A COUNTER-EXAMPLE TO LEVELT-TURRITTIN IN DIMENSION TWO

by

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Let $\mathbb{P}^1$ be the complex projective line covered with two charts $\mathbb{A}^1_0$ (coordinate x) and $\mathbb{A}^1_\infty$ (coordinate y), so that $y = 1/x$ on $\mathbb{A}^1_0 \cap \mathbb{A}^1_\infty$. Let $\mathbb{A}^1$ be the affine line with coordinate z.

We denote by $S_\pm$ the hyperbolas with equation $xz = \pm 1$ in $\mathbb{A}^1_0 \times \mathbb{A}^1$, and by $\overline{S}_\pm$ their closure in $\mathbb{P}^1 \times \mathbb{A}^1$. The intersection of $\overline{S}_\pm$ with $\mathbb{A}^1_\infty \times \mathbb{A}^1$ is nothing but the two lines $y = \pm z$. We set $\overline{S} = \overline{S}_+ \cup \overline{S}_-$ and $S = S_+ \cup S_-$.

Lemma. *The fundamental group $\pi_1(\mathbb{P}^1 \times \mathbb{A}^1 \smallsetminus \overline{S})$ is generated by $\mathbb{Z} \oplus \mathbb{Z}$ (loops around $\overline{S}_+$ and $\overline{S}_-$) modulo the relation that the sum of these loops is zero.*

Proof. Direct application of van Kampen applied to $\mathbb{A}^1_\infty \times \mathbb{A}^1 \smallsetminus \overline{S}$ glued with a neighbourhood of $\{0\} \times \mathbb{A}^1$. $\square$

Let us fix a complex number $\lambda \neq 0, 1$, and let $\overline{\mathscr{L}}_\lambda$ be the rank-one local system on $\mathbb{P}^1 \times \mathbb{A}^1 \smallsetminus \overline{S}$ with local monodromy λ around $\overline{S}_+$, and hence monodromy $1/\lambda$ around $\overline{S}_-$. Since $\lambda \neq 1$, the nearby cycle sheaf $\psi_{y-z}\overline{\mathscr{L}}_\lambda$ and the vanishing cycle sheaf $\phi_{y-z}\overline{\mathscr{L}}_\lambda$ coincide on S_+, and both are equal to the rank-one local system with monodromy $1/\lambda$ on S_+. Similarly, $\psi_{y+z}\overline{\mathscr{L}}_\lambda$ and $\phi_{y+z}\overline{\mathscr{L}}_\lambda$ coincide on S_- and both are equal to the rank-one local system with monodromy λ on S_-.

We denote by $\mathscr{L}_\lambda$ the restriction of $\overline{\mathscr{L}}_\lambda$ to $\mathbb{A}^1_0 \times \mathbb{A}^1 \smallsetminus S$. A similar assertion holds for the nearby and vanishing cycles along $S_\pm$. Let $j_S : \mathbb{A}^1_0 \times \mathbb{A}^1 \smallsetminus S \hookrightarrow \mathbb{A}^1_0 \times \mathbb{A}^1$ denote the inclusion. The minimal extension $\mathscr{F}_\lambda := j_{S,*}\mathscr{L}_\lambda$ is also equal to $j_{S,!}\mathscr{L}_\lambda$ and to $\boldsymbol{R}j_{S,*}\mathscr{L}_\lambda$ since $\lambda \neq 1$, and is an irreducible perverse sheaf on $\mathbb{A}^1_0 \times \mathbb{A}^1$. It corresponds to an irreductible regular holonomic $\mathbb{C}[x,z]\langle\partial_x, \partial_z\rangle$-module M_λ.

This research was supported by the grant ANR-08-BLAN-0317-01 of the Agence nationale de la recherche.

Let ${}^{\mathrm{F}}\!M_\lambda$ be the partial Laplace transform of M_λ with respect to the variable x, which is a $\mathbb{C}[\xi,z]\langle\partial_\xi,\partial_z\rangle$-module. We have ${}^{\mathrm{F}}\!M_\lambda = M_\lambda$ as a $\mathbb{C}[z]\langle z\rangle$-module, and the action of ξ (resp. ∂_ξ) is that of ∂_x (resp. $-x$).

Lemma. *${}^{\mathrm{F}}\!M_\lambda$ is a holonomic $\mathbb{C}[\xi,z]\langle\partial_\xi,\partial_z\rangle$-module with singular set equal to $\{z=0\}\cup\{\xi=0\}\cup\{\xi=\infty\}$. It is generically regular along $\xi=0$. The corresponding local system on $\mathbb{A}^1_\xi\times\mathbb{A}^1\smallsetminus\{z\xi=0\}$ has rank two. Its monodromy around $\xi=0$ is unipotent with one Jordan block. The restriction of ${}^{\mathrm{F}}\!M_\lambda(1/z)$ to $\xi=\xi_o\neq 0$ is irregular with Levelt-Turrittin decomposition*

$$\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}[z]}{}^{\mathrm{F}}\!M_\lambda(1/z)_{|\xi=\xi_o} = (\widehat{\mathscr{E}}^{\xi_o/z}\otimes\widehat{R}_+)\oplus(\widehat{\mathscr{E}}^{-\xi_o/z}\otimes\widehat{R}_-)$$

where R_+ and R_- are $\mathbb{C}(\{z\})$-vector spaces with regular connection corresponding to $\phi_{x-z}\mathscr{L}_\lambda$ and $\phi_{x+z}\mathscr{L}_\lambda$ respectively.

Proof. Analytically away from $z=0$, ${}^{\mathrm{F}}\!M_\lambda$ is a holonomic $\mathscr{D}_{\mathbb{A}^1_\xi\times\mathbb{C}^*}$-module. It is easy to check, by restricting to $z=z_o\neq 0$, that it has regular singularity along $\xi=0$, irregular singularity along $\xi=\infty$, and the corresponding local system on $\xi\neq 0$ has rank two. To compute the monodromy around $\xi=0$, we can also restrict to $z=z_o$. We denote by M^o_λ the corresponding $\mathbb{C}[x]\langle\partial_x\rangle$-module. Then, since M^o_λ is irreducible, so is ${}^{\mathrm{F}}\!M^o_\lambda$, and in particular it is a minimal extension at $\xi=0$.

On the one hand, since the monodromy at infinity of $\mathscr{L}_\lambda$ is equal to the identity by construction, the monodromy T_ξ of the nearby cycle space $\psi_\xi\,\mathrm{DR}\,{}^{\mathrm{F}}\!M^o_\lambda$ has only 1 as an eigenvalue. It is therefore unipotent, and written as $\exp(-2\pi i\mathrm{N})$ for some nilpotent operator N.

On the other hand, since ${}^{\mathrm{F}}\!M^o_\lambda$ is a minimal extension at $\xi=0$ and has regular singularity, $\mathrm{DR}\,{}^{\mathrm{F}}\!M^o_\lambda = j_{\xi,*}(\mathrm{DR}\,{}^{\mathrm{F}}\!M^o_\lambda)_{|\xi\neq 0}$ ($j_\xi:\{\xi\neq 0\}\hookrightarrow\mathbb{A}^1_\xi$), and thus $\phi_\xi\,\mathrm{DR}\,{}^{\mathrm{F}}\!M^o_\lambda$ is identified with the image of N. But $\phi_\xi\,\mathrm{DR}\,{}^{\mathrm{F}}\!M^o_\lambda$ is also identified, by a standard analysis of the local Fourier transform $\mathscr{F}^{(\infty,0)}$ (cf. [**1, 2**]) with the nearby cycle space of $\mathscr{L}_\lambda$ at infinity. In particular, it has dimension one. This forces N to have exactly one Jordan block.

The restriction at $\xi=\xi_o$ of ${}^{\mathrm{F}}\!M_\lambda$ is by definition the push-forward by $p:(x,z)\mapsto z$ of $\mathscr{E}^{x\xi_o}\otimes M^\lambda$. Up to a change of the variable x, we reduce to the case where $\xi_o=1$. We can then apply [**4**, Th. 1 & 2] to obtain the second statement. ☐

Proposition. *The rank-two free $\mathbb{C}[\xi,z][1/\xi z]$-module with connection ${}^{\mathrm{F}}\!M_\lambda(1/\xi z)$ does not have a Levelt-Turrittin decomposition at $(\xi=0,z=0)$, even after a finite ramification along $\xi=0$ and $z=0$.*

Proof. By contradiction. We first prove the non-existence of a Levelt-Turrittin decomposition. We will then give the argument in order to treat ramification.

Assume that $\widehat{N} := \mathbb{C}[\![\xi,z]\!]\otimes_{\mathbb{C}[\xi,z]}{}^{\mathrm{F}}\!M_\lambda(1/\xi z)$ has a Levelt-Turrittin decomposition. We can write it as $\widehat{N} = (\widehat{\mathscr{E}}^{\varphi_+}\otimes\widehat{R}_+)\oplus(\widehat{\mathscr{E}}^{\varphi_-}\otimes\widehat{R}_-)$, with at most two terms since $\widehat{N}$ has rank two. Here, $\varphi_\pm \in \mathbb{C}[\![\xi,z]\!][1/\xi z]/\mathbb{C}[\![\xi,z]\!]$ and $\widehat{R}_\pm$ are free (maybe zero) $\mathbb{C}[\![\xi,z]\!][1/\xi z]$-modules with flat regular connection.

According to [**5**, Prop. I.1.2.4.1], $\varphi_\pm$ are convergent. Moreover, it is easy to find a finite sequence e of point blow up over the origin so that that the pull-back decomposition is *good* at each point of the exceptional divisor. Then, according to [**3**, Prop. 2.19] applied to $\varphi_\pm \circ e$ along the strict transform of $\{z=0\}$ by e, and according to the previous lemma, the restriction of $\varphi_\pm$ to $\xi=\xi_o$, $\xi_o \neq 0$ and small enough, is equal to $\pm\xi_o/z$. This being true for each such ξ_o, this implies that $\varphi_\pm \mp \xi/z$ has a pole at most along $\xi=0$. Arguing similarly with the strict transform of $\{\xi=0\}$, along which generically the connection has a regular singularity, gives $\varphi_\pm = \pm\xi/z$. In particular, the decomposition of $\widehat{N}$ has exactly two terms.

We now consider the moderate nearby cycles of ${}^{\mathrm{F}}M_\lambda$ and $\widehat{N}$ along $\xi=0$, given by the V-filtration construction. The uniqueness of the V-filtration implies that $(\psi_\xi(\widehat{N}),\mathrm{T}_\xi)$ is the formalization with respect to z of $(\psi_\xi({}^{\mathrm{F}}M_\lambda(1/\xi z)),\mathrm{T}_\xi)$, where T_ξ denotes the monodromy operator given by this construction.

On the one hand, when restricted analytically to a punctured disc Δ^* with coordinate z in $\{\xi=0\}$, in the neighbourhood of which ${}^{\mathrm{F}}M_\lambda$ has a regular singularity along $\xi=0$, $(\psi_\xi({}^{\mathrm{F}}M_\lambda),\mathrm{T}_\xi)$ is a bundle with connection on Δ^*, corresponding to a rank-two local system with monodromy operator T_z. Both operators T_ξ and T_z commute, and we recall that T_ξ is unipotent with only one Jordan block. It follows that $(\psi_\xi({}^{\mathrm{F}}M_\lambda),\mathrm{T}_\xi)_{|\Delta^*}$ *does not split* as the direct sum of two objects of rank one of the same kind.

On the other hand, we will prove:

Lemma. *The $\mathbb{C}[\![z]\!]\langle\partial_z\rangle$-module $\psi_\xi(\widehat{\mathscr{E}}^{\pm\xi/z}\otimes\widehat{R}_\pm)$ has a regular singularity at $z=0$.*

Proof. Let us work with the $+$ case for instance. Then $\widehat{\mathscr{E}}^{\xi/z}\otimes\widehat{R}_+$ has a generator e which satisfies $\partial_\xi e=(1/z)e$ and $\partial_z e=[(\alpha z-\xi)/z^2]e$ for some $\alpha\notin\mathbb{Z}$. We consider the V-filtration $U_\bullet(\widehat{\mathscr{E}}^{\xi/z}\otimes\widehat{R}_+)$ (with respect to $\xi=0$) generated by e, so that $e\in U_0(\widehat{\mathscr{E}}^{\xi/z}\otimes\widehat{R}_+)$. Our goal is to show that the class $[e]$ of e in $\mathrm{gr}_0^U(\widehat{\mathscr{E}}^{\xi/z}\otimes\widehat{R}_+)$ satisfies a regular differential equation with respect to z.

We have $[(\alpha z\xi-\xi^2)/z^2]e=\xi\partial_z e\in U_{-1}(\widehat{\mathscr{E}}^{\xi/z}\otimes\widehat{R}_+)$. Then one checks that $(z\partial_z-\alpha)(z\partial_z+1)e\in U_{-1}(\widehat{\mathscr{E}}^{\xi/z}\otimes\widehat{R}_+)$, that is, $(z\partial_z-\alpha)(z\partial_z+1)[e]=0$, as wanted. □

It follows that $\psi_\xi(\widehat{N})(1/z)$ has a regular singularity at $z=0$, hence so has $\psi_\xi({}^{\mathrm{F}}M_\lambda)(1/z)$ and, since $(\psi_\xi(\widehat{N})(1/z),\mathrm{T}_\xi)$ splits as the direct sum of two regular holonomic $\mathbb{C}(\!(z)\!)$-vector spaces of rank one with connection and monodromy operator T_ξ, so does $(\psi_\xi({}^{\mathrm{F}}M_\lambda)(1/z),\mathrm{T}_\xi)$. This leads to a contradiction.

For the ramified case, we are lead to a contradiction in a similar way, by replacing $\widehat{\mathscr{E}}^{\pm\xi/z}$ with $\widehat{\mathscr{E}}^{\pm\xi^p/z^q}$ in the lemma, with $p,q\geqslant 1$. The proof of the corresponding statement is similar. □

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