition 2.28. However, we will use an induction with respect to the rank of $\mathscr{M}^{\mathrm{good}}$ in order to take care of the level structure.

Let us consider the sheaf $\mathscr{H}_{D^{(x_o)}}(\mathscr{M}^{\mathrm{good}}_{|D^{(x_o)}})$ whose sections on an open set $U\subset D^{(x_o)}$ consist of pairs of a germ of a D-meromorphic flat bundle $\mathscr{M}$ on U and an isomorphism $\mathrm{iso}_{\widehat{U}}:\mathscr{M}_{\widehat{U}}\xrightarrow{\sim}\mathscr{M}^{\mathrm{good}}_{\widehat{U}}$ (the formalization is along $D^{(x_o)}$, not along D; that it forms a sheaf and not only a presheaf follows from the uniqueness of isomorphisms compatible with iso). By the same argument and references as in the second proof of Proposition 2.28, we find that this is a locally constant sheaf of pointed sets, whose fiber at x_o is identified by γ^+ with $\mathscr{H}_0(\gamma^+\mathscr{M}^{\mathrm{good}}_{|0})$. We are thus led to proving the following:

(1) If $U\subset D^{(x_o)}$ is simply connected and $x\in U$, then $\mathscr{M}_{\widehat{U}}$ has $\mathscr{M}^{\mathrm{good}}_{|U}$ as a D-*formal* model if and only if a similar assertion holds for $\gamma^+\mathscr{M}_{\widehat{U}}$, where γ is a transversal slice of U at x.

(2) There is a one-to-one correspondence between the set of $\mathrm{iso}_{\widehat{D}|U}$'s lifting $\mathrm{iso}_{\widehat{U}}$ and the corresponding pullbacks by γ^*.

The second point is clear since the sets consist of one element. For the first point, we then do not care about controlling the formal isomorphisms and we argue by induction on the rank of $\mathscr{M}$. If $D(\Phi_{x_o})\neq D_{x_o}$, then the result follows from Example 2.21. Assume now that $D(\Phi_{x_o})=D_{x_o}$. Then $\mathscr{M}_{\widehat{U}}$ has $\mathscr{M}^{\mathrm{good}}_{|U}$ as a D-*formal* model if and only if, for every $c\in C(x_o,\varphi_o)$, $\mathscr{M}_{c|U}$ has $\mathscr{M}^{\mathrm{good}}_{c|U}$ as a D-*formal* model (see Proposition 2.22). Since we can assume $\#C(x_o,\varphi_o)\geqslant 2$ (see Lemma 2.5), the rank of each $\mathscr{M}_c$ is strictly smaller than that of $\mathscr{M}$ and by induction the first point holds for every $\mathscr{M}_{c|U}$. On the other hand, we also have that $\gamma^+\mathscr{M}_{\widehat{U}}$ has $\gamma^+\mathscr{M}^{\mathrm{good}}_{|U}$ as a D-*formal* model if and only if, for every $c\in C(x_o,\varphi_o)$, $\gamma^+\mathscr{M}_{c|U}$ has $\gamma^+\mathscr{M}^{\mathrm{good}}_{c|U}$ as a $\gamma^{-1}(D)$-*formal* model. This gives the first point for $\mathscr{M}_{|U}$. □

2.i. Uniqueness results. We fix $\mathscr{M}^{\mathrm{good}}$ as in (2.14), and we use the notation and definitions of Section 2.a.

Local uniqueness results. We work in a neighbourhood of $x_o\in D$, so that X is the product B^m with coordinates $x_1,\dots,x_m$, where B is a small disc in $\mathbb{C}$, and D is the divisor defined by $x_1\cdots x_\ell=0$ with components $D_i=\{x_i=0\}$. We regard Φ_{x_o} as a finite subset of $\mathbb{C}\{x_1,\dots,x_n\}[(x_1\cdots x_\ell)^{-1}]/\mathbb{C}\{x_1,\dots,x_n\}$, and we assume that it is good. Let $\varpi:\widetilde{X}\to X$ the real oriented blowing up of $D_1,\dots,D_\ell$. Let us also set $D_1^\circ:=D_1\cap D^{\mathrm{smooth}}$. In this setting we have $D^{(x_o)}=\{0\}^\ell\times B^{m-\ell}$ and

$$\widetilde{X}\simeq(S^1)^\ell\times[0,\varepsilon)^\ell\times B^{m-\ell},\qquad \varpi^{-1}(D^{(x_o)})\simeq(S^1)^\ell\times\{0\}^\ell\times B^{m-\ell},$$
$$\partial\widetilde{X}\simeq(S^1)^\ell\times\partial[0,\varepsilon)^\ell\times B^{m-\ell},\qquad \varpi^{-1}(D_1^\circ)\simeq(S^1)^\ell\times\{0\}\times(0,\varepsilon)^{\ell-1}\times B^{m-\ell}.$$

The following result is due to J.-B. Teyssier who proved a much stronger statement, in the sense that it takes into account more structure on the sheaf of Stokes torsors and, moreover, it compares two D-meromorphic flat bundles that have the same good $\mathscr{O}_{\widehat{D^{(x_o)}}}$-model $\mathscr{M}^{\mathrm{good}}$ and that are $\mathscr{O}_{\widehat{D}}$-isomorphic. The present statement will be enough for our purpose, and we will give an independent proof for the sake of completeness.

Proposition 2.33 **(J.-B. Teyssier, [Tey18,** Th. 3 & (2.3.1)]**).** *Two germs $(\mathscr{M},\mathrm{iso}_{\widehat{D}})$, $(\mathscr{M}',\mathrm{iso}'_{\widehat{D}})$ at x_o are isomorphic if and only if their restrictions to a curve transversal to D^{smooth} at one point x near x_o are so.*

As in the second proof of Proposition 2.28, we will use the notion of Stokes-filtered local system. Recall that, in this setting, the restriction functor starting from non-ramified Stokes-filtered local systems $(\mathscr{L},\mathscr{L}_\bullet)$ on $\partial\widetilde{X}$ indexed by Φ_{x_o} (see Convention 2.2) to Stokes-filtered local systems on $\varpi^{-1}(D^{(x_o)})$ indexed by Φ_{x_o} is an equivalence (see [**Moc11b**, Lem. 3.17], see also [**Sab17**, §2.e]). A quasi-inverse functor is obtained by

(a) taking the pullback local system $\mathscr{L}$ by the natural projection forgetting the component $\partial[0,\varepsilon)^\ell$,

(b) taking, for every $\mathscr{L}_{\leqslant\varphi}\subset\mathscr{L}$ ($\varphi\in\Phi_{x_o}$), its pullback in the pullback of $\mathscr{L}$,

(c) and at each point of $\varpi^{-1}(D)$, summing in $\mathscr{L}$ the various $\mathscr{L}_{\leqslant\varphi}$'s for the φ's that coincide near this point.

Proof of Proposition 2.33. We fix $x\in D^{\text{smooth}}$ and denote by D_1° the stratum of x and by $\gamma:(\mathbb{C},0)\to(X,x)$ a curve transversal to D_1° at x.

We first note that it is enough to prove the following statement.

Assertion 2.34. *A germ $\mathscr{M}$ at x_o that satisfies $\mathscr{M}_{\widehat{D},x_o}\simeq\mathscr{M}^{\text{good}}_{\widehat{D},x_o}$ is uniquely determined by its restriction $\mathscr{M}_{\widehat{D}|D_1^\circ}$.*

Indeed, if Assertion 2.34 is proved, assume $\gamma^+(\mathscr{M},\mathrm{iso}_{\widehat{D}})\simeq\gamma^+(\mathscr{M}',\mathrm{iso}'_{\widehat{D}})$. By Corollary 2.29, we obtain that $(\mathscr{M},\mathrm{iso}_{\widehat{D}})_{|D_1^\circ}\simeq(\mathscr{M}',\mathrm{iso}'_{\widehat{D}})_{|D_1^\circ}$. According to the assertion, we then have $\mathscr{M}\simeq\mathscr{M}'$. Let us check that in such a case, $\mathrm{iso}_{\widehat{D}}$ is also uniquely determined by its restriction to D_1°. Since it can be represented by a matrix with entries in $\mathscr{O}_{\widehat{D}}$, it is enough to prove the assertion for sections f of $\mathscr{O}_{\widehat{D}}$ on a neighbourhood of x_o, which follows then from the description of $\mathscr{O}_{\widehat{D}}$ recalled in the proof of Proposition 2.11. Indeed, setting $f=(f_i)_{i=1,\dots,\ell}$, the condition $f_{1|D_1^\circ}=0$ implies $f_1=0$, hence $\widehat{f}_i=0$ in $\mathscr{O}_{\widehat{x}_o}$ for each i, and therefore $f_i=0$ in $\mathscr{O}_{\widehat{D}_i,x_o}$.

We will thus prove Assertion 2.34. Note that, if $D(\Phi_{x_o})\neq D_{x_o}$, Example 2.21 implies that $\mathscr{M}\simeq\mathscr{M}^{\text{good}}$ and there is nothing to prove. However, note that the existence of some $\mathrm{iso}_{\widehat{D}}$ is essential in this case. We will then assume that $D(\Phi_{x_o})=D_{x_o}$. We start with a particular case.

Proof of the assertion in a simple case. We assume here that for all $\varphi\neq\psi\in\Phi_{x_o}$, $\varphi-\psi$ has poles along *each* component of D_{x_o}. Let $(\mathscr{L},\mathscr{L}_\bullet)_{\varpi^{-1}(D_1^\circ)}$ be the Stokes-filtered local system on $\varpi^{-1}(D_1^\circ)$ corresponding to $\mathscr{M}_{|D_1^\circ}$. By our assumption on Φ_{x_o}, it is the pullback by the natural projection

$$\varpi^{-1}(D_1^\circ)=(S^1)^\ell\times\{0\}\times(0,\varepsilon)^{\ell-1}\times B^{m-\ell}\longrightarrow(S^1)^\ell\times\{0\}^\ell\times B^{m-\ell}=\varpi^{-1}(D^{(x_o)})$$

of a Stokes-filtered local system $(\mathscr{L},\mathscr{L}_\bullet)_{\varpi^{-1}(D^{(x_o)})}$ on $\varpi^{-1}(D^{(x_o)})$, because in each summation procedure recalled in (c) above, there is only one term. As a consequence, $(\mathscr{L},\mathscr{L}_\bullet)_{\varpi^{-1}(D^{(x_o)})}$ is uniquely determined from $(\mathscr{L},\mathscr{L}_\bullet)_{\varpi^{-1}(D_1^\circ)}$. We conclude that $\mathscr{M}$ is uniquely determined by its restriction to D_1°. □

Before giving the proof in general, let us recall the level structure of Stokes-filtered local systems indexed by Φ_{x_o} on $\partial\widetilde{X}$ (see [**Moc11a**, §§2.6 & 3.3], [**Sab13**, pp. 41 & 139]). It is parallel to that for D-meromorphic flat bundles considered in Proposition 2.22. In order to simplify the notation, we will assume that $0 \in \Phi_{x_o}$, which we take as base point φ_o, and we use notation as in (2.4). Let us start with a Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ indexed by Φ_{x_o} on $\varpi^{-1}(D^{(x_o)})$. It induces a filtration $\mathscr{L}_{\preceq}$ indexed by $C = C(\varphi_o) \subset \mathbb{C}$, where the order $c \preceq c'$ on the latter set at a point of $\varpi^{-1}(D^{(x_o)})$ is defined as the property that $\exp((c-c')x^{-\boldsymbol{m}_o})$ has moderate growth near this point. By definition, we have

$$\mathscr{L}_{\preceq c} = \sum_{\substack{\psi \in \Phi_{x_o} \\ c(\psi) \leqslant c}} \mathscr{L}_{\leqslant \psi}.$$

Moreover, for every $c \in C$ (recall that we can assume $\#C \geqslant 2$, see Lemma 2.5), the graded sheaf $\operatorname{gr}_c \mathscr{L}$ is a locally constant sheaf, on which the filtration $\mathscr{L}_\bullet$ induces a Stokes-filtration whose jumps are contained in $\Phi_{x_o}(c)$. Lastly, for $\psi \in \Phi_{x_o}(c)$, $\mathscr{L}_{\leqslant\psi}$ is the pullback of $(\operatorname{gr}_c \mathscr{L})_{\leqslant\psi}$ by the projection $\mathscr{L}_{\preceq c} \to \operatorname{gr}_c \mathscr{L}$. By the Riemann-Hilbert correspondence, $(\operatorname{gr}_c \mathscr{L}, (\operatorname{gr}_c \mathscr{L})_\bullet)$ corresponds to $\mathscr{M}_c$ considered in Proposition 2.22.

A similar structure is obtained for a Stokes-filtered local system on $\varpi^{-1}(D_1^\circ)$, by taking pullback and summation of $(\mathscr{L}, \mathscr{L}_\bullet)_{\varpi^{-1}(D_1^\circ)}$ as above.

Proof of Assertion 2.34 *in the general case.* We argue by induction on the rank r of $\mathscr{M}$. It remains to consider the case where $D(\Phi_{x_o}) = D_{x_o}$. Given $\mathscr{M}, \mathscr{M}'$ with $\mathscr{M}_{\widehat{D}} \simeq \mathscr{M}'_{\widehat{D}} \simeq \mathscr{M}^{\text{good}}_{\widehat{D}}$, assume that $\mathscr{M}_{|D_1^\circ} \simeq \mathscr{M}'_{|D_1^\circ}$. It follows that the Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)_{\varpi^{-1}(D_1^\circ)}, (\mathscr{L}', \mathscr{L}'_\bullet)_{\varpi^{-1}(D_1^\circ)}$ are isomorphic, hence we have an isomorphism on $\varpi^{-1}(D_1^\circ)$:

$$\bigl(\mathscr{L}, \mathscr{L}_{\preceq}, (\operatorname{gr}_c \mathscr{L}, (\operatorname{gr}_c \mathscr{L})_\bullet)_{c\in C}\bigr)_{\varpi^{-1}(D_1^\circ)} \overset{(*)}{\simeq} \bigl(\mathscr{L}', \mathscr{L}'_{\preceq}, (\operatorname{gr}_c \mathscr{L}', (\operatorname{gr}_c \mathscr{L}')_\bullet)_{c\in C}\bigr)_{\varpi^{-1}(D_1^\circ)}.$$

Arguing as in the special case for $\mathscr{L}_{\preceq}$ instead of $\mathscr{L}_{\leqslant}$, we obtain an isomorphism $(\mathscr{L}, \mathscr{L}_{\preceq})_{\varpi^{-1}(D^{(x_o)})} \simeq (\mathscr{L}', \mathscr{L}'_{\preceq})_{\varpi^{-1}(D^{(x_o)})}$ and, for each $c \in C$, the induced isomorphism $\operatorname{gr}_c \mathscr{L}_{\varpi^{-1}(D^{(x_o)})} \simeq \operatorname{gr}_c \mathscr{L}'_{\varpi^{-1}(D^{(x_o)})}$ corresponds to that induced by $(*)$ (by pullback by the projection). Since $\mathscr{M}_{c,\widehat{D}} \simeq \mathscr{M}'_{c,\widehat{D}} \simeq \mathscr{M}^{\text{good}}_{c,\widehat{D}}$ have rank $< r$, we conclude by induction that the component

$$(*)_c : (\operatorname{gr}_c \mathscr{L}, (\operatorname{gr}_c \mathscr{L})_\bullet)_{\varpi^{-1}(D_1^\circ)} \xrightarrow{\;\sim\;} (\operatorname{gr}_c \mathscr{L}', (\operatorname{gr}_c \mathscr{L}')_\bullet)_{\varpi^{-1}(D_1^\circ)}$$

of $(*)$ comes from an isomorphism $\mathscr{M}_c \simeq \mathscr{M}'_c$, that is, an isomorphism on $\varpi^{-1}(D^{(x_o)})$:

$$(\operatorname{gr}_c \mathscr{L}, (\operatorname{gr}_c \mathscr{L})_\bullet)_{\varpi^{-1}(D^{(x_o)})} \simeq (\operatorname{gr}_c \mathscr{L}', (\operatorname{gr}_c \mathscr{L}')_\bullet)_{\varpi^{-1}(D^{(x_o)})}.$$

As a consequence, the isomorphism $\mathscr{L}_{\preceq c} \xrightarrow{\sim} \mathscr{L}'_{\preceq c}$ sends $\mathscr{L}_{\leqslant\psi}$ isomorphically to $\mathscr{L}'_{\leqslant\psi}$ on $\varpi^{-1}(D^{(x_o)})$ for any $\psi \in \Phi_{x_o}(c)$: indeed, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{L}_{\preceq c} & \xrightarrow{\sim} & \mathscr{L}'_{\preceq c} \\ \downarrow & & \downarrow \\ \mathrm{gr}_c \mathscr{L} & \xrightarrow{\sim} & \mathrm{gr}_c \mathscr{L}' \\ \cup & & \cup \\ (\mathrm{gr}_c \mathscr{L})_{\leqslant\psi} & \xrightarrow{\sim} & (\mathrm{gr}_c \mathscr{L}')_{\leqslant\psi} \end{array}$$

and $\mathscr{L}_{\leqslant\psi}$ (resp. $\mathscr{L}'_{\leqslant\psi}$) is the pullback of $(\mathrm{gr}_c \mathscr{L})_{\leqslant\psi}$ (resp. $(\mathrm{gr}_c \mathscr{L}')_{\leqslant\psi}$) by the projection $\mathscr{L}_{\preceq c} \to \mathrm{gr}_c \mathscr{L}$ (resp. $\mathscr{L}'_{\preceq c} \to \mathrm{gr}_c \mathscr{L}'$).

We conclude that $(\mathscr{L}, \mathscr{L}_\bullet) \simeq (\mathscr{L}', \mathscr{L}'_\bullet)$ on $\varpi^{-1}(D^{(x_o)})$, as was to be proved. □

Global results. We now go back to the global setting of Section 2.a. In particular, D is connected and $\mathscr{M}^{\mathrm{good}}$ is fixed. The following result is due to J.-B. Teyssier, who has obtained a stronger form.

***Corollary 2.35* (J.-B. Teyssier [Tey18, Proof of Th. 4]).** *Given $\mathscr{M}^{\mathrm{good}}$-marked D-meromorphic flat bundles $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})$ and $(\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ on X, we have $(\mathscr{M}, \mathrm{iso}_{\widehat{D}}) \simeq (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ if and only if there exists a point x in D^{smooth} and a germ of curve $\gamma : (\mathbb{C}, 0) \to (X, x)$ transverse to D at x such that $(\gamma^+\mathscr{M}, \mathrm{iso}_{\widehat{0}}) \simeq (\gamma^+\mathscr{M}', \mathrm{iso}'_{\widehat{0}})$.*

Proof. For any two germs $(\mathscr{M}, \mathrm{iso}_{\widehat{D}}), (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$, let $D' \subset D$ be the subset of points where they are isomorphic. Assume that it is not empty. It is open by definition. Let us show that it is closed. Let $x_o \in \overline{D'}$.

- Either $x_o \in D^{\mathrm{smooth}}$, in which case D' contains a point x' in the component of D^{smooth} containing x_o, hence also contains this whole component according to Corollary 2.29, and therefore $D' \ni x_o$,
- or $x_o \notin D^{\mathrm{smooth}}$, in which case we apply Proposition 2.33 to also conclude that $x_o \in D'$.

Therefore, D' is closed, hence equal to D. The "only if" part of the proposition is clear, and for the "if" part we apply the previous argument, since we know by Corollary 2.29 that $D' \neq \varnothing$. □

We will make use of the following consequence, which is a weaker version of [**Tey18**, Th. 3]. Although its statement is local, it relies on the global result of Corollary 2.35.

***Corollary 2.36* (J.-B. Teyssier).** *Assume that D is smooth. Let $\Psi_{x_o} \subset \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ be a (not necessarily good) finite subset, and, for any $\psi \in \Psi_{x_o}$, let $\mathscr{R}_\psi$ be a germ of regular D-meromorphic flat bundle at x_o. Set $\mathscr{N} = \bigoplus_{\psi \in \Psi_{x_o}} (\mathscr{E}^\psi \otimes \mathscr{R}_\psi)$. Let $(\mathscr{M}, \mathrm{iso}_{\widehat{D}}), (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ be two $\mathscr{N}$-marked D-meromorphic flat bundles in the neighbourhood of x_o and let $\gamma : (\mathbb{C}, 0) \to (X, D)$ be a germ of curve whose image is not* *contained in D. Then*

$$(\mathscr{M}, \mathrm{iso}_{\widehat{D}}) \simeq (\mathscr{M}', \mathrm{iso}'_{\widehat{D}}) \iff \gamma^+(\mathscr{M}, \mathrm{iso}_{\widehat{D}}) \simeq \gamma^+(\mathscr{M}', \mathrm{iso}'_{\widehat{D}}).$$

Proof. The point is to prove the implication $\Leftarrow$ and we assume that the isomorphism of the right-hand side holds for some curve γ. We shall denote by X a small (connected) neighbourhood of x_o where all data are defined. Let us first assume that $\gamma(0) = x_o$. There exists (see [**Sab13**, Lem. 9.11]) a projective modification $e : X' \to X$ that is an isomorphism away from D and the normal crossing divisor $D' := e^{-1}(D)$ such that $e^+\mathscr{N}$ is good. This result is much easier than the general result on the resolution of turning points obtained by K. Kedlaya [**Ked11**] (and T. Mochizuki [**Moc11a**] in the algebraic case). We can moreover assume that γ lifts as a curve γ' that is transverse to $e^{-1}(x_o)$ at a smooth point. We can apply Corollary 2.35 to $e^+(\mathscr{M}, \mathrm{iso}_{\widehat{D}}), e^+(\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$, and deduce that both are isomorphic. It follows from Lemma 2.10 that $(\mathscr{M}, \mathrm{iso}_{\widehat{D}}), (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})$ are isomorphic.

Assume now that $x := \gamma(0) \neq x_o$. The first part of the proof applied at x shows that $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})_{|\mathrm{nb}(x)} \simeq (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})_{|\mathrm{nb}(x)}$. Let U be the open subset of D on which $\mathscr{N}$ is good. This is the complement in D of the union of the zero sets of the meromorphic functions $\psi - \eta$ for $\psi, \eta \in \Psi_{x_o}$ and $\psi \neq \eta$. Since D is smooth, this set is connected. By Corollary 2.29, we deduce an isomorphism $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})_{|U} \simeq (\mathscr{M}', \mathrm{iso}'_{\widehat{D}})_{|U}$. We conclude with Lemma 2.12. □

3. Proof of the main results

We consider the setting and notation of Theorem 1.3.

3.a. The most degenerate case. If all coordinates of t_o coincide, there is only one class $(\gamma^+_{t_o}\mathscr{M}, \gamma^+_{t_o}\mathrm{iso}_{\widehat{T}})$ since, up to a twist by $\mathscr{E}^{t_{o,1}/z}$, $\gamma^+_{t_o}\mathscr{N}$ has a regular singularity at $z = 0$. The restriction induces then a surjective map between the two corresponding sets of isomorphism classes, and the injectivity is a consequence of the following more precise proposition.

Proposition 3.1. *Assume that the connected open subset U contains such a t_o. Let $(\mathscr{M}_U, \mathrm{iso}_{\widehat{U}}) \in \Gamma(U, \mathscr{H}_T(\mathscr{N}))$. Then $\mathrm{iso}_{\widehat{U}}$ can be lifted (in a unique way) as an isomorphism $\mathrm{iso}_U : \mathscr{M}_U \xrightarrow{\sim} \mathscr{N}_{|U}$.*

Proof. We will argue when t_o is the origin of T, the other cases being obtained by an exponential twist. Let $e : X' \to X$ be the blowing-up of the origin in X and let T' be the strict transform of T by e, so that $e_{|T'} : T' \to T$ is nothing but the blowing-up of T at the origin. Let $E = e^{-1}(0) \simeq \mathbb{P}^n$ be the exceptional divisor, so that $D := E \cup T'$ is a divisor with normal crossings.

There is a chart with coordinates $(u_1,\dots,u_n,\zeta)$ such that E is defined as $\zeta=0$ and e is given by $(u_1,\dots,u_n,\zeta)\mapsto(u_1\zeta,\dots,u_n\zeta,\zeta)$, so that $e^+\mathscr{N}$ has regular singularities along E away from $E\cap T'\simeq\mathbb{P}^{n-1}$. On the other hand, one checks that $e^+\mathscr{N}$ is good on a Zariski dense open subset of $E\cap T'$, e.g. by computing the meromorphic functions $e^*((t_i-t_j)/z)$ in a chart with coordinates $(v_1,\dots,v_n,\zeta)$ where e is given (say) by $(v_1,\dots,v_n,\zeta)\mapsto(v_1,v_1v_2,\dots,v_1v_n,v_1\zeta)$.

Let t'_o be a point in this open subset of $E\cap T'$, which we can also consider as a point in X'. At this point, $e^+\mathscr{N}$ has thus regular singularities along one of the components of D passing through t'_o. From Example 2.21 we conclude that $e^+\mathscr{M}_U\simeq e^+\mathscr{N}$ in some open neighbourhood of t'_o. Intersecting this neighbourhood with $U\smallsetminus\Delta$ gives a non-empty open subset in $U\smallsetminus\Delta$ where $\mathrm{iso}_{\widehat{U}}$ can be lifted as an isomorphism $\mathscr{M}_U\xrightarrow{\sim}\mathscr{N}$. We deduce from Corollary 2.30, applied to the connected open subset $U\smallsetminus\Delta$, that $\mathrm{iso}_{\widehat{U}}$ can be lifted as an isomorphism $\mathrm{iso}_{U\smallsetminus\Delta}:\mathscr{M}_{|U\smallsetminus\Delta}\xrightarrow{\sim}\mathscr{N}_{|U\smallsetminus\Delta}$. We now conclude with Lemma 2.12. □

3.b. Proof of a variant of Theorem 1.3 in a special case. As an example for the method of proof of Theorem 1.3, we develop in this section a low-dimensional case, obtained by a generic two-dimensional slice of the pair (T,Δ) considered in the general case.

The setting is as follows. We have $\dim T=2$, with coordinates $t=(t_1,t_2)$. We fix $a,b\in\mathbb{C}$ such that $a,b,a-b\neq0$ and we consider the functions

$$f_1(t)=t_1,\quad f_2(t)=t_1+t_2,\quad f_3(t)=t_1+a,\quad f_4(t)=t_2+b,$$

and the elementary T-meromorphic flat bundle

$$\mathscr{N}=\bigoplus_{i=1}^4(\mathscr{E}^{-f_i(t)/z}\otimes\mathscr{R}_i),$$

with $\mathscr{R}_i$ regular along $T=\{z=0\}$. Then $\mathscr{N}$ is good away from $\Delta=\{t_2=0\}$, so that the stratum is reduced to Δ and we set $U=T$. We also set

$$\mathscr{N}_o=\bigl(\mathscr{E}^{-f_1(t)/z}\otimes(\mathscr{R}_1\oplus\mathscr{R}_2)\bigr)\oplus\bigoplus_{i=3}^4(\mathscr{E}^{-f_i(t)/z}\otimes\mathscr{R}_i),$$

so that $\mathscr{N}_o$ is a good decomposed T-meromorphic flat bundle.

We denote by $e_1:X_1\to X$ the blowing-up of the origin in X. We set $E_1=e_1^{-1}(0)\simeq\mathbb{P}^2$, we denote by T_1 the strict transform of T by e_1, so that $e_{1|T_1}:T_1\to T$ is the blowing-up of the origin in T. The strict transform Δ_1 of Δ is contained in T_1 and intersects $e_{1|T_1}^{-1}(0)=E_1\cap T_1\simeq\mathbb{P}^1$ at one point δ_1. The restriction $e_{1|\Delta_1}:\Delta_1\to\Delta$ is an isomorphism.

We denote by $e_2:X_2\to X_1$ the blowing-up of Δ_1 in X_1. We set

$$E_2=e_2^{-1}(\Delta_1)\simeq\mathbb{P}^1\times\Delta_1,$$

we denote by $E_{1,2}$ the strict transform of E_1, so that $e_{2|E_{1,2}}:E_{1,2}\to E_1$ is the blowing-up of δ_1 in E_1 and $e_{2|E_{1,2}}^{-1}(\delta_1)\simeq\mathbb{P}^1\times\{\delta_1\}$. It is standard to check that $E_{1,2}$,

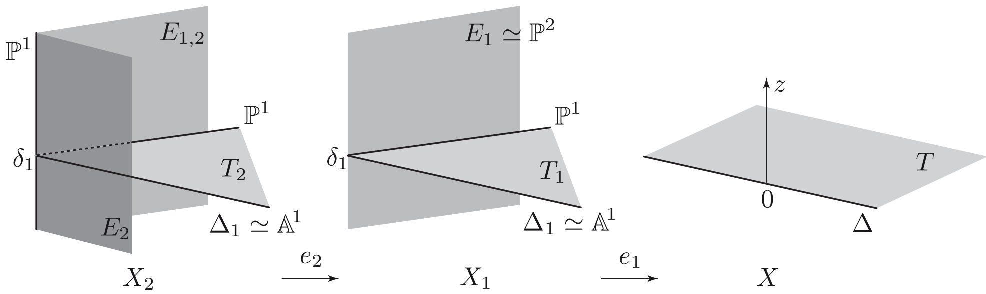


FIGURE 1. Simplified representation of the blowing-ups e_1 and e_2.

being the complex blow-up of $\mathbb{P}^2$ at one point, is simply connected.(4) Let T_2 be the strict transform of T_1. Then $e_{2|T_2} : T_2 \to T_1$ is an isomorphism since it is the blowing-up of Δ_1 of codimension one in T_1. We also have $T_2 \cap E_{1,2} \simeq \mathbb{P}^1$ since it is the blow-up of $\mathbb{P}^1$ at δ_1. We thus regard δ_1 as a point in X_2 and Δ_1 as the subset $T_2 \cap E_2$ in X_2. The geometric setting at δ_1 in X_2 is pictured in Figure 1.

We set $e = e_1 \circ e_2 : X_2 \to X$. We denote by $\varpi : \widetilde{X}_2 \to X_2$ the real oriented blowing-up of X_2 along the components $(T_2, E_{1,2}, E_2)$ of $D := e^{-1}(T)$.

Lemma 3.2. *The sheaf* $\mathcal{S}t\mathrm{T}_D(e^+\mathscr{N}_o)$ *is locally constant on* $e^{-1}(T)$.

Proof. Using the notation of Section 2.f, we note that the pushforward e_* induces an isomorphism $\mathscr{H}_D(e^+\mathscr{N}_o) \xrightarrow{\sim} e^{-1}\mathscr{H}_T(\mathscr{N}_o)$. By Proposition 2.28, $\mathscr{H}_T(\mathscr{N}_o)$ is a locally constant sheaf on T, hence so is $\mathscr{H}_D(e^+\mathscr{N}_o)$ on D. The assertion follows then from Theorem 2.26. □

Lemma 3.3. *The D-meromorphic flat bundle $e^+\mathscr{N}$ is good and the sheaf* $\mathcal{S}t\mathrm{T}_D(e^+\mathscr{N})$ *is constant when restricted to* $E_{1,2}$.

Proof. Since $E_{1,2}$ is simply connected, it is enough to prove local constancy. Let us write down the charts of the various blow-ups. We cover X_1 by three charts $X_1(i)$ $(i = 1, 2, 3)$, with coordinates (u_i, v_i, ζ_i) so that e_1 is given respectively by the formulas:

$$X_1(1) : \begin{cases} t_1 = u_1\zeta_1, \\ t_2 = v_1\zeta_1, \\ z = \zeta_1, \end{cases} \qquad X_1(2) : \begin{cases} t_1 = u_2v_2, \\ t_2 = v_2, \\ z = v_2\zeta_2, \end{cases} \qquad X_1(3) : \begin{cases} t_1 = u_3, \\ t_2 = u_3v_3, \\ z = u_3\zeta_3. \end{cases}$$

We note that the charts $X_1(1)$ and $X_1(2)$ can be regarded as contained in X_2, since they do not intersect the center $\Delta_1 = \{u_3 = \zeta_3 = 0\}$ of the blowing-up e_2.

(4) One can regard $E_{1,2}$ as the union of $\mathbb{C}^2$ and the union of two $\mathbb{P}^1$ that intersect at one point; both sets are simply connected, and the second one has an open neighbourhood W that retracts onto it, hence is also simply connected. Since $W \cap \mathbb{C}^2$ is connected, the van Kampen theorem gives the result.

(1) In the chart $X_1(1)$, we have $X_1(1) \cap T_1 = \varnothing$ and $X_1(1) \cap E_1 = \{\zeta_1 = 0\}$. One checks that $e_1^+\mathscr{N}$ is good there, and more precisely $\exp e_1^*(-t_2/z)$ is holomorphic there and the block-diagonal morphism $(\mathrm{Id}, \exp e_1^*(-t_2/z)\,\mathrm{Id}, \mathrm{Id}, \mathrm{Id})$ induces an isomorphism between $e_1^+\mathscr{N}_o$ and $e_1^+\mathscr{N}$.

(2) In the chart $X_1(2)$, we have $X_1(2) \cap T_1 = \{\zeta_2 = 0\}$ and $X_1(2) \cap E_1 = \{v_2 = 0\}$. One checks that $e_1^+\mathscr{N}$ is good there. Although $\exp e_1^*(-t_2/z) = \exp(-1/\zeta_2)$ is not holomorphic near $E_1 \cap T_1$, we claim that the block-diagonal morphism corresponding to $(\mathrm{Id}, \exp e_1^*(-t_2/z)\,\mathrm{Id}, \mathrm{Id}, \mathrm{Id})$ induces an isomorphism between $\mathscr{A}ut^{\mathrm{rd}\,D}(e_1^+\mathscr{N}_o)$ and $\mathscr{A}ut^{\mathrm{rd}\,D}(e_1^+\mathscr{N})$ on $\varpi^{-1}(E_1 \cap T_1)$. Indeed, this morphism only affects the blocks ij with $i \neq j$ and i or j equal to 2. The blocks 12 and 21 of a section of $\mathscr{A}ut^{\mathrm{rd}\,D}(e_1^+\mathscr{N})$ are zero, since $\exp e_1^*((f_2 - f_1)/z) = \exp e_1^*(-t_2/z)$ is nowhere of rapid decay, and so are the blocks 12 and 21 of $\mathscr{A}ut^{\mathrm{rd}\,D}(e_1^+\mathscr{N}_o)$. For the blocks 23, 32, 24 and 34, which may be nonzero, multiplying by $\exp e_1^*(-t_2/z)$ does not affect the leading term of the exponential, hence neither does it affect the rapid decay condition.

At this step, we have proved that $\mathscr{A}ut^{\mathrm{rd}\,D}(e_1^+\mathscr{N}_o)$ and $\mathscr{A}ut^{\mathrm{rd}\,D}(e_1^+\mathscr{N})$ are isomorphic on $\varpi^{-1}(E_{1,2}) \smallsetminus \varpi^{-1}(E_2)$, hence so are $\mathcal{S}\mathrm{tT}_D(e^+\mathscr{N}_o)$ and $\mathcal{S}\mathrm{tT}_D(e^+\mathscr{N})$ on $E_{1,2} \smallsetminus E_2$.

(3) We now blow up the chart $X_1(3)$ along the ideal (v_3, ζ_3) of Δ_1, giving rise to the charts $X_2(3\mathrm{a})$ and $X_2(3\mathrm{b})$ of X_2, with respective coordinates $(u_3, w_1, \eta_1), (u_3, w_2, \eta_2)$ satisfying

$$X_2(3\mathrm{a}) : \begin{cases} v_3 = w_1\eta_1, \\ \zeta_3 = \eta_1, \end{cases} \qquad X_2(3\mathrm{b}) : \begin{cases} v_3 = w_2, \\ \zeta_3 = w_2\eta_2. \end{cases}$$

Then one checks that $e^+\mathscr{N}$ is good and the same argument as given in (2) above gives an isomorphism on $\varpi^{-1}(E_{1,2})$ between $\mathscr{A}ut^{\mathrm{rd}\,D}(e^+\mathscr{N}_o)$ and $\mathscr{A}ut^{\mathrm{rd}\,D}(e^+\mathscr{N})$, hence $\mathcal{S}\mathrm{tT}_D(e^+\mathscr{N}_o)$ and $\mathcal{S}\mathrm{tT}_D(e^+\mathscr{N})$ are isomorphic on $E_{1,2}$. The lemma follows from Lemma 3.2. $\square$

Proof of the surjectivity of $\gamma_{t_o}^+$. Recall that, here, $S_o = \Delta$ and that we have fixed $t_o \in \Delta$.

Step 1: extension to $E_{1,2}$. The curve γ_{t_o} lifts as a curve $\gamma_{t'_o} : (\mathbb{C}, 0) \to (X_2, t'_o)$ transverse to $E_{1,2}$ at a point t'_o in D^{smooth}. We have $\mathscr{H}_0(i_{t_o}^+\mathscr{N}) = \mathscr{H}_0(i_{t'_o}^+e^+\mathscr{N})$. By Proposition 2.28, this set is identified with $i_{t'_o}^{-1}\mathscr{H}_D(e^+\mathscr{N})$. Since $\mathscr{H}_D(e^+\mathscr{N})_{|E_{1,2}}$ is constant, according to Lemma 3.3 and Theorem 2.26, this set is also identified with $\Gamma(E_{1,2}, \mathscr{H}_D(e^+\mathscr{N}))$. An element of $\mathscr{H}_0(i_{t_o}^+\mathscr{N})$ defines thus a pair $(\mathscr{M}, \mathrm{iso}_{\widehat{D}})$ on some neighbourhood of $E_{1,2}$ in X_2.

Step 2: extension to $E_{1,2} \cup \Delta_1$. Now we apply Proposition 2.32 to our section of $\mathscr{H}_D(e^+\mathscr{N})_{|E_{1,2}}$. Since Δ_1 retracts to a neighbourhood of δ_1, the section extends to a neighbourhood of Δ_1 in a unique way.

Step 3*: extension to* $E_{1,2}\cup E_2$. The restriction to $(E_{1,2}\cap E_2)\smallsetminus\{\delta_1\}$ of the section of $\mathscr{H}_D(e^+\mathscr{N})_{|E_{1,2}}$ constructed in Step 1 extends uniquely as a section of $\mathscr{H}_D(e^+\mathscr{N})_{E_2\smallsetminus\Delta_1}$ by Proposition 2.28. The restriction of the section constructed in Step 2 to a punctured neighbourhood of Δ_1 in E_2 coincides with the restriction to this punctured neighbourhood of this new extension, by uniqueness, since they coincide in the neighbourhood of δ_1.

Step 4*: extension to* $D=E_{1,2}\cup E_2\cup T_2$. The section constructed in Step 3 exists in some neighbourhood of $E_{1,2}\cup E_2$ in D. There exists a fundamental basis of neighbourhoods U of $E_{1,2}\cup E_2$ in D such that the inclusion $U\subset D$ induces an isomorphism $\pi_1(U)\xrightarrow{\sim}\pi_1(D)$, as seen by taking the pullback by e of a suitable basis of neighbourhoods of Δ in T. By arguing like in Corollary 2.31, the section constructed in Step 3, defined on such a neighbourhood U, extends in a unique way to a section of $\mathscr{H}_D(e^+\mathscr{N})$ defined on D. Lemma 2.10 enables us to conclude. □

3.c. Proof of Theorem 1.3. We now consider the general case in Theorem 1.3. Our first aim is to prove, in the context of Theorem 1.3, an analogue of Propositions 2.28 and 2.32 on each stratum of Δ. Let $t_o\in\Delta\subset T$ and let $S(t_o)$ be its stratum.

Proposition 3.4. *When restricted to* $S(t_o)$*, the sheaf* $\mathscr{H}_T(\mathscr{N})$ *is a locally constant sheaf of pointed sets, and its fiber at* t_o *is in bijection with* $\mathscr{H}_0(\gamma_{t_o}^+\mathscr{N})$*.*

Proof. We fix $t_o\in\Delta\subset T$ and we work locally at t_o. We can decompose $\{1,\dots,n\}$ as $\bigsqcup_{r\in R}I_r$ such that, for every $r\in R$, we have $\{i,j\}\subset I_r$ if and only if $t_{o,i}=t_{o,j}$. For each $r\in R$, we choose an element in I_r that we denote by r and we set $I'_r=I_r\smallsetminus\{r\}$. We then set $p=\#R=\dim S(t_o)$, $m=n-p$, and

$$\mathscr{N}_o=\bigoplus_{r\in R}(\mathscr{E}^{-t_r/z}\otimes\mathscr{R}_{I_r}),\quad \mathscr{R}_{I_r}:=\bigoplus_{i\in I_r}\mathscr{R}_i. \tag{3.5}$$

We fix a neighbourhood of t_o in T of the form $V\times W$, such that V has the coordinates $(t_r)_{r\in R}$ and W has coordinates $(\tau_i^r)_{r\in R,\,i\in I'_r}$, so that for $r\in R$ and $i\in I'_r$, we have $t_i=t_r+\tau_i^r$, and small enough such that $\Delta\cap(V\times W)$ is given by the equations $\tau_i^r=0$ ($r\in R$, $i\in I'_r$) and $\tau_i^r-\tau_j^r$ ($r\in R$, $i\neq j\in I'_r$). We have $S(t_o)\cap(V\times W)=V\times\{0\}$. We now denote this neighbourhood by T, and set $X=(V\times W)\times\mathbb{C}_z=:V\times Y$.

Let $e_1:X_1\to X$ be the blowing-up of this stratum, i.e., that of the ideal

$$\big((\tau_i^r)_{r\in R,\,i\in I'_r},z\big).$$

We have $X_1=V\times Y_1$ with obvious notation. Every object below is a product of V with the corresponding object in Y_1. The pullback $D_1:=e_1^{-1}(T)$ is the union of the exceptional divisor $E_1=e_1^{-1}(S(t_o))$ (in this local setting, $E_1\simeq S(t_o)\times\mathbb{P}^m$), and the strict transform T_1 of T. The exceptional divisor of $e_{|T_1}:T_1\to T$ is equal to $E_1\cap T_1\simeq S(t_o)\times\mathbb{P}^{m-1}$. Moreover, T_1 is a disc-bundle in the normal bundle of $E_1\cap T_1$ in T_1. Similarly, the strict transform $\Delta_1\subset T_1$ of Δ is a disc-bundle over $\delta_1:=E_1\cap\Delta_1$ (the product of an arrangement of projective hyperplanes in $\mathbb{P}^{m-1}$ with $S(t_o)$).

We denote by Δ'_1 the union of the two-by-two intersections of the components of Δ_1 (it has codimension three in X_1) and by δ'_1 its intersection with E_1. Lastly, we set $X_1^\circ = X_1 \smallsetminus \Delta'_1$, and similarly for the other objects. In particular, δ_1° is non-singular and Δ_1° is a disc-bundle over it.

We now argue as for the simple case of Section 3.b. We denote by $e_2 : X_2^\circ \to X_1^\circ$ the blowing-up of Δ_1° in X_1° and we obtain $D_2^\circ := E_2^\circ \cup E_{1,2}^\circ \cup T_2^\circ$. Then Lemmas 3.2 and 3.3 hold in this setting. The isomorphism to be considered is block diagonal with blocks indexed by R, each block having the form $(\mathrm{Id}, (\exp(-e^*\tau_i^r)\,\mathrm{Id})_{i\in I'_r})$.

For the surjectivity of $\gamma_{t_o}^+$, there is no change to be done in Step 1 of Section 3.b. For the other steps, we use the disc-bundle structure over the intersection with $E_{1,2}^\circ$ of all the objects involved, instead of the structure of a product with $\mathbb{C}$. The same homotopy argument applies.

Let us now conclude with the surjectivity of $\gamma_{t_o}^+$. Given $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$ with model $\gamma_{t_o}^+\mathscr{N}$, corresponding to an element of

$$\mathscr{H}_0(\gamma_{t_o}^+\mathscr{N}) \simeq \mathscr{H}_0(\gamma_{t'_o}^+ e^+\mathscr{N}) \simeq \mathscr{H}_{D_2^{\mathrm{smooth}}}(e^+\mathscr{N})_{t'_o},$$

we have extended it in a unique way as a section of $\mathscr{H}_{D_2^\circ}(e^+\mathscr{N})$, which corresponds thus to a pair $(\mathscr{M}_2^\circ, \mathrm{iso}_{\widehat{D_2^\circ}})$ with model $e^+\mathscr{N}$ satisfying $\gamma_{t'_o}^+(\mathscr{M}_2^\circ, \mathrm{iso}_{\widehat{D_2^\circ}}) = (\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$. Applying e_{2+} and according to Lemma 2.10, we obtain a pair $(\mathscr{M}_1^\circ, \mathrm{iso}_{\widehat{D_1^\circ}})$ on X_1°. Owing to the theorem of B. Malgrange (Proposition 2.11), this pair extends in a unique way as a pair $(\mathscr{M}_1, \mathrm{iso}_{\widehat{D_1}})$ on X_1. Lastly, applying e_{1+} and according to Lemma 2.10, we obtain a pair $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ as wanted.

On the other hand, injectivity of $\gamma_{t_o}^+$ is given by the proof above, since there is no choice in any extension procedure. This shows that, on V, the sheaf $\mathscr{H}_T(\mathscr{N})_{|S(t_o)}$ is constant, with fibre given by applying $\gamma_{t_o}^+$. □

End of the proof of Theorem 1.3. We first define the notion of a star-shaped open set. Let S_o be a stratum of Δ and set $S_o^\star = \bigcup_{\overline{S}\supset S_o} \overline{S}$ be its star (where S varies in the set of strata of the natural stratification of Δ, so that $\overline{S}$ is a linear subspace). By choosing the coordinates as in the proof of Proposition 3.4, we find a product decomposition $S_o^\star \simeq S_o \times \mathbb{C}^m$, and we consider the corresponding projection $p_o : S_o^\star \to S_o$. We endow $\mathbb{C}^m$ with its standard Euclidean metric.

Definition 3.6. An open subset $U \subset T$ is said to be *star shaped* with respect to $U \cap S_o$ if U contains the Euclidean ball centered at the origin of the linear subspace $\overline{S(t)} \cap p_o^{-1}(p_o(t))$ containing t, for any $t \in U$.

For a star-shaped open set U with respect to $U \cap S_o$, the flow of the radial vector field in each stratum S containing S_o in its closure induces a deformation retraction of $U \cap S$ to $U \cap S_o$. On the other hand, there exists a fundamental system of open neighbourhoods V_S of $U \cap S_o$ in $U \cap \overline{S}$ that are star shaped. So $V_S \cap S \subset U \cap S$ induces an isomorphism of fundamental groups.

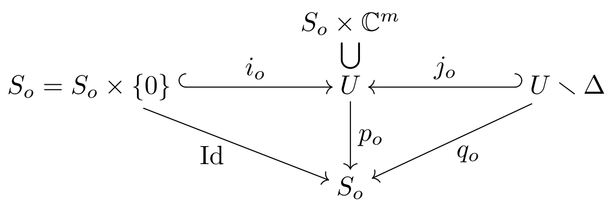
Assume we are given $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$ with model $\gamma_{t_o}^+\mathscr{N}$ for $t_o \in S_o$. Since $\mathscr{H}_T(\mathscr{N})_{|S_o}$ is locally constant with fibre $\mathscr{H}_0(\gamma_{t_o}^+\mathscr{N})$ (Proposition 3.4) and $U \cap S_o$ is simply connected, $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$ extends in a unique way as a section of $\mathscr{H}_T(\mathscr{N})_{|U\cap S_o}$, and we find $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ defined in some neighbourhood of $U \cap S_o$ in T. Given a stratum S with $\overline{S} \supset S_o$, $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ is defined on some V_S as above and defines a section of $\mathscr{H}_T(\mathscr{N})_{|V_S\cap S}$. Since $\mathscr{H}_T(\mathscr{N})_{|S}$ is locally constant and since $\pi_1(V_S \cap S, \star) \to \pi_1(U \cap S, \star)$ is an isomorphism, this section extends to $U \cap S$ (see Corollary 2.31) and we obtain $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ on $U \cap S$, hence on a neighbourhood of $U \cap S$ in U. Since U does not cut any stratum S' such that $\overline{S'} \supsetneq S_o$, it is covered by the strata $U \cap S$ with S such that $\overline{S} \supset S_o$. By uniqueness (Corollary 2.36), the extensions on the neighbourhoods of the various strata glue together, and give rise to $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ on U. □

3.d. Proof of Corollaries 1.4 and 1.6. The first part of Corollary 1.4 is contained in the theorem. For the second part, let $t_o \in V$ and let $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})_V$ and $(\mathscr{M}', \mathrm{iso}'_{\widehat{T}})_V$ be two $\mathscr{N}$-marked T-meromorphic flat bundles on $V \times (\mathbb{C}_z, 0)$ whose restrictions at t_o are equal to $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$. Then, by Theorem 1.3, $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})_V$ and $(\mathscr{M}', \mathrm{iso}'_{\widehat{T}})_V$ coincide on some $\mathrm{nb}(t_o) \subset T$, hence on a nonempty open set in $V \smallsetminus (\Delta \cap V)$. Since the latter is connected, they coincide on $V \smallsetminus (\Delta \cap V)$, by an argument similar to that used in Corollary 2.30. We conclude with Lemma 2.12. □

Proof of Corollary 1.6. The surjectivity of $\gamma_{y_o}^+$ and the injectivity of f^+ are obvious from Theorem 1.3, since we can identify $(\mathscr{M}^{y_o}, \mathrm{iso}_{\widehat{0}})$ with $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$, due to the identification $\gamma_{y_o}^+\mathscr{N}_Y = \gamma_{t_o}^+\mathscr{N}$, and we have $\gamma_{t_o}^+ = \gamma_{y_o}^+ \circ f^+$. It is then enough to prove the injectivity of $\gamma_{y_o}^+$. This is an immediate consequence of Corollary 2.36. □

3.e. Application to the sheaf of Stokes torsors. We note that the restriction of $\mathscr{N}$ to any stratum of the natural stratification of T compatible with Δ is good, hence for each such stratum S, the sheaf $\mathscr{H}_S(i_S^+\mathscr{N}) \simeq \mathcal{S}\mathrm{tT}_S(i_S^+\mathscr{N})$ is a locally constant sheaf. On the other hand, we do not have much information on the sheaf $\mathcal{S}\mathrm{tT}_T(\mathscr{N})$, except on the open dense stratum $T \smallsetminus \Delta$. However, the sheaf $\mathscr{H}_T(\mathscr{N})$ is better behaved, and this will enable us to compare the various sheaves $\mathcal{S}\mathrm{tT}_S(i_S^+\mathscr{N})$. Indeed, Theorem 1.3 can be interpreted as a constructibility theorem for the sheaf $\mathscr{H}_T(\mathscr{N})$.

Let us fix a stratum S_o of Δ in T, let U be an open subset of T containing S_o and satisfying 1.3(b) and (c), and let us consider the following diagram:



(see the notation in the proof of Proposition 3.4). We also denote by i_o, j_o the complementary inclusions of S_o and $T \smallsetminus S_o$ in T.

***Corollary 3.7* (Constructibility of $\mathscr{H}_T(\mathscr{N})$).** *The sheaf $\mathscr{H}_T(\mathscr{N})$ is constructible, and more precisely, for each stratum S_o of Δ, each of the natural morphisms*

$$(3.7\,*)\qquad p_{o*}\mathscr{H}_T(\mathscr{N})_{|U} \longrightarrow i_o^{-1}\mathscr{H}_T(\mathscr{N}) \xrightarrow{\;i_o^+\;} \mathscr{H}_{S_o}(i_{S_o}^+\mathscr{N})$$

is an isomorphism (the left one comes from the sheaf-theoretic adjunction $\mathrm{Id}\to i_{o*}i_o^{-1}$), and the natural composed morphism

$$\mathscr{H}_{S_o}(i_{S_o}^+\mathscr{N}) \xrightarrow{\sim} i_o^{-1}\mathscr{H}_T(\mathscr{N}) \longrightarrow i_o^{-1}j_{o*}j_o^{-1}\mathscr{H}_T(\mathscr{N})$$

is injective. Moreover, the natural restriction morphism

$$p_{o*}\mathscr{H}_T(\mathscr{N})_{|U} \longrightarrow q_{o*}\mathscr{H}_T(\mathscr{N})_{|U\smallsetminus\Delta}$$

is injective. Lastly, the natural morphism that one deduces from it together with (3.7 $*$) *and the sheaf-theoretic adjunction $q_o^{-1}q_{o*} \to \mathrm{Id}$:*

$$(3.7\,**)\qquad q_o^{-1}\mathscr{H}_{S_o}(i_o^+\mathscr{N}) \longrightarrow \mathscr{H}_T(\mathscr{N})_{|U\smallsetminus\Delta}$$

is also injective. The image subsheaf of (3.7 $**$) *is characterized as follows: given $t \in U \smallsetminus \Delta$, setting $t_o = q_o(t) \in S_o$, a germ of section $(\mathscr{M}^t, \mathrm{iso}_{\widehat{T},t}) \in \mathscr{H}_{T\smallsetminus\Delta}(\mathscr{N})_t \simeq \mathscr{H}_0(i_t^+\mathscr{N})$ belongs to (the image of) $[q_o^{-1}\mathscr{H}_{S_o}(i_o^+\mathscr{N})]_t \simeq \mathscr{H}_{S_o}(i_o^+\mathscr{N})_{t_o}$ if and only if there exists a neighbourhood $\mathrm{nb}(t_o)$ of t_o in S_o and $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ on $p_o^{-1}(\mathrm{nb}(t_o))$ whose germ at t is $(\mathscr{M}^t, \mathrm{iso}_{\widehat{T},t})$.*

Proof. The statement is local on S_o. Each $t_o \in S_o$ has a fundamental system of 1-connected neighbourhoods V in S_o. On the other hand, each such V has a fundamental system of neighbourhoods U' in T satisfying the properties 1.3(a)–(c). Let us denote by i_o the inclusion $S_o \hookrightarrow T$. We wish to prove that for each such pair (V, U'), the restriction map $i_o^+ : \Gamma(U', \mathscr{H}_T(\mathscr{N})) \to \Gamma(V, \mathscr{H}_{S_o}(i_o^+\mathscr{N}))$ is a bijection, as the case $U' = p_o^{-1}(V)$ also implies that the composed morphism (3.7 $*$) is an isomorphism.

Since V is simply connected and $\mathscr{H}_{S_o}(i_o^+\mathscr{N})$ is locally constant (Proposition 2.28), we have $\Gamma(V, \mathscr{H}_{S_o}(i_o^+\mathscr{N})) = i_{t_o}^{-1}\mathscr{H}_{S_o}(i_o^+\mathscr{N})$. Moreover, the same proposition identifies $i_{t_o}^{-1}\mathscr{H}_{S_o}(i_o^+\mathscr{N})$ with $\mathscr{H}_0(i_{t_o}^+\mathscr{N})$ via $\gamma_{t_o}^+$. Lastly, Theorem 1.3 implies that $\gamma_{t^0}^+ : \Gamma(U', \mathscr{H}_T(N)) \to \mathscr{H}_0(i_{t_o}^+\mathscr{N})$ is a bijection. We conclude that $\Gamma(U', \mathscr{H}_T(\mathscr{N})) \to \Gamma(V, \mathscr{H}_{S_o}(i_o^+\mathscr{N}))$ is also a bijection. We then use Corollary 2.13 to conclude the first assertion.

For the second assertion, we need to prove that if $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ and $(\mathscr{M}', \mathrm{iso}'_{\widehat{T}})$ defined on $p_o^{-1}(V)$ (V as above) coincide on $q_o^{-1}(V)$, then they coincide. This follows from Hartogs theorem (Lemma 2.12).

For the (3.7 $**$), we are left with proving that the adjunction morphism

$$q_o^{-1}q_{o*}\mathscr{H}_T(\mathscr{N})_{|U\smallsetminus\Delta} \longrightarrow \mathscr{H}_T(\mathscr{N})_{|U\smallsetminus\Delta}$$

is injective. This amounts to proving that, if two sections of $\mathscr{H}_T(\mathscr{N})$ on $q_o^{-1}(V)$ (V, t_o as above) coincide at $t \in q_o^{-1}(t_o)$, they coincide everywhere on $q_o^{-1}(V)$. Since $q_o^{-1}(V)$ is also connected by the star-shaped property of U, we can apply the same argument as in Corollary 2.29 to obtain the desired property.

Let us prove the final assertion of the corollary. That $(\mathscr{M}^t, \mathrm{iso}_{\widehat{T},t}) \in \mathscr{H}_{T\smallsetminus\Delta}(\mathscr{N})_t$ belongs to the image of $[q_o^{-1}\mathscr{H}_{S_o}(i_o^+\mathscr{N})]_t$ means that there exists $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{T},t_o}) \in \mathscr{H}_{S_o}(i_o^+\mathscr{N})_{t_o}$, which we consider as a section $\sigma_o : S_o \supset \mathrm{nb}(t_o) \to \mathscr{H}_{S_o}(i_o^+\mathscr{N})^{\text{ét}}$ of the sheaf (étalé) space $\mathscr{H}_{S_o}(i_o^+\mathscr{N})^{\text{ét}} \to S_o$ (which is a covering since $\mathscr{H}_{S_o}(i_o^+\mathscr{N})$ is locally constant) such that, if $\sigma : \mathrm{nb}(t) \to \mathscr{H}_{T\smallsetminus\Delta}(\mathscr{N})^{\text{ét}}$ denotes the section corresponding to $(\mathscr{M}^t, \mathrm{iso}_{\widehat{T},t})$, we have $\sigma(t) = (\sigma_o \circ q_o)(t)$, regarding $\sigma_o \circ q_o$ as a section

$$q_o^{-1}(\mathrm{nb}(t_o)) \longrightarrow [q_o^{-1}\mathscr{H}_{S_o}(i_o^+\mathscr{N})]^{\text{ét}} \longrightarrow \mathscr{H}_{T\smallsetminus\Delta}(\mathscr{N})^{\text{ét}}$$

of the sheaf space $\mathscr{H}_{T\smallsetminus\Delta}(\mathscr{N})^{\text{ét}}$ over $T \smallsetminus \Delta$ (which is also a covering). By definition, $\sigma_o \circ q_o$ corresponds to the restriction to $q_o^{-1}(\mathrm{nb}(t_o))$ of $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ constructed by Theorem 1.3 on $p_o^{-1}(\mathrm{nb}(t_o))$ from $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$ corresponding to σ_o. Then the germ $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})_t$ corresponds to σ by the correspondence (2.25).

The converse is obtained similarly. ☐

On the one hand, let us consider the sheaf $\mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N}) = \mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N}_o)$ (see (3.5)). Since $\mathscr{N}_o$ is good on S_o, this is a locally constant sheaf of pointed sets on S_o. On the other hand, we have the locally constant sheaf $\mathcal{S}\mathrm{tT}_{T\smallsetminus\Delta}(\mathscr{N})$ since $\mathscr{N}$ is good on $T\smallsetminus\Delta$.

***Corollary 3.8* (Comparison of sheaves of Stokes torsors).** *There exist natural injective morphisms*

$$\mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N}) \hookrightarrow i_o^{-1}j_{o*}\,\mathcal{S}\mathrm{tT}_{T\smallsetminus\Delta}(\mathscr{N}). \tag{3.8 $*$}$$

and

$$q_o^{-1}\,\mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N}) \hookrightarrow \mathcal{S}\mathrm{tT}_{T\smallsetminus\Delta}(\mathscr{N})_{|U\smallsetminus\Delta}. \tag{3.8 $**$}$$

The image subsheaf of (3.8 $**$) *is characterized as follows: given $t \in U \smallsetminus \Delta$, setting $t_o = q_o(t) \in S_o$, a germ of section $\sigma_t \in \mathcal{S}\mathrm{tT}_{T\smallsetminus\Delta}(\mathscr{N})_t \simeq \mathcal{S}\mathrm{tT}_0(i_t^+\mathscr{N})$ belongs to (the image of) $[q_o^{-1}\,\mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N})]_t \simeq \mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N})_{t_o}$ if and only if there exists a neighbourhood* $\mathrm{nb}(t_o)$ *of t_o in S_o and $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ on $p_o^{-1}(\mathrm{nb}(t_o))$ whose germ at t corresponds to σ_t via* (2.25).

Proof. The morphism (3.8 $*$) is defined by the following diagram:

$$\begin{array}{ccc}
\mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N}) & \dashrightarrow & i_o^{-1}j_{o*}\,\mathcal{S}\mathrm{tT}_{T\smallsetminus\Delta}(\mathscr{N}) \\
\wr\downarrow & & \downarrow\wr \\
i_o^{-1}\mathscr{H}_T(\mathscr{N}) & \hookrightarrow & i_o^{-1}j_{o*}j_o^{-1}\mathscr{H}_T(\mathscr{N})
\end{array}$$

The left vertical isomorphism is obtained by composing the isomorphism of Theorem 2.26 on S_o ($\mathcal{S}\mathrm{tT}_{S_o}(i_o^+\mathscr{N}) \simeq \mathscr{H}_{S_o}(i_o^+\mathscr{N})$) together with the isomorphism of

Corollary 3.7. The right vertical isomorphism is given by Theorem 2.26 on $T \setminus \Delta$. Lastly, the lower horizontal morphism is injective, according to Corollary 2.13. We define (3.8 $**$) similarly and use (3.7 $**$) for its injectivity. The last assertion is obtained similarly from the last assertion in Corollary 3.7. □

4. Isomonodromic deformations

Notation 4.1.

(1) For a given square matrix M, we denote by $M = M' + M''$ its decomposition into the diagonal and non-diagonal parts.

(2) For $t_o \in \Delta$, we decompose $\{1, \dots, n\} = \bigsqcup_{r \in R} I_r$ such that, for every $r \in R$, we have $\{i, j\} \subset I_r$ if and only if $t_{o,i} = t_{o,j}$.

4.a. Universal deformation of a Birkhoff normal form. For $t \in T$, we denote by $\Lambda(t)$ the matrix $\mathrm{diag}(t_1, \dots, t_n)$. For $t_o \in T$, say that a system with matrix $(\Lambda(t_o)/z + B(z))\mathrm{d}z/z$, with $B(z)$ holomorphic, is in *Birkhoff normal form* if $B(z) = A_1$ is constant, that is,

$$\Big(\frac{\Lambda(t_o)}{z} + A_1\Big)\frac{\mathrm{d}z}{z}. \tag{4.2}$$

For the system (4.2), if $t_o \notin \Delta$, a theorem by B. Malgrange [**Mal83a, Mal86**] asserts that there exists a universal integrable deformation of this system in the neighbourhood of t_o (see also [**Sab07**, §VI.3]). In particular (see [**Sab07**, §VI.3.f]), there exists a holomorphic matrix $F_1''(t)$ near t_o with zeros on the diagonal, such that the system of the form (1.1)

$$\Big(\frac{\Lambda(t)}{z} + [\Lambda(t), F_1''(t)] + A_1'\Big)\frac{\mathrm{d}z}{z}, \tag{4.3}$$

where A_1' is the diagonal part of A_1, is integrable and restricts to (4.2) at $t = t_o$. The integrable connection (on the trivial bundle) has matrix (see [**Sab07**, VI (3.12)])

$$-\,\mathrm{d}(\Lambda(t)/z) + \big([\Lambda(t), F_1''(t)] + A_1'\big)\frac{\mathrm{d}z}{z} - [\mathrm{d}\Lambda(t), F_1''(t)] \tag{4.4}$$

and is a universal integrable deformation of its restriction at each point of the neighbourhood where it exists. Moreover, there exists a z-formal base change that transforms (4.4) to the system

$$-\,\mathrm{d}(\Lambda(t)/z) + A_1'\frac{\mathrm{d}z}{z}. \tag{4.5}$$

The following two questions are natural:

(4.6) If $t_o \in \Delta$, can we find an integrable deformation (4.4) of the Birkhoff normal form (4.2) with z-formal normal form (4.5)?

(4.7) Given a system (1.1) defined on an open set V of T (so that its restriction at every $t \in V$ is in Birkhoff normal form), and given a matrix $F_1''(t)$, holomorphic on V, assume that, in a small neighbourhood W of $t_o \in V \smallsetminus \Delta$, (1.1) underlies a universal integrable deformation (4.4) of its restriction at $t = t_o$. Does this integrable deformation extend on V, in particular at points $t \in \Delta \cap V$, and has z-formal normal form (4.5) on V?

As an application of Theorem 1.3 we give an answer to (4.6) in Section 4.b. On the other hand, the results of [**CDG19**, Cor. 1.1] and [**CG18**, Cor. 2.1] concern (4.7).

4.b. Deformation of a Birkhoff normal form. We will show that Question (4.6) for (4.2) has a positive answer provided that Assumptions 4.8(a) and (b) below are fulfilled by A_1. Moreover, we make more precise the domain of existence of the integrable deformation.

Let $t_o \in \Delta \subset T$ and let us fix a Birkhoff normal form (4.2).

Lemma 4.8. *With the following assumptions*

(a) $A''_1 \in \operatorname{Im}\operatorname{ad}(\Lambda(t_o))$,[(5)]

(b) A'_1 *is partially non-resonant, i.e.,*

$$\forall\, r \in R,\ \forall\, i,j \in I_r, \quad A'_{1ii} - A'_{1jj} \notin \mathbb{Z} \smallsetminus \{0\},$$

which are always fulfilled if $t_o \notin \Delta$, there exists a z-formal base change of the form $\operatorname{Id} + zP_1 + \cdots$ *that transforms the matrix of* ∇^o

$$A(t_o, z) = \Big(\frac{\Lambda(t_o)}{z} + A_1\Big)\frac{\mathrm{d}z}{z}$$

to the diagonal matrix

$$A'(t_o, z) = \Big(\frac{\Lambda(t_o)}{z} + A'_1\Big)\frac{\mathrm{d}z}{z}.$$

Proof. The result is standard (see e.g. [**CDG19**, Prop. 4.2]), but we will give a proof valid on any algebraic closed field of characteristic zero instead of $\mathbb{C}$, i.e., not depending on the transcendental notion of fundamental solution. Firstly, one can find a formal base change $\widehat{F}_o = \operatorname{Id} - zF''_1(t_o) + \cdots$ that transforms $A(t_o, z)$ to the formal matrix

$$\widetilde{A}(t_o, z) = \Big(\frac{\Lambda(t_o)}{z} + A'_1 + zA_2 + \cdots\Big)\frac{\mathrm{d}z}{z},$$

where $A_2, \dots$ are block diagonal with respect to $r \in R$, so that $\widetilde{A}(t_o, z)$ is also block diagonal. Each block ($r \in R$) is written as

$$\widetilde{A}^{(r)}(t_o, z) = \Big(\frac{t_{o,r}\operatorname{Id}^{(r)}}{z} + A'^{(r)}_1 + zA^{(r)}_2 + \cdots\Big)\frac{\mathrm{d}z}{z}.$$

Since $A'^{(r)}_1$ is non-resonant, there exists a base change $\operatorname{Id}^{(r)} + zQ^{(r)}_1 + \cdots$ that transforms $\widetilde{A}^{(r)}(t_o, z)$ to the rth block of $A'(t_o, z)$. □

(5) Recall that $\operatorname{ad}(\Lambda(t_o))(B) = [\Lambda(t_o), B]$ so that $A \in \operatorname{Im}\operatorname{ad}(\Lambda(t_o))$ iff $A_{ij} = 0$ whenever $i, j \in I_r$ for some $r \in R$.

Theorem 4.9. *Let $t_o \in \Delta$ and let U be a connected open subset of T satisfying* 1.3(a)–(c) *with respect to $S(t_o)$. Under Assumptions* 4.8(a) *and* (b) *on A_1, there exists a holomorphic hypersurface Θ in U not containing t_o and a holomorphic matrix $F_1''(t)$ on $U \smallsetminus \Theta$, such that the meromorphic connection on the trivial bundle with matrix* (4.4) *is integrable, restricts to* (4.2) *at t_o, and is formally equivalent at $z = 0$ to the matrix connection*

$$-\mathrm{d}(\Lambda(t)/z) + A_1' \frac{\mathrm{d}z}{z}.$$

Proof. The proof is very similar to that of [**Mal83a, Mal86**]. We set $T = \{z = 0\} \subset X = \mathbb{C}^n \times \mathbb{C}_z$. We denote by $\mathscr{N}$ the T-meromorphic flat bundle $\bigoplus_{i=1}^n (\mathscr{E}^{-t_i/z} \otimes \mathscr{R}_i)$, where $\mathscr{R}_i = (\mathscr{O}_X(*T), \mathrm{d}+A'_{1\,ii}\,\mathrm{d}z/z)$. By Lemma 4.8, (4.2) defines an object $(\mathscr{M}^{t_o}, \mathrm{iso}_{\widehat{0}})$ in $\mathscr{H}_0(\gamma_{t_o}^+\mathscr{N})$. From Theorem 1.3 we deduce an $\mathscr{N}$-marked T-meromorphic flat bundle $(\mathscr{M}, \mathrm{iso}_{\widehat{T}})$ on $U \times (\mathbb{C}_z, 0)$. We can now apply [**Sab07**, Th. VI.2.1] and obtain a hypersurface $\Theta \subset U$ and a basis $\boldsymbol{\varepsilon}$ of $\mathscr{M}(*(\Theta \times \mathbb{C}_z))$ in which the matrix of the integrable connection takes the form

$$\Big(\frac{A_0(t)}{z} + A_1\Big)\frac{\mathrm{d}z}{z} + \frac{C(t)}{z}$$

with $A_0(t)$ conjugate to $\Lambda(t)$ for each $t \in U \smallsetminus \Theta$ and $C(t) \in \Gamma(U, \Omega_T^1(*\Theta))$. One can then apply the results in [**Sab07**, §VI.3.f], since the regularity property of $\Lambda(t)$ (i.e., the fact that the eigenvalues are pairwise distinct) is not needed at this point. The change of notation with respect to loc. cit. is as follows: Δ_0 is $\Lambda(t)$, Δ_∞ is A_1', τ is z and the non-diagonal part T'' is $F_1''(t)$. □

4.c. Application to the construction of Frobenius manifolds. It is known (see [**Dub96**, Th. 3.2, p. 223]) that, given suitable initial data consisting of a diagonal matrix $\Lambda(t_o)$ with pairwise distinct eigenvalues (i.e., $t_o \notin \Delta$), of a matrix A_1 such that $A_1 - (w/2)\,\mathrm{Id}$ is skew-symmetric for some integer w, and an eigenvector ω_o of A_1 that has no zero entry, one can construct a Frobenius manifold structure on the complement of some hypersurface Θ in the universal covering $\widetilde{T \smallsetminus \Delta}$ (see also [**Sab07**, §VII.4.a]). It is the universal model at a semisimple point of a Frobenius manifold.

Theorem 4.9 enables us to relax the regularity assumption on $\Lambda(t_o)$, provided that Assumptions 4.8(a) and (b) are fulfilled for A_1. Let $t_o \in \Delta$ and set $S_o = S(t_o)$. Assume that $(\Lambda(t_o), A_1, \omega_o)$ satisfy the following properties.

(a) $A_1'' \in \mathrm{Im}\,\mathrm{ad}(\Lambda(t_o))$,
(b) $A_1 - (w/2)\,\mathrm{Id}$ is skew-symmetric (so that $A_1' = (w/2)\,\mathrm{Id}$ is non-resonant),
(c) ω_o is an eigenvector of A_1 whose entries are all nonzero.

Let U be an open subset of T containing S_o and satisfying 1.3(b) and (c), and let $\widetilde{U}$ be its universal covering. By Theorem 4.9, there exists an integrable deformation of the Birkhoff normal form (4.2), which exists on $\widetilde{U} \smallsetminus \Theta$ for some complex hypersurface Θ of $\widetilde{U}$. The vector ω_o can be extended flatly as a vector function ω on $\widetilde{U}$, meromorphic along Θ and its entries do not vanish away from some hypersurface Θ_{ω_o}. To ω is then associated an infinitesimal period mapping (see [**Sab07**, §VII 3.a]).

Corollary 4.10. *Under Assumptions* (a), (b) *and* (c) *above, the infinitesimal period mapping associated with* ω *endows the manifold* $\widetilde{U} \smallsetminus (\Theta \cup \Theta_{\omega_o})$ *with a Frobenius structure.* □

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Added on April 21, 2021. Concerning question (4.7), a new proof has been given by D. Guzzetti (Isomonodromic Laplace transform with coalescing eigenvalues and confluence of Fuchsian singularities, `arXiv:2101.03397`), and another proof has later been given by the author (A short proof of a theorem of Cotti, Dubrovin and Guzzetti, `arXiv:2103.16878`). Concerning Corollary 4.10, another proof has been given by G. Cotti (Degenerate Riemann-Hilbert-Birkhoff problems, semisimplicity, and convergence of WDVV-potentials, `arXiv:2011.04498`).

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