ERRATUM TO "ON THE IRREGULAR HODGE FILTRATION OF EXPONENTIALLY TWISTED MIXED HODGE MODULES"

by

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Abstract. We correct the proof of [**SY15**, (5.3)]. The mistake is in the formula for the filtration $V_\alpha^{s'}(i_+\mathcal{M}')$ on the top of Page 21 of loc. cit. The new proof is a minor modification of the published one. We thank Takahiro Saito for noticing the mistake.

It is enough to check

$$i_+\big(\mathcal{M}\otimes\mathcal{E}^t, F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}\otimes\mathcal{E}^t)\big) = \big((i_+\mathcal{M})\otimes\mathcal{E}^s, F^{\mathrm{Del}}_{\alpha+\bullet}((i_+\mathcal{M})\otimes\mathcal{E}^s)\big), \tag{5.3}$$

and the question is non obvious in the charts $t'=1/t$ and $s'=1/s$. Let us set $\delta=\delta(s'-t')$. Let us first recall that, by definition,

$$\begin{aligned} i_+\mathcal{M}' &= \bigoplus_{k\geqslant 0} i_*\mathcal{M}'\otimes\partial_{s'}^k\delta,\\ F_p(i_+\mathcal{M}') &= \bigoplus_{k\geqslant 0} i_*F_{p-k-1}\mathcal{M}'\otimes\partial_{s'}^k\delta, \end{aligned}\tag{5.4}$$

(the shift by one comes from the left-to-right transformation on $R_F\mathscr{D}$-modules) and, concerning the V-filtration, one checks that[(1)]

$$V_\alpha^{s'}(i_+\mathcal{M}') = \sum_{k\geqslant 0}\partial_{t'}^k(V_\alpha^{t'}\mathcal{M}'\otimes\delta).$$

We set $G_p(i_+\mathcal{M}') = \bigoplus_{0\leqslant k\leqslant p} i_*\mathcal{M}'\otimes\partial_{s'}^k\delta$. Let $\sum_{k=0}^{\ell}\partial_{t'}^k(m_{\alpha,k}\otimes\delta)\in V_\alpha^{s'}(i_+\mathcal{M}')$. Then this term is contained in $G_\ell(i_+\mathcal{M}')$, and its image in $G_\ell(i_+\mathcal{M}')/G_{\ell-1}(i_+\mathcal{M}')$ is the class of $m_{\alpha,\ell}\otimes\partial_{s'}^\ell\delta$, hence it is nonzero if and only if $m_{\alpha,\ell}\neq 0$. In particular, $V_\alpha^{s'}(i_+\mathcal{M}')\cap G_0(i_+\mathcal{M}') = V_\alpha^{t'}\mathcal{M}'\otimes\delta$.

We recall (see [**ESY17**, (3.1.1)]):

$$F^{\mathrm{Del}}_{\alpha+p}(\mathcal{M}'\otimes\mathcal{E}^{1/t'}) = \sum_{k\geqslant 0}\partial_{t'}^k t'^{-1}\big[(F_{p-k}\mathcal{M}'\cap V_\alpha^{t'}\mathcal{M}')\otimes\mathcal{E}^{1/t'}\big],$$

and, by the analogue of (5.4),

$$i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p = F^{\mathrm{Del}}_{\alpha+p-1}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\otimes\delta + \partial_{s'}\big[i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_{p-1}\big].$$

(1) The formula in the published version is not correct.

On the other hand,

$$\begin{aligned}F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)&=\sum_{k\geqslant0}\partial^k_{s'}s'^{-1}\big[(F_{p-k}(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}'))\otimes\mathcal{E}^{1/s'}\big]\\&=s'^{-1}\big(F_p(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')\big)\otimes\mathrm{e}^{1/s'}\\&\qquad+\partial_{s'}F^{\mathrm{Del}}_{\alpha+p-1}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big).\end{aligned}$$

We prove (5.3) by induction on p. Let p_o be such that $F_{p_o-2}\mathcal{M}'=0$. Then $F_{p_o}(i_+\mathcal{M}')=F_{p_o-1}\mathcal{M}'\otimes\delta\subset G_0(i_+\mathcal{M}')$, $F^{\mathrm{Del}}_{\alpha+p_o-1}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)=0$ and

$$\begin{aligned}F^{\mathrm{Del}}_{\alpha+p_o}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)&=s'^{-1}\big(F_{p_o}(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')\big)\otimes\mathrm{e}^{1/s'}\\&=s'^{-1}\big(F_{p_o-1}\mathcal{M}'\cap V^{t'}_\alpha\mathcal{M}'\big)\otimes(\delta\otimes\mathrm{e}^{1/s'})\\&=\big(t'^{-1}(F_{p_o-1}\mathcal{M}'\cap V^{t'}_\alpha\mathcal{M}')\otimes\mathrm{e}^{1/t'}\big)\otimes\delta\\&=i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_{p_o}.\end{aligned}$$

We now assume that (5.3) holds for $p-1$. Let us first show that

$$i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p\subset F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big).$$

By induction and the above formula for $i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p$, it is enough to check

$$i_*F^{\mathrm{Del}}_{\alpha+p-1}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\otimes\delta\subset F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big).$$

We have

$$F^{\mathrm{Del}}_{\alpha+p-1}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})=t'^{-1}(F_{p-1}\mathcal{M}'\cap V^{t'}_\alpha\mathcal{M}')\otimes\mathrm{e}^{1/t'}+\partial_{t'}\big(F^{\mathrm{Del}}_{\alpha+p-2}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big).$$

Then on the one hand, by induction,

$$\begin{aligned}i_*\partial_{t'}\big(F^{\mathrm{Del}}_{\alpha+p-2}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)\otimes\delta&\subset\partial_{t'}\big[i_*\big(F^{\mathrm{Del}}_{\alpha+p-2}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)\otimes\delta\big]\\&\qquad+\partial_{s'}\big[i_*\big(F^{\mathrm{Del}}_{\alpha+p-2}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)\otimes\delta\big]\\&\subset\partial_{t'}F^{\mathrm{Del}}_{\alpha+p-1}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)+\partial_{s'}F^{\mathrm{Del}}_{\alpha+p-1}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)\\&\subset F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big).\end{aligned}$$

On the other hand, $i_*\big[t'^{-1}(F_{p-1}\mathcal{M}'\cap V^{t'}_\alpha\mathcal{M}')\otimes\mathrm{e}^{1/t'}\big]\otimes\delta$ is the degree zero term (w.r.t. $G_\bullet$) in $F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)$.

Let us now prove the reverse inclusion

$$F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)\subset i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p.$$

By induction and the above formula for $F^{\mathrm{Del}}_{\alpha+p}\big((i_+\mathcal{M}')\otimes\mathcal{E}^{1/s'}\big)$, it is enough to prove

$$(*)\qquad s'^{-1}\big(F_p(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')\big)\otimes\mathrm{e}^{1/s'}\subset i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p.$$

Let $m=\sum_{j=0}^\ell\partial^j_{t'}(m_{\alpha,j}\otimes\delta)$ be in $F_p(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')$. We wish to prove that it belongs to the right-hand side of $(*)$. Assume $m_{\alpha,\ell}\neq0$, so that $m_{\alpha,\ell}\in F_{p-1-\ell}\mathcal{M}'\cap V^{t'}_\alpha(\mathcal{M}')$. Then $m_{\alpha,\ell}\otimes\delta\in F_{p-\ell}(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')$ and $\partial^\ell_{t'}(m_{\alpha,j}\otimes\delta)\in F_p(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')$. It follows that each nonzero term in the sum expressing m also belongs to $F_p(i_+\mathcal{M}')\cap V^{s'}_\alpha(i_+\mathcal{M}')$, and it is enough to prove the desired inclusion for $m=\partial^\ell_{t'}(m_{\alpha,\ell}\otimes\delta)$, that is,

$$(**)\qquad \partial^\ell_{t'}t'^{-1}(m_{\alpha,\ell}\otimes\delta\otimes\mathrm{e}^{1/s'})\in i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p.$$

The left-hand side of $(**)$ reads $\partial_{t'}^{\ell} t'^{-1}\big[(m_{\alpha,\ell}\otimes \mathrm{e}^{1/t'})\otimes\delta\big]$ and is a sum of terms

$$\big[\partial_{t'}^{\ell-j} t'^{-1}(m_{\alpha,\ell}\otimes \mathrm{e}^{1/t'})\big]\otimes\partial_{s'}^{j}\delta,\quad j=0,\dots,\ell.$$

We have

$$\partial_{t'}^{\ell-j} t'^{-1}(m_{\alpha,\ell}\otimes \mathrm{e}^{1/t'})\in\partial_{t'}^{\ell-j} t'^{-1}(F_{p-1-\ell}\mathcal{M}'\cap V_{\alpha}^{t'}\mathcal{M}')\otimes \mathrm{e}^{1/t'}\subset F^{\mathrm{Del}}_{\alpha+p-1-j}(\mathcal{M}'\otimes\mathcal{E}^{1/t'}),$$

and thus

$$\big[\partial_{t'}^{\ell-j} t'^{-1}(m_{\alpha,\ell}\otimes \mathrm{e}^{1/t'})\big]\otimes\partial_{s'}^{j}\delta\in F^{\mathrm{Del}}_{\alpha+p-1-j}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\otimes\partial_{s'}^{j}\delta\subset i_+\big(F^{\mathrm{Del}}_{\alpha+\bullet}(\mathcal{M}'\otimes\mathcal{E}^{1/t'})\big)_p,$$

as wanted. □

References

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