Introduction to algebraic theory of linear systems of differential equations

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Introduction

In this set of notes we shall study properties of linear differential equations of a complex variable with coefficients in either the ring convergent (or formal) power series or in the polynomial ring. The point of view that we have chosen is intentionally algebraic. This explains why we shall consider the $\mathcal{D}$-module associated with such equations. In order to solve an equation (or a system) we shall begin by finding a "normal form" for the corresponding $\mathcal{D}$-module, that is by expressing it as a direct sum of elementary $\mathcal{D}$-modules. It is then easy to solve the corresponding equation (in a similar way, one can solve a system of linear equations on a vector space by first putting the matrix in a simple form, *e.g.* triangular form or (better) Jordan canonical form).

This lecture is supposed to be an introduction to $\mathcal{D}$-module theory. That is why we have tried to set out this classical subject in a way that can be generalized in the case of many variables. For instance we have introduced the notion of *holonomy* (which is not difficult in dimension one), we have emphasized the connection between $\mathcal{D}$-modules and meromorphic connections. Because the notion of a normal form does not extend easily in higher dimension, we have also stressed upon the notions of nearby and vanishing cycles.

The first chapter consists of a local algebraic study of $\mathcal{D}$-modules. The coefficient ring is either $\mathbf{C}\{x\}$ (convergent power series) or $\mathbf{C}[\![x]\!]$ (formal power series). The main results concern however the latter case. The second chapter deals with the extension of these results to the former case. Many analytic results will then be necessary. The most difficult one will not be proved here and concerns the resolution of some non linear equation. Finally, the third chapter deals with the global study of linear differential equations on the Riemann sphere.

The results contained in these notes may all be found (at least in some form) in the literature on the subject, given in the bibliography. We have tried to make these notes self-contained and readable without (or with very few) prerequisites. We have given references for the statements (with some comments) in a special chapter. Let us only mention here that we have greatly benefitted of handwritten notes of a future book by Bernard Malgrange [33] (see now [35]).

Chapter I
Algebraic methods

1. The algebra $\mathcal{D}$

1.1. Some results of commutative algebra

General references are [1, 2, 3, 4]. We shall consider differential operators in one variable with coefficient in one of the following three rings: $\mathbf{C}[x]$ (polynomials in one variable), $\mathbf{C}\{x\}$ (convergent power series), $\mathbf{C}[\![x]\!]$ (formal power series), which satisfy the following elementary properties:

1. $\mathbf{C}[x]$ is graded by the degree of polynomials, and this graduation induces an increasing filtration. It is a noetherian ring, all the ideals of which are of the type $(x-a)\mathbf{C}[x]$, $a \in \mathbf{C}$.
2. If $\mathfrak{m}$ is the maximal ideal of $\mathbf{C}[x]$ generated by x, one has
$$\mathbf{C}[\![x]\!] = \varprojlim_k \mathbf{C}[x]/\mathfrak{m}^k.$$
The ring $\mathbf{C}[\![x]\!]$ is a noetherian local ring: every power series with a nonzero constant term is invertible. This ring is also a discrete valuation ring. The filtration by powers of the maximal ideal (also called the $\mathfrak{m}$-adic filtration) is the filtration $\mathfrak{m}^k = \{f \in \mathbf{C}[\![x]\!] \,|\, v(f) \geq k\}$. One has $\mathrm{gr}_{\mathfrak{m}}(\mathbf{C}[\![x]\!]) = \mathbf{C}[x]$.
3. $\mathbf{C}\{x\} \subset \mathbf{C}[\![x]\!]$ is the subring of power series whose radius of convergence is strictly positive. Except the definition by projective limit, it satisfies the same properties as $\mathbf{C}[\![x]\!]$.
4. One can also consider intermediate rings between $\mathbf{C}\{x\}$ and $\mathbf{C}[\![x]\!]$ (formal power series with Gevrey type conditions).

1.2. Definition of $\mathcal{D}$

1.2.1. — The rings considered previously come equipped with a derivation operator $\partial/\partial x$. Let $f \in \mathbf{C}[x]$ (*resp.* $\mathbf{C}\{x\}$, *resp.* $\mathbf{C}[\![x]\!]$). One has the following commutation relation between *the derivation operator* $\partial/\partial x$ and *the multiplication operator* f:

$$\left[\frac{\partial}{\partial x}, f\right] = \frac{\partial f}{\partial x} \tag{1.2.2}$$

where the R.H.S. is the multiplication operator by $\partial f/\partial x$; this means that for all $g \in \mathbf{C}[x]$ one has:

$$\left[\frac{\partial}{\partial x}, f\right] \cdot g = \frac{\partial fg}{\partial x} - f\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} \cdot g.$$

One defines the Weyl algebra $A_1(\mathbf{C})$ (*resp.* the algebra $\mathcal{D}$ of linear differential operators with coefficients in $\mathbf{C}\{x\}$, *resp.* the algebra $\widehat{\mathcal{D}}$ (coefficients in $\mathbf{C}[\![x]\!]$)) as the quotient algebra of the free algebra generated by the coefficient ring and an element denoted by ∂_x, quotient by the relations (1.2.2) for all f in the ring of coefficients. We shall use the following notation: $A_1(\mathbf{C}) = \mathbf{C}[x]\langle\partial_x\rangle$ ...

Remark. — $A_1(\mathbf{C})$ is a non commutative algebra. By definition, $\mathbf{C}[x]$ is a left module on $A_1(\mathbf{C})$: ∂_x operates as $\partial/\partial x$, and $f \in \mathbf{C}[x]$ operates as mutliplication by f, the product in $A_1(\mathbf{C})$ is then induced by the composition law of operators.

PROPOSITION 1.2.3. — *Every element in $A_1(\mathbf{C})$ (resp. $\mathcal{D}$, $\widehat{\mathcal{D}}$) can be written in a unique way as $\sum_{i=0}^{n} a_i(x)\partial_x^i$, where for all i, $a_i(x) \in \mathbf{C}[x]$ (resp. ...).*

Proof. — Every element in $A_1(\mathbf{C})$ is a sum of "monomials" which are products of powers of ∂_x and elements of $\mathbf{C}[x]$. Such a monomial can be written $\alpha_1(x)\partial_x^{i_1} \cdots \alpha_k(x)\partial_x^{i_k}$. One uses the commutation relations (1.2.2) in order to write it in the expected form. One has to show now that an element $P = \sum_{i=0}^{n} a_i(x)\partial_x^i$ is not zero in $A_1(\mathbf{C})$ if and only if one of the coefficients a_i is not zero. In order to do that, it suffices to let $A_1(\mathbf{C})$ act on $\mathbf{C}[x]$ (on the left) in the way explained before and to find a nonzero $f \in \mathbf{C}[x]$ such that $P(f) \neq 0$ (left as an exercise). □

1.2.4. Some formulae. — The following (easy to prove) formulae may be useful:

$$\begin{aligned}
\left[\partial_x, x^k\right] &= kx^{k-1} \\
[\partial_x^j, x] &= j\partial_x^{j-1} \\
\left[\partial_x^j, x^k\right] &= \sum_{i \geq 1} \frac{k(k-1)\cdots(k-i+1)\cdot j(j-1)\cdots(j-i+1)}{i!} x^{k-i}\partial_x^{j-i}
\end{aligned}$$

where by convention negative powers are zero and zero powers equal 1.

1.3. Some properties

1.3.1. — Denote now by A one of the three rings of coefficients considered above. The ring $A\langle\partial_x\rangle$ comes equipped with an increasing filtration denoted by $F(A\langle\partial_x\rangle)$: an operator P is in F_k if the maximal total power of ∂_x that appears in a monomial of P is less than or equal to k. The degree of P (denoted by $\deg P$) with respect to $FA\langle\partial_x\rangle$ is the unique integer k such that $P \in F_kA\langle\partial_x\rangle - F_{k-1}A\langle\partial_x\rangle$.

This does not depend on the way one writes P. One has $F_kA\langle\partial_x\rangle = \{0\}$ for $k < 0$ and $F_0A\langle\partial_x\rangle = A$. Moreover, for all i and j one has $F_iA\langle\partial_x\rangle \cdot F_jA\langle\partial_x\rangle = F_{i+j}A\langle\partial_x\rangle$.

LEMMA 1.3.2. — *If* $\deg P = k$, $\deg Q = \ell$, *then the commutator* $[P, Q]$ *has degree less than or equal to* $k + \ell - 1$. □

PROPOSITION 1.3.3. — *The ring* $A\langle\partial_x\rangle$ *is simple (it does not contain any proper two-sided ideal).*

Proof. — Let I be a nonzero two-sided ideal of $A\langle\partial_x\rangle$ and $P \in I - \{0\}$. One may write $P = \sum_{i=1}^n a_i(x)\partial_x^i$ with $a_n \neq 0$. Then $Q \overset{\text{def}}{=} [P, x]$ belongs to I and $Q = \sum_{i=0}^n ia_i(x)\partial_x^{i-1}$. Hence, if $n \geq 1$, there exists Q in I of degree $n-1$. Continuing this process, one proves that I contains a nonzero element R of A. Assume that $A = \mathbf{C}[x]$. One then considers commutators with ∂_x to prove that I contains a nonzero constant, hence $I = A\langle\partial_x\rangle$. If $A = \mathbf{C}\{x\}$ or $A = \mathbf{C}[\![x]\!]$ one can argue as follows: if $k = v(R)$ (the valuation of R), then I contains x^k (by dividing R by a unit) then one commutes k times with ∂_x to show that I contains a constant. □

PROPOSITION 1.3.4. — *The graded ring*

$$\operatorname{gr}^F A\langle\partial_x\rangle \overset{\text{def}}{=} \bigoplus_{k=0}^{\infty} F_kA\langle\partial_x\rangle / F_{k-1}A\langle\partial_x\rangle$$

is isomorphic (as a graded ring) to the ring of polynomials in one variable ξ *on* A *(with the graduation given by the degree in* ξ*), the variable* ξ *being the class of* ∂_x.

Proof. — Immediate consequence of the previous lemma. □

COROLLARY 1.3.5. — $A\langle\partial_x\rangle$ *is a left and right noetherian ring.*

Proof. — One has to show that every left (or right) ideal I in $A\langle\partial_x\rangle$ admits a finite number of generators. Consider the filtration of I induced by $FA\langle\partial_x\rangle$: $F_k(I) = F_kA\langle\partial_x\rangle \cap I$. Then $\operatorname{gr}^F(I)$ is an ideal of $\operatorname{gr}^F A\langle\partial_x\rangle = A[\xi]$ and admits a finite number of generators p_i $(i = 1, \dots, r)$. One may assume that each generator p_i is homogeneous with respect to F (simply replace the set of generators by the set of their homogeneous components). For each i, let $P_i \in I$ an element with principal symbol equal to p_i. Then I is generated by the set $\{P_i | i = 1, \dots, r\}$: by induction on j one proves that $F_j(I)$ is contained in the left ideal generated by this set. In fact, let $P \in F_j(I) - F_{j-1}(I)$ and let $\sigma_j(P) \in \operatorname{gr}_j^F$ its principal symbol. One has a homogeneous relation:

$$\sigma_j(P) = \sum a_i p_i$$

with $a_i \in \operatorname{gr}_{j-\deg p_i}(A\langle\partial_x\rangle)$. Therefore $P - \sum a_i P_i \in F_{j-1}(I)$, so one gets the result. □

2. The structure of left ideals of $\mathcal{D}$

In this section, we shall consider only a local situation. The ring of coefficients will then be either $\mathbf{C}\{x\}$ or $\mathbf{C}[\![x]\!]$. We shall only consider the case of convergent coefficients, the case of formal coefficients being completely analogous.

2.1. Division by an operator

Let $P \in \mathcal{D}$ written as $P = a_d(x)\partial_x^d + \cdots + a_0(x)$, with $a_d \neq 0$. By definition, the *exponent* of P is the pair of integers

$$\exp(P) = (v(a_d), d) \in \mathbf{N}^2$$

where $d = \deg P$ and $v(a_d)$ is the valuation of a_d. This exponent is additive under product:

$$\exp(PQ) = \exp(P) + \exp(Q)$$

(sum in $\mathbf{N}^2$). In fact, if $Q = b_e(x)\partial_x^e + \cdots + b_0(x)$, one can write $PQ = a_d b_e \partial_x^{d+e} + R$ with $\deg R < d + e$. One deduces of this a "division statement":

Proposition 2.1.1. — *Let $A \in \mathcal{D}$ and $P \in \mathcal{D}$ with* $\exp(P) = (v, d)$. *There exists a unique pair (Q, R) of elements of $\mathcal{D}$ such that*

1. $A = PQ + R$
2. $R = \sum_{k=0}^{v-1} \sum_{\ell=d}^{\deg A} u_{k,\ell}(x) x^k \partial_x^\ell + S$ *with* $\deg S < d$ *and $u_{k,\ell}$ is a unit.*

Remark. — One has an analogous statement (division on the right): $A = PQ' + R'$.

Proof. — The proof of the proposition is elementary (induction on $\deg A$). The pairs (k, ℓ) which appear in the expansion of R are contained in the dotted part of *fig.* 1.

2.2. Division basis

Given an ideal I, the set $\mathrm{Exp}\,(I)$ is defined as the subset of $\mathbf{N}^2$ consisting of all $\exp(P)$ for $P \in I$. One has

$$\mathrm{Exp}\,(I) + \mathbf{N}^2 = \mathrm{Exp}\,(I)$$

because I is an ideal of $\mathcal{D}$. Such a set has the form indicated in *fig.* 2. The dotted part of the boundary of $\mathrm{Exp}\,(I)$ is denoted by $\mathrm{ES}\,(I)$ and will be called *the stairs* of I. We shall use the following notation:

$$\mathrm{ES}\,(I) = \Big\{(\alpha_p, p), (\alpha_{p+1}, p+1), \ldots, (\alpha_q, q)\Big\}$$

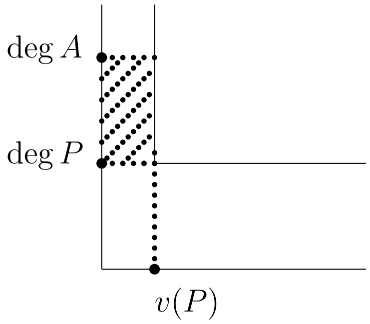

Figure 1

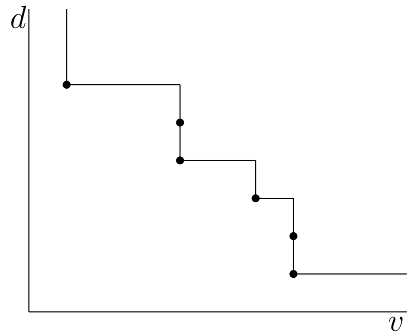

Figure 2

where p is the minimal degree of elements in I and in general α_j is the minimal valuation of the elements of I which have degree j. These assertions are easily deduced from the division statement above. One then has (as can be seen on the picture)

$$\operatorname{Exp}(I) = \bigcup_{p \leq j < q} \Big\{(\alpha_j, j) + \mathbf{N} \times \{0\}\Big\} \cup \Big\{(\alpha_q, q) + \mathbf{N}^2\Big\}$$

Definition 2.2.1. — *A division basis of the ideal I consists of the data for all $(\alpha_j, j) \in \operatorname{ES}(I)$ of an element P_j of I with $\exp(P_j) = (\alpha_j, j)$.*

Remark. — Take a minimal system $(\alpha_{j_1}, j_1), \ldots, (\alpha_{j_r}, j_r)$ of elements of $\operatorname{ES}(I)$ such that $\operatorname{Exp}(I) = \bigcup_{k=1}^r \Big\{(\alpha_{j_k}, j_k) + \mathbf{N}^2\Big\}$ (these are the vertices of the stairs where the angle is directed toward the origin). It is then easy to construct a division basis $P_p, \ldots, P_q$ knowing only $P_{j_1}, \ldots, P_{j_r}$ (of course one must have $j_1 = p$ and $j_r = q$).

The previous terminology comes from the following proposition:

Proposition 2.2.2. — *Let I be a proper ideal of $\mathcal{D}$ and let $P_p, \ldots, P_q$ be a division basis of I.*

1. *For all* $A \in \mathcal{D}$ *there exist unique elements* $Q_p, Q_{p+1}, \dots, Q_{q-1} \in \mathbf{C}\{x\}$, $Q_q \in \mathcal{D}$ *and* $R \in \mathcal{D}$ *such that*
$$A = Q_p P_p + \cdots + Q_q P_q + R$$
with
$$R = \sum_{\ell=p}^{\deg A} \sum_{k=0}^{\alpha_\ell - 1} u_{k,\ell} x^k \partial_x^\ell + S$$
and $\deg S < p$.
2. *With these notations, one has* $A \in I$ *if and only if* $R = 0$.

Consequently, all monomials appearing in R are essentially under the stairs of I (up to units $u_{k,\ell}$). Remark that one can give an analogous statement using only the minimal system $P_{j_1}, \dots, P_{j_r}$, but in that case the operators $Q_{j_1}, \dots, Q_{j_r}$ are in $\mathcal{D}$.

Proof. — The existence of such a division comes from the division statement 2.1.1: one divides first A by P_q, then one divides the part of the remainder which is of degree less than q by P_{q-1}, and so on.

Moreover one gets that $A \in I$ if and only if $R \in I$. But if $R \neq 0$ one has $\exp(R) \notin \mathrm{Exp}\,(I)$, hence $R \in I$ if and only if $R = 0$. This gives the second part of the proposition and also the uniqueness of R.

In order to show the uniqueness of $Q_p, \dots, Q_q$ it is sufficient to prove that if
$$Q_p P_p + \cdots + Q_q P_q = 0$$
with $Q_p, \dots, Q_{q-1} \in \mathbf{C}\{x\}$ and $Q_q \in \mathcal{D}$ then all these operators are zero. This can be easily proved by considering the highest degree term of such a sum. □

Corollary 2.2.3. — *Let* $I' \subset I$ *be two ideals of* $\mathcal{D}$ *with* $\mathrm{ES}\,(I') = \mathrm{ES}\,(I)$. *Then one has* $I' = I$.

Proof. — A division basis $P'_p, \dots, P'_q$ for I' is also one for I. The criterion for an operator to belong to I' is then the same as the one for I, because of the previous proposition. □

2.3. An ideal is generated by two elements

A division basis is then a system of generators of I, which is sufficiently redundant in order that the previous proposition is true. However let us now prove

Proposition 2.3.1. — *If* $P_p, \dots, P_q$ *is a division basis for* I *then* I *is generated by only* P_p *and* P_q.

Proof. — Let $J = \mathcal{D}\cdot P_p + \mathcal{D}\cdot P_q \subset I$ and consider the left $\mathcal{D}$-module I/J. This is a finite type module over $\mathcal{D}$ but also over $\mathbf{C}\{x\}$: it is actually generated over $\mathbf{C}\{x\}$ by the classes of $P_{p+1},\dots,P_{q-1}$ due to 2.2.2. The following lemma will be useful:

LEMMA 2.3.2. — *The left $\mathcal{D}$-module $I/\mathcal{D}\cdot P_p$ is a torsion module, i.e. for every $A\in I$ there exits k such that $x^kA\in\mathcal{D}\cdot P_p$.*

In order to prove this lemma, one remarks that in the elementary division of A by P, if one has $\exp(A)\in\exp(P)+\mathbf{N}^2$ then the remainder R has degree less than $\deg A$. Choose n_1 such that $\exp(x^{n_1}A)\in\exp(P_p)+\mathbf{N}^2$. One has

$$x^{n_1}A = B_1P_p + R_1 \qquad \deg R_1 < \deg A$$

and by induction one finds n such that

$$x^nA = BP_p + R$$

with $\deg R < \deg P_p = p$. But R belongs to I so $R=0$. □

One concludes from this lemma that I/J is also a torsion module. In order to prove that $I/J=\{0\}$, one uses the following lemma:

LEMMA 2.3.3. — *Let $\mathcal{M}$ be a left $\mathcal{D}$-module of finite type, which is also of finite type over $\mathbf{C}\{x\}$. Then $\mathcal{M}$ is a free $\mathbf{C}\{x\}$-module.*

Indeed, if $\mathcal{M}$ is of finite type over $\mathbf{C}\{x\}$, the $\mathbf{C}$-vector space $\mathcal{M}/x\mathcal{M}$ has finite dimension. Let $\overline{e_1},\dots,\overline{e_n}$ be a basis of this vector space and $e_1,\dots,e_n$ a set of representatives in $\mathcal{M}$. Nakayama's lemma shows that this set is a set of generators of $\mathcal{M}$ over $\mathbf{C}\{x\}$. Hence one gets a surjective mapping

$$\mathbf{C}\{x\}^n \longrightarrow \mathcal{M} \longrightarrow 0$$

which induces an isomorphism modulo the maximal ideal. Let K be the kernel of this mapping. We shall prove that $K=0$. An element in K can be written $\sum a_i\varepsilon_i$ with $a_i\in\mathbf{C}\{x\}$ and $(\varepsilon_i)_{i=1,\dots,n}$ is the canonical basis of $\mathbf{C}\{x\}^n$. One then has $\sum a_ie_i=0$ in $\mathcal{M}$. Hence, because $\mathcal{M}$ is also a left $\mathcal{D}$-module, one has $\partial_x(\sum a_ie_i)=0$. Put $\partial_xe_j=\sum b_{ji}e_i$. Then for all i one gets $\sum_i\left(\partial a_i/\partial x+\sum_j a_jb_{ji}\right)e_i=0$ and so $\sum_i\left(\partial a_i/\partial x+\sum_j a_jb_{ji}\right)\varepsilon_i\in K$. Continuing this process one is led to the case where there exists i such that $v(a_i)=0$ (*i.e.* a_i is a unit). Considering now the class modulo the maximal ideal one obtains a contradiction if $K\neq 0$. This concludes the proof of the lemma and consequently the proof of the proposition. □

As an easy consequence one gets

COROLLARY 2.3.4. — *If $\mathcal{M}$ is a left $\mathcal{D}$-module of finite type which is also a finite dimensional vector space, then $\mathcal{M} = \{0\}$.* □

This is a way to express *Bernstein inequality.* One can give an easy direct proof of this: one considers the trace of the commutator $[\partial_x, x]$ on $\mathcal{M}$. It must be zero because $\mathcal{M}$ is a finite dimensional vector space and $\dim \mathcal{M}$ because $\mathcal{M}$ is a left $\mathcal{D}$-module. Hence $\mathcal{M} = \{0\}$.

2.3.5. Exercises.

1. Show that if a left ideal I of $\mathcal{D}$ is generated by one element P then P is a division basis for I.

2. Let
$$P = x^4(1+x)\partial_x^5 + x^3(1+x^2)\partial_x^3 + 1.$$
Divide P by $x\partial_x + 3$ and also by $x^2\partial_x^3 + 1$.

3. Let I be a left ideal of $\mathcal{D}$ given by a set of generators $(P_1, \dots, P_r)$. Imagine a simple algorithm in order to construct a division basis for I.

3. Holonomic $\mathcal{D}$-modules

3.1. Differential systems, solutions

3.1.1. — Let P be a linear differential operator with coefficients in $\mathbf{C}\{x\}$, written as $P = \sum_{i=1}^{d} a_i(x)\partial_x^i$, with $a_i \in \mathbf{C}\{x\}$. One says that 0 is a singular point for P if $a_d(0) = 0$. If 0 is not singular then P admits d solutions in $\mathbf{C}\{x\}$ and these are independant over $\mathbf{C}$.

Generally speaking, if u is a solution for P in some functional space then u is also a solution for any operator of the form $Q \cdot P$ with $Q \in \mathcal{D}$. Hence the solutions depend only on the left ideal I of $\mathcal{D}$ generated by P.

More precisely, let $\mathcal{F}$ be a function space acted on by differential operators. A solution u of P in $\mathcal{F}$ defines a left $\mathcal{D}$-linear morphism $\mathcal{D}/I \to \mathcal{F}$ given by $Q \mapsto Q(u)$. The space of solutions (which is a $\mathbf{C}$-vector space) depends only on the left $\mathcal{D}$-module $\mathcal{D}/I$. We are therefore led to the following definition:

DEFINITION 3.1.2. — *A linear differential system is a left $\mathcal{D}$-module of finite type.*

Because $\mathcal{D}$ is left noetherian, one verifies

LEMMA 3.1.3. — *A differential system $\mathcal{M}$ is of finite presentation, i.e. there exists an exact sequence*

$$\mathcal{D}^q \xrightarrow{\varphi} \mathcal{D}^p \longrightarrow \mathcal{M} \longrightarrow 0$$

where φ is given by the right *multiplication by some matrix Φ with coefficients in $\mathcal{D}$.* □

3.1.4. Some function spaces.

1. $\mathbf{C}\{x\}$, $\mathbf{C}[\![x]\!]$
2. $K \overset{\text{def}}{=} \mathbf{C}\{x\}\,[x^{-1}]$, $\widehat{K} \overset{\text{def}}{=} \mathbf{C}[\![x]\!]\,[x^{-1}]$
3. $\mathcal{N}_{\alpha,p} \overset{\text{def}}{=} \mathcal{D}/\mathcal{D}\cdot(x\partial_x - \alpha)^{p+1}$, with $\alpha \in \mathbf{C}$ and $p \in \mathbf{N}$. This $\mathcal{D}$-module is the set of Nilsson class functions isomorphic as a $\mathbf{C}\{x\}$-module to $\bigoplus_{k=0}^{p} K \otimes x^\alpha(\log x)^k/k!$. These are germs of multi-valued functions of the variable x.
4. $\widetilde{\mathcal{O}}$ is the space of germs of all multi-valued functions.

3.2. Good filtrations, characteristic variety

Let $\mathcal{M}$ be a differential system. If $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$, one may associate with $\mathcal{M}$ the graded $\mathrm{gr}^F\mathcal{D}$-module $\mathrm{gr}^F\mathcal{D}/\mathrm{gr}^F\mathcal{D}\cdot\sigma(P)$. We want now to associate such a graded module to each differential system.

DEFINITION 3.2.1. — *Let $F\mathcal{M}$ be an increasing filtration of $\mathcal{M}$ indexed by $\mathbf{Z}$.*

1. *We say that $(\mathcal{M}, F\mathcal{M})$ is a filtered module over the filtered ring $(\mathcal{D}, F\mathcal{D})$ if $\mathcal{M} = \bigcup F_k\mathcal{M}$ and there exists ℓ such that $F_k\mathcal{M} = 0$ for $k \leq \ell$. Moreover one asks that for all k and ℓ $F_\ell\mathcal{D}\cdot F_k\mathcal{M} \subset F_{k+\ell}\mathcal{M}$.*
2. *We say that the filtered module is good (or that the filtration $F\mathcal{M}$ is good) if moreover one has the following two properties: $F_k\mathcal{M}$ is a $\mathbf{C}\{x\}$ module of finite type for all k and there exists k_0 such that for all $k \geq k_0$ and for all ℓ one has*
$$F_\ell\mathcal{D}\cdot F_k\mathcal{M} = F_{k+\ell}\mathcal{M}.$$

Remark that if $(\mathcal{M}, F\mathcal{M})$ is a filtered module, then the associated graded module

$$\mathrm{gr}^F\mathcal{M} = \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}^F_k\mathcal{M} \overset{\text{def}}{=} \bigoplus_{k\in\mathbf{Z}} F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

is naturally equipped with a structure of module over $\mathrm{gr}^F\mathcal{D}$. One has the following caracterization of a good filtration:

PROPOSITION 3.2.2. — *Let $(\mathcal{M}, F\mathcal{M})$ be a filtered left $\mathcal{D}$-module. Then the following conditions are equivalent:*

1. *$F\mathcal{M}$ is a good filtration of $\mathcal{M}$ (relatively to $F\mathcal{D}$).*

2. $\mathrm{gr}^F\mathcal{M}$ *is a* $\mathrm{gr}^F\mathcal{D}$*-module of finite type.*
3. *There exists a surjective morphism* $\mathcal{D}^p \to \mathcal{M} \to 0$*, a decomposition* $p = \sum_i p_i$ *(and* $\mathcal{D}^p = \bigoplus_i \mathcal{D}^{p_i}$*) and integers* n_i *such that* $F\mathcal{M}$ *is the filtration of* $\mathcal{M}$ *induced by* $\bigoplus_i F\mathcal{D}^{p_i}[n_i]$*, where one denotes by* $F[n]$ *the filtration defined by* $F[n]_k = F_{n+k}$*.*

Moreover, if $\mathcal{M}$ *admits a good filtration then* $\mathcal{M}$ *is of finite type over* $\mathcal{D}$*.*

Proof. — We shall just give a sketch of proof, the details are left as an exercise. First, given some filtered $\mathcal{D}$-module $(\mathcal{M}, F\mathcal{M})$, any submodule $\mathcal{M}'$ and any quotient $\mathcal{M}''$ come equipped with a structure of filtered module: just use the naturally induced filtration $F_k\mathcal{M}' = F_k\mathcal{M} \cap \mathcal{M}'$ and $F_k\mathcal{M}'' = \mathrm{Im}\,(F_k\mathcal{M} \to \mathcal{M}'')$. One shows easily that a good filtration of $\mathcal{M}$ induces a good filtration on a quotient module. So the third property implies the first. The converse is proven by taking generators of $\mathcal{M}$ wich are adapted to the filtration (take generators of each $F_k\mathcal{M}$ over $\mathbf{C}\{x\}$ for $k \leq k_0$). The third property implies the second one, just by grading the morphism $\mathcal{D}^p \to \mathcal{M}$. The converse is proven in the same way as in 1.3.5. □

Remark however that condition 2 holds because one knows that $F_k\mathcal{M} = 0$ for $k \ll 0$. Remark also that a differential system admits many good filtrations (so at least one).

PROPOSITION 3.2.3. — *Let* $F\mathcal{M}$ *and* $F'\mathcal{M}$ *be two good filtrations of* $\mathcal{M}$*. Then there exists* ℓ_0 *such that for all* k *one has*

$$F_{k-\ell_0} \subset F'_k \subset F_{k+\ell_0}.$$

Proof. — Let k'_0 such that $F_\ell\mathcal{D} \cdot F'_k\mathcal{M} = F'_{k+\ell}\mathcal{M}$ for all ℓ and all $k \geq k'_0$. There exists ℓ_0 such that $F'_{k'_0}\mathcal{M} \subset F_{\ell_0}\mathcal{M}$: indeed for all k, $F_\ell\mathcal{M} \cap F'_{k'_0}\mathcal{M}$ is a sub-$\mathbf{C}\{x\}$-module of finite type of $F'_{k'_0}\mathcal{M}$ and one has

$$\bigcup_\ell \left[F_\ell\mathcal{M} \cap F'_{k'_0}\mathcal{M}\right] = F'_{k'_0}\mathcal{M}$$

hence this sequence (indexed by ℓ) is stationary. Therefore one concludes that for all $k \leq k'_0$ one has $F'_k\mathcal{M} \subset F_{k+\ell_0}\mathcal{M}$ and for $k \geq k'_0$

$$\begin{aligned} F'_k\mathcal{M} = F'_{k-k'_0+k'_0}\mathcal{M} &= F_{k-k'_0}\mathcal{D} \cdot F_{k'_0}\mathcal{M} \\ &\subset F_{k-k'_0}\mathcal{D} \cdot F_{\ell_0}\mathcal{M} \\ &\subset F_{k-k'_0+\ell_0}\mathcal{M} \\ &\subset F_{k+\ell_0}\mathcal{M}. \quad \square \end{aligned}$$

3.2.4. — The ring $\mathrm{gr}^F\mathcal{D}$ is isomorphic to $\mathbf{C}\{x\}[\xi]$ as we have seen before. Let M be a noetherian $\mathrm{gr}^F\mathcal{D}$-module. One defines the *support* of M as the set of prime ideals of $\mathbf{C}\{x\}[\xi]$ associated with the annihilator ideal of M:

$$\sqrt{\operatorname{Ann} M} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r.$$

If M is a graded module, then its annihilator is a graded ideal as well as each $\mathfrak{p}_i$.

Definition 3.2.5. — *Let $\mathcal{M}$ be a linear differential system and $F\mathcal{M}$ a good filtration of $\mathcal{M}$. The characteristic variety of $\mathcal{M}$ is the support of* $\mathrm{gr}^F\mathcal{M}$.

3.2.6. — This definition infers that the support of $\mathrm{gr}^F\mathcal{M}$ does not depend on the choice of a good filtration but only on $\mathcal{M}$. Let us verify this point: assume that we are given two good filtrations $F\mathcal{M}$ and $G\mathcal{M}$. We know that there exists ℓ_0 such that for all k one has

$$F_{k-\ell_0}\mathcal{M} \subset G_k\mathcal{M} \subset F_{k+\ell_0}\mathcal{M}.$$

We shall show that

$$\sqrt{\operatorname{Ann} \mathrm{gr}^F\mathcal{M}} = \sqrt{\operatorname{Ann} \mathrm{gr}^G\mathcal{M}}.$$

Let $\varphi \in \sqrt{\operatorname{Ann} \mathrm{gr}^F\mathcal{M}}$. There exists an a such that $\varphi^a \in \operatorname{Ann} \mathrm{gr}^F\mathcal{M}$, *i.e.* for all k one has $\varphi^a F_k\mathcal{M} \subset F_{k-1}\mathcal{M}$. One deduces of this (by iterating the process) that there exists a b such that for all k one has

$$\varphi^b F_{k+\ell_0}\mathcal{M} \subset F_{k-\ell_0}\mathcal{M}$$

and hence for all k

$$\varphi^b G_k\mathcal{M} \subset G_{k-1}\mathcal{M}.$$

Consequently one obtains that $\varphi \in \sqrt{\operatorname{Ann} \mathrm{gr}^G\mathcal{M}}$. By symmetry the other inclusion is also satisfied. $\square$

Examples.

1. Let $P = \sum_{i=0}^d a_i(x)\partial_x^i$, with $a_d \neq 0$, $a_d(0) = 0$ and $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$. Then

$$\mathrm{gr}^F\mathcal{M} = \mathrm{gr}^F\mathcal{D}/\mathrm{gr}^F\mathcal{D}\cdot\sigma(P)$$

   where $F\mathcal{M}$ is the filtration naturally induced on $\mathcal{M}$, $\operatorname{Ann} \mathrm{gr}^F\mathcal{M} = \sigma(P)$ and $\sigma(P) = a_d(x)\xi^d$. So $\mathrm{gr}^F\mathcal{D}\cdot\sigma(P) = \mathrm{gr}^F\cdot(x^v\xi^d)$ and $\sqrt{\operatorname{Ann} \mathrm{gr}^F\mathcal{M}} = (x)\cap(\xi)$. The variety associated with $\sqrt{\operatorname{Ann} \mathrm{gr}^F\mathcal{M}}$ is the subset of the space T (which is the germ along $\{0\}\times\mathbf{C}$ of $\mathbf{C}^2$) defined by the equation $x\xi = 0$.

2. Let $\mathcal{M} = \mathcal{D}/I$ where I is a proper ideal of $\mathcal{D}$. Choose a division basis for I $\{P_p,\dots,P_q\}$ and consider the corresponding presentation of $\mathcal{M}$:

$$\begin{array}{ccccccc} \mathcal{D}^{p-q+1} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\ (Q_p,\dots,Q_q) & \mapsto & \sum Q_jP_j & & & & \end{array}$$

   and the filtration induced by $F\mathcal{D}$.

LEMMA 3.2.7. — $\mathrm{gr}^F \mathcal{M} = \mathrm{gr}^F \mathcal{D} \,/(\sigma(P_p), \ldots, \sigma(P_q))\,.$

Proof. — By definition one has a surjective mapping

$$\mathrm{gr}^F \mathcal{D} \longrightarrow \mathrm{gr}^F \mathcal{M} \longrightarrow 0$$

which kernel is equal to $\mathrm{gr}^F I$ if one defines $F_k I = F_k \mathcal{D} \cap I$. We have to show that $\mathrm{gr}^F I$ is the ideal generated by the symbols of the division basis. This follows directly from the definition of $\mathrm{Exp}\,(I)$: indeed, if $P \in I$ then $\exp(P) \in \mathrm{Exp}\,(I)$ and $\exp(P) = \exp(P_i) + (\beta_i, m_i)$ for some $i \in \{p, \ldots, q\}$ and some $(\beta_i, m_i) \in \mathbf{N}^2$. Put $\exp(P_i) = (\alpha_i, i)$, $\sigma(P_i) = u_i(x) x^{\alpha_i} \xi^i$ where u_i is a unit and $\sigma(P) = u(x) x^\alpha \xi^d$ where u is a unit. Then $\sigma(P) = (u(x)/u_i(x))\, x^{\beta_i} \xi^{m_i} \sigma(P_i)$. □

The annihilator of $\mathrm{gr}^F \mathcal{M}$ is then equal to $(\sigma(P_p), \ldots, \sigma(P_q))$ and its radical is the intersection of the ideals (x) and (ξ), *i.e.* the ideal $(x\xi)$.

3.2.8. Some properties of the characteristic variety.

- $\mathrm{Car}\,\mathcal{M}$ is defined by equations which are homogeneous with respect to the variable ξ (namely of the form $a(x)\xi^d$), because $\mathrm{gr}^F \mathcal{M}$ is a graded $\mathrm{gr}^F \mathcal{D}$-module.
- Here are the different possibilities for $\mathrm{Car}\,\mathcal{M}$:
  - $\mathrm{Car}\,\mathcal{M} = T$, *i.e.* $\sqrt{\mathrm{Ann}\,\mathrm{gr}^F \mathcal{M}} = \{0\}$ which is obtained for instance with $\mathcal{M} = \mathcal{D}$ or $\mathcal{D}^p$ or $\mathcal{D}^p + \mathcal{M}'$, $\mathcal{M}'$ any system.
  - $\mathrm{Car}\,\mathcal{M} = \{\xi = 0\}$. Example: $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$ with $P = \sum_{i=o}^{d} a_i(x) \partial_x^i$ and $a_d(0) \neq 0$.
  - $\mathrm{Car}\,\mathcal{M} = \{x = 0\}$. Example: $P = a_0(x)$, with $a_0(0) = 0$.
  - $\mathrm{Car}\,\mathcal{M} = \{x = 0\} \cup \{\xi = 0\}$. This is the general case.
- $\mathrm{Car}\,\mathcal{M}$ cannot be supported on a point (Bernstein inequality). Indeed, if this would be the case, then for any good filtration $F\mathcal{M}$ there would exist v such that $x^v \in \mathrm{Ann}\,\mathrm{gr}^F \mathcal{M}$ and d such that $\xi^d \in \mathrm{Ann}\,\mathrm{gr}^F \mathcal{M}$. Assume for instance that $F_k \mathcal{M} = 0$ for $k < 0$. Then for all $k \geq 0$ one has $x^{vk} F_k \mathcal{M} = 0$ and $\partial_x^{dk} F_k \mathcal{M} = 0$. Choose generators $m_1, \ldots, m_r$ of $\mathcal{M}$ and put $F_k \mathcal{M} = \sum_i F_k \mathcal{D} \cdot m_i$. One obtains

  $$x^v m_1 = \cdots = x^v m_r = 0 \qquad \text{and} \qquad \partial_x^d m_1 = \cdots = \partial_x^d m_r = 0.$$

  This implies that $\mathcal{M}$ is a finite dimensional $\mathbf{C}$-vector space, so $\mathcal{M} = \{0\}$ and $\mathrm{Car}\,\mathcal{M} = \emptyset$.

3.3. Holonomic systems

DEFINITION 3.3.1. — *Let $\mathcal{M}$ be a linear differential system. One says that $\mathcal{M}$ is holonomic if $\mathcal{M} = 0$ or if* $\mathrm{Car}\,\mathcal{M} \subset \{x = 0\} \cup \{\xi = 0\}$.

Examples. — $\mathcal{D}^p$ is not holonomic and $\mathcal{D}/I$ is holonomic as soon as I is not equal to 0.

3.3.2. Some properties. — Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of left $\mathcal{D}$-modules.

1. $\mathcal{M}$ is of finite type over $\mathcal{D}$ if and only if so are $\mathcal{M}'$ and $\mathcal{M}''$.
2. Moreover, $\mathcal{M}$ is holonomic if and only if so are $\mathcal{M}'$ and $\mathcal{M}''$.

Proof. — The first point is clear. Let us prove the second one. In fact one has the following equality:

$$\operatorname{Car}\mathcal{M} = \operatorname{Car}\mathcal{M}' \cup \operatorname{Car}\mathcal{M}''. \tag{3.3.3}$$

Indeed let $F\mathcal{M}$ be a good filtration of $\mathcal{M}$. The image filtration $F\mathcal{M}''$ is good and so is the induced filtration $F\mathcal{M}'$: by definition one can use the exact sequence $0 \to \operatorname{gr}^F\mathcal{M}' \to \operatorname{gr}^F\mathcal{M} \to \operatorname{gr}^F\mathcal{M}'' \to 0$ in conjunction with proposition 3.2.2. This exact sequence also shows that

$$\sqrt{\operatorname{Ann}\operatorname{gr}^F\mathcal{M}} = \sqrt{\operatorname{Ann}\operatorname{gr}^F\mathcal{M}'} \cap \sqrt{\operatorname{Ann}\operatorname{gr}^F\mathcal{M}''}.$$

Remark also that $\mathcal{M}$ is holonomic if and only if $\mathcal{M}$ is a successive extension of modules isomorphic to $\mathcal{D}/I$ with nonzero ideals I of $\mathcal{D}$ (take an element m of $\mathcal{M}$ and consider the $\mathcal{D}$-module $\mathcal{D}\cdot m$ which is of the form $\mathcal{D}/I$, then consider the quotient $\mathcal{M}/\mathcal{D}\cdot m$ and repeat the process; conclude by noetherianity). But one can give a more precise statement.

THEOREM 3.3.4. — *Let $\mathcal{M}$ be a left $\mathcal{D}$-module of finite type. Then $\mathcal{M}$ is holonomic if and only if $\mathcal{M}$ is of finite length over $\mathcal{D}$, i.e. every decreasing sequence of left $\mathcal{D}$-modules is stationary.*

COROLLARY 3.3.5. — *Let $\mathcal{M}$ be a holonomic left $\mathcal{D}$-module. Then there exists an element $m \in \mathcal{M}$ which generates $\mathcal{M}$, i.e. there exists an isomorphism $\mathcal{M} \simeq \mathcal{D}/I$ for some nonzero ideal I of $\mathcal{D}$.*

This corollary is a direct consequence of the previous theorem and the following one, which is due to Stafford:

THEOREM 3.3.6. — *Let A be a simple ring, which is of infinite length as a left A-module. Then every left A-module of finite length is generated by one element.*

The fact that $\mathcal{D}$ is simple was proven in 1.3.3 and the fact that $\mathcal{D}$ is of infinite length is left as an exercise.

Proof of theorem 3.3.4. — We shall use some results of commutative algebra as explained in the following subsection 3.4. Let $\mathcal{M}$ be a left $\mathcal{D}$-module of finite type and $F\mathcal{M}$ be a good filtration.

PROPOSITION 3.3.7. — *The dimension* $\dim_{\mathrm{gr}^F\mathcal{D},0} \mathrm{gr}^F\mathcal{M}$ *and the multiplicity* $e_{\mathrm{gr}^F\mathcal{D},0}\mathrm{gr}^F\mathcal{M}$ *do not depend on the good filtration* F

Let us just give a sketch of proof. Let $F\mathcal{M}$ and $G\mathcal{M}$ be two good filtrations. First one may assume that for all k one has $F_k\mathcal{M} \subset G_k\mathcal{M}$ (the shift of a filtration does not affect dimension and multiplicity of the corresponding graded module). One obtains a morphism $\mathrm{gr}^F\mathcal{M} \to \mathrm{gr}^G\mathcal{M}$ which kernel and cokernel have support at one point, so of dimension 0. One concludes by using the fact that $\dim_{\mathrm{gr}^F\mathcal{D},0} \mathrm{gr}^F\mathcal{M} \geq 1$ (Bernstein inequality) and the same for G: the difference between the two Hilbert functions is then of degree 0, so these two Hilbert functions have the same dominant term.

We have the following characterization of holonomic $\mathcal{D}$-module.

LEMMA 3.3.8. — $\mathcal{M}$ *is holonomic if and only if* $\dim_{\mathrm{gr}^F\mathcal{D},0} \mathrm{gr}^F\mathcal{M} = 1$.

Proof. — It is enough to prove that $\dim_{\mathrm{gr}^F\mathcal{D},0} \mathrm{gr}^F\mathcal{M} = 2$ if $\mathcal{M} = \mathcal{D}$ and $\dim_{\mathrm{gr}^F\mathcal{D},0} \mathrm{gr}^F\mathcal{M} = 1$ if $\mathcal{M} = \mathcal{D}/I$ with $I \neq 0$ because every module of finite type is a finite extension of such modules. For the first case the result is clear (*cf.* 3.4) and for the second one computes the dimension with a division basis and $\mathrm{Exp}\,(I)$.

Let now $\mathcal{M}$ be a holonomic $\mathcal{D}$-module and consider a decreasing sequence $\mathcal{M} \supset \mathcal{M}_1 \supset \cdots$. The dimension of every corresponding graded module is one so the multiplicity is a decreasing function. It must then be stationary. The corresponding value must be 0 because of the additivity property of multiplicity (*cf.* 3.4). Conversely, if $\mathcal{M}$ is of finite length, no factor isomorphic to $\mathcal{D}$ can appear as a subquotient of $\mathcal{M}$. Hence $\mathcal{M}$ is holonomic. □

Proof of theorem 3.3.6. — Let M be a left A-module of finite length. We shall use induction on the length of M denoted by ℓ (this is the maximal length of a chain of submodules of M). If $\ell = 1$, M has no proper submodule hence is generated by any nonzero element. In particular M is cyclic. Assume that ℓ is bigger than 1 and that the result is true for any module of length less than ℓ. It is then possible to find $m' \in M$ such that $M' = A \cdot m'$ is of length 1 (because M is of finite length). Hence M/M' is of length less than ℓ and therefore can be generated by one element. Choose $m'' \in M$ which class in M/M' is a generator and put $M'' = A \cdot m''$. We conclude that $M = M' + M''$. We shall show that M is generated by an element of the form $m'' + \alpha m'$ for some $\alpha \in A$.

Consider the following surjective morphisms $A \to M'$ and $A \to M''$ (product on the right by m' or m'') and their kernel K' and K''. Because M (hence M' and M'') is of finite length and A of infinite length, one has $K' \neq 0$ and $K'' \neq 0$ (and these left ideals are different from A). Because A is simple, the two-sided ideal $K''A$ is equal to A. Let $\alpha \in A$ such that $K''\alpha \neq K'$ and consider $m = m'' + \alpha m'$. We shall show that $M = A \cdot m$.

Let us first prove that $K' + K''\alpha = A$. In fact, $K' + K''\alpha$ is an ideal strictly containing K' and one has a surjective morphism

$$M' = A/K' \to A/K' + K''\alpha \to 0.$$

The kernel $K' + K''\alpha/K'$ is a submodule of M' which is nonzero. Hence it is equal to M', which proves the assertion.

Put $1 = \lambda' + \lambda''\alpha$ with $\lambda' \in K'$ and $\lambda'' \in K''$. One sees that (because $K' \cdot m' = 0$)

$$m' = \lambda' m' + \lambda''\alpha m' = \lambda''\alpha m'$$

hence (because $K'' \cdot m'' = 0$)

$$m' = \lambda'' m'' + \lambda''\alpha m' = \lambda'' m.$$

Consequently $m' \in A \cdot m$ and $m'' \in A \cdot m$. □

3.4. Appendix on dimension and multiplicities

Recall some well known facts (see for instance [2, Chap.8, §4.3] or the last chapter of [1]). Let $A = \mathbf{C}\{x\}[\xi]$ and let M be a noetherian A-module. Define the Hilbert function

$$H_M(T) = \sum_{n=0}^{\infty} \dim_{\mathbf{C}} \Big[(x,\xi)^n M/(x,\xi)^{n+1} M\Big]\, T^n.$$

Proposition 3.4.1. — *There exist $d \in \mathbf{N}$ and $R \in \mathbf{Z}[T, T^{-1}]$ such that*

$$H_M(T) = \frac{R(T)}{(1-T)^d} \quad .$$

Moreover one has $R(1) > 0$. □

The integer d is called the dimension of M over A at 0 and is denoted by $\dim_{A,0} M$. The integer $R(1)$ is denoted by $e_{A,0}(M)$ and is called the multiplicity of M at 0. Let us give some properties without proof:

1. $$\dim_{\mathbf{C}} M/(x,\xi)^n M = e_{A,0}(M)\frac{n^d}{d!} + \beta_n n^{d-1}$$

   and $\lim\limits_{n\to\infty} \beta_n$ exists.

2. $\dim_{A,0} M \geq 0$ and this dimension is zero if and only if M is supported on $\{0\}$.
3. $e_{A,0}(M) = 0$ if and only if $M = \{0\}$.
4. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. Then one has $\dim_{A,0} M = \max\big(\dim_{A,0} M', \dim_{A,0} M''\big)$ and if all these dimensions are equal then $e_{A,0}(M) = e_{A,0}(M') + e_{A,0}(M'')$.

4. Meromorphic connections

Let M be a $\mathbf{C}\{x\}$-module of finite type. Then, if M is nonzero, multiplication by $x : M \to M$ is not bijective. This follows immediately from Nakayama's lemma (see the references in commutative algebra). We shall now see that "Nakayama's lemma" is not true over the ring $\mathcal{D}$. In fact there exist many non trivial holonomic $\mathcal{D}$-modules on which (left) multiplication by x is bijective: these are called meromorphic connections.

4.1. Localization of a $\mathbf{C}\{x\}$-module

Let $\mathbf{C}\{x\}[x^{-1}]$ be the ring of Laurent series. This is a field and will be denoted by K. In the same way one denotes by $\widehat{K}$ the field of formal Laurent series $\mathbf{C}[\![x]\!][x^{-1}]$. Let M be a $\mathbf{C}\{x\}$-module. We denote by $M[x^{-1}]$ the K-vector space $M \otimes_{\mathbf{C}\{x\}} K$. In general, if M is of finite type over $\mathbf{C}\{x\}$, $M[x^{-1}]$ is of finite dimension over K. Remark however that $M[x^{-1}]$ is not of finite type over $\mathbf{C}\{x\}$ in general. We have a $\mathbf{C}\{x\}$-linear mapping

$$M \longrightarrow M \otimes K$$

given by $m \mapsto m \otimes 1$. Remark that this mapping is an isomorphism if and only if multiplication by x on M is bijective. The localization satisfies the following two properties, which proof is left as an exercise (or see the references in commutative algebra).

Lemma 4.1.1. — *The kernel of this mapping is the submodule of torsion elements of M, i.e. the set of $m \in M$ such that there exists k with $x^k \cdot m = 0$. The cokernel is also a torsion module.* □

Lemma 4.1.2. — *Localization is an exact functor, namely if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $\mathbf{C}\{x\}$-modules, then the corresponding sequence $0 \to M' \otimes K \to M \otimes K \to M'' \otimes K \to 0$ is also exact.* □

4.2. Localization of a holonomic $\mathcal{D}$-module

Let $\mathcal{M}$ be a left $\mathcal{D}$-module. First we consider it only as a $\mathbf{C}\{x\}$-module and let $\mathcal{M}[x^{-1}]$ be the localized module.

Proposition 4.2.1. — *$\mathcal{M}[x^{-1}]$ admits a natural structure of a left $\mathcal{D}$-module and the natural mapping $\mathcal{M} \to \mathcal{M}[x^{-1}]$ is left $\mathcal{D}$-linear.*

Proof. — One defines the action of ∂_x on $\mathcal{M}[x^{-1}]$ with the help of Leibniz rule:

$$\partial_x(m \otimes x^{-k}) = \big((\partial_x m) \otimes x^{-k}\big) - km \otimes x^{-k-1}.$$

We leave as an exercise the proof of $\mathcal{D}$-linearity. □

COROLLARY 4.2.2. — *The kernel and the cokernel of $\mathcal{M} \to \mathcal{M}[x^{-1}]$ are also left $\mathcal{D}$-modules.* □

Remark. — As for $\mathbf{C}\{x\}$-modules of finite type, $\mathcal{M}[x^{-1}]$ may not be of finite type over $\mathcal{D}$ (being however of finite type over the ring $\mathcal{D}[x^{-1}]$ of linear differential operators with coefficients in K). For instance, $\mathcal{D}[x^{-1}]$ is not of finite type over $\mathcal{D}$. The main result of this section will be however the following:

THEOREM 4.2.3. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. Then $\mathcal{M}[x^{-1}]$ is of finite type over $\mathcal{D}$ and even is holonomic.*

4.2.4. Filtration $V\mathcal{D}$. — Before giving the proof of this theorem, we shall introduce a new filtration of $\mathcal{D}$, which will appear to be useful later on. This filtration is denoted by $V\mathcal{D}$. This is an increasing filtration, indexed by $\mathbf{Z}$. Let $P = \sum a_i(x)\partial_x^i \in \mathcal{D}$. We say that $P \in V_k\mathcal{D}$ if $\max_i(i - v(a_i)) \leq k$. Let us give first some elementary properties of this filtration.

1. $V_0\mathcal{D} = \mathbf{C}\{x\}\langle x\partial_x\rangle$, $x \in V_{-1}\mathcal{D}$ and $\partial_x \in V_1\mathcal{D}$.
2. The filtration $V\mathcal{D}$ induces on $\mathbf{C}\{x\} \subset \mathcal{D}$ the opposite filtration to the $\mathfrak{m}$-adic filtration, *i.e.* $V_k\mathcal{D} \cap \mathbf{C}\{x\} = \mathfrak{m}^{-k}$ for all k, if one puts $\mathfrak{m}^{-k} = \mathbf{C}\{x\}$ for $k \geq 0$. The graded ring obtained from $\mathbf{C}\{x\}$ is then isomorphic to $\mathbf{C}[x]$ (with its oppposite usual grading).
3. For all $k \geq 0$, one has:
$$V_{-k}\mathcal{D} = x^k V_0\mathcal{D} = V_0\mathcal{D}x^k$$
and
$$V_k\mathcal{D} = \partial_x^k V_0\mathcal{D} + V_{k-1}\mathcal{D} = V_0\mathcal{D}\partial_x^k + V_{k-1}\mathcal{D}.$$
4. A monomial $x^{\alpha_1}\partial_x^{\beta_1}x^{\alpha_2}\partial_x^{\beta_2}\cdots x^{\alpha_r}\partial_x^{\beta_r}$ is in $V_k\mathcal{D}$ if $k \geq \beta_1+\cdots+\beta_r-(\alpha_1+\cdots+\alpha_r)$.
5. If $P \in V_k$, then $(x\partial_x + k)\cdot P - P\cdot(x\partial_x) \in V_{k-1}$.
6. For all $k \geq 0$, multiplication on the left (or on the right) by x induces a bijective mapping
$$V_{-k}\mathcal{D} \longrightarrow V_{-k-1}\mathcal{D}.$$
7. The graded ring $\mathrm{gr}^V\mathcal{D}$ is isomorphic to the Weyl algebra $\mathbf{C}[x]\langle\partial_x\rangle$, and this isomorphism is compatible with graduation if one takes on the Weyl algebra the grading induced by the V-filtration.

Proposition 4.2.5. — *Let $P \in V_0\mathcal{D}$. Then right multiplication by P*

$$\mathcal{D} \xrightarrow{\cdot P} \mathcal{D}$$

is strict with respect to $V\mathcal{D}$.

Proof. — One has to show that $V_k\mathcal{D} \cdot P = V_k\mathcal{D} \cap \mathcal{D} \cdot P$. Let $A \in \mathcal{D} \cdot P$, written as $A = QP$ and assume that $A \in V_k\mathcal{D}$ and $Q \in V_{k+\ell}\mathcal{D}$ for some $\ell \geq 0$. Then we have $[Q][P] = 0$ in $\mathrm{gr}^V_{k+\ell}\mathcal{D}$ if $\ell > 0$. But right multiplication by $[P]$, $\mathrm{gr}^V_{k+\ell}\mathcal{D} \xrightarrow{\cdot [P]} \mathrm{gr}^V_{k+\ell}\mathcal{D}$ is injective (exercise). Then $[Q] = 0$ in $\mathrm{gr}^V_{k+\ell}\mathcal{D}$, so by induction $Q \in V_k\mathcal{D}$. □

Corollary 4.2.6. — *Let $P \in V_0\mathcal{D}$ and consider the exact sequence*

$$0 \longrightarrow \mathcal{D} \xrightarrow{\cdot P} \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{D} \cdot P \longrightarrow 0.$$

Consider on $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$ the filtration induced by $V\mathcal{D}$:

$$U_k\mathcal{M} \stackrel{\text{def}}{=} V_k\mathcal{D}/V_k\mathcal{D} \cap \mathcal{D} \cdot P.$$

Then the sequence

$$0 \longrightarrow V_k\mathcal{D} \xrightarrow{\cdot P} V_k\mathcal{D} \longrightarrow U_k\mathcal{M} \longrightarrow 0$$

is exact for all k. □

Proof of theorem 4.2.3. — We shall first reduce the proof to the case where $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$. We know that $\mathcal{M}$ is an extension of modules of the form $\mathcal{D}/I$ for some proper ideals I. Because localization is an exact functor, it is sufficient to prove the theorem for such modules (one could also use the theorem proved in the previous section saying that $\mathcal{M}$ is in fact isomorphic to such a module, but it is not necessary). Choose a division basis $\{P_p, \ldots, P_q\}$ for I . One has a surjective morphism

$$\mathcal{D}/\mathcal{D} \cdot P_p \longrightarrow \mathcal{D}/I \longrightarrow 0$$

which kernel is equal to $I/\mathcal{D} \cdot P_p$. We have yet seen that this is a torsion module (see lemma 2.3.2). Hence the induced morphism

$$\mathcal{D}/\mathcal{D} \cdot P_p[x^{-1}] \longrightarrow \mathcal{D}/I[x^{-1}]$$

is an isomorphism. One is then reduced to the case of one generator and one relation.

Consider $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$ for some nonzero $P \in \mathcal{D}$. We shall reduce to the case where $P \in V_0\mathcal{D} - V_{-1}\mathcal{D}$. Assume that $P \in V_k\mathcal{D} - V_{k-1}\mathcal{D}$ with $k > 0$. Put $P' = x^k P$. Then $\mathcal{D} \cdot P' \subset \mathcal{D} \cdot P$ and we get a surjective mapping

$$\mathcal{D}/\mathcal{D} \cdot P' \longrightarrow \mathcal{D}/ \cdot P \longrightarrow 0$$

which kernel $\mathcal{D} \cdot P/\mathcal{D} \cdot P'$ is a torsion module (if $Q \in \mathcal{D}$ and $k > 0$, there exists some N such that $x^N Q = Q' x^k$ for some Q'; hence $x^N QP = Q' x^k P \in \mathcal{D} \cdot P'$). Consequently these two modules have isomorphic localized modules. Assume now that $k < 0$. Then we know that $P = x^k P'$ for some $P' \in V_0\mathcal{D}$, so we can apply the same argument as above.

We shall then assume now that $P = b(x\partial_x) + xP'$ where b is a nonzero polynomial in one variable with constant coefficients and $P' \in V_0\mathcal{D}$ (this is equivalent to saying that $P \in V_0\mathcal{D} - V_{-1}\mathcal{D}$). We can now state:

LEMMA 4.2.7. — *Left multiplication by x*

$$\mathcal{D}/\mathcal{D}\cdot P \xrightarrow{x\cdot} \mathcal{D}/\mathcal{D}\cdot P$$

is bijective (i.e. $\mathcal{D}/\mathcal{D}\cdot P = \mathcal{D}/\mathcal{D}\cdot P[x^{-1}]$) if and only if one has $b(k) \neq 0$ for all $k \in \mathbf{N}$.

Proof. — We shall use the filtration $U(\mathcal{D}/\mathcal{D}\cdot P)$ introduced above. First, independently of the hypothesis that $b(k) \neq 0$ for $k \in \mathbf{N}$, one sees that left multiplication by x

$$U_0(\mathcal{D}/\mathcal{D}\cdot P) \xrightarrow{x\cdot} U_{-1}(\mathcal{D}/\mathcal{D}\cdot P)$$

is bijective. Indeed, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_0\mathcal{D} & \xrightarrow{\cdot P} & V_0\mathcal{D} & \longrightarrow & U_0(\mathcal{D}/\mathcal{D}\cdot P) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle x\cdot} & & \downarrow{\scriptstyle x\cdot} & & \downarrow{\scriptstyle x\cdot} & & \\
0 & \longrightarrow & V_{-1}\mathcal{D} & \xrightarrow{\cdot P} & V_{-1}\mathcal{D} & \longrightarrow & U_{-1}(\mathcal{D}/\mathcal{D}\cdot P) & \longrightarrow & 0
\end{array}$$

We know from 4.2.6 that the two horizontal lines are exact and we have also seen that the two left vertical maps are bijective. We then obtain the desired assertion by the snake lemma.

In order to prove the lemma, we must now see that under the hypothesis that we have made,

$$\mathcal{M}/U_0\mathcal{M} \xrightarrow{x\cdot} \mathcal{M}/U_{-1}\mathcal{M}$$

with $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$ is bijective. It is enough to prove that for all $k \geq 1$,

$$U_k\mathcal{M}/U_{k-1}\mathcal{M} \xrightarrow{x\cdot} U_{k-1}\mathcal{M}/U_{k-2}\mathcal{M}$$

is bijective. Remark the following fact: for all $k \in \mathbf{Z}$ we have

$$b(x\partial_x + k)\cdot U_k\mathcal{M} \subset U_{k-1}\mathcal{M}.$$

Indeed, let $Q \in V_k\mathcal{D}$. It is enough to see that

$$b(x\partial_x + k)\cdot Q \in V_{k-1}\mathcal{D} + \mathcal{D}\cdot P.$$

We have (because of the properties of $V\mathcal{D}$ that we have mentioned):

$$\begin{array}{rcll}
b(x\partial_x + k)\cdot Q & = & Q\cdot b(x\partial_x) + Q' & Q' \in V_{k-1}\mathcal{D}\\
 & = & QP - QxP' + Q' & \\
 & = & QP + Q'' & Q'' \in V_{k-1}\mathcal{D}
\end{array}$$

which gives the desired result. One deduces from this that the operator induced by left multiplication by $x\partial_x + 1$ on $U_k\mathcal{M}/U_{k-1}\mathcal{M}$ admits a minimal polynomial which divides $b(s+k-1)$. If $b(k-1) \neq 0$ for all $k \geq 1$, one deduces that $x\partial_x + 1 = \partial_x x$ is invertible on $U_k\mathcal{M}/U_{k-1}\mathcal{M}$ for all $k \geq 1$. This implies that left multiplication by x

$$U_k\mathcal{M}/U_{k-1}\mathcal{M} \xrightarrow{x\cdot} U_{k-1}\mathcal{M}/U_{k-2}\mathcal{M}$$

is injective. Surjectivity is obtained in the same way. □

It follows from this lemma that if the condition on the zeros of b is satisfied, then $\mathcal{M}$ is isomorphic to its localized module, so in particular the latter is of finite type over $\mathcal{D}$ and also holonomic, because so is $\mathcal{M}$. What happens when the condition of the lemma is not satisfied ? Let $P = b(x\partial_x) + xP'$ as above. There always exits $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$ one has $b(k) \neq 0$. Consider then the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}\,(b(x\partial_x + 1) + P'x)$. One has a $\mathcal{D}$-linear mapping

$$\mathcal{D}/\mathcal{D}\,(b(x\partial_x) + xP') \longrightarrow \mathcal{D}/\mathcal{D}\,(b(x\partial_x + 1) + P'x)$$

given by *right* multiplication by x. Choose $P'' \in V_0\mathcal{D}$ such that $xP'' = P'x$. In the same way we get a $\mathcal{D}$-linear mapping

$$\mathcal{D}/\mathcal{D}\,(b(x\partial_x) + xP') \longrightarrow \mathcal{D}/\mathcal{D}\,(b(x\partial_x + 2) + P''x)$$

and iterating the process we get

$$\mathcal{D}/\mathcal{D}\,(b(x\partial_x) + xP') \longrightarrow \mathcal{D}/\mathcal{D}\left(b(x\partial_x + k_0 + 1) + P^{(k_0+1)}x\right).$$

So by the previous lemma, the last module obtained is isomorphic to its localized module. We can conclude if we know that the kernel and cokernel of this mapping are torsion modules. This is done step by step and left as an exercise. □

Examples. — Let $P = x\partial_x + \alpha$ with $\alpha \in \mathbf{C}$. Then $\mathcal{M} = \mathcal{M}[x^{-1}]$ if and only if $\alpha \notin \mathbf{N}$. If $P = x^2\partial_x + \alpha$, then $\mathcal{M} = \mathcal{M}[x^{-1}]$ if and only if $\alpha \neq 0$.

As a corollary of the proof we have also obtained

COROLLARY 4.2.8. — *Let $\mathcal{M}$ be a holonomic module. Then its localized module is isomorphic to $\mathcal{D}/\mathcal{D}\cdot P$ for some nonzero $P \in \mathcal{D}$ (in fact even in $V_0\mathcal{D} - V_{-1}\mathcal{D}$).*

Proof. — We know that $\mathcal{M}$ is isomorphic to some $\mathcal{D}/I$, so

$$\mathcal{M}[x^{-1}] \simeq \mathcal{D}/\mathcal{D}\cdot P_p[x^{-1}] \simeq \mathcal{D}/\mathcal{D}\cdot P$$

for some P constructed as above. □

4.2.9. Exercise. — Let $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot(x\partial_x - \alpha)^p$ with $\alpha \in \mathbf{C}$ and $p \in \mathbf{N} - \{0\}$.

1. Determine the set $A \in \mathbf{C}$ of complex numbers α for which $\mathcal{M}$ is a meromorphic connection.

2. Show that $\mathcal{M}$ admits a submodule isomorphic to $\mathcal{D}/\mathcal{D}\cdot(x\partial_x - \alpha)$ and a quotient isomorphic to $\mathcal{D}/\mathcal{D}\cdot(x\partial_x - \alpha)$.

3. Let $\alpha, \beta \in A$. Show that there exists a non trivial left $\mathcal{D}$-linear homomorphism from $\mathcal{D}/\mathcal{D}\cdot(x\partial_x - \alpha)^p$ in $\mathcal{D}/\mathcal{D}\cdot(x\partial_x - \beta)^q$ if and only if $\alpha - \beta \in \mathbf{Z}$ (one can begin with the case where $p = q = 1$).

4.3. Meromorphic connections and localized holonomic $\mathcal{D}$-modules

We are now in position to compare two *a priori* distinct notions: from one side the notion of a holonomic $\mathcal{D}$-module isomorphic to its localized module (*i.e.* on which left multiplication by x is invertible) and from the other side the notion of a meromorphic connection.

Definition 4.3.1. — *A meromorphic connection $\mathcal{M}_K$ is a K-vector space of finite dimension equipped with a derivation ∂_x:*

1. *$\partial_x : \mathcal{M}_K \to \mathcal{M}_K$ is $\mathbf{C}$-linear.*
2. *For all $f \in K$ and all $m \in \mathcal{M}_K$ one has $\partial_x(fm) = (\partial f/\partial x)m + f\partial_x m$.*

Choose a basis $\boldsymbol{e} = (e_1, \dots, e_d)$ of $\mathcal{M}_K$ over K. The matrix A of ∂_x in this basis has coefficients in K. Put

$$\partial_x e_i = \sum_j a_{i,j} e_j$$

with $a_{i,j} \in K$. The meromorphic connexion $\mathcal{M}_K$ corresponds to the linear system of linear differential equations $\partial/\partial x - A$. If one changes the basis by a matrix $B \in \mathrm{Gl}_d(K)$ the new matrix of ∂_x is $A' = BAB^{-1} + \partial B/\partial x B^{-1}$.

Theorem 4.3.2. — *A meromorphic connection determines a holonomic localized $\mathcal{D}$-module and conversely.*

Proof. — Let us give some explanation. Let $\mathcal{M}$ be a holonomic localized module. Then $\mathcal{M}$ is also a module of finite type over the ring $\mathcal{D}[x^{-1}] = K\langle\partial_x\rangle$ so is a K-vector space with a derivation ∂_x. It is asserted that this vector space is finite dimensional. Conversely, let $\mathcal{M}_K$ be a meromorphic connection. Then $\mathcal{M}_K$ is a $K\langle\partial_x\rangle$-module of finite type. It is asserted that it is also of finite type over $\mathcal{D}$.

So let $\mathcal{M}$ be a holonomic localized $\mathcal{D}$-module. We may assume that $\mathcal{M}$ is generated by one element m (one can reduce to that case by successive extensions or use directly 3.3.5). Then m is also a generator of $\mathcal{M}$ over $K\langle\partial_x\rangle$. Because $\mathcal{M}$ is holonomic there exists a nonzero P such that $P \cdot m = 0$, hence $\partial_x^d m$ is in the K-vector space generated by $m, \partial_x m, \ldots, \partial_x^{d-1} m$. One deduces that $\mathcal{M}$ is a K-vector space of dimension $\leq d$.

Conversely, let $\mathcal{M}_K$ be a meromorphic connection. This also a $K\langle\partial_x\rangle$-module of finite type. Choose an increasing filtration by sub-$\mathcal{D}$-modules of finite type $\mathcal{M}_0 \subset \cdots \subset \mathcal{M}_\ell \subset \cdots \subset \mathcal{M}_K$ (for instance, $\mathcal{M}_0$ is the sub-$\mathcal{D}$-module generated by a basis of $\mathcal{M}_K$ over K and $\mathcal{M}_{\ell+1}$ is the sub-$\mathcal{D}$-module generated by $\mathcal{M}_\ell$ and $(1/x)\mathcal{M}_\ell$). Each $\mathcal{M}_\ell$ is holonomic: otherwise, $\mathcal{M}_K$ would have a subquotient isomorphic to $\mathcal{D}$, and also, because localization is exact, a subquotient isomorphic to $\mathcal{D}[x^{-1}]$; but this module is certainly not a finite dimensional K-vector space. We conclude that each $\mathcal{M}_\ell[x^{-1}]$ is $\mathcal{D}$-holonomic, contained in $\mathcal{M}_K$ and by the previous result a finite dimensional vector space. The filtration of $\mathcal{M}_K$ by the notherian $K\langle\partial_x\rangle$-modules $\mathcal{M}_\ell[x^{-1}]$ must be stationary, so $\mathcal{M}_K = \mathcal{M}_\ell[x^{-1}]$ for ℓ big enough. □

We know that a holonomic localized $\mathcal{D}$-module is isomorphic to $\mathcal{D}/\mathcal{D} \cdot P$ for some P. We shall now give the corresponding statement for meromorphic connections (known as "lemme du vecteur cyclique").

PROPOSITION 4.3.3. — *Let $\mathcal{M}_K$ be a meromorphic connection. There exists an element $m \in \mathcal{M}_K$ and an integer d such that $m, \partial_x m, \ldots, \partial_x^{d-1} m$ is a K-basis of $\mathcal{M}_K$.*

Proof. — We can adapt the proof above and use the fact that $\mathcal{M}_K \simeq \mathcal{D}/\mathcal{D} \cdot P$ to get the result. We shall give another proof. Let $m_1, \ldots, m_d$ be a K-basis of $\mathcal{M}_K$. One adds a new parameter s: one considers the ring $K[s]$ and $\mathcal{M}_K[s] = \oplus s^k \mathcal{M}_K = K[s] \otimes_K \mathcal{M}_K$ and one puts $\partial_x(s) = 0$ ($\mathcal{M}_K[s]$ is a trivial family of meromorphic connections parametrized by $\mathbf{C}[s]$). Consider for all $i = 1, \ldots, d$ and $p \geq d$,

$$\mu_i = \sum_{k=0}^{p} \frac{(s-x)^k}{k!} \partial_x^k m_i \in \mathcal{M}_K[s].$$

Then

$$\partial_x \mu_i = -\sum_{k=1}^{p} \frac{(s-x)^{k-1}}{(k-1)!} \partial_x^k m_i + \sum_{k=0}^{p} \frac{(s-x)^k}{k!} \partial_x^{k+1} m_i = \frac{(s-x)^p}{k!} \partial_x^{p+1} m_i$$

and $\partial_x^k \mu_i \equiv 0 \bmod (s-x)^{p-k+1}$. Put

$$\mu(s) = \mu = \mu_1 + (x-s)\mu_2 + \cdots + \frac{(x-s)^{d-1}}{(d-1)!} \mu_{d-1}.$$

If one computes modulo $(s-x)$ on gets $\mu \equiv \mu_1$, $\partial_x \mu \equiv \mu_2, \ldots, \partial_x^{d-1} \mu \equiv \mu_d$. Hence $\mu \wedge \partial_x \mu \wedge \cdots \wedge \partial_x^{d-1} \mu \not\equiv 0$. We deduce that for general $\sigma \in K$ the restriction of this element to $s = \sigma$ is nonzero, which means that $\mu(\sigma)$ satisfies the requirement. □

5. Formal structure of meromorphic connections

In this section, we shall work with the ring $\widehat{\mathcal{D}}[x^{-1}] = \widehat{K}\langle\partial_x\rangle$ also denoted by $\mathcal{D}_{\widehat{K}}$ where $\widehat{K} = \mathbf{C}[\![x]\!]\,[x^{-1}]$. The results of the previous section are also valid when $\mathcal{D}$ is replaced with $\widehat{\mathcal{D}}$. However the main results of this section (theorems 5.3.1 and 5.4.7) will not be true over $\mathcal{D}$.

5.1. Slopes and Newton polygon

Let $\mathcal{M}_{\widehat{K}}$ be a meromorphic connection over $\widehat{K}$ (we shall say that $\mathcal{M}_{\widehat{K}}$ is a formal meromorphic connection). We have given two proofs of the fact that $\mathcal{M}_{\widehat{K}}$ is isomorphic to $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P$ for some $P \in \mathcal{D}_{\widehat{K}}$. One may assume moreover that all the coefficients of P belong to $\mathbf{C}[\![x]\!]$ and that at least one of them is a unit when P is expressed in term of $x\partial_x$: just multiply P by the right power of x (this does not affect $\mathcal{M}_{\widehat{K}}$). Write $P = \sum a_i(x)(x\partial_x)^i$ with $a_i \in \mathbf{C}[\![x]\!]$ and at least one a_i is a unit.

For every "monomial" $a_i(x)(x\partial_x)^i$ associate the point $(i, -v(a_i))$ of $\mathbf{N}\times\mathbf{Z}$ and define the *Newton polygon* $N(P)$ of P to be the convex hull of the set

$$\bigcup_i \Big\{(i, -v(a_i)) - \mathbf{N}^2\Big\}.$$

This Newton polygon does not depend on the way one writes P. We get the following picture:

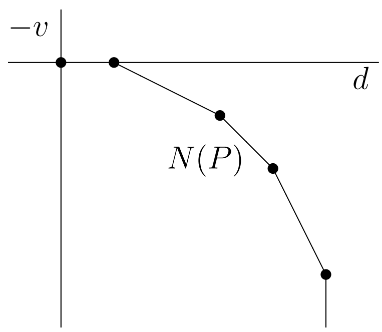

Figure 3

Definition 5.1.1. — *We shall say that P is regular (or has a regular singularity at 0) if $N(P)$ is a quadrant. We shall say that a meromorphic connection $\mathcal{M}_{\widehat{K}}$ is regular if it is isomorphic to $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P$ with P regular.*

Remarks.

1. One may define the notion of regularity of an operator P in $\mathcal{D}$ or in $\widehat{\mathcal{D}}$. The Newton polygon is defined in the same way: if $P = \sum b_i(x)\partial_x^i$ then $N(P)$ is the convex hull of the union of the sets $(i, i - v(b_i)) - \mathbf{N}^2$. The only difference

is that we may not assume here that the origin is a point in the boundary of $N(P)$.

2. The fact that P is regular depends only on the associated meromorphic connection $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P$. In fact, the Newton polygon of P depends only on the associated meromorphic connection, as we shall see below.

3. $P=\sum a_i(x)(x\partial_x)^i$ with $a_i\in\mathbf{C}[\![x]\!]$ is regular if and only if the coefficient a_d of the dominant term is a unit.

We shall now consider with more details the notion of slopes of the Newton polygon. Fix a linear form L of two variables with coefficients λ_0 and λ_1 in $\mathbf{N}$ and relatively prime: $L(s_0,s_1)=\lambda_0 s_0+\lambda_1 s_1$. We shall define for each such L a new filtration of $\mathcal{D}_{\widehat{K}}$ (one can do the same for $\mathcal{D}$, $\widehat{\mathcal{D}}$, $\mathcal{D}_K$) which interpolates between $F\mathcal{D}_{\widehat{K}}$ and $V\mathcal{D}_{\widehat{K}}$ (these two filtrations are defined in the same way as in 1.3.1 and 4.2.4). Let $P\in\mathcal{D}_{\widehat{K}}$. If P is a monomial $x^a\partial_x^b$ with $a\in\mathbf{Z}$ and $b\in\mathbf{N}$, we put

$$\operatorname{ord}_L(P)=L(b,b-a)$$

and if $P=\sum_{i=0}^d b_i(x)\partial_x^i$ with $b_i\in\widehat{K}$ we put

$$\operatorname{ord}_L(P)=\max_i L(i,i-v(a_i)).$$

We now define the increasing filtration ${}^LV\mathcal{D}_{\widehat{K}}$ indexed by $\mathbf{Z}$ as

$${}^LV_\lambda\mathcal{D}_{\widehat{K}}=\left\{P\in\mathcal{D}_{\widehat{K}}|\operatorname{ord}_L(P)\le\lambda\right\}.$$

One has the following properties:

1. If $L=L_0$ with $L_0(s_0,s_1)=s_0$ then ${}^{L_0}V\mathcal{D}_{\widehat{K}}=F\mathcal{D}_{\widehat{K}}$ and if $L=L_1$ with $L_1(s_0,s_1)=s_1$ then ${}^{L_1}V\mathcal{D}_{\widehat{K}}=V\mathcal{D}_{\widehat{K}}$.

2. $\operatorname{ord}_L(\partial_x)=L(1,1)=\lambda_0+\lambda_1$, $\operatorname{ord}_L(x)=-\lambda_1$ and $\operatorname{ord}_L(x^{-1})=\lambda_1$.

3. One has $\operatorname{ord}_L(PQ)=\operatorname{ord}_L(P)+\operatorname{ord}_L(Q)$ and *if* $\lambda_0\neq 0$ one has also the inequality $\operatorname{ord}_L([P,Q])\le\operatorname{ord}_L(P)+\operatorname{ord}_L(Q)-1$. Consequently, if $\lambda_0\neq 0$ the graded ring $\operatorname{gr}^{LV}\mathcal{D}_{\widehat{K}}\overset{\text{def}}{=}\bigoplus_{\lambda\in\mathbf{Z}}\operatorname{gr}^{LV}_\lambda\mathcal{D}_{\widehat{K}}$ is a commutative ring. Denote by ξ the class of ∂_x in this ring. Then this ring is isomorphic to $\widehat{K}[\xi]$. This isomorphism is compatible with graduations if one puts on $\widehat{K}[\xi]$ the twisted graduation defined by L as above (this is left as an exercise). One denotes by $\sigma_L(P)$ (L-symbol of P) the class of P in $\operatorname{gr}^{LV}_{\operatorname{ord}_L(P)}\mathcal{D}_{\widehat{K}}$. For instance, $\sigma_L(x^a\partial_x^b)=x^a\xi^b$.

Example. — Let $P=x^2\partial_x+1$ and L as above with $\lambda_0\neq 0$. Let us compute $\sigma_L(P)$. One has

$$\begin{array}{lllll}\operatorname{ord}_L(x^2\partial_x)&=&L(1,-1)&=&\lambda_0-\lambda_1\\ \operatorname{ord}_L(1)&=&L(0,0)&=&0\end{array}$$

so one may distinguish three cases:

- $\lambda_0 - \lambda_1 > 0$, $\sigma_L(P) = x^2\xi$ in $\mathrm{gr}^{LV}_{\lambda_0-\lambda_1}\mathcal{D}_{\widehat{K}}$.
- $\lambda_0 = \lambda_1 (= 1)$, $\sigma_L(P) = x^2\xi + 1$ in $\mathrm{gr}_0^{LV}\mathcal{D}_{\widehat{K}}$.
- $\lambda_0 - \lambda_1 < 0$, $\sigma_L(P) = 1$.

5.1.2. — Let $\mathcal{M}_{\widehat{K}}$ be a formal meromorphic connection. For each L as above, one may define the notion of a good filtration (relative to ${}^LV\mathcal{D}_{\widehat{K}}$): a filtration ${}^LU\mathcal{M}_{\widehat{K}}$ is good if it is induced by a direct sum of shifted ${}^LV\mathcal{D}_{\widehat{K}}$-filtrations. Remark however that in order for a filtration to be good, it is not sufficient that the graded module is of finite type over the graded ring $\mathrm{gr}^{LV}\mathcal{D}_{\widehat{K}}$, because filtrations ${}^LU$ do not satisfy ${}^LU_\lambda = 0$ for $\lambda \ll 0$.

5.1.3. Exercise. — Denote by $\mathcal{R}_L(\mathcal{D}_{\widehat{K}})$ the Rees ring associated with the filtration ${}^LV\mathcal{D}_{\widehat{K}}$: this ring is the subring of $\mathcal{D}_{\widehat{K}}[u, u^{-1}]$ (where u is a new variable) defined by

$$\mathcal{R}_L(\mathcal{D}_{\widehat{K}}) = \bigoplus_{k\in\mathbf{Z}} {}^LV_k\mathcal{D}_{\widehat{K}} \cdot u^k$$

(verify first that this is a subring). Let ${}^LU\mathcal{M}_{\widehat{K}}$ be a filtration of $\mathcal{M}_{\widehat{K}}$ indexed by $\mathbf{Z}$.

1. Show that $(\mathcal{M}_{\widehat{K}}, {}^LU\mathcal{M}_{\widehat{K}})$ is a filtered module over the filtered ring $(\mathcal{D}_{\widehat{K}}, {}^LV_k\mathcal{D}_{\widehat{K}})$ if and only if
$$\mathcal{R}_L(\mathcal{M}_{\widehat{K}}) \overset{\mathrm{def}}{=} \bigoplus_{k\in\mathbf{Z}} {}^LU_k\mathcal{M}_{\widehat{K}} \cdot u^k$$
is a module over $\mathcal{R}_L(\mathcal{D}_{\widehat{K}})$ (in a natural way).

2. Show that the Rees ring $\mathcal{R}_L(\mathcal{D}_{\widehat{K}})$ is a (left and right) noetherian ring (one can adapt the proof of corollary 1.3.5).

3. Show that the filtration ${}^LU\mathcal{M}_{\widehat{K}}$ is good if and only if the Rees module $\mathcal{R}_L(\mathcal{M}_{\widehat{K}})$ is of finite type over $\mathcal{R}_L(\mathcal{D}_{\widehat{K}})$.

4. Conclude that if $\mathcal{M}'_{\widehat{K}}$ is a submodule of $\mathcal{M}_{\widehat{K}}$ and if ${}^LU\mathcal{M}_{\widehat{K}}$ is a good filtration, then the induced filtration ${}^LU\mathcal{M}'_{\widehat{K}} = {}^LU\mathcal{M}_{\widehat{K}} \cap \mathcal{M}'_{\widehat{K}}$ is good, and an analogous statement for a quotient.

5. Given a good filtration ${}^LU\mathcal{M}_{\widehat{K}}$, show that a filtration ${}^LU'\mathcal{M}_{\widehat{K}}$ is good if and only if there exists $\ell \in \mathbf{N}$ such that for all $k \in \mathbf{Z}$ one has
$${}^LU_{k-\ell}\mathcal{M}_{\widehat{K}} \subset {}^LU'_k\mathcal{M}_{\widehat{K}} \subset {}^LU_{k+\ell}\mathcal{M}_{\widehat{K}}.$$

6. Verify that all the previous results are also valid over the rings $\mathcal{D}$, $\widehat{\mathcal{D}}$, $\mathbf{C}[x]\langle\partial_x\rangle$.

One proves in the same way as in 3.2.6 the following

Proposition 5.1.4. — *Let L be a linear form with $\lambda_0 \neq 0$. Denote by* $\mathrm{Car}^L(\mathcal{M}_{\widehat{K}})$ *the support of* $\mathrm{gr}^{LU}\mathcal{M}_{\widehat{K}}$ *defined by* $\sqrt{\mathrm{Ann}\,\mathrm{gr}^{LU}\mathcal{M}_{\widehat{K}}}$*. Then this support does not depend on the choice of the good filtration* ${}^{L}U\mathcal{M}_{\widehat{K}}$*.* □

Example. — If $\mathcal{M}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P$ then the annihilator ideal is equal to the ideal generated by $\sigma_L(P)$. The previous example shows that this ideal is homogeneous with respect to the L-grading but not always bi-homogeneous (this did not happen for the filtration F).

Definition 5.1.5. — *A linear form L as above with $\lambda_0 \neq 0$ is called as slope for $\mathcal{M}_{\widehat{K}}$ if* $\mathrm{Car}^L(\mathcal{M}_{\widehat{K}})$ *is not defined by a monomial. This means that the ideal* $\sqrt{\mathrm{Ann}\,\mathrm{gr}^{LU}\mathcal{M}_{\widehat{K}}}$ *is not equal to one of the following ideals of $\mathcal{D}_{\widehat{K}}$: $(x\xi)$, (ξ), (x), (1).*

Of course, in $\mathcal{D}_{\widehat{K}}$ the first two ideals coincide as well as the other ones, but as given here this definition can be extended to the rings used in the previous sections without any change. We have then obtained an intrinsic definition of slopes. If $\mathcal{M}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot P$ and $\lambda_0, \lambda_1 \neq 0$ then L is a slope for $\mathcal{M}_{\widehat{K}}$ iff $\sigma_L(P)$ is not a monomial. Remark that the linear form L_0 (for which $\lambda_1 = 0$) is never a slope for $\mathcal{M}_{\widehat{K}}$, because the F-symbol of P is always a monomial (up to a unit). This exactly means that L is a slope iff L is the direction of a side of $N(P)$ which is not vertical.

We shall now consider the linear form L_1 which is not treated in the same way because $\mathrm{gr}^V\mathcal{D}_{\widehat{K}}$ is not commutative.

Definition 5.1.6. — *Given a meromorphic connection $\mathcal{M}_{\widehat{K}}$, we shall say that L_1 is a slope for $\mathcal{M}_{\widehat{K}}$ if for all linear form L with $\lambda_0 \neq 0$ one has* $\mathrm{Car}^L(\mathcal{M}_{\widehat{K}}) \neq \emptyset$*, i.e.* $\sqrt{\mathrm{Ann}\,\mathrm{gr}^{LU}\mathcal{M}_{\widehat{K}}} \neq \widehat{K}[\xi]$*.*

Let us explain this definition in terms of $N(P)$. Assume as we did before that $P \in V_0\widehat{\mathcal{D}} - V_{-1}\widehat{\mathcal{D}}$, which means that P has coefficients in $\mathbf{C}[\![x]\!]$ and one of them is a unit. Then one proves easily (left as an exercise):

Lemma 5.1.7. — *L_1 is a slope for $\mathcal{M}_{\widehat{K}}$ if and only if $N(P)$ has a vertex on the horizontal axis different from the origin.*

This means exactly that $N(P)$ has a horizontal side (see *fig.* 4).

We shall denote by $\mathcal{P}(\mathcal{M}_{\widehat{K}})$ the set of all slopes of $\mathcal{M}_{\widehat{K}}$. These correspond to the directions of the non vertical sides of $N(P)$. Let us give some elementary properties of $\mathcal{P}(\mathcal{M}_{\widehat{K}})$ which are proven by using for instance the behaviour of $\mathrm{Car}^L(\mathcal{M}_{\widehat{K}})$ under a morphism (analogous to 3.3.3).

1. $\mathcal{P}(\mathcal{M}_{\widehat{K}})$ is non empty if $\mathcal{M}_{\widehat{K}} \neq \{0\}$.

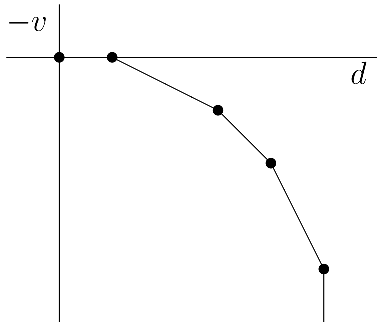

Figure 4: L_1 is a slope

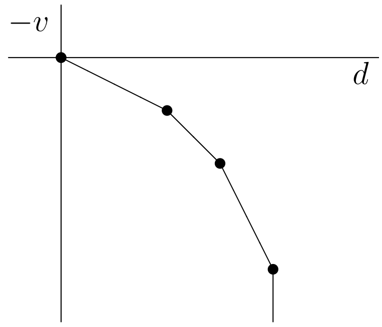

Figure 5: L_1 is not a slope

2. In an exact sequence of connections $0 \to \mathcal{M}'_{\widehat{K}} \to \mathcal{M}_{\widehat{K}} \to \mathcal{M}''_{\widehat{K}} \to 0$ one has $\mathcal{P}(\mathcal{M}_{\widehat{K}}) = \mathcal{P}(\mathcal{M}'_{\widehat{K}}) \cup \mathcal{P}(\mathcal{M}''_{\widehat{K}})$.
3. Let $\varphi : \mathcal{M}'_{\widehat{K}} \to \mathcal{M}_{\widehat{K}}$ a $\mathcal{D}_{\widehat{K}}$-linear morphism. If $\mathcal{P}(\mathcal{M}_{\widehat{K}}) \cap \mathcal{P}(\mathcal{M}'_{\widehat{K}}) = \emptyset$, then $\varphi = 0$.
4. $\mathcal{M}_{\widehat{K}}$ is regular iff $\mathcal{P}(\mathcal{M}_{\widehat{K}}) = \{L_1\}$.

5.1.8. Exercise. — Give an intrinsic definition of the Newton polygon (*i.e.* in term of $\mathcal{M}_{\widehat{K}}$ only and good filtrations).

5.2. Formal structure of regular connections

Let $\mathcal{M}_{\widehat{K}}$ be a regular formal meromorphic connection.

LEMMA 5.2.1. — *There exists a basis of $\mathcal{M}_{\widehat{K}}$ over $\widehat{K}$ such that the matrix of $x\partial_x$ has entries in $\mathbf{C}[\![x]\!]$.*

Proof. — Choose a cyclic vector m and consider the basis $m, \partial_x m, \dots, \partial_x^{d-1} m$. Then m satisfies an equation of the form $\partial_x^d m + \sum_{i=0}^{d-1} b_i(x)\partial_x^i m = 0$. We may in fact write $b_i(x) = x^i b'_i(x)$ with $b'_i \in \mathbf{C}[\![x]\!]$, because of regularity. This implies that $m, x\partial_x m, \dots, (x\partial_x)^{d-1} m$ is also a basis of $\mathcal{M}_{\widehat{K}}$. The matrix of $x\partial_x$ in this basis has entries in $\mathbf{C}[\![x]\!]$. □

THEOREM 5.2.2. — *Let $\mathcal{M}_{\widehat{K}}$ be a regular formal meromorphic connection. Then there exists a basis for which the matrix of $x\partial_x$ is constant.*

Proof. — Start with a basis $m_1, \dots, m_d$ for which the matrix $A(x)$ of $x\partial_x$ has entries in $\mathbf{C}[\![x]\!]$. Assume first that the following condition is satisfied: two distinct eigenvalues of A_0 (constant part of A) do not differ by an integer. We shall find a basis $\boldsymbol{m}' = (m'_1, \dots, m'_d)$ for which the matrix is A_0. Let $B(x) \in \mathrm{Gl}(d, \mathbf{C}[\![x]\!])$ and put

$$m'_i = \sum_j b_{ij}(x) m_j.$$

The matrix of $x\partial_x$ in the basis $\boldsymbol{m}'$ is

$$BAB^{-1} + x(\partial B/\partial x)B^{-1}. \tag{5.2.3}$$

One then search for a B such that

$$BAB^{-1} + x\frac{\partial B}{\partial x}B^{-1} = A_0$$

that is

$$x\frac{\partial B}{\partial x} = A_0 \cdot B - B \cdot A.$$

Put $B = \sum_0^\infty x^k B_k$ and $A = \sum_0^\infty x^k A_k$. We shall find B_k inductively. In degree 0 one must have

$$0 = A_0 B_0 - B_0 A_0$$

so one may take $B_0 = \mathrm{Id}$. In degree $p \geq 1$ one can write

$$(p\mathrm{Id} - A_0)B_p + B_p A_0 = \Phi\left(B_0, \dots, B_{p-1}; A_0, \dots, A_p\right)$$

where Φ is a polynomial. If one knows $B_0, \dots, B_{p-1}$ one may then solve for B_p thanks to the following lemma

Lemma 5.2.4. — *Let $P \in \mathrm{End}(\mathbf{C}^p)$ and $Q \in \mathrm{End}(\mathbf{C}^q)$ be given. Then the following equation*

$$XP - QX = Y$$

has a unique solution $X \in \mathrm{Hom}(\mathbf{C}^p, \mathbf{C}^q)$ for all $Y \in \mathrm{Hom}(\mathbf{C}^p, \mathbf{C}^q)$ if and only if P and Q have no common eigenvalue.

One applies the lemma to $P = p\mathrm{Id} - A_0$ and $Q = -A_0$.

Proof of the lemma. — Let $\varphi : \mathrm{Hom}(\mathbf{C}^p, \mathbf{C}^q) \to \mathrm{Hom}(\mathbf{C}^p, \mathbf{C}^q)$ given by $\varphi(X) = XP - QX$. One shows that the eigenvalues of φ are precisely the differences $\lambda_i - \mu_j$ where λ_i is an eigenvalue of P and μ_j one of Q. Hence φ is bijective iff for all i, j the difference $\lambda_i - \mu_j$ is nonzero. □

One has now to get rid of the hypothesis made on the eigenvalues of A_0. Start with a basis $\boldsymbol{m}$ of $\mathcal{M}_{\widehat{K}}$ for which the matrix of $x\partial_x$ has entries in $\mathbf{C}[\![x]\!]$. Let $\{\lambda_1, \dots, \lambda_p\}$ the set of eigenvalues of A_0.

Lemma 5.2.5. — *There exists a basis $\boldsymbol{m}'$ of $\mathcal{M}_{\widehat{K}}$ for which the matrix A' has entries in $\mathbf{C}[\![x]\!]$ and A'_0 admits $\lambda_1 + 1, \lambda_2, \dots, \lambda_p$ as eigenvalues.*

By induction this lemma gives a basis of $\mathcal{M}_{\widehat{K}}$ for which the hypothesis made above is satisfied. □

Proof of the lemma. — Let $B \in \mathrm{Gl}(d, \widehat{K})$ the transition matrix. Then A' is given by 5.2.3. Remark first that if B has constant coefficients, the second term in 5.2.3 vanishes. So we may assume that A_0 is a Jordan matrix. We may write

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} + xA^{(1)}$$

where Q contains only the Jordan blocks associated with λ_1. Put

$$B = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & x\mathrm{Id} \end{pmatrix}.$$

Then A'_0 admits $\lambda_1 + 1, \lambda_2, \dots, \lambda_p$ as eigenvalues. □

Corollary 5.2.6. — *If $\mathcal{M}_{\widehat{K}}$ is a regular formal meromorphic connection, then $\mathcal{M}_{\widehat{K}}$ is isomorphic to a direct sum of elementary such connections, namely those isomorphic to $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot (x\partial_x - \alpha)^p$ (with $\alpha \in \mathbf{C}$ and $p \in \mathbf{N}$).*

Proof. — One chooses a basis for which the matrix of $x\partial_x$ is constant and then by a constant change of basis one gets a Jordan matrix. □

5.3. Factorization into one slope terms

We shall now consider the general case of a formal meromorphic connection, without hypothesis of regularity.

Theorem 5.3.1. — *Let $\mathcal{M}_{\widehat{K}}$ be a formal meromorphic connection and let $\{L^{(1)}, \dots, L^{(r)}\}$ be the set of its slopes. There exists a unique (up to permutation) splitting $\mathcal{M}_{\widehat{K}} = \bigoplus_{i=1}^r \mathcal{M}_{\widehat{K}}^{(i)}$ into meromorphic connections with $\mathcal{P}(\mathcal{M}_{\widehat{K}}^{(i)}) = \{L^{(i)}\}$.*

Proof of uniqueness. — Consider two such splittings $\mathcal{M}_{\widehat{K}} = \bigoplus_{i=1}^r \mathcal{M}_{\widehat{K}}^{(i)} = \bigoplus_{i=1}^r \mathcal{M}_{\widehat{K}}^{\prime(i)}$. Each element in $\mathcal{M}_{\widehat{K}}$ is the sum of its projections in each $\mathcal{M}_{\widehat{K}}^{(i)}$. The restriction to $\mathcal{M}_{\widehat{K}}^{\prime(i)}$ of the projection $\mathcal{M}_{\widehat{K}} \to \mathcal{M}_{\widehat{K}}^{(j)}$ is zero for $i \neq j$, because the two modules have no common slope. This proves that $\mathcal{M}_{\widehat{K}}^{\prime(i)} \subset \mathcal{M}_{\widehat{K}}^{(i)}$. The converse inclusion is obtained in the same way. □

Proof of existence. — Put $\mathcal{M}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot P$ and assume that $N(P)$ has at least two non vertical sides. Split $N(P)$ in two parts N_1 and N_2 (see *fig.* 6).

Lemma 5.3.2. — *There exists a splitting $P = P_1P_2$ with*

1. *$N(P_1) \subset N_1$ and $N(P_2) \subset N_2$,*
2. *A is a vertex of $N(P_1)$ and the origin is a vertex of $N(P_2)$.*

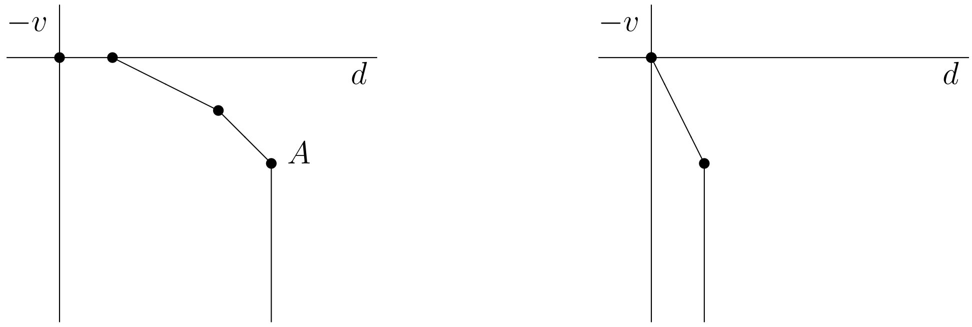

Figure 6

Proof. — Let L be a linear form such that the line $L = L(A)$ has contact with $N(P)$ in A only. Put $a = L(A)$. Put $\sigma_L(P) = \alpha x^\ell \xi^k$ with $\alpha \in \mathbf{C}$ and $(k, k-\ell) = A$. If $P = P_1 P_2$ one must have $\sigma_L(P) = \sigma_L(P_1)\sigma_L(P_2)$. So put $\sigma_L(P_1) = \sigma_L(P)$ and $\sigma_L(P_2) = 1$. One then looks for P_1 and P_2 in the following form: P_1 is of degree k and

$$\begin{aligned} P_1 &= P_1^{(a)} + P_1^{(a-1)} + P_1^{(a-2)} + \cdots \\ P_2 &= 1 + P_2^{(-1)} + P_2^{(-2)} + \cdots \end{aligned}$$

with $P_1^{(a)} = \alpha x^\ell \partial_x^k$ and each term in the sum is L-homogeneous, the exponents giving the L-degree. These homogeneous parts must then satisfy

$$\begin{aligned} P^{(a-1)} &= \alpha x^\ell \partial_x^k P_2^{(-1)} + P_1^{(a-1)} \\ P^{(a-2)} &= \alpha x^\ell \partial_x^k P_2^{(-2)} + P_1^{(a-2)} + P_1^{(a-1)} P_2^{(-1)} \\ \vdots &= \vdots \end{aligned}$$

Argue now by induction: one wants to prove that for all h one may find $P_1^{(a-h)}$ and $P_2^{(-h)}$ such that these equations are satisfied and $N(P_1^{(a-h)}) \subset N_1$, $N(P_2^{(-h)}) \subset N_2$. For $h = 1$ this is elementary division: one may write $P^{(a-1)} = Q^{(a-1)} + R^{(a-1)}$ where $Q^{(a-1)}$ (*resp.* $R^{(a-1)}$) is the sum of monomials contained in N_1 (*resp.* $N_2 + A$), so $R^{(a-1)}$ can be written $\alpha x^\ell \partial_x^k R'$. Put $P_1^{(a-1)} = Q^{(a-1)}$ and $P_2^{(-1)} = R'$.

One must have

$$P^{(a-h-1)} = \alpha x^\ell \partial_x^k P_2^{(-h-1)} + P_1^{(a-h-1)} + \sum_{j=1}^{h} P_1^{(a-j)} P_2^{(j-h-1)}$$

so one divides $P^{(a-h-1)} - \sum_{j=1}^{h} P_1^{(a-j)} P_2^{(j-h-1)}$ by $\alpha x^\ell \partial_x^k$ and take the L-homogeneous part to get $P_1^{(a-h-1)}$ and $P_2^{(-h-1)}$. $\square$

As a consequence of this lemma, one gets an exact sequence

$$0 \longrightarrow \mathcal{D}_{\widehat{K}} \cdot P_2 / \mathcal{D}_{\widehat{K}} \cdot P \longrightarrow \mathcal{D}_{\widehat{K}} / \mathcal{D}_{\widehat{K}} \cdot P \longrightarrow \mathcal{D}_{\widehat{K}} / \mathcal{D}_{\widehat{K}} \cdot P_2 \longrightarrow 0.$$

Lemma 5.3.3. — *Right multiplication by P_2 induces an isomorphism of left $\mathcal{D}_{\widehat{K}}$-modules*

$$\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P_1 \xrightarrow{\cdot P_2} \mathcal{D}_{\widehat{K}}\cdot P_2/\mathcal{D}_{\widehat{K}}\cdot P.$$

Proof. — Surjectivity is clear. For the injectivity, let $Q\in\mathcal{D}_{\widehat{K}}$ be such that $QP_2\in\mathcal{D}_{\widehat{K}}\cdot P$. One has to prove that $Q\in\mathcal{D}_{\widehat{K}}\cdot P_1$. One has $QP_2=RP_1P_2$ hence $(Q-RP_1)P_2=0$. But right multiplication by P_2 on $\mathcal{D}_{\widehat{K}}$ is injective (consider the symbols for the filtration $F\mathcal{D}_{\widehat{K}}$). So $Q-RP_1=0$ and $Q\in\mathcal{D}_{\widehat{K}}\cdot P_1$. □

We have now obtained an exact sequence

$$0\longrightarrow\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P_1\longrightarrow\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P\longrightarrow\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P_2\longrightarrow 0.$$

Corollary 5.3.4. — $\mathcal{P}(P)=\mathcal{P}(P_1)\cup\mathcal{P}(P_2)$ *and* $\mathcal{P}(P_1)\cap\mathcal{P}(P_2)=\emptyset$.

The second equality comes from the properties of the Newton polygons of P_1 and P_2. In fact, one can prove that $N(P_1)=N_1$ and $N(P_2)=N_2$ (first one proves that $N(P)=N(P_1P_2)=N(P_1)+N(P_2)$, then one deduces the equality from the inclusions given in lemma 5.3.2).

We shall now prove that the exact sequence above splits, *i.e.* we have a splitting

$$\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P=\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P_1\oplus\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P_2.$$

Lemma 5.3.5. — *With P_1 and P_2 as above, there exist $Q,R\in\mathcal{D}_{\widehat{K}}$ such that*

$$QP_1+P_2R=1.$$

Assume that this lemma is proven. Consider then the $\mathcal{D}_{\widehat{K}}$-linear morphism

$$\mathcal{D}_{\widehat{K}}\longrightarrow\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P$$

which sends 1 to the class of $1-RP_2$ and hence each $T\in\mathcal{D}_{\widehat{K}}$ to the class of $T(1-RP_2)$. The image of the left ideal $\mathcal{D}_{\widehat{K}}\cdot P_2$ is zero: let TP_2 be an element of this ideal. Its image is then the class of $TP_2(1-RP_2)$. But we have

$$TP_2(1-RP_2)=T(1-P_2R)P_2=T(QP_1)P_2=TQP\in\mathcal{D}_{\widehat{K}}\cdot P.$$

We have then constructed a section of the projection $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P\to\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}\cdot P_2$.

Proof of lemma 5.3.5. — One uses here also the decomposition into L-homogeneous components. We shall impose that Q has L-order 0 and R L-order $-a-1$. So write

$$\begin{aligned} Q &= 1 + Q^{(-1)} + Q^{(-2)} + \cdots \\ R &= R^{(-a-1)} + R^{(-a-2)} + \cdots \end{aligned}$$

and these L-homogeneous components must satisfy

$$\begin{aligned} 1 &= 1 \\ 0 &= Q^{(-1)} + P_1^{(-1)} + P_2^{(a)} R^{(-a-1)} \\ 0 &= Q^{(-2)} + Q^{(-1)} P_1^{(-1)} + P_2^{(a)} R^{(-a-2)} + P_2^{(a-1)} R^{(-a-1)} \\ \vdots &= \vdots \end{aligned}$$

In general one obtains that $P_2^{(a)} R^{(-a-h)} + Q^{(-h)}$ is equal to something known by induction. One gets Q and R by an inductive process. □

5.3.6. Exercise. — Let $P = x(x\partial_x)^2 + x\partial_x + 1/2$.

1. Show that $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$ is a meromorphic connection.
2. Draw the Newton polygon of P and find the formal decomposition of $\mathcal{M}_{\widehat{K}}$ without computation.
3. Show that $\mathcal{M}$ cannot be decomposed into a direct sum of two $\mathcal{D}$-modules:
   (a) Show that there exists a product decomposition
   $$P = (x(x\partial_x) + v(x)) \cdot (x\partial_x + u(x))$$
   with $u, v \in \mathbf{C}[\![x]\!]$.
   (b) Compute by induction the coefficients of u.
   (c) Show that $u \notin \mathbf{C}\{x\}$ and conclude.

5.3.7. Exercise. — Let $\mathcal{M}_{\widehat{K}}$ and $\mathcal{M}'_{\widehat{K}}$ be two meromorphic connections. Show that the tensor product $\mathcal{M}_{\widehat{K}} \otimes_{\widehat{K}} \mathcal{M}'_{\widehat{K}}$ comes equipped with a natural structure of $\mathcal{D}_{\widehat{K}}$-module. Show that if $\mathcal{M}'_{\widehat{K}}$ is regular and nonzero, then the set of slopes of $\mathcal{M}_{\widehat{K}} \otimes_K \mathcal{M}'_{\widehat{K}}$ is exactly the set of slopes of $\mathcal{M}_{\widehat{K}}$ (use the classification of formal meromorphic connections).

5.4. Formal structure (general case)

5.4.1. Ramification. — Let $\pi : \mathbf{C} \to \mathbf{C}$ be the function defined by $t \mapsto t^q = x$. This function induces a mapping denoted by $\pi^* : \mathbf{C}\{x\} \hookrightarrow \mathbf{C}\{t\}$ (and $\mathbf{C}[\![x]\!] \hookrightarrow \mathbf{C}[\![t]\!]$) by putting $\pi^* f = f \circ \pi$, *i.e.* $\pi^*\left(\sum a_n x^n\right) = \sum a_n t^{qn}$. In the same way one obtains $\pi^* : K \hookrightarrow L = \mathbf{C}\{t\}[t^{-1}]$ and $\widehat{K} \hookrightarrow \widehat{L} = \mathbf{C}[\![t]\!][t^{-1}]$. Hence L (*resp.* $\widehat{L}$) is a finite extension of K (*resp.* $\widehat{K}$).

5.4.2. Exercise. — The Galois group of L/K (*resp.* $\widehat{L}/\widehat{K}$) is the cyclic group $\mathbf{Z}/q\mathbf{Z}$.

Let $\mathcal{M}_{\widehat{K}}$ be a formal meromorphic connexion. One first defines $\pi^*\mathcal{M}_{\widehat{K}}$ as a vector space over $\widehat{L}$: $\pi^*\mathcal{M}_{\widehat{K}} = \widehat{L} \otimes_{\widehat{K}} \mathcal{M}_{\widehat{K}}$. Then one defines the action of ∂_t by: $t\partial_t \cdot (1 \otimes m) = q(1 \otimes (x\partial_x \cdot m))$ and hence

$$t\partial_t \cdot (\varphi \otimes m) = q(\varphi \otimes (x\partial_x \cdot m)) + \left((t\frac{\partial\varphi}{\partial t}) \otimes m\right).$$

One deduces from this formula the action of $\partial_t = t^{-1}(t\partial_t)$.

Lemma 5.4.3. — *Let $\mathcal{P}(\mathcal{M}_{\widehat{K}}) = \left\{L^{(1)}, \dots, L^{(r)}\right\}$ be the set of slopes of $\mathcal{M}_{\widehat{K}}$. Then $\mathcal{P}(\mathcal{M}_{\widehat{L}}) = \left\{L'^{(1)}, \dots, L'^{(r)}\right\}$ with $L(s_0, s_1) = \lambda_0 s_0 + \lambda_1 s_1$ and $L'(s_0, s_1) = \lambda_0 s_0 + (\lambda_1/q)s_1$.*

Proof. — Let $\mathcal{M}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot P$. Then $\mathcal{M}_{\widehat{L}} = \mathcal{D}_{\widehat{L}}/\mathcal{D}_{\widehat{L}} \cdot P'$ where, if one writes $P = \sum a_i(x)(x\partial_x)^i$, one puts $P' = \sum q^i a_i(t^q)(t\partial_t)^i$. Consequently, $N(P')$ can be obtained from $N(P)$ by a dilatation of the vertical axis in a ratio of 1 to q. If L is a slope of $\mathcal{M}_{\widehat{K}}$ we know that $\lambda_1 \neq 0$ so that we may associate with L the number λ_0/λ_1 which is also called a slope. This number is then multiplied by q after ramification. □

Remark that as a consequence of this lemma, there always exists a ramification such that all the slopes of $\pi^*\mathcal{M}_{\widehat{K}}$ are integers. Remark also that $\deg P' = \deg P$.

5.4.4. Elementary meromorphic connections. — Let $R(z) = \sum_{i=1}^k \alpha_i z^i$ be a $\deg k$ polynomial without constant term with coefficients in $\mathbf{C}$. We shall denote by $\mathcal{F}^R_{\widehat{K}}$ the following meromorphic connection: the $\widehat{K}$-vector space is isomorphic to $\widehat{K}$ with a basis denoted by $e(R)$. The action of $x\partial_x$ is defined by

$$x\partial_x(\varphi \cdot e(R)) = \left[(x\frac{\partial\varphi}{\partial x}) + \varphi x \frac{\partial R(x^{-1})}{\partial x}\right] \cdot e(R).$$

This means that $e(R)$ plays the role of $\exp R(x^{-1})$ (verify that this defines a meromorphic connection).

Definition 5.4.5. — *An elementary meromorphic connection (over $\widehat{K}$) is a connection isomorphic to $\mathcal{F}^R_{\widehat{K}} \otimes_{\widehat{K}} \mathcal{G}_{\widehat{K}}$ where $\mathcal{G}_{\widehat{K}}$ is an elementary regular meromorphic connection.*

5.4.6. Exercise. — Let $\mathcal{G}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot (x\partial_x - \alpha)^p$. Find P such that $\mathcal{F}^R_{\widehat{K}} \otimes_{\widehat{K}} \mathcal{G}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot P$. Show that this connection has only one slope and compute it.

We can now state the main theorem of this section.

THEOREM 5.4.7. — *Let $\mathcal{M}_{\widehat{K}}$ be a formal meromorphic connection. There exists an integer q such that the connection $\pi^*\mathcal{M}_{\widehat{K}} = \mathcal{M}_{\widehat{L}}$ is isomorphic to a direct sum of elmentary formal meromorphic connections.*

Remarks.

1. This result is analogous to Puiseux theorem for plane algebraic (or algebroid) curves.

2. If at least one slope of $\mathcal{M}_{\widehat{K}}$ is not an integer then the ramification is necessary, because the slope of an elementary formal connection is an integer (see exercise above).

Proof of the theorem 5.4.7. — The proof is done by induction on the lexicographically ordered pair $(\dim_{\widehat{K}} \mathcal{M}_{\widehat{K}}, \kappa)$ where $\kappa \in \mathbf{N} \cup \{+\infty\}$ is equal to the biggest slope of $\mathcal{M}_{\widehat{K}}$, *i.e.* the biggest ratio λ_0/λ_1 (this represents the most vertical side of $N(P)$ which is not vertical) if this slope is an integer, and is equal to $+\infty$ if not. Remark that $\dim_{\widehat{K}} \mathcal{M}_{\widehat{K}} = \deg P$ if $\mathcal{M}_{\widehat{K}} = \mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}} \cdot P$.

We may first assume that $\mathcal{M}_{\widehat{K}}$ has only one slope L by applying induction to each term of the splitting into one slope terms. One calls $\sigma_L(P) \in \widehat{K}[\xi]$ the *determinant equation* of P. Because L is a slope, $\sigma_L(P)$ has at least two monomials and is homogeneous of degree $\operatorname{ord}_L(P) = 0$ because P is chosen with coefficients in $\mathbf{C}[\![x]\!]$, one of them being a unit. Write

$$\sigma_L(P) = \sum_{L(i,j)=0} \alpha_{ij} x^j (x\xi)^i.$$

Let $\theta = x^{\lambda_0}(x\xi)^{\lambda_1}$ where we assume that λ_0 and λ_1 are relatively prime integers. Then we can write

$$\sigma_L(P) = \sum_{k \geq 0} \alpha_k \theta^k$$

with $\alpha_0 \neq 0$ because the origin is a vertex of $N(P)$.

First case: $\lambda_1 = 1$. — This means that the slope is an integer (hence $\kappa < +\infty$). Consider the factorization

$$\sigma_L(P) = * \prod_{\beta} (\theta - \beta)^{\gamma_\beta}$$

where $*$ is constant and choose a root β_0 of this polynomial. Put $R(z) = (\beta_0/(\lambda_0 + 1))z^{\lambda_0+1}$ and consider $\mathcal{M}_{\widehat{K}} \otimes \mathcal{F}^R_{\widehat{K}}$. If e is a cyclic vector for $\mathcal{M}_{\widehat{K}}$ then $e \otimes e(R)$ is a cyclic vector for $\mathcal{M}_{\widehat{K}} \otimes \mathcal{F}^R_{\widehat{K}}$. If $P(x, x\partial_x) \cdot e = 0$ then

$$P\left(x, x\partial_x - x\frac{\partial R(x^{-1})}{\partial x}\right) \cdot e \otimes e(R) = 0$$

and here we have $x \cdot \partial R(x^{-1})/\partial x = \beta_0 x^{-(\lambda_0+1)}$. Denote $P' = P(x, x\partial_x + \beta x^{-(\lambda_0+1)})$. Then one verifies that P' has coefficients in $\mathbf{C}[\![x]\!]$. Moreover, $\sigma_L(P') = \sum_{k\geq 0} \alpha_k(\theta + \beta_0)^k$. Distinguish two cases:

1. *The determinant equation has only one root β_0.* Then $\sigma_L(P') = *\theta^r$. This implies that L is not a slope for P'. Moreover, one verifies in the same way that if L' is another linear form with $\lambda'_0/\lambda'_1 > \lambda_0/\lambda_1 (= \lambda_0)$ then L' is not a slope for P' (one proves that for such a linear form one has $\sigma_{L'}(P') = \sigma_{L'}(P)$). Hence either $\kappa(\mathcal{M}_{\widehat{K}} \otimes \mathcal{F}^R_{\widehat{K}}) < \kappa(\mathcal{M}_{\widehat{K}})$ or $\kappa(\mathcal{M}_{\widehat{K}} \otimes \mathcal{F}^R_{\widehat{K}}) = +\infty$. In the first case, apply induction to obtain a splitting after ramification of $\mathcal{M}_{\widehat{L}} \otimes \mathcal{F}^R_{\widehat{L}}$. Tensor the corresponding direct sum of elementary meromorphic connections with $\mathcal{F}^{-R}_{\widehat{L}}$ to obtain the splitting of $\mathcal{M}_{\widehat{L}}$. In the other case, apply the case $\kappa = +\infty$ below.

2. *The determinant equation has more than one root.* Then L is a slope of P' and the same argument as above shows that if L' is such that $\lambda'_0/\lambda'_1 > \lambda_0$ then L' is not a slope for P'. However, P' has at least one more slope because $\sigma_L(P') = \theta^{\beta_0}Q(\theta)$ vanishes for $\theta = 0$, which means that the segment between the origin and the point of coordinates $(\deg P', -\lambda_0 \deg P')$ (which is a vertex of $N(P')$) cannot be a side of $N(P')$. One can now split $\mathcal{M}_{\widehat{K}} \otimes \mathcal{F}^R_{\widehat{K}}$ into one slope terms and argue as above.

Second case: $\lambda_1 > 1$ (i.e. $\kappa = +\infty$). — The slope is not an integer so one ramifies in degree $q = \lambda_1$. Consider $\mathcal{M}_{\widehat{L}} = \mathcal{D}_{\widehat{L}}/\mathcal{D}_{\widehat{L}} \cdot P'$. Then P' has only one slope L' as explained in 5.4.3 and $\sigma_{L'}(P') = \prod(\theta - \beta)^{\gamma_\beta}$ with $\theta = \left[t^{\lambda_0}(t\partial_t)\right]^{\lambda_1} = \theta'^{\lambda_1}$. This implies that $\sigma_{L'}(P')$ has at least two distinct roots when considered as a polynomial in the variable θ'. One then applies the second possibility of the first case treated above. □

6. Formal structure of holonomic $\widehat{\mathcal{D}}$-modules

We shall now extend the results of the previous section to holonomic $\widehat{\mathcal{D}}$-modules. We know that a torsion module has a simple structure, so what we have to do is to describe in simple terms the extension of a meromorphic connection by a torsion module. The notion of (moderate) nearby and vanishing cycles will be useful for that purpose.

6.1. Moderate nearby and vanishing cycles

The results of this section apply to modules over $\mathcal{D}$, $\widehat{\mathcal{D}}$, or $\mathbf{C}[x]\langle\partial_x\rangle$. We shall only consider the case of $\mathcal{D}$. Let $U\mathcal{M}$ be a filtration of $\mathcal{M}$ good for $V\mathcal{D}$ (*see* §4.2.4 and exercise 5.1.3). We shall develop below the notions introduced in §4.2.4.

PROPOSITION 6.1.1. — *There exists a polynomial $B \in \mathbf{C}[s] - \{0\}$ such that for all $k \in \mathbf{Z}$ one has*

$$B(x\partial_x + k) \cdot U_k\mathcal{M} \subset U_{k-1}\mathcal{M}.$$

This polynomial B depends on the filtration $U\mathcal{M}$. A minimal polynomial satisfying this property is called *a Bernstein polynomial* for the filtration $U\mathcal{M}$.

Proof. — We leave it as an exercise, but we shall indicate the main steps.

1. If the property is satisfied for one good filtration, it is satisfied for all ones (changing the polynomial B). One uses for that 5.1.3–5.

2. In an exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$, the property is satisfied for $\mathcal{M}$ if and only if it is satisfied for $\mathcal{M}'$ and $\mathcal{M}''$. One uses for that a good filtration on $\mathcal{M}$ and one induces it on $\mathcal{M}'$ and $\mathcal{M}''$ (*see* 5.1.3–4).

3. One is then reduced to the case where $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$. If $\mathcal{M}$ is local, one takes $P \in V_0\mathcal{D} - V_{-1}\mathcal{D}$ (corollary 4.2.8) and one takes the filtration induced by $V\mathcal{D}$ (as in corollary 4.2.6). The case where $\mathcal{M}$ is a torsion module is treated in the same way.

If one considers Bernstein polynomials B (for $U\mathcal{M}$) and B' (for $U'\mathcal{M}$), the inclusions 5.1.3–5 show that $B'(s)$ divides the product $\prod_{j=-\ell+1}^{\ell} B(s+j)$ and conversely. In other words, the roots of B and the roots of B' coincide modulo $\mathbf{Z}$. We shall now construct a filtration for which the roots of B are contained in the set (*see fig.* 7)

$$\Sigma = \{\alpha \in \mathbf{C} \,|\, -1 \leq \operatorname{Re}\alpha \leq 0,\ \operatorname{Im}\alpha \geq 0 \text{ if } \operatorname{Re}\alpha = -1,\ \operatorname{Im}\alpha < 0 \text{ if } \operatorname{Re}\alpha = 0\}$$

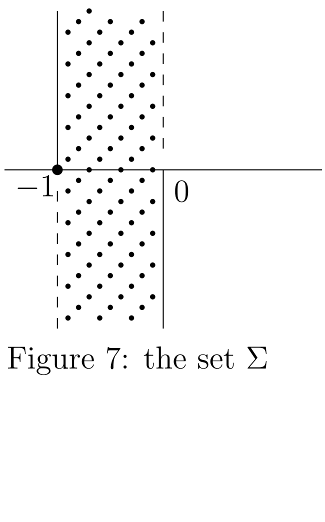

Figure 7: the set Σ

What is important here is that if $\alpha, \beta \in \Sigma$, then $\alpha - \beta \in \mathbf{Z} \Leftrightarrow \alpha = \beta$. One could do the same construction for any section $\sigma : \mathbf{C}/\mathbf{Z} \to \mathbf{C}$ of the projection $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$.

Proposition 6.1.2. — *There exists a unique filtration, denoted by $V\mathcal{M}$, satisfying the following properties:*

1. *$V\mathcal{M}$ is a filtration (indexed by $\mathbf{Z}$) good for $V\mathcal{D}$,*
2. *the roots of the Bernstein polynomial b of $V\mathcal{M}$ are contained in Σ.*

Before giving the proof of this proposition, let us state the following corollary:

Corollary 6.1.3. — *Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of holonomic modules. Then $V\mathcal{M}' = V\mathcal{M} \cap \mathcal{M}'$ and $V\mathcal{M}'' = \mathrm{Image}(V\mathcal{M} \to \mathcal{M}'')$.*

Indeed, put for instance $U\mathcal{M}' = V\mathcal{M} \cap \mathcal{M}'$. This is a good filtration of $\mathcal{M}'$, due to Artin-Rees lemma (5.1.3–4) and its Bernstein polynomial divides the one of $V\mathcal{M}$, hence has its roots in Σ. From unicity one deduces that $U\mathcal{M}' = V\mathcal{M}'$. □

One concludes also that any morphism of holonomic $\mathcal{D}$-modules is *strictly compatible* with the V-filtration: if $\varphi : \mathcal{M} \to \mathcal{M}_1$ is such a morphism, then $\mathrm{Ker}\,\varphi$ and $\mathrm{Im}\,\varphi$ are holonomic and one has

$$V\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\varphi \cap V\mathcal{M} \qquad , \qquad V\mathrm{Im}\,\varphi = \mathrm{Im}\,\varphi \cap V\mathcal{M}_1 = \varphi(V\mathcal{M}).$$

In other words, if one denotes by $\mathrm{gr}^V\varphi : \mathrm{gr}^V\mathcal{M} \to \mathrm{gr}^V\mathcal{M}_1$ the graded morphism associated with φ, one has

$$\mathrm{Ker}\,\mathrm{gr}^V\varphi = \mathrm{gr}^V(\mathrm{Ker}\,\varphi) \qquad \text{and} \qquad \mathrm{Coker}\,\mathrm{gr}^V\varphi = \mathrm{gr}^V(\mathrm{Coker}\,\varphi)\,.$$

Proof of proposition 6.1.2. — Start with a good filtration $U\mathcal{M}$ with Bernstein polynomial B. Put $B = B_1 \cdot B_2$. Consider the following filtration $U'\mathcal{M}$: put for all $k \in \mathbf{Z}$

$$U'_k\mathcal{M} = U_{k-1}\mathcal{M} + B_2(x\partial_x + k) \cdot U_k\mathcal{M}.$$

This defines a good filtration (5.1.3–5) and its Bernstein polynomial divides $B_1(s) \cdot B_2(s-1)$. Continuing this process, one obtains a good filtration $U'\mathcal{M}$ for which the roots of the Bernstein polynomial are contained in the set $\Sigma + \ell$ for some $\ell \in \mathbf{Z}$. Put then $V_k\mathcal{M} = U'_{k-\ell}\mathcal{M}$.

We shall now give some properties of this filtration $V\mathcal{M}$.

1. For each $k \in \mathbf{Z}$, $\mathrm{gr}_k^V \mathcal{M}$ is a module of finite type over the ring $\mathrm{gr}_0^V \mathcal{D} = \mathbf{C}[x\partial_x]$. Moreover, the endomorphism $x\partial_x$ admits a minimal polynomial $b(x\partial_x + k)$ on this space, which implies that $\mathrm{gr}_k^V \mathcal{M}$ is a finite dimensional $\mathbf{C}$-vector space.

2. Left multiplication by $x : V_k\mathcal{M} \to V_{k-1}\mathcal{M}$ (remember that $x \in V_{-1}\mathcal{D}$) induces a $\mathbf{C}$-linear mapping
$$x : \mathrm{gr}_k^V \mathcal{M} \longrightarrow \mathrm{gr}_{k-1}^V \mathcal{M}.$$
In the same way, $\partial_x : V_k\mathcal{M} \to V_{k-1}\mathcal{M}$ induces
$$\partial_x : \mathrm{gr}_k^V \mathcal{M} \longrightarrow \mathrm{gr}_{k+1}^V \mathcal{M}.$$
The first mapping is invertible as soon as $k \neq 0$ and the second one as soon as $k \neq -1$: indeed, for $k \neq 0$ the composed mapping $\partial_x \cdot x : \mathrm{gr}_k^V \mathcal{M} \to \mathrm{gr}_k^V \mathcal{M}$ is invertible because its minimal polynomial $\beta(s)$ is equal to $b(s+k-1)$ and hence $\beta(0) = b(k-1) \neq 0$ if $k \neq 0$. In the same way, the composed mapping $x \cdot \partial_x : \mathrm{gr}_k^V \mathcal{M} \to \mathrm{gr}_k^V \mathcal{M}$ is invertible when $k \neq -1$.

3. When $k \leq -1$, left multiplication by $x : V_k\mathcal{M} \to V_{k-1}\mathcal{M}$ is invertible. This property does not follow directly from the previous one, but as a consequence of this one, it is enough to show that there exists $\ell \leq -1$ for which $x : V_\ell\mathcal{M} \to V_{\ell-1}\mathcal{M}$ is invertible, and it is also enough to show that this last propety is valid for some good filtration $U\mathcal{M}$. Use then a presentation of $\mathcal{M}$ and argue as in the proof of lemma 4.2.7.

6.1.4. Exercises.

1. Show that the natural morphism $\mathcal{M} \to \mathcal{M}[x^{-1}]$ induces an isomorphism
$$V_k\mathcal{M} \longrightarrow V_k\mathcal{M}[x^{-1}]$$
for all $k \leq -1$.

2. $\mathcal{M}$ is a torsion module if and only if $V_{-1}\mathcal{M} = \{0\}$.

3. If $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$ with $P \in V_0\mathcal{D} - V_{-1}\mathcal{D}$, $P = b(x\partial_x) + xP'$, and if the roots of b are contained in Σ, then one has for all $k \in \mathbf{Z}$
$$V_k\mathcal{M} = V_k\mathcal{D}/\mathcal{D}\cdot P \cap V_k\mathcal{D}.$$

6.1.5. — We may now introduce the "nearby cycles" and the "vanishing cycles" of $\mathcal{M}$. For a holonomic module $\mathcal{M}$ put
$$\begin{aligned} \Psi(\mathcal{M}) &= \mathrm{gr}_{-1}^V \mathcal{M} \\ \Phi(\mathcal{M}) &= \mathrm{gr}_0^V \mathcal{M}. \end{aligned}$$

Each of these vector spaces comes equipped with a "monodromy operator"

$$T = \exp\left(-2i\pi x\partial_x\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-2i\pi x\partial_x\right)^n$$

and the following diagrams commute:

$$\begin{array}{ccc} \Psi(\mathcal{M}) & \xrightarrow{\partial_x} & \Phi(\mathcal{M}) \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle T} \\ \Psi(\mathcal{M}) & \xrightarrow{\partial_x} & \Phi(\mathcal{M}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Psi(\mathcal{M}) & \xleftarrow{x} & \Phi(\mathcal{M}) \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle T} \\ \Psi(\mathcal{M}) & \xleftarrow{x} & \Phi(\mathcal{M}) \end{array}$$

One considers the following mappings

$$\mathrm{can} : \Psi(\mathcal{M}) \to \Phi(\mathcal{M}) \quad \text{for "canonical"}$$

$$\mathrm{var} : \Phi(\mathcal{M}) \to \Psi(\mathcal{M}) \quad \text{for "variation"}$$

which satisfy

$$\begin{aligned} \mathrm{can} \circ \mathrm{var} &= T - \mathrm{Id} : \Phi(\mathcal{M}) \to \Phi(\mathcal{M}) \\ \mathrm{var} \circ \mathrm{can} &= T - \mathrm{Id} : \Psi(\mathcal{M}) \to \Psi(\mathcal{M}) \end{aligned}$$

and which are defined as follows:

$$\mathrm{can} = \partial_x$$

and

$$\mathrm{var} = \sum_{n=1}^{\infty} \frac{(-2i\pi)^n}{n!}(x\partial_x)^{n-1} \cdot x = x \cdot \sum_{n=1}^{\infty} \frac{(-2i\pi)^n}{n!}(\partial_x x)^{n-1}.$$

6.1.6. Exercises.

1. Show that there exists no torsion submodule of $\mathcal{M}$ if and only if $\mathrm{var} : \Phi(\mathcal{M}) \to \Psi(\mathcal{M})$ is injective.
2. Show that $\mathcal{M}$ admits no torsion quotient if and only if $\mathrm{can} : \Psi(\mathcal{M}) \to \Phi(\mathcal{M})$ is onto.
3. Show that $\mathcal{M} = \mathcal{M}[x^{-1}]$ if and only if $\mathrm{var} : \Phi(\mathcal{M}) \to \Psi(\mathcal{M})$ is bijective.

6.2. Regular holonomic $\widehat{\mathcal{D}}$-modules

6.2.1. — We have seen in corollary 5.2.6 that a regular connection over $\widehat{K}$ is isomorphic to a direct sum of elementary such connections $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}(x\partial_x - \alpha)^p$. It follows from exercise 4.2.9 that two such connections, corresponding to α, p and β, q, are isomorphic if and only if $\alpha - \beta \in \mathbf{Z}$ and $p = q$. Hence the datum of a regular meromorphic connection over $\widehat{K}$ (up to isomorphism) is equivalent to the datum of a $\mathbf{C}$-vector space equipped with an automorphism (up to isomorphism): one associates with the connection $\mathcal{D}_{\widehat{K}}/\mathcal{D}_{\widehat{K}}(x\partial_x - \alpha)^p$ the space $\mathbf{C}^p$ with the automorphism corresponding to the Jordan block of size p with eigenvalue $\exp(-2i\pi\alpha)$. We shall now extend this correspondence to regular $\widehat{\mathcal{D}}$-modules.

Definition 6.2.2. — *We shall say that a $\widehat{\mathcal{D}}$-module is regular if the associated meromorphic connection $\widehat{\mathcal{M}}[x^{-1}]$ is regular (i.e. $\widehat{\mathcal{M}}[x^{-1}]_i = 0$).*

Every submodule and every quotient of a regular module is also regular. A module which is extension of two regular modules is regular. Every torsion module is regular.

6.2.3. — Consider the following category $\mathcal{C}$: the objects are the symbols $E \underset{v}{\overset{c}{\rightleftarrows}} F$ where E and F are finite dimensional vector spaces, c and v are two linear mappings which satisfy the property that

$$T_E \overset{\text{def}}{=} cv - \mathrm{Id}_E : E \to E \qquad \text{and} \qquad T_F \overset{\text{def}}{=} vc - \mathrm{Id}_F : F \to F$$

are invertible. A morphism between two such symbols consists of a pair (e, f), $e : E \to E'$ and $f : F \to F'$ linear, such that the following diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{c} & F \\ \big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle f} \\ E' & \xrightarrow{c'} & F' \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xleftarrow{v} & F \\ \big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle f} \\ E' & \xleftarrow{v'} & F' \end{array}$$

and in particular $e \circ T_E = T_{E'} \circ e$ and $f \circ T_F = T_{F'} \circ f$. One verifies easily that (e, f) is an isomorphism if and only if e and f are isomorphisms.

6.2.4. Exercises.

1. Define the kernel and the cokernel of a morphism as objects in the category $\mathcal{C}$.
2. Show that any object for which v is an isomorphism is isomorphic to an object of the form

$$E \underset{\mathrm{Id}_E}{\overset{T_E - \mathrm{Id}_E}{\rightleftarrows}} E$$

and the datum of such an object is equivalent to the datum of the space E equipped with the automorphism T_E.

3. If $E = \{0\}$, then $c = v = 0$ and $T_F = \mathrm{Id}_F$

Theorem 6.2.5. — *The correspondence which associates with each holonomic $\widehat{\mathcal{D}}$-module the object $\Psi(\widehat{\mathcal{M}}) \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \Phi(\widehat{\mathcal{M}})$ of the category $\mathcal{C}$ is an equivalence of categories.*

Remark. — Under this correspondence, $\widehat{\mathcal{M}}$ is equal to its localized module if and only if var is invertible and the corresponding object of $\mathcal{C}$ is isomorphic to the object

$$\Psi(\mathcal{M}) \underset{\mathrm{Id}}{\overset{T - \mathrm{Id}}{\rightleftarrows}} \Psi(\mathcal{M}).$$

In the same way, $\widehat{\mathcal{M}}$ is a torsion module if and only if $\Psi(\mathcal{M}) = \{0\}$.

Proof. — It is done in two steps. One considers first the correspondence which associates with each $\widehat{\mathcal{M}}$ its V-graded module $\mathrm{gr}^V\widehat{\mathcal{M}}$, which is a holonomic *graded* module over the ring $\mathrm{gr}^V\widehat{\mathcal{D}}=\mathbf{C}[x]\langle\partial_x\rangle$. □

LEMMA 6.2.6. — *If $\widehat{\mathcal{M}}$ is regular holonomic, there exists a functorial isomorphism*

$$\widehat{\mathcal{M}}\simeq\mathbf{C}[\![x]\!]\otimes_{\mathbf{C}[x]}\mathrm{gr}^V\widehat{\mathcal{M}}.$$

Proof. — Consider for each $\alpha\in\mathbf{C}$ the following vector subspace

$$P_\alpha(\widehat{\mathcal{M}})=\bigcup_{n\in\mathbf{N}}\mathrm{Ker}\,(x\partial_x+\alpha+1)^n.$$

One has $P_\alpha(\widehat{\mathcal{M}})\subset V_k\widehat{\mathcal{M}}$ if $\alpha\leq k$ (take an element $m\in P_\alpha(\widehat{\mathcal{M}})$, consider the submodule $\widehat{\mathcal{D}}\cdot m$ and verify that $m\in V_k(\widehat{\mathcal{D}}\cdot m)$) and $P_\alpha(\widehat{\mathcal{M}})\cap P_\beta(\widehat{\mathcal{M}})=\{0\}$ if $\alpha\neq\beta$; hence one obtains a mapping

$$\bigoplus_{\{\alpha|\,-(\alpha+1)\in\Sigma+k\}}P_\alpha(\widehat{\mathcal{M}})\longrightarrow\mathrm{gr}_k^V\widehat{\mathcal{M}}\tag{6.2.7}$$

for all $k\in\mathbf{Z}$. We shall show that this mapping is invertible, which will prove in particular that $P_\alpha(\widehat{\mathcal{M}})$ is a finite dimensional $\mathbf{C}$-vector space. Let α be such that $-(\alpha+1)\in\Sigma+k$. We shall show that $P_\alpha(\widehat{\mathcal{M}})\cap V_{k-1}\widehat{\mathcal{M}}=\{0\}$, which will prove injectivity. If $m\in P_\alpha(\widehat{\mathcal{M}})\cap V_{k-1}\widehat{\mathcal{M}}$, there exists n such that $(x\partial_x+\alpha+1)^n\cdot m=0$ and one has also $b(x\partial_x+k-1)\cdot m\in V_{k-2}\widehat{\mathcal{M}}$. But the polynomials $(s+\alpha+1)^n$ and $b(s+k-1)$ have no common factor, so $m\in P_\alpha(\widehat{\mathcal{M}})\cap V_{k-2}\widehat{\mathcal{M}}$. One can iterate this process. It is then enough to prove that

$$\bigcap_{k\in\mathbf{Z}}V_k\widehat{\mathcal{M}}=\{0\}\,.\tag{6.2.8}$$

Since $V_k\widehat{\mathcal{M}}=V_k\widehat{\mathcal{M}}[x^{-1}]$ for $k\leq-1$, one may assume that $\widehat{\mathcal{M}}=\widehat{\mathcal{M}}[x^{-1}]$ and one may use the formal classification of regular formal connections to be reduced to the case of elementary ones. This case can be checked easily.

Before proving surjectivity, put

$$P(\widehat{\mathcal{M}})=\bigoplus_{k\in\mathbf{Z}}\bigoplus_{\{\alpha|\,-(\alpha+1)\in\Sigma+k\}}P_\alpha(\widehat{\mathcal{M}}).$$

One verifies that $P(\widehat{\mathcal{M}})$ is a sub-$\mathbf{C}[x]\langle\partial_x\rangle$-module of $\widehat{\mathcal{M}}$ (one has $x\cdot P_\alpha(\widehat{\mathcal{M}})\subset P_{\alpha-1}(\widehat{\mathcal{M}})$ and $\partial_x\cdot P_\alpha(\widehat{\mathcal{M}})\subset P_{\alpha+1}(\widehat{\mathcal{M}})$). If $\widehat{\mathcal{M}}'\subset\widehat{\mathcal{M}}$ is a submodule, one has $P(\widehat{\mathcal{M}}')=P(\widehat{\mathcal{M}})\cap\widehat{\mathcal{M}}'$ and if $\widehat{\mathcal{M}}''$ is a quotient of $\widehat{\mathcal{M}}$, the image of $P(\widehat{\mathcal{M}})$ in $\widehat{\mathcal{M}}''$ is contained in $P(\widehat{\mathcal{M}}'')$. Consider now the inclusion

$$i:\mathbf{C}[\![x]\!]\otimes_{\mathbf{C}[x]}P(\widehat{\mathcal{M}})\hookrightarrow\widehat{\mathcal{M}}.\tag{6.2.9}$$

This inclusion is an isomorphism: this is easily verified for regular meromorphic connections, due to formal classification, as well as for torsion modules. Let $\widehat{\mathcal{M}}_1$ be the image of $\widehat{\mathcal{M}}$ in $\widehat{\mathcal{M}}[x^{-1}]$. Because $P(\widehat{\mathcal{M}}_1) = P(\widehat{\mathcal{M}}[x^{-1}]) \cap \widehat{\mathcal{M}}_1$, one deduces that (6.2.9) is an isomorphism for $\widehat{\mathcal{M}}_1$. Let us show that the mapping $P(\widehat{\mathcal{M}}) \to P(\widehat{\mathcal{M}}_1)$ is surjective. Let $m \in \widehat{\mathcal{M}}$. Assume that the image of m in $\widehat{\mathcal{M}}_1$ is contained in $P(\widehat{\mathcal{M}}_1)$. Then there exists a polynomial b_m such that $b_m(x\partial_x) \cdot m \in \operatorname{Ker}(\widehat{\mathcal{M}} \to \widehat{\mathcal{M}}_1)$, *i.e.* there exists k such that $x^k b_m(x\partial_x) \cdot m = 0$, hence $\partial_x^k x^k b_m(x\partial_x) \cdot m = 0$. Hence $m \in P(\widehat{\mathcal{M}})$, which proves the assertion. One concludes from that and from the previous cases that (6.2.9) is an isomorphism in general.

The isomorphism (6.2.9) induces an isomorphism at the graded level:

$$P(\widehat{\mathcal{M}}) \xrightarrow{\sim} \operatorname{gr}^V \widehat{\mathcal{M}}.$$

This gives the surjectivity of (6.2.7). The isomorphism in the lemma is then obtained as follows:

$$\begin{array}{ccc} \mathbf{C}[\![x]\!] \otimes_{\mathbf{C}[x]} P(\widehat{\mathcal{M}}) & \xrightarrow{i} & \widehat{\mathcal{M}} \\ \Big\downarrow {\scriptstyle 1\otimes \operatorname{gr}^V i} & & \\ \mathbf{C}[\![x]\!] \otimes_{\mathbf{C}[x]} \operatorname{gr}^V \widehat{\mathcal{M}} & & \end{array}$$

Functoriality comes from the fact that if $\varphi : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}'$ is a morphism of holonomic $\widehat{\mathcal{D}}$-modules, one has $\varphi(P(\widehat{\mathcal{M}})) \subset P(\varphi(\widehat{\mathcal{M}}))$. □

This lemma proves that the correspondence which associates with each regular $\widehat{\mathcal{M}}$ its graded module $\operatorname{gr}^V \widehat{\mathcal{M}}$ is an equivalence between the category of regular holonomic $\widehat{\mathcal{D}}$-modules and the category of graded holonomic $\operatorname{gr}^V \widehat{\mathcal{D}}$-modules.

The second step consists in showing that the latter category is equivalent to the category $\mathcal{C}$. Indeed, starting with an object in $\mathcal{C}$, one may construct a graded $\operatorname{gr}^V \widehat{\mathcal{D}}$-module using formulas for can and var to define the action of x and ∂_x. Details are left as an exercise.

6.3. Holonomic $\widehat{\mathcal{D}}$-modules

We want to apply the previous results to holonomic $\widehat{\mathcal{D}}$-modules. Let $\widehat{\mathcal{M}}$ be such a module. One has an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \widehat{\mathcal{M}} \xrightarrow{\varphi} \widehat{\mathcal{M}}[x^{-1}] \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{K}$ and $\mathcal{C}$ are torsion $\widehat{\mathcal{D}}$-modules: each element $m \in \mathcal{K}$ (*resp.* $\in \mathcal{C}$) satisfies $x^n m = 0$ for some n. One knows the structure of $\widehat{\mathcal{M}}[x^{-1}]$ because of the correspondence of §4.3. So one may write $\widehat{\mathcal{M}}[x^{-1}] = \widehat{\mathcal{M}}[x^{-1}]_r \oplus \widehat{\mathcal{M}}[x^{-1}]_i$ where the first summand (regular part) corresponds to the terms with horizontal slope and the second summand to the terms with non horizontal slope (irregular part) in the splitting of $\widehat{\mathcal{M}}[x^{-1}]$. The structure of holonomic $\widehat{\mathcal{D}}$-modules is given by the following theorem:

THEOREM 6.3.1. — *Let $\widehat{\mathcal{M}}$ be a holonomic $\widehat{\mathcal{D}}$-module. Then one has a unique splitting $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_r \oplus \widehat{\mathcal{M}}_i$ where $\widehat{\mathcal{M}}_r$ is regular and $\widehat{\mathcal{M}}_i \simeq \widehat{\mathcal{M}}[x^{-1}]_i$.*

This theorem describes completely the structure of holonomic $\widehat{\mathcal{D}}$-modules with the help of the results obtained in the previous sections.

Proof. — Define $\widehat{\mathcal{M}}_r$ as the inverse image in $\widehat{\mathcal{M}}$ of $\widehat{\mathcal{M}}[x^{-1}]_r$. This is a regular module because $\mathcal{K}$ is a torsion module. One has now to find $\widehat{\mathcal{M}}_i$. In order to do that, we have two possibilities: either use an argument analogous to the one given in lemma 5.3.5 or use the following construction. Let $U\widehat{\mathcal{M}}$ be a filtration good with respect to the filtration $V\widehat{\mathcal{D}}$. Put

$$\mathrm{Rad}_U(\widehat{\mathcal{M}}) = \bigcap_{\lambda \in \mathbf{Z}} U_\lambda \widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}.$$

This object has the following properties which proofs are left as an exercise:

1. $\mathrm{Rad}_U(\widehat{\mathcal{M}})$ does not depend on the choice of the good filtration U. It will be denoted by $\mathrm{Rad}(\widehat{\mathcal{M}})$.
2. $\mathrm{Rad}(\widehat{\mathcal{M}})$ is a sub-$\widehat{\mathcal{D}}$-module of finite type of $\widehat{\mathcal{M}}$.
3. If $\widehat{\mathcal{M}}'$ is a submodule of $\widehat{\mathcal{M}}$ then $\mathrm{Rad}(\widehat{\mathcal{M}}') = \mathrm{Rad}(\widehat{\mathcal{M}}) \cap \widehat{\mathcal{M}}'$.
4. $\widehat{\mathcal{M}} = \mathrm{Rad}(\widehat{\mathcal{M}})$ if and only if for each $m \in \widehat{\mathcal{M}}$ there exists $P \in V_{-1}\widehat{\mathcal{D}}$ such that $m = P \cdot m$.
5. Left multiplication by x is invertible on $\mathrm{Rad}(\widehat{\mathcal{M}})$.
6. One has $\mathrm{Rad}(\widehat{\mathcal{M}}[x^{-1}]_i) = \widehat{\mathcal{M}}[x^{-1}]_i$.
7. $\mathrm{Rad}(\widehat{\mathcal{M}}_r) = \{0\}$ (*see* proof of (6.2.8)).

Put now $\widehat{\mathcal{M}}_i = \mathrm{Rad}(\widehat{\mathcal{M}})$ (note that this is the right definition when $\widehat{\mathcal{M}}$ is a meromorphic connection). We have $\widehat{\mathcal{M}}_i \cap \widehat{\mathcal{M}}_r = \{0\}$ and we have to show that the induced morphism

$$\widehat{\mathcal{M}}_i \longrightarrow \widehat{\mathcal{M}}[x^{-1}]_i$$

is surjective. Let $\widehat{\mathcal{M}}_1$ be the image of $\widehat{\mathcal{M}}$ in $\widehat{\mathcal{M}}[x^{-1}]$. Then we have $\mathrm{Rad}(\widehat{\mathcal{M}}_1) = \mathrm{Rad}(\widehat{\mathcal{M}}[x^{-1}]) = \widehat{\mathcal{M}}[x^{-1}]_i$ because the cokernel $\mathcal{C}$ is a torsion module and we have an exact sequence

$$0 \longrightarrow \mathrm{Rad}(\widehat{\mathcal{M}}_1) \longrightarrow \mathrm{Rad}(\widehat{\mathcal{M}}[x^{-1}]) \longrightarrow \mathrm{Rad}(\mathcal{C})$$

the last term being zero.

The morphism $\widehat{\mathcal{M}} \to \widehat{\mathcal{M}}_1$ induces an isomorphism $\mathrm{Rad}(\widehat{\mathcal{M}}) \simeq \mathrm{Rad}(\widehat{\mathcal{M}}_1)$: we know that the induced morphism is injective, because $\mathrm{Rad}(\mathcal{K}) = 0$. Let us show that it is surjective. Let $m \in \widehat{\mathcal{M}}$ such that its image in $\widehat{\mathcal{M}}_1$ belong to $\mathrm{Rad}(\widehat{\mathcal{M}}_1)$. Then there exists $P \in V_{-1}\widehat{\mathcal{D}}$ such that $m = P \cdot m + m'$ for some $m' \in \mathcal{K}$. Hence there exists k such that $x^k m = x^k P \cdot m$. Write $x^k P = Qx^k$ for some $Q \in V_{-1}\widehat{\mathcal{D}}$. One concludes that $x^k m \in \mathrm{Rad}(\widehat{\mathcal{M}})$. But left multiplication by x is bijective on $\mathrm{Rad}(\widehat{\mathcal{M}})$ so $m \in \mathrm{Rad}(\widehat{\mathcal{M}})$. □

6.3.2. Exercise. — Show that the mapping (6.2.7) is injective even when $\widehat{\mathcal{M}}$ is not regular: show that $P_\alpha(\widehat{\mathcal{M}}) \cap \mathrm{Rad}(\widehat{\mathcal{M}}) = \{0\}$ for each $\alpha \in \mathbf{C}$ (use the fourth property of Rad given above).

Chapter II
Analytic structure of holonomic $\mathcal{D}$-modules

1. Regularity and irregularity

1.1. Structure of regular meromorphic connections

We shall now give the statements for regular meromorphic connections defined over K which are analogous to theorem I–5.2.2 and corollary I–5.2.6.

THEOREM 1.1.1. — *Let $\mathcal{M}_K$ be a regular meromorphic connection. There exists a basis of $\mathcal{M}_K$ over K for which the matrix of $x\partial_x$ is constant and has Jordan normal form. Consequently, $\mathcal{M}_K$ is isomorphic to a direct sum of elementary regular meromorphic connections.*

Proof. — One has to show, following the proof of I–5.2.2 that if $A = \sum_{k\geq 0} A_k x^k$ has convergent entries, then the matrix B which is solution of the system

$$B_0 = \mathrm{Id} \qquad , \qquad x\frac{\partial B}{\partial x} = A_0 B - BA$$

has also convergent entries (one may assume that eigenvalues of A_0 do not differ by a nonzero integer). In order to do that, we shall use the following

PROPOSITION 1.1.2. — *Let $\mathcal{M}_K$ be a meromorphic connection with a K-basis for which the matrix of $x\partial_x$ has entries in $\mathbf{C}\{x\}$. Then every formal solution of $\mathcal{M}_K$ is in fact convergent.*

Proof. — Consider a linear system of dimension d for which the matrix $C(x)$ has entries in $\mathbf{C}\{x\}$. Let $\sum_{k=0}^{\infty} a_k x^k$ be a formal solution, with $a_k \in \mathbf{C}^d$ and put $C(x) = \sum_{j>0} C_j x^j$. One must have

$$C_0 \cdot a_0 = 0 \qquad , \qquad (C_0 - k\mathrm{Id}\,)\, a_k = -\sum_{j=1}^{k} C_j \cdot a_{k-j}.$$

Let ℓ be such that $C_0 - k\mathrm{Id}$ is invertible for all $k \geq \ell$. There exists a constant c independent of k such that for all $k \geq \ell$ one has

$$\left\| (C_0 - k\mathrm{Id}\,)^{-1} \right\| \leq c \tag{1.1.3}$$

where for a matrix $A = (a_{ij})$ we put $\|A\| = \max \sum_j \left| a_{ij} \right|$.

1.1.4. Exercise. — Prove this assertion (use the fact that if C is a matrix such that $\|C\| < 1$ then the series $\log(\mathrm{Id} - C) = -\sum_{j\geq 1}(1/j)C^j$ converges and

$$\mathrm{Id} - C = \exp\left(-\sum_{j\geq 1}(1/j)C^j \right);$$

deduce that

$$\left\| (\mathrm{Id} - C)^{-1} \right\| \leq (1 - \|C\|)^{-1}$$

if $\|C\| < 1$ and conclude).

From the inequality 1.1.3 one deduces that for all $k \geq \ell$ one has

$$\|a_k\| \leq c \sum_{j=1}^{k} \|C_j\| \left\| a_{k-j} \right\|.$$

Consider now the series

$$\varphi(x) = \sum_{j=1}^{+\infty} \|C_j\| \, x^j.$$

If the series $C(x)$ converges for $|x| < \rho$, then so does the series $\varphi(x)$ (indeed one knows that for $\rho' < \rho$ the sequence $C_j \rho'^j$ is bounded by $M_{\rho'} < +\infty$, hence $\|C_j\| \leq M_{\rho'} \rho'^{-j}$ and the series $\sum_{j=1}^{+\infty} \|C_j\| \, x^j$ converges as soon as the series $M_{\rho'} \sum_{j=1}^{+\infty} |x|^j / \rho'^j$ converges, *i.e.* when $|x| < \rho'$).

Put now

$$\alpha_k = \begin{cases} \|a_k\| & \text{for } k < \ell \\ \\ c \sum_{j=1}^{k} \|C_j\| \, \alpha_{k-j} & \text{for } k \geq \ell. \end{cases}$$

It is easily seen by induction over k that for all k one has $\|a_k\| \leq \alpha_k$. We shall show that there exists $\rho_1 > 0$ such that the series $\sum \alpha_k x^k$ is convergent for $|x| < \rho_1$, which will imply that the series $\sum a_k x^k$ does so.

Lemma 1.1.5. — *One has*

$$\sum_{k=0}^{\infty} \alpha_k x^k = (1 - c\varphi(x))^{-1} \left[\|a_0\| + \sum_{j=1}^{\ell} \left(\|a_j\| - c \sum_{i=1}^{j} \|C_i\| \left\| a_{j-i} \right\| \right) \cdot x^j \right].$$

Proof. — Exercise.

Because $\varphi(0) = 0$ the series $(1 - c\varphi(x))^{-1}$ is convergent for $|x| < \rho_1$ for some $\rho_1 > 0$. The other term of the product being a polynomial, this implies that the series $\sum \alpha_k x^k$ is convergent for $|x| < \rho_1$ and this proves the proposition. □

Let us end the proof of theorem 1.1.1. Since B is solution of a linear system where the matrix of $x\partial_x$ has entries in $\mathbf{C}\{x\}$, one deduces from the previous proposition that B has entries in $\mathbf{C}\{x\}$. This gives the result when the eigenvalues of A_0 do not differ by a nonzero integer. When this is not so, one may apply the method given in lemma I–5.2.5, because the matrix which was used there has entries in K. □

We shall now give various regularity criteria for a meromorphic connection.

COROLLARY 1.1.6. — *Let $\mathcal{M}_K$ be a meromorphic connection. The following conditions are equivalent:*

1. *$\mathcal{M}_K$ is regular (i.e. isomorphic to $\mathcal{D}_K/\mathcal{D}_K \cdot P$ and $N(P)$ is a quadrant).*
2. *The set of slopes $\mathcal{P}(\mathcal{M}_K)$ is equal to $\{L_1\}$ (horizontal slope).*
3. *There exists a basis of $\mathcal{M}_K$ for which the matrix of $x\partial_x$ has entries in $\mathbf{C}\{x\}$.*
4. *There exists a basis of $\mathcal{M}_K$ for which the matrix of $x\partial_x$ is constant.*
5. *$\mathcal{M}_K$ is isomorphic to a (finite) direct sum of elementary regular meromorphic connections.*
6. *The formal meromorphic connection associated with $\mathcal{M}_K$ is regular.*

Proof. — Equivalence between 1 and 2 has been shown in §5.1 over the field $\widehat{K}$ but the proof is the same over K. That 2 implies 3 is evident, and the implication $3 \Rightarrow 4$ is given by the previous theorem. The fact that $4 \Rightarrow 5$ comes from the Jordan canonical form of a matrix with entries in $\mathbf{C}$, $5 \Rightarrow 1$ is trivial and $1 \Leftrightarrow 6$ follows from the definition of regularity. □

1.2. Regular holonomic $\mathcal{D}$-modules

Let now $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. We shall say that $\mathcal{M}$ is regular if the corresponding meromorphic connection $\mathcal{M}_K = K \times_{\mathbf{C}\{x\}} \mathcal{M} = \mathcal{M}[x^{-1}]$ is regular. Every sub-$\mathcal{D}$-module and every quotient module of a regular holonomic $\mathcal{D}$-module is so. We shall give various regularity criteria. As was stated after definition I–5.1.1 one may define the set $\mathcal{P}(\mathcal{M})$ by using the characteristic varieties $\operatorname{Car}^L(\mathcal{M})$. Hence definitions I–5.1.1 and I–5.1.5 may be extended to $\mathcal{D}$-modules of finite type. Given a holonomic $\mathcal{D}$-module we shall denote by $\mathcal{P}'(\mathcal{M}) \subset \mathcal{P}(\mathcal{M})$ the set of slopes which are not horizontal (*i.e.* $\mathcal{P}'(\mathcal{M}) = \mathcal{P}(\mathcal{M}) - \{L_1\}$ if $L_1 \in \mathcal{P}(\mathcal{M})$ and $\mathcal{P}'(\mathcal{M}) = \mathcal{P}(\mathcal{M})$ if $L_1 \notin \mathcal{P}(\mathcal{M})$).

Proposition 1.2.1. — *A holonomic $\mathcal{D}$-module $\mathcal{M}$ is regular if and only if $\mathcal{P}(\mathcal{M}) = \{L_1\}$.*

Proof. — If $\mathcal{N}$ is a torsion holonomic $\mathcal{D}$-module one verifies easily that $\mathcal{P}(\mathcal{M}) = \{L_1\}$. Indeed, it is enough to compute $\mathrm{Car}^{\,L}(\mathcal{N})$ when $\mathcal{N} = \mathcal{D}/\mathcal{D}\cdot x^\ell$. One deduces that

$$\mathcal{P}'(\mathcal{M}) = \mathcal{P}'(\mathcal{M}\left[x^{-1}\right]).$$

Since $\mathcal{M}\left[x^{-1}\right]$ is regular if and only if $\mathcal{P}'(\mathcal{M}\left[x^{-1}\right]) = \emptyset$, one gets the result. □

Proposition 1.2.2. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module isomorphic to $\mathcal{D}/I$ where I is a nonzero left ideal of $\mathcal{D}$. Let $\{P_p, \dots, P_q\}$ be a division basis of I. The following conditions are equivalent:*

1. *$\mathcal{M}$ is regular;*
2. *there exists $P \in I - \{0\}$ such that the Newton polygon $N(P)$ is a quadrant;*
3. *the Newton polygon $N(P_p)$ is a quadrant;*
4. *for each element P_j of the division basis $N(P_j)$ is a quadrant.*

Proof. — Equivalence between 1 and 3 is clear because $\mathcal{M}$ and $\mathcal{D}/\mathcal{D}\cdot P_p$ have the same localized module. That 3 implies 2 is trivial and the fact that 2 implies 1 comes from the fact that $\mathcal{M}$ is a quotient of $\mathcal{D}/\mathcal{D}\cdot P$, which is a regular module (because the non horizontal slopes $\mathcal{P}'(\mathcal{D}/\mathcal{D}\cdot P)$ are the non horizontal slopes of $N(P)$). Let us now prove that 3 implies 4. By definition, if one puts $\exp(P_j) = (\alpha_j, j)$, one can divide $x^{\alpha_{j-1}-\alpha_j}P_j$ by $P_p, \dots, P_{j-1}$ and more precisely one has a relation

$$x^{\alpha_{j-1}-\alpha_j}P_j = \left(\partial_x + u_{j,j-1}\right)P_{j-1} + u_{j,j-2}P_{j-2} + \cdots + u_{j,p}P_p$$

with $u_{j,k} \in \mathbf{C}\{x\}$. Assume by induction that $P_p, \dots, P_{j-1}$ are regular. This implies that the RHS is also regular and hence the LHS is so. This implies that P_j is regular. □

1.2.3. — Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. A *lattice* of $\mathcal{M}$ is a sub-$\mathbf{C}\{x\}$-module of finite type which generates $\mathcal{M}$ over $\mathcal{D}$. For instance, if $\mathcal{M}$ is generated over $\mathcal{D}$ by $m_1, \dots, m_r$, the $\mathbf{C}\{x\}$-module $\mathbf{C}\{x\}\cdot m_1 + \cdots + \mathbf{C}\{x\}\cdot m_r$ is a lattice. Recall that if $V\mathcal{D}$ denotes the filtration introduced in §4.2.4 one has $\mathbf{C}\{x\} \subset V_0\mathcal{D} = \mathbf{C}\{x\}\langle x\partial_x\rangle$.

Proposition 1.2.4. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. The following properties are equivalent:*

1. *$\mathcal{M}$ is regular;*
2. *there exists a lattice stable under $x\partial_x$;*
3. *there exists a sub-$V_0\mathcal{D}$-module of finite type which generates $\mathcal{M}$ and which is also of finite type over $\mathbf{C}\{x\}$;*

4. *every sub-$V_0\mathcal{D}$-module of finite type is also of finite type over* $\mathbf{C}\{x\}$.

Remark that the second property is analogous to property 3 of corollary 1.1.6.

Proof. — 2 and 3 are two formulations of the same property. Let us show that $3 \Rightarrow 4$. Let U satisfies 3 and let U' be a sub-$V_0\mathcal{D}$-module of finite type of $\mathcal{M}$. For all $k \geq 0$ put

$$U_k = U + \partial_x U + \cdots + \partial_x^k U \subset \mathcal{M}.$$

Then U_k is of finite type over $V_0\mathcal{D}$ and also over $\mathbf{C}\{x\}$. Moreover there exists k such that $U' \subset U_k$ because $\mathcal{M} = \bigcup_k U_k$. This implies that U' is of finite type over $\mathbf{C}\{x\}$.

Equivalence between 1 and 2: it is enough to see that $\mathcal{M}$ satisfies 2 if and only if $\mathcal{M}[x^{-1}]$ does so. Remark first that in an exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$, $\mathcal{M}$ satisfies 2 if and only if $\mathcal{M}'$ and $\mathcal{M}''$ do so. It is then enough to show that if $\mathcal{M}$ is a torsion module, then $\mathcal{M}$ satisfies property 2. Such a module is a finite extension of modules isomorphic to $\mathcal{D}/\mathcal{D} \cdot x^k$ for some k and property 2 is clearly satisfied for such modules (use the fact that such a module is obtained by successive extensions of modules isomorphic to $\mathcal{D}/\mathcal{D} \cdot x$). □

1.2.5. — Results of §1.1 show that the category of regular meromorphic connections (over K) is equivalent to the category of formal ones. One can show, in the same way as in theorem 6.2.5 that the category of regular holonomic $\mathcal{D}$-modules is equivalent to the category $\mathcal{C}$, hence also to the category of regular $\widehat{\mathcal{D}}$-modules.

1.3. Irregularity

We shall now give a numerical criterion for a holonomic $\mathcal{D}$-module to be regular. We shall define an index $i(\mathcal{M})$, which vanishes exactly when $\mathcal{M}$ is regular. This index behaves in an additive way in exact sequences.

1.3.1. — Let $\mathcal{M}$ and $\mathcal{N}$ be two left $\mathcal{D}$-modules. Consider the $\mathbf{C}$-vector space of $\mathcal{D}$-linear morphisms from $\mathcal{M}$ to $\mathcal{N}$, denoted by $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$. In general this space does not carry any natural structure (*e.g.* a structure of $\mathbf{C}\{x\}$ or $\mathcal{D}$-module). Remark however that if $\mathcal{N}$ admits a structure of right $\mathcal{D}$-module (for instance if $\mathcal{N} = \mathcal{D}$) then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$ comes equipped with such a structure. As we mentioned earlier, this space $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$ is the space of solutions of $\mathcal{M}$ in $\mathcal{N}$. If $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$ with $P \in \mathcal{D}$ then

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) = \mathrm{Ker}\, [P\cdot : \mathcal{N} \longrightarrow \mathcal{N}]\,.$$

Given an operator P, one is interested not only on solutions of $P \cdot u = 0$ with $u \in \mathcal{N}$ but also on solutions of $P \cdot u = f$ for a given $f \in \mathcal{N}$. The cokernel of left multiplication by P in $\mathcal{N}$ is the set of (classes of) elements f in $\mathcal{N}$ for which the equation $P \cdot u = f$ has no solution in $\mathcal{N}$. This is a first explanation of why one is interested not only on the kernel but also on the cokernel of $P : \mathcal{N} \to \mathcal{N}$.

Here is another explanation. Let $\mathcal{N} = \mathbf{C}\{x\}$ and consider the operator $P = x$, corresponding to the torsion module $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot x$. Then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) = 0$ because (left) multiplication $x : \mathbf{C}\{x\} \to \mathbf{C}\{x\}$ is injective. However the cokernel is isomorphic to $\mathbf{C}$ and the class of $1 \in \mathbf{C}\{x\}$, denoted by δ, satisfies then the equation $x \cdot \delta = 0$. This class plays the role of the Dirac distribution at the origin. We have then introduced in the theory an object analogous to the Dirac distribution, keeping inside the holomorphic frame. This leads us to the following

Definition 1.3.2. — *Let $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$ and $\mathcal{N}$ be a left $\mathcal{D}$-module. The solution complex of $\mathcal{M}$ in $\mathcal{N}$ is the two term complex*

$$\mathcal{N} \xrightarrow{P\cdot} \mathcal{N}$$

where the differential is equal to left multiplication by P. The solutions of P in $\mathcal{N}$ are the two cohomology groups of this complex, namely $\mathrm{Ker}\, P$ *and* $\mathrm{Coker}\, P$.

1.3.3. — We shall now generalize this definition to the case of a left $\mathcal{D}$-module of finite type. In order to do that, we shall first recall some facts of homological algebra. Let $\mathcal{M}$ be such a module. Then $\mathcal{M}$ admits a *resolution* by free $\mathcal{D}$-modules:

$$\cdots \xrightarrow{\varphi_i} \mathcal{D}^{p_i} \xrightarrow{\varphi_{i-1}} \mathcal{D}^{p_1} \xrightarrow{\varphi_0} \mathcal{D}^{p_0} \longrightarrow \mathcal{M} \longrightarrow 0$$

where the φ_i are left $\mathcal{D}$-linear morphisms (hence are given by right multiplication by a matrix with entries in $\mathcal{D}$) and satisfy $\varphi_{i+1} \circ \varphi_i = 0$ and moreover $\mathrm{Ker}\, \varphi_i = \mathrm{Im}\, \varphi_{i+1}$. Recall that this resolution is constructed step by step by taking generators of $\mathrm{Ker}\, \varphi_i$ at each step. With such a complex one may construct a new complex:

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{p_0}, \mathcal{N}) \xrightarrow{\psi_0} \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{p_1}, \mathcal{N}) \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{i-1}} \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{p_i}, \mathcal{N}) \xrightarrow{\psi_i} \cdots \tag{1.3.4}$$

where ψ_i is the mapping obtained by composing with φ_i. One has $\mathrm{Ker}\, \psi_0 = \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$. The cohomology of this last complex is denoted by $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$. One then has

$$\begin{aligned} \mathrm{Ext}^0_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) &= \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) \\ \mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) &= \mathrm{Ker}\, \psi_i / \mathrm{Im}\, \psi_{i-1} \qquad \text{for } i \geq 1 \end{aligned}$$

A short exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}'', \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}', \mathcal{N}) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}'', \mathcal{N}) \longrightarrow \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}', \mathcal{N}) \longrightarrow \cdots \end{aligned}$$

Proposition 1.3.5. — *Let $\mathcal{M}$ be a left $\mathcal{D}$-module of finite type. Then we have* $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) = 0$ *for $i \geq 2$ and every left $\mathcal{D}$-module $\mathcal{N}$.*

Proof. — Given a short exact sequence as before, if the result is true for $\mathcal{M}'$ and $\mathcal{M}''$, it is also true for $\mathcal{M}$, because of the long exact sequence above. Since $\mathcal{M}$ is obtained by successive extensions of $\mathcal{D}$-module isomorphic to $\mathcal{D}/I$, it is enough to prove the result for such modules.

If $I = \{0\}$ then $\mathcal{M}$ is free and $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) = 0$ for $i \geq 1$. If $I \neq \{0\}$ then $\mathcal{M}$ is holonomic. If I is generated by one element P, then $\mathcal{M} \simeq \mathcal{D}/\mathcal{D} \cdot P$ and $\mathcal{M}$ admits a resolution

$$0 \longrightarrow \mathcal{D} \xrightarrow{\cdot P} \mathcal{D} \longrightarrow \mathcal{M} \longrightarrow 0$$

hence the result is true. If $\mathcal{M}$ is a torsion module, then $\mathcal{M}$ is obtained by successive extensions of modules isomorphic to $\mathcal{D}/\mathcal{D} \cdot x^k$ hence the result is also true for such modules. In general one uses the two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M}[x^{-1}] & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \end{array}$$

where $\mathcal{M}_1 = \mathrm{Image}\,(\mathcal{M} \to \mathcal{M}[x^{-1}])$. Since $\mathcal{M}[x^{-1}]$ is isomorphic to $\mathcal{D}/\mathcal{D} \cdot P$ for some P and since $\mathcal{C}$ is a torsion module, the result is true for $\mathcal{M}_1$ (use the long exact sequence above). Since $\mathcal{K}$ is a torsion module, the result is then true for $\mathcal{M}$. □

1.3.6. Exercise.

1. Let $\mathcal{M}$ be a left $\mathcal{D}$-module of finite type.

   (a) Show that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ can be equipped with a natural structure of *right* $\mathcal{D}$-module and that it is then of finite type over $\mathcal{D}$.

   (b) Same question for $\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ (one can use in both cases a free presentation of $\mathcal{M}$).

   (c) Show that if $\mathcal{M}$ is holonomic, one has $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = \{0\}$.

2. Let $\mathcal{N}$ be a right $\mathcal{D}$-module of finite type. Show that if one puts

$$\begin{aligned} \partial_x \cdot n &\overset{\text{def}}{=} n \cdot (-\partial_x) \\ x \cdot n &\overset{\text{def}}{=} n \cdot x \end{aligned}$$

   for each $n \in \mathcal{N}$, one defines a structure of *left* $\mathcal{D}$-module on $\mathcal{N}$, and in that way $\mathcal{N}$ becomes a left $\mathcal{D}$-module of finite type.

3. If $\mathcal{M}$ is a left holonomic $\mathcal{D}$-module, one denotes by $\mathcal{M}^*$ the left $\mathcal{D}$-module associated with $\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$. Show that if

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

   is an exact sequence of holonomic left $\mathcal{D}$-modules, then the (dual) sequence

$$0 \longrightarrow \mathcal{M}''^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{M}'^* \longrightarrow 0$$

   is also exact.

4. If $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$ with $P \neq 0$, Show that there exists an operator P^* such that $\mathcal{M}^* = \mathcal{D}/\mathcal{D}\cdot P^*$. Compute it in terms of P. Show that P and P^* have same Newton polygon.

5. Show that if $\mathcal{M}$ is a torsion module, then $\mathcal{M}^*$ also. Give an example for which $\mathcal{M}$ is equal to its localized module, but not $\mathcal{M}^*$.

Definition 1.3.7. — *The complex* (1.3.4) *is called the solution complex of $\mathcal{M}$ in $\mathcal{N}$. The cohomology groups of this complex (namely* $\operatorname{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ *and* $\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ *if $\mathcal{M}$ is of finite type) are the solutions spaces of $\mathcal{M}$ in $\mathcal{N}$.*

Remark. — As notation brings in evidence, the groups $\operatorname{Ext}^i_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ do not depend on the choice of a resolution of $\mathcal{M}$ by free $\mathcal{D}$-modules. However the solution complex of $\mathcal{M}$ in $\mathcal{N}$ does depend on such a resolution. What is independent of such a resolution is some equivalence class of complexes in a derived category. We shall not use this here.

The dimension of the usual solution space of $P\cdot u = 0$ in $\mathcal{N}$ is given by $\dim\operatorname{Ker}\,[P:\mathcal{N}\to\mathcal{N}]$. With the previously introduced notion of solutions, the dimension becomes an index:

Definition 1.3.8. — *Let $\mathcal{M}$ be a left $\mathcal{D}$-module of finite type and $\mathcal{N}$ be a left $\mathcal{D}$-module. If the spaces of solutions* $\operatorname{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ *and* $\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M},\mathcal{N})$ *have finite dimension over* $\mathbf{C}$*, we say that the index of $\mathcal{M}$ in $\mathcal{N}$ is defined and we put*

$$\chi(\mathcal{M},\mathcal{N}) = \dim_{\mathbf{C}}\operatorname{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{N}) - \dim_{\mathbf{C}}\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M},\mathcal{N}).$$

From the long exact sequence above one deduces

Proposition 1.3.9. — *Let $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ be a short exact sequence of left $\mathcal{D}$-module of finite type. If $\chi(\mathcal{M}',\mathcal{N})$ and $\chi(\mathcal{M}'',\mathcal{N})$ are defined, then so is $\chi(\mathcal{M},\mathcal{N})$ and one has*

$$\chi(\mathcal{M},\mathcal{N}) = \chi(\mathcal{M}',\mathcal{N}) + \chi(\mathcal{M}'',\mathcal{N}). \quad \square$$

Remark. — Given a short exact sequence $0\to\mathcal{N}'\to\mathcal{N}\to\mathcal{N}''\to 0$ one has an analogous statement

$$\chi(\mathcal{M},\mathcal{N}) = \chi(\mathcal{M},\mathcal{N}') + \chi(\mathcal{M},\mathcal{N}'')$$

if both right terms are defined. This is proven by using the corresponding (covariant) long exact sequence.

In the following we shall be interested on the indices of $\mathcal{M}$ in one of the following left $\mathcal{D}$-module: $\mathbf{C}\{x\}$, $\mathbf{C}[\![x]\!]$, $\mathbf{C}[\![x]\!]/\mathbf{C}\{x\}$, K, $\widehat{K}$, $\widehat{K}/K$. The principal result is the following

Theorem 1.3.10. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module.*

1. *The indices of $\mathcal{M}$ in one of the above $\mathcal{D}$-modules are defined.*
2. $\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}) = 0$ *(hence $\chi(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}) \geq 0$).*
3. $\operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M}, \widehat{K}/K) = 0$ *(hence $\chi(\mathcal{M}, \widehat{K}/K) \geq 0$) and*
$$\chi(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}) = \chi(\mathcal{M}, \widehat{K}/K).$$
4. $$\chi(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}) = \chi(\mathcal{M}[x^{-1}], \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}).$$
5. *$\mathcal{M}$ is regular if and only if $\chi(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}) = 0$.*

The (Malgrange-Komatsu) irregularity of $\mathcal{M}$ is the number
$$i(\mathcal{M}) = \chi(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}).$$

Proof. — Consider first the case where $\mathcal{M}$ is a torsion module. It is enough to consider the case where $\mathcal{M} \simeq \mathcal{D}/\mathcal{D}\cdot x^k$ for some $k \geq 0$. The solution complex in $\mathbf{C}\{x\}$ is then equal to
$$\mathbf{C}\{x\} \xrightarrow{x^k} \mathbf{C}\{x\}$$
hence $\operatorname{Hom}_{\mathcal{D}}(\mathcal{M}, \mathbf{C}\{x\}) = 0$ and $\dim_{\mathbf{C}} \operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathbf{C}\{x\}) = k$. The same result holds for $\mathbf{C}[\![x]\!]$. This implies that $i(\mathcal{M}) = 0$. Moreover solutions in K or $\widehat{K}$ are zero because $x^k : K \to K$ is bijective. All the statements of the theorem are then satisfied for such modules.

We shall now prove the theorem when $\mathcal{M} = \mathcal{M}[x^{-1}]$. This will imply that statements 1, 2, 3 and 4 are true in general, by using the exact sequence of localization and the result for torsion modules. The statement 5 will then be obtained using 4 and the local case. We may now assume that $\mathcal{M} \simeq \mathcal{D}/\mathcal{D}\cdot P$ for some P. We may write $P = b(x\partial_x) + xP'$ with $P' \in V_0/cD$ and $b(k) \neq 0$ for $k \in \mathbf{N}$ (lemma I–4.2.7). We may also write $P = \sum_{i=0}^d a_i(x)(x\partial_x)^i$ with $a_d \neq 0$ and at least one of the coefficients a_i is a unit. We shall show the following properties:

1. The index of $P : \mathbf{C}\{x\} \to \mathbf{C}\{x\}$ is defined and $\chi(\mathcal{M}, \mathbf{C}\{x\}) = -v(a_d)$.
2. The index of $P : K \to K$ is defined and $\chi(\mathcal{M}, K) = -v(a_d)$.
3. The kernel and cokernel of $P : \mathbf{C}[\![x]\!] \to \mathbf{C}[\![x]\!]$ are zero, hence we have $\chi(\mathcal{M}, \mathbf{C}[\![x]\!]) = 0$.
4. The index of $P : \widehat{K} \to \widehat{K}$ is defined and $\chi(\mathcal{M}, \widehat{K}) = 0$.

These results imply that $i(\mathcal{M}) = v(a_d)$ and this is enough to prove the theorem. Before going further, it is interesting to give the geometrical meaning of $i(\mathcal{M})$ with the Newton polygon $N(P)$ (*see fig.* 8).

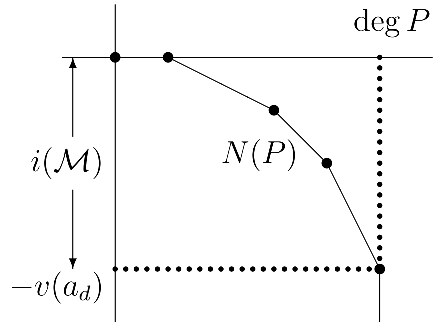

Figure 8

We shall first show assertions (c) and (d). One has $(x\partial_x)^i(x^\ell) = \ell^i x^\ell$ for all $\ell \in \mathbf{Z}$ hence

$$P(x^\ell) = \left(\sum_{i=0}^{d} a_i(x)\ell^i\right) x^\ell.$$

We may then write $P(x^\ell) = b(\ell)x^\ell + \text{terms of order} > \ell$. If $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\mathbf{C}[\![x]\!]$, one deduces that for all $\ell \geq 0$ P induces an isomorphism

$$\widehat{\mathfrak{m}}^\ell/\widehat{\mathfrak{m}}^{\ell+1} \xrightarrow{P} \widehat{\mathfrak{m}}^\ell/\widehat{\mathfrak{m}}^{\ell+1}.$$

Consequently P induces an isomorphism

$$\mathbf{C}[\![x]\!]/\widehat{\mathfrak{m}}^\ell \longrightarrow \mathbf{C}[\![x]\!]/\widehat{\mathfrak{m}}^\ell$$

for all $\ell \geq 0$, and assertion (c) is obtained by taking the projective limit. Let us prove (d) in the same way: it is enough to show that

$$P : \widehat{K}/\mathbf{C}[\![x]\!] \longrightarrow \widehat{K}/\mathbf{C}[\![x]\!]$$

has an index and that this index is zero. Filter the quotient $\widehat{K}/\mathbf{C}[\![x]\!]$ by the images of $x^\ell\mathbf{C}[\![x]\!]$ for $\ell \leq 0$. Then P is compatible with this filtration as we have yet seen and induces multiplication by $b(\ell)$ on the graded part of order ℓ. So for $\ell \ll 0$ P induces an isomorphism on this graded part. This implies that the kernel and cokernel of $P : \widehat{K}/\mathbf{C}[\![x]\!] \to \widehat{K}/\mathbf{C}[\![x]\!]$ are equal to the kernel and cokernel of

$$P : x^\ell\mathbf{C}[\![x]\!]/\mathbf{C}[\![x]\!] \longrightarrow x^\ell\mathbf{C}[\![x]\!]/\mathbf{C}[\![x]\!]$$

for some $\ell \ll 0$. Because $x^\ell\mathbf{C}[\![x]\!]/\mathbf{C}[\![x]\!]$ has finite dimension, the kernel and cokernel have also finite dimension and these dimensions are equal, which proves (d).

Now (b) can be deduced from (a), (c) and (d): indeed, one has

$$K/\mathbf{C}\{x\} \simeq \widehat{K}/\mathbf{C}[\![x]\!]$$

hence $\chi(\mathcal{M}, K/\mathbf{C}\{x\})$ is defined and is equal to 0. Consequently, assuming (a), $\chi(\mathcal{M}, K)$ is defined and is equal to $\chi(\mathcal{M}, \mathbf{C}\{x\})$.

We shall now show statement (a). Remark first that

$$\operatorname{Ker}\,[P : \mathbf{C}\{x\} \longrightarrow \mathbf{C}\{x\}] = \{0\}$$

because this is true when $\mathbf{C}\{x\}$ is replaced by $\mathbf{C}[\![x]\!]$. We are then reduced to show that

$$\dim_{\mathbf{C}} \operatorname{Coker}\,[P : \mathbf{C}\{x\} \longrightarrow \mathbf{C}\{x\}] = v(a_d).$$

Let Δ_r be a closed disk in $\mathbf{C}$ centered at zero and with radius r. Consider the space $B^m(\Delta_r)$ of functions which are $\mathcal{C}^\infty$ in some open neighborhood of Δ_r and which are holomorphic in the interior of Δ_r. This is a Banach space for the norm

$$\|f\|_m = \sup_{|\alpha|\leq m} \sup_{\Delta_r} \left| \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} u \partial^{\alpha_2} v} \right|$$

where we have put $x = u + iv$, $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$. We shall use the following results:

Proposition 1.3.11. — *For all $m \geq 0$ the injection $B^{m+1}(\Delta_r) \hookrightarrow B^m(\Delta_r)$ is compact.* □

Theorem 1.3.12. — *Let $U, V : E \to F$ be continuous linear operators between two Banach spaces. If the index of U is defined and if V is compact then the index of $U + V$ is defined and is equal to the index of V.* □

Now P defines a continuous operator $B^d(\Delta_r) \to B^0(\Delta_r)$ for each $r > 0$ sufficiently small. We shall prove that this operator has an index, and that this index is equal to $-v(a_d)$. Remark that $P = a_d(x)(x\partial_x)^d + Q$ with $\deg Q < d$. Because of the previous proposition, Q induces a compact operator $B^d(\Delta_r) \to B^0(\Delta_r)$, and we are reduced to show that the index of $a_d(x)(x\partial_x)^d$ is equal to $-v(a_d)$. This comes from the fact that ∂_x has index 1 and x has index -1. □

1.3.13. Exercise. — Let $P = x^2\partial_x + 1$.

1. Find a formal solution f of the equation

$$P \cdot f = x$$

   and show that $f \notin \mathbf{C}\{x\}$.

2. Let $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot P$. Compute

$$\dim \operatorname{Hom}_{\mathcal{D}}(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\}) \qquad \text{and} \qquad \dim \operatorname{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathbf{C}[\![x]\!]/\mathbf{C}\{x\})$$

   and give a basis of each of these vector spaces.

1.4. The local index theorem

We shall formulate the previous results in a form which can be generalized directly for holonomic $\mathcal{D}$-modules in more than one variable. So let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. We have defined earlier the characteristic variety $\operatorname{Car}\mathcal{M}$ as a subset of T (cotangent space to the germ $(\mathbf{C},0)$ with coordinates (x,ξ)). Recall that $\operatorname{Car}\mathcal{M}$ is contained in the union of the two subvarieties $\{x=0\}$ and $\{\xi=0\}$. We shall now define the *characteristic cycle* $\operatorname{Ch}\mathcal{M}$: in order to do that we shall associate with each component of $\operatorname{Car}\mathcal{M}$ a multiplicity.

Let $F\mathcal{M}$ be a good filtration of $\mathcal{M}$. Consider the graded module $\operatorname{gr}^F\mathcal{M}$ on the graded ring $\operatorname{gr}^F\mathcal{D}=\mathbf{C}\{x\}[\xi]$. We shall localize at the generic point of the x-axis. Recall some properties of localization at a prime ideal $\mathfrak{p}$ of a commutative ring A (*see* the references in commutative algebra): $A_\mathfrak{p}$ denotes the ring obtained from A by inverting all elements in A which are not in $\mathfrak{p}$. Then $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$. If M is a A-module, one can define in the same way the module $M_\mathfrak{p}$ and one has $M_\mathfrak{p}\simeq A_\mathfrak{p}\otimes_A M$. Consider now the prime ideal (ξ) in $\operatorname{gr}^F\mathcal{D}$ and the localized module $\operatorname{gr}^F\mathcal{M}_{(\xi)}$ over rhe localized ring $\operatorname{gr}^F\mathcal{D}_{(\xi)}$. We may then define as in §3.4 the multiplicity of this module at the origin of the local ring (*i.e.* by changing the module $\operatorname{gr}^F\mathcal{M}$ by $\operatorname{gr}^F\mathcal{M}_{(\xi)}$ and the ideal (x,ξ) by the ideal $(\xi)\operatorname{gr}^F\mathcal{D}_{(\xi)}$). We shall denote by $e_{\{\xi=0\}}(\mathcal{M})$ this multiplicity. In an analogous way one defines the multiplicity $e_{\{x=0\}}(\mathcal{M})$. The *characteristic cycle* $\operatorname{Ch}\mathcal{M}$ will now be defined as the formal linear combination

$$\operatorname{Ch}\mathcal{M}=e_{\{\xi=0\}}(\mathcal{M})\cdot\{\xi=0\}+e_{\{x=0\}}(\mathcal{M})\cdot\{\xi=0\}\,.$$

One can show the following properties of the characteristic cycle, using results on multiplicities given in §3.4 and the fact that $A_\mathfrak{p}$ is flat over A:

1. $e_{\{x=0\}}(\mathcal{M})$ and $e_{\{\xi=0\}}(\mathcal{M})$ do not depend on the choice of the good filtration used to define them.

2. $e_{\{x=0\}}(\mathcal{M})$ and $e_{\{\xi=0\}}(\mathcal{M})$ are non negative integers, not both equal to zero if $\mathcal{M}\neq 0$ (Bernstein inequality).

3. In an exact sequence $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ of holonomic $\mathcal{D}$-modules one has
$$\operatorname{Ch}\mathcal{M}=\operatorname{Ch}\mathcal{M}'+\operatorname{Ch}\mathcal{M}''$$
equality which means that $e_{\{x=0\}}(\mathcal{M})=e_{\{x=0\}}(\mathcal{M}')+e_{\{x=0\}}(\mathcal{M}'')$ and the same for $e_{\{\xi=0\}}(\mathcal{M})$.

Definition 1.4.1. — *The algebraic index of $\mathcal{M}$, denoted by $\chi_{\mathrm{alg}}(\mathcal{M})$ is the index of its characteristic cycle, namely*

$$\chi_{\mathrm{alg}}(\mathcal{M})=\chi(\operatorname{Ch}\mathcal{M})\overset{\mathrm{def}}{=}e_{\{\xi=0\}}(\mathcal{M})-e_{\{x=0\}}(\mathcal{M}).$$

The algebraic index is an additive function (in exact sequences). This index can take positive or negative values.

Examples.

1. If $\mathcal{M}$ is of finite type over $\mathbf{C}\{x\}$ then $\mathcal{M}$ is $\mathbf{C}\{x\}$-free and $e_{\{x=0\}}(\mathcal{M}) = 0$, $e_{\{\xi=0\}}(\mathcal{M}) = \mathrm{rank}_{\mathbf{C}\{x\}}\mathcal{M}$.

2. If $\mathcal{M}$ is a torsion module, $e_{\{\xi=0\}}(\mathcal{M}) = 0$ and $e_{\{x=0\}}(\mathcal{M}) \neq 0$.

3. If $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$ with $P \in \mathcal{D}$, $P = \sum_{i=0}^d a_i(x)(x\partial_x)^i$, $a_d \neq 0$ then $\chi_{\mathrm{alg}}(\mathcal{M}) = -v(a_d)$. Indeed one has

$$\mathrm{gr}^F\mathcal{M} \simeq \mathbf{C}\{x\}\,[\xi]/\mathbf{C}\{x\}\,[\xi]x^v(x\xi)^d$$

with $v = v(a_d)$. Hence $e_{\{\xi=0\}}(\mathcal{M}) = d$ and $e_{\{x=0\}}(\mathcal{M}) = v + d$.

We can now formulate the *local index theorem*:

Theorem 1.4.2. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. One has the equality*

$$\chi(\mathcal{M}, \mathbf{C}\{x\}) = \chi_{\mathrm{alg}}(\mathcal{M})(= \chi(\mathrm{Ch}\,\mathcal{M})).$$

Proof. — It was essentially done in the previous section. Indeed, since both terms behave in an additive way in exact sequences, it is enough to prove the equality when $\mathcal{M} = \mathcal{M}[x^{-1}]$ or when $\mathcal{M}$ is a torsion module. In the first case, both terms are equal to $-v(a_d)$ and in the second case one verifies that both terms are equal to $-k$ when $\mathcal{M} \simeq \mathcal{D}/\mathcal{D}\cdot x^k$. □

1.4.3. Index of a $\widehat{\mathcal{D}}$-module. — It follows from the previous computations that the index of a holonomic $\mathcal{D}$-module depends only on its associated formalized module $\widehat{\mathcal{M}} \overset{\mathrm{def}}{=} \mathbf{C}[\![x]\!] \otimes_{\mathbf{C}\{x\}} \mathcal{M}$. Indeed, if $F\mathcal{M}$ is a good filtration of $\mathcal{M}$ one may verify that the filtration $F\widehat{\mathcal{M}} \overset{\mathrm{def}}{=} \mathbf{C}[\![x]\!] \otimes_{\mathbf{C}\{x\}} F\mathcal{M}$ is a good filtration of $\widehat{\mathcal{M}}$ and that $\mathrm{gr}^F\widehat{\mathcal{M}} \simeq \mathbf{C}[\![x]\!] \otimes_{\mathbf{C}\{x\}} \mathrm{gr}^F\mathcal{M}$. This implies that the associated multiplicities are the same for $\mathcal{M}$ and for $\widehat{\mathcal{M}}$. Consequently one has $\chi_{\mathrm{alg}}(\widehat{\mathcal{M}}) = \chi_{\mathrm{alg}}(\mathcal{M})$ (if for instance $\mathcal{M} = \mathcal{D}/\mathcal{D}\cdot P$ as above, then $\widehat{\mathcal{M}} = \widehat{\mathcal{D}}/\widehat{\mathcal{D}}\cdot P$ and the two numbers are equal to $-v(a_d)$).

If now $\widehat{\mathcal{M}}$ is a holonomic $\widehat{\mathcal{D}}$-module (which does not necessarily come from a holonomic $\mathcal{D}$-module) on may define the algebraic index of $\widehat{\mathcal{M}}$ by $\chi_{\mathrm{alg}}(\widehat{\mathcal{M}}) = \chi(\mathrm{Ch}\,\widehat{\mathcal{M}})$. However the index of holomorphic solutions does not have any meaning because $\mathbf{C}\{x\}$ is not a $\widehat{\mathcal{D}}$-module. Moreover the index of formal solutions is not very interesting because if for instance $\widehat{\mathcal{M}}$ is a connection, this index is equal to 0.

1.4.4. Algebraic computation of irregularity. — We have seen how to compute the algebraic index when $\mathcal{M}$ is a connection or a torsion module. We shall give a general formula for this index. Let us begin with a formula for multiplicity.

Proposition 1.4.5. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module (or a holonomic $\widehat{\mathcal{D}}$-module). Let $F\mathcal{M}$ be a good filtration of $\mathcal{M}$. Then for k sufficiently large*

1. *the graded part $F_k\mathcal{M}/F_{k-1}\mathcal{M}$ is a finite dimensional $\mathbf{C}$-vector space,*
2. *the induced mapping $\partial_x : F_k\mathcal{M}/F_{k-1}\mathcal{M} \to F_{k+1}\mathcal{M}/F_k\mathcal{M}$ is bijective,*
3. *the dimension of $F_k\mathcal{M}/F_{k-1}\mathcal{M}$ (which does not depend on k) does not depend on the good filtration $F\mathcal{M}$ and behaves in an additive way in exact sequences. This dimension is equal to $e_{\{x=0\}}(\mathcal{M})$.*

Proof. — Let $F\mathcal{M}$ be a good filtration of $\mathcal{M}$. Then $F_k\mathcal{M}[x^{-1}]$ is an increasing filtration of K-vector spaces in $\mathcal{M}[x^{-1}]$. Since this one has finite dimension over K, one deduces that there exists k_0 such that for $k \geq k_0$ one has $F_k\mathcal{M}[x^{-1}] = F_{k-1}\mathcal{M}[x^{-1}]$. This implies that for such k the quotient $F_k\mathcal{M}/F_{k-1}\mathcal{M}$ is a $\mathbf{C}\{x\}$-torsion module, hence a finite dimensional vector space. This proves the first assertion.

The second is proven by using the fact that if $F\mathcal{M}$ is a good filtration, there exists k_1 such that for $k \geq k_1$ one has $F_k\mathcal{M} = F_{k-1}\mathcal{M} + \partial_x \cdot F_{k-1}\mathcal{M}$ (hence the mapping induced by ∂_x is onto for $k \geq k_1$ and bijective for k sufficiently large, because of the first assertion).

In order to prove the third assertion, it is enough to compare the dimensions for two good filtrations $F\mathcal{M}$ and $G\mathcal{M}$ satisfying for all k

$$F_k\mathcal{M} \subset G_k\mathcal{M} \subset F_{k+\ell_0}\mathcal{M}$$

for some $\ell_0 \geq 0$ and which satisfy the first two assertions for all $k \geq 0$ (a shift in a filtration does not affect the asymptotic value of the dimension of the graded parts). Put $P(F\mathcal{M}/F_0\mathcal{M}, k) = \dim_{\mathbf{C}} F_k\mathcal{M}/F_0\mathcal{M}$. With the previous assumptions, one has

$$P(F\mathcal{M}/F_0\mathcal{M}, k) = d_F \cdot (k-1)$$

where $d_F = \dim_{\mathbf{C}} F_k/F_{k-1}$. We see that $P(F\mathcal{M}/F_0\mathcal{M}, k)$ is a degree one polynomial in k, with dominating coefficient equal to d_F. In the same way $P(G\mathcal{M}/G_0\mathcal{M}, k)$ satisfies the same property, with dominating coefficient equal to d_G. We have assumed that $F_0\mathcal{M}[x^{-1}] = \mathcal{M}[x^{-1}] = G_0\mathcal{M}[x^{-1}]$, so $\dim_{\mathbf{C}} G_0/F_0 < \infty$. Consequently $P(G\mathcal{M}/F_0\mathcal{M}, k)$ satisfies the same property with dominating coefficient d_G. Moreover one has

$$P(F\mathcal{M}/F_0\mathcal{M}, k) \leq P(G\mathcal{M}/F_0\mathcal{M}, k) \leq P(F\mathcal{M}/F_0\mathcal{M}, k+\ell_0)$$

hence the dominating terms of these polynomials are equal.

The additive behaviour in exact sequences is now clear, by taking the filtrations induced by the central one.

Let us compute the number that we have obtained. If $\mathcal{M}$ is a torsion module, for instance if $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot x^k$, this number is equal to k (take the natural filtration on this quotient to compute it). If $\mathcal{M} = \mathcal{M}[x^{-1}] = \mathcal{D}/\mathcal{D} \cdot P$ with $P \in V_0\mathcal{D} - V_{-1}\mathcal{D}$ then this number is equal to $v(a_d) + d$ (same computation). This ends the proof of the proposition. □

In order to obtain in a similar way the index of $\mathcal{M}$, we shall start with a sub-$V_0\mathcal{D}$-module of finite type $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{D} \cdot \mathcal{N} = \mathcal{M}$ (for instance, if $U\mathcal{M}$ is a filtration good for $V\mathcal{M}$ one can take $\mathcal{N} = U_k\mathcal{M}$ for k large enough). The ring $V_0\mathcal{D}$ may be identified with $\mathbf{C}\{x\}[x\partial_x]$. Consider on this ring the filtration $FV_0\mathcal{D}$ by the degree with respect to $x\partial_x$. One can define the notion of a good filtration $F\mathcal{N}$.

Proposition 1.4.6. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module and $\mathcal{N}$ be a sub-$V_0\mathcal{D}$-module which generates $\mathcal{M}$ over $\mathcal{D}$ (similar statement over $\widehat{\mathcal{D}}$). The number* $\dim_{\mathbf{C}} F_k\mathcal{N}/F_{k-1}\mathcal{N}$ *is finite and independent of k for large k, independent of the good filtration $F\mathcal{N}$ and independent of the sub-$V_0\mathcal{D}$-module $\mathcal{N}$ (which generates $\mathcal{M}$). It is additive in exact sequences and is in fact equal to the irregularity $i(\mathcal{M})$.*

Proof. — For fixed $\mathcal{N}$ the proof is identical to the previous one (one uses the mapping induced by $x\partial_x$ instead of the one induced by ∂_x). The proof that the number is the same for two sub-$V_0\mathcal{D}$-modules generating $\mathcal{M}$ is left as an exercise. Additivity in exact sequences is an easy consequence. So one is reduced to compute the number for torsion modules (where it is equal to 0, because $F_k\mathcal{N}$ is constant for large k for $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot x^\ell$) and for connections (where one can take the natural sub-$V_0\mathcal{D}$-module $\mathcal{N} = V_0\mathcal{D}/V_0\mathcal{D} \cdot P$ since $P \in V_0/\mathcal{D}$ to obtain $v(a_d)$). □

Remark. — We have obtained another proof of the fact that $\mathcal{M}$ is regular if and only if any such $\mathcal{N}$ is of finite type over $\mathbf{C}\{x\}$, because $i(\mathcal{M}) = 0$ is then equivalent to the fact that $F_k\mathcal{N} = \mathcal{N}$ for large k.

2. Analytic structure of meromorphic connections

Given a meromorphic connection $\mathcal{M}_K$, the structure of the corresponding formal connection $\mathcal{M}_{\widehat{K}} = \widehat{K} \otimes_K \mathcal{M}_K$ is essentially known: after a convenient ramification, the connection $\mathcal{M}_{\widehat{L}}$ can be decomposed into a direct sum of elementary formal connections. It may happen (*see* exercise I–5.3.6) that this splitting cannot be lifted at the analytic level. The main result of this section is that, for any half-line in the complex plane, starting from the origin, intersecting the unit circle in a point θ, this splitting can be lifted if the coefficients are germs at the origin of holomorphic functions defined in some small sectorial domain around this line. In order to prove this, we shall construct first an analytic model $\mathcal{M}^1_K$ of $\mathcal{M}_{\widehat{K}}$ and we shall try to lift in such sectors the formal isomorphism $\mathcal{M}_{\widehat{K}} \xrightarrow{\sim} \mathcal{M}^1_{\widehat{K}}$. We shall first explain some results about morphisms betwenn two connections.

2.1. Morphisms between two connections

Let $\mathcal{M}_K$ and $\mathcal{N}_K$ be two meromorphic connections.

PROPOSITION 2.1.1. — *$\mathrm{Hom}_K(\mathcal{M}_K, \mathcal{N}_K)$ may be naturally equipped with a structure of a meromorphic connection. Let $\varphi \in \mathrm{Hom}_K(\mathcal{M}_K, \mathcal{N}_K)$. Then φ is left $\mathcal{D}$-linear if and only if $\partial_x \cdot \varphi = 0$.*

Proof. — $\mathrm{Hom}_K(\mathcal{M}_K, \mathcal{N}_K)$ is a finite dimensional K-vector space. One defines the action of ∂_x on this space by

$$(\partial_x \varphi)(m) = \partial_x[\varphi(m)] - \varphi(\partial_x m)$$

for all $m \in \mathcal{M}_K$ and one verifies easily that this defines a structure of a meromorphic connection. It is then clear that

$$\partial_x \varphi = 0 \qquad \Longleftrightarrow \qquad \forall m \in \mathcal{M}_K, \quad \partial_x[\varphi(m)] = \varphi(\partial_x m). \quad \square$$

The proof of the following proposition is left as an exercise.

PROPOSITION 2.1.2. — *Let $\mathcal{F}_K^R$ be an elementary formal connection (*see §I–5.4.4*). Then the K-linear isomorphism*

$$\begin{aligned} \mathrm{Hom}_K(\mathcal{F}_K^R \otimes_K \mathcal{M}_K, \mathcal{N}_K) &\longrightarrow \mathrm{Hom}_K(\mathcal{M}_K, \mathcal{F}_K^{-R} \otimes_K \mathcal{N}_K) \\ \varphi &\longmapsto e(-R(1/x)) \cdot \varphi \cdot e(R(1/x)) \end{aligned}$$

is left $\mathcal{D}$-linear. $\square$

PROPOSITION 2.1.3.

$$\widehat{K} \otimes_K \mathrm{Hom}_K(\mathcal{M}_K, \mathcal{N}_K) = \mathrm{Hom}_{\widehat{K}}(\mathcal{M}_{\widehat{K}}, \mathcal{N}_{\widehat{K}}). \quad \square$$

2.1.4. Exercises.

1. Show that

$$\begin{aligned} \mathrm{Ker}\, \partial_x : \mathrm{Hom}_K\left(\mathcal{D}_K/\mathcal{D}_K \cdot (x\partial_x - \alpha)^p, \mathcal{D}_K/\mathcal{D}_K \cdot (x\partial_x - \beta)^q\right) \longrightarrow \\ \longrightarrow \mathrm{Hom}_K\left(\mathcal{D}_K/\mathcal{D}_K \cdot (x\partial_x - \alpha)^p, \mathcal{D}_K/\mathcal{D}_K \cdot (x\partial_x - \beta)^q\right) \end{aligned}$$

   is zero if $\alpha - \beta \notin \mathbf{Z}$ and is a $\mathbf{C}$-vector space of dimension $\inf(p, q)$ otherwise.

2. Take the notations of §I–5.4.4. Show that

$$\mathrm{Ker}\, \partial_x : \mathrm{Hom}_K\left(\mathcal{F}_K^R \otimes \mathcal{G}_K, \mathcal{F}_K^{R'} \otimes \mathcal{G}'_K\right) \to \mathrm{Hom}_K\left(\mathcal{F}_K^R \otimes \mathcal{G}_K, \mathcal{F}_K^{R'} \otimes \mathcal{G}'_K\right)$$

   is zero if $R \neq R'$ and is equal to

$$\mathrm{Ker}\, \partial_x : \mathrm{Hom}_K\left(\mathcal{G}_K, \mathcal{G}'_K\right) \to \mathrm{Hom}_K\left(\mathcal{G}_K, \mathcal{G}'_K\right)$$

   if $R = R'$.

3. Compute
$$\operatorname{Ker}\partial_x : \operatorname{Hom}_{\widehat{L}}\left(\mathcal{M}_{\widehat{L}}, \mathcal{N}_{\widehat{L}}\right) \to \operatorname{Hom}_{\widehat{L}}\left(\mathcal{M}_{\widehat{L}}, \mathcal{N}_{\widehat{L}}\right)$$
when the extension $\widehat{L}$ of $\widehat{K}$ is chosen such that both $\mathcal{M}_{\widehat{L}}$ and $\mathcal{N}_{\widehat{L}}$ admit a splitting into elementary connections.

4. Compute the irregularity of $\operatorname{Hom}_K(\mathcal{M}_K, \mathcal{N}_K)$.

2.2. Asymptotic expansions

2.2.1. — Let Δ_r denote the open disc centered at 0 in $\mathbf{C}$ with radius $r > 0$. Let U be an open interval on the unit circle (for some statements, it will be important that U is not equal to the unit circle itself). Denote by $\Delta_r^*(U) \subset \Delta_r$ the open set defined as
$$\Delta_r^*(U) = \left\{z \in \Delta_r | z = \rho e^{i\theta}, 0 < \rho < r, \theta \in U\right\}.$$

Let f be a holomorphic function on $\Delta_r^*(U)$ and let $\widehat{\varphi} \in \widehat{K}$ be a formal Laurent series. Put
$$\widehat{\varphi} = \sum_{n \geq -n_0} a_n x^n$$
with $a_n \in \mathbf{C}$.

Definition 2.2.2. — *We say that $\widehat{\varphi}$ is an asymptotic expansion for f at 0 if for all $m \in \mathbf{N}$ one has*
$$\lim_{x \to 0, x \in \Delta_r^*(U)} \left|x^{-m}\right| \cdot \left|x^{n_0} f(x) - \sum_{0 \leq n \leq m} a_n x^n\right| = 0.$$

We shall take the following notations: $\overline{\mathcal{A}}(U, r) \subset \mathcal{O}(\Delta_r^*(U))$ denotes the set of functions which admits an asymptotic power series, $\mathcal{A}(U, r) = \bigcap_V \overline{\mathcal{A}}(V, r)$ where V is a relatively compact open subset of U and $\mathcal{A}(U) = \bigcup_r \mathcal{A}(U, r)$.

2.2.3. Exercise. — Show that the correspondence $U \mapsto \mathcal{A}(U)$ defines a *sheaf* on the unit circle.

2.2.4. Some elementary properties.

1. If $\widehat{\varphi}$ is an asymptotic expansion for f then one has
$$a_0 = \lim_{x \to 0, x \in \Delta_r^*(U)} x^{n_0} f(x)$$
and for $m > 0$,
$$a_m = \lim_{x \to 0, x \in \Delta_r^*(U)} x^{-m} \left[x^{n_0} f(x) - \sum_{0 \leq n \leq m-1} a_n x^n\right].$$

In particular this asymptotic expansion is unique: indeed, remark first that f admits an identically zero asymptotic expansion if and only if for all $p \in \mathbf{Z}$ one has

$$\lim_{x\to 0, x\in\Delta_r^*(U)} x^p f(x) = 0.$$

So if $f \in \overline{\mathcal{A}}(U,r)$, and if f admits a non identically zero asymptotic expansion, the coefficients are given by the previous formulae.

2. One may now define a mapping $\mathcal{A}(U) \to \widehat{K}$ denoted by $f \mapsto \widehat{f}$. One verifies that $\mathcal{A}(U)$ is a subring of $\mathcal{O}(\Delta_r^*(U))$ and that this mapping is a morphism of rings.

3. Denote by $\mathcal{A}^{<0}(U)$ the kernel of this morphism. For instance the function $x \mapsto e^{-1/x}$ has a zero asymptotic expansion in some sector around $\theta = 0$. If $f \in \mathcal{A}(U) - \mathcal{A}^{<0}(U)$ then f is invertible in $\mathcal{A}(U)$ and one has $\widehat{f}^{-1} = \widehat{f^{-1}}$. Indeed one may assume that $\lim_{x\to 0} f(x) \neq 0$ after multiplication by some power of x. This implies that f^{-1} is holomorphic in some open set $\Delta_r^*(U)$ for r sufficiently small.

4. $\mathcal{A}(U)$ is stable under derivation. In fact, let $f \in \overline{\mathcal{A}}(U)$ and let $\widehat{f} = \sum_{-n_0\leq n} a_n x^n$ its asymptotic expansion. One may easily assume that $n_0 = 0$. One has for each $m \geq 0$

$$f(x) = \sum_{n=0}^{m} a_n x^n + R_m(x)x^m$$

with

$$\lim_{x\to 0, x\in\Delta_r^*(U)} R_m(x) = 0.$$

This implies that R_m is holomorphic in $\Delta_r^*(U)$. One has

$$f'(x) = \sum_{n=0}^{m} n a_n x^{n-1} + m x^{m-1} R_m(x) + x^m R_m'(x).$$

If C_ρ is a circle centered at x with radius ρ contained in $\Delta_r^*(U)$ one has

$$|R_m'(x)| \leq \frac{1}{\rho} \cdot \max_{z\in C_\rho} |R(z)|$$

because of Cauchy theorem. Let V be a relatiely compact open set in U. There exists a positive number α such that for all $x \in \Delta_r^*(V)$ one has $C_{\alpha|x|} \subset \Delta_r^*(U)$. This proves that $\widehat{f}'$ is an asymptotic expansion for f' in $\Delta_r^*(V)$ at 0.

5. $\mathcal{A}(U)$ contains K as a subfield.

The following lemma, known as Borel-Ritt lemma, will be useful later on.

Lemma 2.2.5. — *If U is a proper open interval of the unit circle the mapping*

$$\mathcal{A}(U) \longrightarrow \widehat{K}$$

is onto.

It follows from this lemma that one has an exact sequence

$$0 \longrightarrow \mathcal{A}^{<0}(U) \longrightarrow \mathcal{A}(U) \longrightarrow \widehat{K} \longrightarrow 0.$$

Proof. — One first reduces the proof to the case where U is an interval of the form $]-\gamma,\gamma[$ on the unit circle, with $0<\gamma\leq\pi$ by composing with a rotation. Let $\sum_{n\geq 0} a_n x^n \in \widehat{K}$ (it is enough to prove the result when this series is in $\mathbf{C}[\![x]\!]$). We shall consider a series of the form

$$\sum_{n\geq 0} a_n \alpha_n(x) x^n$$

where $\alpha_n(x) = 1-\exp(-b_n/x^\beta)$ for some $b_n>0$ and $0<\beta<1$ to be defined. In fact one puts

$$\beta = \pi/2\gamma$$

so that $\beta\theta \in]-\pi/2,\pi/2[$ for all $\theta\in]-\gamma,\gamma[$ and

$$b_n = \begin{cases} 1/\left|a_n\right| & \text{if } a_n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The maximum principle implies that for $z\in\mathbf{C}$ such that $\mathrm{Re}\,(z)<0$ one has $|1-e^z|<|z|$. This implies that

$$|a_n\alpha_n(x)x^n| \leq |a_n|\,|b_n|\left|x^{n-\beta}\right|$$

for $\mathrm{Arg}x\in U$ hence the series $\sum a_n\alpha_n(x)x^n$ is dominated by the series

$$\sum_{n=1}^{\infty} |x|^{n-\beta}$$

which converges as soon as $|x|<1$ so defines a holomorphic function in $\Delta_1^*(U)$. We shall now compute the asymptotic expansion for f. We have for all $m\geq 0$

$$x^{-m}\left[f(x) - \sum_{n=0}^{m} a-nx^n\right] = -\sum_{n=0}^{m} a_n \exp(-b_n/x^\beta)x^{-(m-n)} + \sum_{n\geq m+1} a_n\alpha_n(x)x^{n-m}.$$

The first term goes to zero when $x\to 0, x\in\Delta_1^*(U)$ because one has $\mathrm{Re}\,(-b_n/x^\beta)<0$ on that domain. The second term is dominated by $\sum_{n\geq m+1}|x|^{n-m-\beta}$ in this domain, so tends to zero also. □

2.3. Statement of the main result and consequences

Theorem 2.3.1. — *Let $\mathcal{M}_K$ be a meromorphic connection. There exists an integer $q\geq 1$ such that, after the ramification $x=t^q$, one has, for all $\theta\in S^1$ and each sufficiently small interval V centered at θ*

$$\mathcal{A}_L(V)\otimes_L \mathcal{M}_L \simeq \mathcal{A}_L(V)\otimes_L\left(\mathcal{F}_L^R\otimes\mathcal{G}_L\right).$$

Corollary 2.3.2. — *Let $\mathcal{M}_K$ be a meromorphic connection. For all $\theta \in S^1$ and each sufficiently small interval U centered at θ, the formal decomposition into one slope terms $\mathcal{M}_{\widehat{K}} = \oplus \mathcal{M}_{\widehat{K}}^{(L_i)}$ can be lifted into a decomposition*

$$\mathcal{A}(U) \otimes_K \mathcal{M}_K \simeq \oplus \mathcal{M}_{\mathcal{A}(U)}^{(L_i)}.$$

Proof. — The property is clear after some ramification, as a consequence of the theorem. Choose a K-basis $\boldsymbol{m}$ of $\mathcal{M}_K$. There exists a matrix $B_L(t)$ with entries in $\mathcal{A}_L(V)$ such that the basis $\widehat{B_L(t)} \cdot \boldsymbol{m}$ is adapted to the splitting of $\mathcal{M}_{\widehat{L}}$ into one slope terms. Let $B_K(x)$ be the matrix obtained from $B_L(t)$ by taking the trace of each entry in $B_L(t)$. Then the basis $\widehat{B_K(x)} \cdot \boldsymbol{m}$ is adapted to the splitting of $\mathcal{M}_{\widehat{K}}$ into one slope terms. □

Corollary 2.3.3. — *Let $\mathcal{M}_K$ and $\mathcal{M}'_K$ be two meromorphic connections. Assume that we are given a left $\mathcal{D}_{\widehat{K}}$-linear morphism $\widehat{\varphi} : \mathcal{M}_{\widehat{K}} \to \mathcal{M}'_{\widehat{K}}$. Then, for all $\theta \in S^1$ and each sufficiently small interval U centered at θ, $\widehat{\varphi}$ can be lifted into a $\mathcal{D}_{\mathcal{A}(U)}$-linear morphism*

$$\varphi_U : \mathcal{A}(U) \otimes_K \mathcal{M}_K \longrightarrow \mathcal{A}(U) \otimes_K \mathcal{M}'_K.$$

If moreover $\widehat{\varphi}$ is an isomorphism, then so is φ_U.

Proof. — Consider the connection $\mathcal{N}_K = \operatorname{Hom}_K(\mathcal{M}_K, \mathcal{M}'_K)$ and the corresponding formal connection $\mathcal{N}_{\widehat{K}} = \operatorname{Hom}_{\widehat{K}}(\mathcal{M}_{\widehat{K}}, \mathcal{M}'_{\widehat{K}})$. We have seen above that a $\mathcal{D}_K$-linear morphism can be interpreted as an horizontal element of $\mathcal{N}_K$ (*i.e.* an element killed by ∂_x). The first part is then a consequence of the following

Lemma 2.3.4. — *Let $\mathcal{N}_K$ be a meromorphic connection. Then for all $\theta \in S^1$ and each sufficiently small interval U centered at θ, the natural mapping*

$$\operatorname{Ker}\left[\partial_x : \mathcal{N}_{\mathcal{A}(U)} \to \mathcal{N}_{\mathcal{A}(U)}\right] \longrightarrow \operatorname{Ker}\left[\partial_x : \mathcal{N}_{\widehat{K}} \to \mathcal{N}_{\widehat{K}}\right]$$

is onto.

If moreover $\widehat{\varphi}$ is an isomorphism and if A_U is the matrix of φ_U for given basis of $\mathcal{M}_K$ and $\mathcal{M}'_K$, then $\widehat{A_U}$ is the matrix of $\widehat{\varphi}$ in these bases. Because $\det \widehat{A_U} = \widehat{\det A_U}$ is not zero in $\widehat{K}$, so is $\det A_U$ in K. □

Proof of the lemma. — Consider a lifting $\oplus \mathcal{N}_{\mathcal{A}(U)}^{(L_i)}$ of the splitting of $\mathcal{N}_K$ into one slope terms, as given by the previous corollary. If L_i is not horizontal (*i.e.* if $\mathcal{N}_{\widehat{K}}^{(L_i)}$ is irregular), then one has

$$\operatorname{Ker}\left[\partial_x : \mathcal{N}_{\widehat{K}}^{(L_i)} \to \mathcal{N}_{\widehat{K}}^{(L_i)}\right] = \{0\}$$

because an element in this kernel defines a $\mathcal{D}_{\widehat{K}}$-linear morphism $\widehat{K} \to \mathcal{N}_{\widehat{K}}^{(L_i)}$ and the trivial connection has no non horizontal slope. It is then enough to show that

$$\operatorname{Ker}\left[\partial_x : \mathcal{N}_{\mathcal{A}(U)}^{(L_1)} \to \mathcal{N}_{\mathcal{A}(U)}^{(L_1)}\right] \longrightarrow \operatorname{Ker}\left[\partial_x : \mathcal{N}_{\widehat{K}}^{(L_1)} \to \mathcal{N}_{\widehat{K}}^{(L_1)}\right]$$

is onto when L_1 is the horizontal slope. Because of theorem 2.3.1 there exists a regular meromorphic connection $\mathcal{G}_K$ such that

$$\mathcal{N}_{\widehat{K}}^{(L_1)} \simeq \mathcal{G}_{\widehat{K}} \qquad \text{and} \qquad \mathcal{N}_{\mathcal{A}(U)}^{(L_1)} \simeq \mathcal{A}(U) \otimes_K \mathcal{G}_K.$$

It is then enough to prove that

$$\operatorname{Ker}\left[\partial_x : \mathcal{G}_K \to \mathcal{G}_K\right] \longrightarrow \operatorname{Ker}\left[\partial_x : \mathcal{G}_{\widehat{K}} \to \mathcal{G}_{\widehat{K}}\right]$$

is onto. But this mapping is in fact bijective, by applying the results in the regular case. □

2.4. Proof of the main result

Let $\mathcal{M}_K$ be a meromorphic connection. Choose first a ramified covering $t \mapsto x = t^q$ in order to apply theorem I–5.4.7:

$$\mathcal{M}_{\widehat{L}} \simeq \oplus \mathcal{F}_{\widehat{L}}^R \otimes \mathcal{G}_{\widehat{L}}.$$

Put $\mathcal{M}_{\widehat{L}} = \mathcal{M}'_{\widehat{L}} \oplus \mathcal{M}''_{\widehat{L}}$ where $\mathcal{M}'_{\widehat{L}}$ is the sum of terms $\mathcal{F}_{\widehat{L}}^R \otimes \mathcal{G}_{\widehat{L}}$ for which R has maximal degree (say k) and with fixed dominating coefficient (say $\alpha \in \mathbf{C}$), $\mathcal{M}''_{\widehat{L}}$ being the sum of the other terms. The proof will be done by induction, by showing that this splitting can be lifted to $\mathcal{A}_L(V)$ when V is sufficiently small. However, the terms $\mathcal{M}'_{\mathcal{A}_L(V)}$ and $\mathcal{M}''_{\mathcal{A}_L(V)}$ do not come necessarily from modules defined over L. That is why the proof has to be done for modules defined over $\mathcal{A}_L(V)$.

LEMMA 2.4.1. — *Let $\mathcal{M}_{\mathcal{A}_L(V)}$ be a free $\mathcal{A}_L(V)$-module equipped with a connection and let $\mathcal{M}_{\widehat{L}}$ its formalized module. A splitting of $\mathcal{M}_{\widehat{L}}$ as above can be lifted to a splitting of $\mathcal{M}_{\mathcal{A}_L(V)}$ when V is sufficiently small.*

Once this lemma is proven, one may argue by induction on k (maximal degree of polynomials R which appear in the splitting of $\mathcal{M}_{\widehat{L}}$, *i.e.* largest slope of $\mathcal{M}_{\widehat{L}}$) and $\dim_{\widehat{L}} \mathcal{M}_{\widehat{L}}$. If $\mathcal{M}''_{\widehat{L}} \neq 0$ one may apply induction to both terms $\mathcal{M}'_{\mathcal{A}_L(V)}$ and $\mathcal{M}''_{\mathcal{A}_L(V)}$. If $\mathcal{M}''_{\mathcal{A}_L(V)} = 0$, one may apply induction to $\mathcal{F}_{\mathcal{A}_L(V)}^{(-\alpha t^{-k})} \otimes \mathcal{M}_{\mathcal{A}_L(V)}$. The first step is the case where $\mathcal{M}_{\widehat{L}}$ is regular. One must prove a result analogous to theorem I–5.2.2:

LEMMA 2.4.2. — *Let $\mathcal{M}_{\mathcal{A}_L(V)}$ be a free $\mathcal{A}_L(V)$-module equipped with a connection such that $\mathcal{M}_{\widehat{L}}$ is regular. There exists a $\mathcal{A}_L(V)$-basis $\boldsymbol{m}$ of $\mathcal{M}_{\mathcal{A}_L(V)}$ such that the matrix of $t\partial_t$ in this basis is constant.*

Proof. — Let $\boldsymbol{m}'$ be a basis of $\mathcal{M}_{\mathcal{A}_L(V)}$ and denote by $\widehat{\boldsymbol{m}'}$ the corresponding basis of $\mathcal{M}_{\widehat{L}}$. We know that there exists a matrix $\widehat{C} \in \mathrm{Gl}(d, \widehat{L})$ such that the matrix of $t\partial_t$ in the basis $\widehat{\boldsymbol{m}'}$ is constant. Because of lemma 2.2.5 there exists a matrix $C \in \mathrm{Gl}(d, \mathcal{A}_L(V))$ which lifts $\widehat{C}$. Consider now the basis $\boldsymbol{m} = C \cdot \boldsymbol{m}'$. The matrix of $t\partial_t$ in this basis may hence be written as $A_0 + A'$ where A_0 is constant and $\widehat{A'} = 0$. We shall find a matrix $B \in \mathrm{Gl}(d, \mathcal{A}_L(V))$ such that the matrix of $t\partial_t$ in the basis $B \cdot \boldsymbol{m}$ is equal to A_0. As in the proof of theorem I–5.2.2 one must have:

$$t\frac{dB}{dt} = A_0 B - BA_0 - BA'$$

and one asks that $\widehat{B} = \mathrm{Id}$. From a general point of view, consider a linear differential system defined on $\mathcal{A}_L(V)$ equipped with a basis for which the matrix of $t\partial_t$ can be written $C = C_0 + C'$ where C_0 is constant and $\widehat{C'} = 0$. We want to show that every formal solution of the corresponding formal system can be lifted to a solution of the original system (this is analogous to proposition 1.1.2). After taking the tensor product with the system corresponding to t^{-C_0} (one chooses some determination of $\log t$ in $\Delta_r^*(V)$) one is reduced to the case where $C_0 = 0$. In $\Delta_r^*(V)$ a fundamental matrix of solutions is given by a primitive of $C'(t)/t$. Since this matrix admits an asymptotic expansion (equal to zero), every primitive matrix also (equal to a constant matrix), and this proves the lemma. □

Proof of lemma 2.4.1. — Begin as in the previous lemma: there exists a basis $\boldsymbol{m}$ of $\mathcal{M}_{\mathcal{A}_L(V)}$ such that the corresponding basis $\widehat{\boldsymbol{m}}$ is compatible with the splitting of $\mathcal{M}_{\widehat{L}}$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be the matrix of $t\partial_t$ in this basis, with $\widehat{A_{12}} = \widehat{A_{21}} = 0$ and $\widehat{A_{11}}, \widehat{A_{22}}$ given. We shall try to find a matrix C with the following form

$$C = \begin{pmatrix} \mathrm{Id} & C_{12} \\ C_{21} & \mathrm{Id} \end{pmatrix}$$

with entries in $\mathcal{A}_L(V)$ and with $\widehat{C_{12}} = \widehat{C_{21}} = 0$, such that in the basis $C \cdot \boldsymbol{m}$, $t\partial_t$ has matrix

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}$$

The matrices C_{12} and C_{21} must satisfy the following equations

$$\begin{aligned} A_{11} + C_{12}A_{21} &= A'_{11} \\ A_{22} + C_{21}A_{12} &= A'_{22} \end{aligned}$$

and

$$\begin{aligned} d/dtC_{12} + A_{12} + C_{12}A_{22} &= A'_{11}C_{12} \\ d/dtC_{21} + A_{21} + C_{21}A_{11} &= A'_{22}C_{21} \end{aligned}$$

and so in particular the following equations must be satisfied

$$\begin{aligned} d/dtC_{12} + A_{12} + C_{12}A_{22} - A_{11}C_{12} - C_{12}A_{21}C_{12} &= 0 \\ d/dtC_{21} + A_{21} + C_{21}A_{11} - A_{22}C_{21} - C_{21}A_{12}C_{21} &= 0 \end{aligned}$$

with $\widehat{C_{12}} = \widehat{C_{21}} = 0$, and in fact if these equations are satisfied, one deduces that the previous ones also. We now have obtained for C_{12} (as well as for C_{21}) a non linear equation of a special type. The existence of a solution for such an equation with a zero asymptotic expansion in some sector will follow from the theorem below, which will not been proven here.

THEOREM 2.4.3. — *Consider a system of non linear differential equations*

$$t^k \frac{du_i}{dt} = \lambda_i u_i + f_i(t, u_1, \dots, u_r)$$

for $i = 1, \dots, r$, where for each i one has

- $\lambda_i \in \mathbf{C}, \lambda_i \neq 0$,
- *f_i is a polynomial in $u_1, \dots, u_r$ with coefficients in $\mathcal{A}_L(V)$ for some sector V,*
- *the coefficients of these polynomials have an asymptotic expansion of order ≥ 0,*
- *moreover, the coefficients of the terms of degree ≤ 1 have positive order.*

Then if the associated formal system is linear there exists in every sufficiently small subsector U of V a set of solutions $u_1, \dots, u_r$ in $\mathcal{A}_L(U)$ with $\widehat{u_i} = 0$ for $i = 1, \dots, r$. □

By applying the theorem to the previous equations one obtains the required matrices C_{12} and C_{21}, and this proves the existence of the splitting of $\mathcal{M}_{\mathcal{A}_L(V)}$ for V sufficiently small. □

Chapter III
Holonomic $\mathcal{D}$-modules on the Riemann sphere

We shall consider in this chapter global properties of $\mathcal{D}$-modules. Many results in this chapter are valid for a compact Riemann surface. We shall restrict however for simplicity to the case of the Riemann sphere $\mathbf{P}^1(\mathbf{C})$.

1. Algebraic properties

1.1. Holonomic modules over the Weyl algebra $\mathbf{C}[x]\langle\partial_x\rangle$

Many properties have yet been seen in chapter I.

1.1.1. — Let $P \in \mathbf{C}[x]\langle\partial_x\rangle$. Write $P = \sum_{i=0}^d a_i(x)\partial_x^i$ with $a_i \in \mathbf{C}[x]$ and $a_d \not\equiv 0$. The singular points of the operator P are defined as the zeros of a_d. We shall use the following division statement:

Lemma 1.1.2. — *Let $A \in \mathbf{C}[x]\langle\partial_x\rangle$. One may write in a unique way*

$$A = PQ + R = Q'P + R'$$

with $(Q, R), (Q', R') \in \mathbf{C}[x, a_d^{-1}(x)]\langle\partial_x\rangle$ and $\deg R < \deg P, \deg R' < \deg P$. □

The proof is elementary. Let now I be a (left) ideal of the ring $\mathbf{C}[x]\langle\partial_x\rangle$. There exists in I an operator P of minimal degree. If P and P' are two such operators, written as $P = \sum_{i=0}^d a_i(x)\partial_x^i$, $P' = \sum_{i=0}^d a'_i(x)\partial_x^i$ the operator $a'_dP - a_dP'$ has degree less than d and is contained in I, so is equal to zero. Moreover such a P generates the ideal I over the ring $\mathbf{C}[x, a_d^{-1}(x)]\langle\partial_x\rangle$. This is easily deduced from the previous lemma.

1.1.3. — Let M be a $\mathbf{C}[x]\langle\partial_x\rangle$-module of finite type. This module is then equal to a successive extension of modules isomorphic to $\mathbf{C}[x]\langle\partial_x\rangle/I$ for some ideals I. We say that M is holonomic if every ideal which appears in such an extension is nonzero.

Remark. — Let $\mathcal{O}_a$ be the ring of germs of holomorphic functions at a point $a \in \mathbf{C}$ and

$$\mathcal{D}_a = \mathcal{O}_a\langle\partial_x\rangle = \mathcal{O}_a \otimes_{\mathbf{C}[x]} \mathbf{C}[x]\langle\partial_x\rangle.$$

Then M is holonomic if and only if for all $a \in \mathbf{C}$, the $\mathcal{D}_a$-module of finite type $M_a = \mathcal{O}_a \otimes_{\mathbf{C}[x]} M$ is holonomic.

Theorem I–3.3.6 may be applied to holonomic $\mathbf{C}[x]\langle\partial_x\rangle$-modules, so one deduces that every such module is isomorphic to $\mathbf{C}[x]\langle\partial_x\rangle/I$. The following proposition follows now from the previous results:

Proposition 1.1.4. — *Let M be a holonomic $\mathbf{C}[x]\langle\partial_x\rangle$-module. There exists an operator $P \in \mathbf{C}[x]\langle\partial_x\rangle$ and a surjective morphism*

$$\mathbf{C}[x]\langle\partial_x\rangle/\mathbf{C}[x]\langle\partial_x\rangle \cdot P \longrightarrow M \longrightarrow 0$$

which kernel is supported on points (i.e. is a torsion module over $\mathbf{C}[x]$) and these points are singular points of P. □

We can now analyse the local structure of a holonomic $\mathbf{C}[x]\langle\partial_x\rangle$-module.

Proposition 1.1.5. — *Let M be a holonomic $\mathbf{C}[x]\langle\partial_x\rangle$-module. There exists a finite set $\Sigma \subset \mathbf{C}$ such that the restriction of M to the open set $U = \mathbf{C} - \Sigma$ is a projective module of finite type over the ring of regular functions over U, or equivalently (*see [4] *) for every $a \in U$ the $\mathcal{D}_a$-module M_a is free of finite type over $\mathcal{O}_a$.*

Proof. — Let P as above and put $\Sigma = a_d^{-1}(o)$. Then

$$\mathbf{C}[x, a_d^{-1}(x)]\langle\partial_x\rangle/\mathbf{C}[x, a_d^{-1}(x)]\langle\partial_x\rangle \cdot P \xrightarrow{\sim} \mathbf{C}[x, a_d^{-1}(x)] \otimes M$$

hence it is enough to show the result for the first module. When the dominating coefficient of P does not vanish in a, the module $\mathcal{D}_a/\mathcal{D}_a \cdot P$ is of finite type over $\mathcal{O}_a$, hence free of rank $d = \deg P$ (lemma I–2.3.3). □

The set of points $a \in \mathbf{C}$ such that M_a is not free of finite type over $\mathcal{O}_a$ is the set of singular points of M on $\mathbf{C}$. it is a finite set of points.

1.2. Algebraic $\mathcal{D}_{\mathbf{P}^1}$-modules

1.2.1. — Let U be an affine open set of $\mathbf{P}^1$. We can define the ring of algebraic differential operators on U without choosing a coordinate on U. In order to do that, consider the $\mathcal{O}(U)$-module of derivations of $\mathcal{O}(U)$ (which is the ring of regular functions on U): A derivation D is a $\mathbf{C}$-linear operator on $\mathcal{O}(U)$ which satisfies Leibniz rule. If $U = \mathbf{C} - \Sigma$, where Σ is a finite set of points, if one chooses a coordinate x on U and an equation f for Σ, we have

$$\mathcal{O}(U) = \mathbf{C}[x, f(x)^{-1}]$$

and the module of derivations is generated by ∂_x. The ring $\mathcal{D}(U)$ of differential operators is the ring generated by $\mathcal{O}(U)$ and the module of derivations, subject to the relations

$$[D, g] = D(g)$$

for all $g \in \mathcal{O}(U)$. If $V \subset U$ is an affine open subset, one has

$$\mathcal{D}(V) = \mathcal{O}(V) \otimes_{\mathcal{O}(U)} \mathcal{D}(U).$$

In this way one defines an algebraic sheaf $\mathcal{D}_{\mathbf{P}^1}$ on $\mathbf{P}^1$.

1.2.2. — We shall refer to [5] for elementary notions in algebraic geometry that will be used below. Let $\mathcal{M}$ be an algebraic (sheaf of) $\mathcal{D}_{\mathbf{P}^1}$-module. We shall say that $\mathcal{M}$ is *coherent* if locally (for the Zariski topology) $\mathcal{M}$ has finite presentation over $\mathcal{D}_{\mathbf{P}^1}$. Since for every affine open set U of $\mathbf{P}^1$, the ring $\mathcal{D}_{\mathbf{P}^1}(U)$ is noetherian (*see* corollary I–1.3.5), one deduces that $\mathcal{M}$ is coherent if and only if for each affine open set U of $\mathbf{P}^1$ the module $\mathcal{M}(U)$ is of finite type over $\mathcal{D}(U)$. In the same way, we shall say that $\mathcal{M}$ is $\mathcal{D}_{\mathbf{P}^1}$-holonomic if for each affine open set the $\mathcal{D}(U)$-module is so, *i.e.* $\mathcal{M}(U)$ is equal to an extension of modules isomorphic to $\mathcal{D}(U)/I$ where I is a nonzero ideal of $\mathcal{D}(U)$.

Examples.

1. Let $U = \mathbf{P}^1 - \{0, \infty\} = \mathbf{C}^*$. If one chooses a coordinate x which vanishes at 0 one has $\mathcal{O}(U) = \mathbf{C}[x, x^{-1}]$ and $\mathcal{D}_{\mathbf{P}^1}(U) = \mathbf{C}[x, x^{-1}]\langle\partial_x\rangle$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_{\mathbf{P}^1|U}$-module (*i.e.* $\mathcal{M}(U)$ is $\mathcal{D}_{\mathbf{P}^1}(U)$-noetherian). Let $j : U \hookrightarrow \mathbf{P}^1$ denote the inclusion. We shall construct a $\mathcal{D}_{\mathbf{P}^1}$-module denoted by $j_+\mathcal{M}$. Consider the two charts $U_0 = \mathbf{P}^1 - \{\infty)$ and $U_\infty = \mathbf{P}^1 - \{0\}$, with $U = U_0 \cap U_\infty$. The chart $U_0 \simeq \mathbf{C}$ comes equipped with the coordinate x and the chart U_∞ with the coordinate z: on U one has $z = x^{-1}$ and $z\partial_z = -x\partial_x$. We shall define $j_+\mathcal{M}(U_0)$, $j_+\mathcal{M}(U_\infty)$ as well as the restriction mappings to U. Put $j_+\mathcal{M}(U_0) = \mathcal{M}(U)$ considering $\mathcal{M}(U)$ as a module over $\mathbf{C}[x]\langle\partial_x\rangle$. In the same way put $j_+\mathcal{M}(U_\infty) = \mathcal{M}(U)$ as a module over $\mathbf{C}[z]\langle\partial_z\rangle$. The restriction mappings are equal to identity.

LEMMA 1.2.3. — *$j_+\mathcal{M}$ is a coherent $\mathcal{D}_{\mathbf{P}^1}$-module.*

Proof. — It is enough to show that $\mathcal{M}(U)$ is coherent over the ring $\mathbf{C}[x]\langle\partial_x\rangle$ (and the same for the variable z). We know that $\mathcal{M}(U)$ is generated by one element over $\mathbf{C}[x, x^{-1}]\langle\partial_x\rangle$ (theorem I–3.3.6) so we may write

$$\mathcal{M}(U) \simeq \mathbf{C}[x, x^{-1}]\langle\partial_x\rangle/I$$

and moreover we may assume that the generators $P_1, \ldots, P_r$ of I are contained in $\mathbf{C}[x]\langle\partial_x\rangle$ (after multiplication by some power of x). Put

$$\mathcal{N}_0 = \mathbf{C}[x]\langle\partial_x\rangle/\mathbf{C}[x]\langle\partial_x\rangle \cdot (P_1, \ldots, P_r).$$

Then $\mathcal{N}_0$ is holonomic over $\mathbf{C}[x]\langle\partial_x\rangle$ as well as $\mathcal{N}_0[x^{-1}]$ (theorem I–4.2.3). Moreover one has $\mathcal{N}_0[x^{-1}] = \mathcal{M}(U)$ as $\mathbf{C}[x]\langle\partial_x\rangle$-modules, which proves the lemma. □

2. Let $P \in \mathbf{C}[x]\langle\partial_x\rangle$ be a nonzero differential operator and $\mathcal{M}_0$ be the associated holonomic module ($\mathcal{M}_0 = \mathbf{C}[x]\langle\partial_x\rangle/\mathbf{C}[x]\langle\partial_x\rangle \cdot P$). There exists a unique $k \in \mathbf{Z}$ such that

$$x^k P = \sum_{i=0}^{d} a_i(x^{-1})(x\partial_x)^i$$

with $a_i \in \mathbf{C}[x^{-1}]$ and $a_i(\infty) \neq 0$ for at least one i. With U as above, one has

$$\begin{aligned} \mathcal{M}_0(U) &= \mathbf{C}[x, x^{-1}]\langle\partial_x\rangle/\mathbf{C}[x, x^{-1}]\langle\partial_x\rangle \cdot (x^k P) \\ &= \mathbf{C}[z, z^{-1}]\langle\partial_z\rangle/\mathbf{C}[z, z^{-1}]\langle\partial_z\rangle \cdot (\textstyle\sum_{i=0}^{d} a_i(z)(-z\partial_z)^i). \end{aligned}$$

One obtains in that way an extension of $\mathcal{M}_0$ to a $\mathcal{D}_{\mathbf{P}^1}$-module which satisfies

$$\mathcal{M}(U_\infty) = \mathbf{C}[z]\langle\partial_z\rangle/\mathbf{C}[z]\langle\partial_z\rangle \cdot (\sum_{i=0}^{d} a_i(z)(-z\partial_z)^i).$$

1.2.4. Exercise. — Let $P \in \mathbf{C}[x]\langle\partial_x\rangle$ be a nonzero operator. Put

$$P = \sum_{i=0}^{d} a_{ij} x^j \partial_x^i.$$

Let $N_{0,\infty}(P)$ be the boundary of the convex hull of the set

$$\bigcup_{\{i,j \mid a_{ij} \neq 0\}} [(i, i-j) - (\mathbf{N} \times \{0\})]$$

1. Show how to recover the Newton polygon of P at 0 from $N_{0,\infty}$.

2. Same question for the Newton polygon at infinity (use the construction given in the previous example 2).

3. Show that P is regular at 0 and ∞ iff $N_{0,\infty}$ is a rectangle.

4. Compute the length of the vertical side (which is not on the vertical axis) of $N_{0,\infty}$ in terms of the characteristic cycle of P at its singular points.

1.3. The algebraic de Rham complex

1.3.1. — Let $\Omega^1_{\mathbf{P}^1} = \Omega^1$ be the sheaf of algebraic differential forms on $\mathbf{P}^1$. If $\mathcal{O}(1)$ denotes the canonical line bundle on $\mathbf{P}^1$ (*see* [5]) one has an isomorphism $\Omega^1 \simeq \mathcal{O}(-2)$ (this means that the differential form which is equal to dx in the chart $\mathbf{P}^1 - \{\infty\}$ has a double pole at infinity, because of the relation $dx = d(1/z) = -dz/z^2$). The algebraic de Rham complex on $\mathbf{P}^1$ is the complex of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1} \xrightarrow{d} \Omega^1_{\mathbf{P}^1} \longrightarrow 0$$

where d is the usual differential.

1.3.2. — Let $\mathcal{M}$ be a (sheaf of) $\mathcal{D}_{\mathbf{P}^1}$-module. One may then twist the usual de Rham complex:

$$0 \longrightarrow \mathcal{M} \xrightarrow{d_{\mathcal{M}}} \Omega^1_{\mathbf{P}^1} \otimes_{\mathcal{O}_{\mathbf{P}^1}} \mathcal{M} \longrightarrow 0$$

The differential $d_{\mathcal{M}}$ is defined as follows: in an affine chart U with coordinate x, dx is a basis of $\Omega^1(U)$ and if $m \in \mathcal{M}(U)$ one puts

$$d_{\mathcal{M}}(m) = dx \otimes (\partial_x m).$$

1.3.3. Exercise. — Show that $d_{\mathcal{M}}$ does not depend on the choice of the coordinate in the chart U and hence defines a $\mathbf{C}$-linear morphism of sheaves.

Remark. — It is important to notice that $d_{\mathcal{M}}$ is only $\mathbf{C}$-linear and hence $\operatorname{Ker} d_{\mathcal{M}}$ and $\operatorname{Coker} d_{\mathcal{M}}$ are sheaves of $\mathbf{C}$-vector spaces (without any other structure in general). If $\mathcal{M} = \mathcal{O}_{\mathbf{P}^1}$ one recovers the usual algebraic de Rham complex. If $\mathcal{M} = \mathcal{D}_{\mathbf{P}^1}$ one verifies (by computing in any affine chart) that $\operatorname{Ker} d_{\mathcal{D}_{\mathbf{P}^1}} = \{0\}$ and that $\operatorname{Coker} d_{\mathcal{D}_{\mathbf{P}^1}} = \Omega^1_{\mathbf{P}^1}$.

1.3.4. Exercise. — Show that $\Omega^1_{\mathbf{P}^1}$ admits a natural structure of *right* $\mathcal{D}_{\mathbf{P}^1}$-module (use Lie derivative). Show that when $\mathcal{D}_{\mathbf{P}^1}$ is equipped with its natural structure of right $\mathcal{D}_{\mathbf{P}^1}$-module then $\operatorname{Coker} d_{\mathcal{D}_{\mathbf{P}^1}}$ admits also such a structure and that the isomorphism above is compatible with this structure of right $\mathcal{D}_{\mathbf{P}^1}$-module.

1.3.5. — Let U be an affine open set of $\mathbf{P}^1$ and consider the complex of $\mathbf{C}$-vector spaces consisting of global sections over U of the algebraic de Rham complex

$$0 \longrightarrow \mathcal{M}(U) \xrightarrow{d_{\mathcal{M}(U)}} \Omega^1_{\mathbf{P}^1}(U) \otimes_{\mathcal{O}_{\mathbf{P}^1}(U)} \mathcal{M}(U) \longrightarrow 0$$

If $\mathcal{M}$ is equal to $\mathcal{D}_{\mathbf{P}^1}$ then as we have seen above $\operatorname{Coker} d_{\mathcal{M}(U)}$ is equal to $\Omega^1_{\mathbf{P}^1}(U)$ and hence is not a finite dimensional vector space over $\mathbf{C}$. We shall see below that this cannot happen when $\mathcal{M}$ is holonomic.

THEOREM 1.3.6. — *When U is an affine open set of $\mathbf{P}^1$ and $\mathcal{M}$ is $\mathcal{D}_{\mathbf{P}^1}$-holonomic, the complex*

$$0 \longrightarrow \mathcal{M}(U) \xrightarrow{d_{\mathcal{M}(U)}} \Omega^1_{\mathbf{P}^1}(U) \otimes_{\mathcal{O}_{\mathbf{P}^1}(U)} \mathcal{M}(U) \longrightarrow 0$$

has finite dimensional cohomology.

Proof. — Remark first that the snake lemma shows that in an exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

of $\mathcal{D}_{\mathbf{P}^1}$-modules, if the theorem is true for two of these modules, it is also true for the third one. Proposition 1.1.4 (applied to the ring $\mathcal{D}_{\mathbf{P}^1}(U)$) reduces then the theorem to the case where $\mathcal{M}(U) = \mathcal{D}(U)/\mathcal{D}(U)\cdot P$ for some $P \in \mathcal{D}(U)$. Choose a coordinate x on U so that $\mathcal{D}(U) = \mathcal{O}(U)\langle\partial_x\rangle$. The algebraic de Rham complex is equal to the complex

$$0 \longrightarrow \mathcal{M}(U) \xrightarrow{\partial_x\cdot} \mathcal{M}(U) \longrightarrow 0.$$

It is easy to verify that the cokernel has finite dimension: put $P = a_0 + \partial_x P'$ with $a_0 \in \mathcal{O}(U)$ and $P' \in \mathcal{D}(U)$. The cokernel is equal to

$$\mathcal{D}(U)/\mathcal{D}(U)\cdot P + \partial_x \cdot \mathcal{D}(U).$$

We then get a surjective mapping

$$\mathcal{O}(U) \longrightarrow \mathcal{D}(U)/\mathcal{D}(U)\cdot P + \partial_x \cdot \mathcal{D}(U)$$

and the ideal $\mathcal{O}(U)\cdot a_0$ is sent to 0. Hence the dimension of the cokernel is less than or equal to $\dim_{\mathbf{C}}\mathcal{O}(U)/(a_0)$.

In order to estimate the dimension of the kernel, one may use the presentation of $\mathcal{M}(U)$: one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}(U) & \xrightarrow{\cdot P} & \mathcal{D}(U) & \longrightarrow & \mathcal{M}(U) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \partial_x\cdot} & & \Big\downarrow{\scriptstyle \partial_x\cdot} & & \Big\downarrow{\scriptstyle \partial_x\cdot} & & \\ 0 & \longrightarrow & \mathcal{D}(U) & \xrightarrow{\cdot P} & \mathcal{D}(U) & \longrightarrow & \mathcal{M}(U) & \longrightarrow & 0 \end{array}$$

and the first two vertical maps are injective. One may identify the kernel of these maps to $\mathcal{O}(U)$. So let $Q\in\mathcal{D}(U)$ such that $QP=\partial_x R$ for some $R\in\mathcal{D}(U)$. Put $Q=q_0+\partial_x Q'$ with $q_0\in\mathcal{O}(U)$. So one gets $q_0P=\partial_x R'$ (and $[Q]=[q_0]$ in $\mathcal{D}(U)/\partial_x\mathcal{D}(U)$). Because $q_0P=Pq_0-P(q_0)$ one obtains the following equation for q_0 (with the notations above):

$$(\partial_x P')(q_0)=\partial_x\cdot R''$$

for some R'' in $\mathcal{D}(U)$. This implies in fact that q_0 is a solution of the differential equation $(\partial_x P')(q_0)=0$. The space of such solutions is finite dimensional, so this implies that the kernel of $\partial_x:\mathcal{M}(U)\to\mathcal{M}(U)$ is also finite dimensional.

Remark. — It is simpler to show that the complex

$$0\longrightarrow\mathcal{M}(U)\xrightarrow{x\cdot}\mathcal{M}(U)\longrightarrow 0$$

has finite dimensional cohomology. Take for instance $U=\mathbf{P}^1-\{\infty\}$ and consider the filtration $V\mathcal{M}(U)$ obtained like in §I–6.1. The previous complex is quasi-isomorphic to the complex

$$0\longrightarrow\mathrm{gr}_0^V\mathcal{M}(U)\xrightarrow{x\cdot}\mathrm{gr}_{-1}^V\mathcal{M}(U)\longrightarrow 0$$

and each term in this complex (hence the cohomology) is a finite dimensional $\mathbf{C}$-vector space. One can show the previous theorem (for the mapping ∂_x) by using a Fourier transform: put $\xi=\partial_x$ and $\partial_\xi=-x$. One then has $\mathbf{C}[x]\langle\partial_x\rangle=\mathbf{C}[\xi]\langle\partial_\xi\rangle$ and one has the relation

$$[\partial_\xi,\xi]=[-x,\partial_x]=1.$$

Now $\mathcal{M}(U)$ is also holonomic as a $\mathbf{C}[\xi]\langle\partial_\xi\rangle$-module and one may apply the above computation.

1.3.7. — With an algebraic sheaf $\mathcal{F}$ of $\mathbf{C}$-vector spaces on $\mathbf{P}^1$ are associated cohomology groups $H^*(\mathbf{P}^1,\mathcal{F})$ and with a complex $\mathcal{F}^\bullet$ of such sheaves are associated hypercohomology groups $\boldsymbol{H}^*(\mathbf{P}^1,\mathcal{F}^\bullet)$. These are $\mathbf{C}$-vector spaces (*see* for instance [5]). We shall now be interested on the hypercohomology of the algebraic de Rham complex $\Omega^\bullet_{\mathbf{P}^1}\otimes\mathcal{M}$ when $\mathcal{M}$ is holonomic.

Theorem 1.3.8. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbf{P}^1}$-module. Then*

$$\dim_{\mathbf{C}} \boldsymbol{H}^i(\mathbf{P}^1, \Omega^\bullet_{\mathbf{P}^1} \otimes \mathcal{M}) \begin{cases} = 0 & \text{if } i \neq 0,1 \\ < +\infty & \text{if } i = 0,1 \end{cases}$$

Remarks.

1. It may be more natural to consider the de Rham complex shifted by 1, that is $\Omega^\bullet_{\mathbf{P}^1} \otimes \mathcal{M}[1]$. This complex has nonzero hypercohomology at most in degrees -1 and 0.

2. The theorem above is valid for every coherent $\mathcal{D}_{\mathbf{P}^1}$-module unlike theorem 1.3.6. This comes from the fact that $\mathbf{P}^1$ is a *compact* Riemann surface. We shall not give the proof of this general statement and refer for it to [10] or [13]. An easy exercise consits in showing that the theorem is true when $\mathcal{M} = \mathcal{D}_{\mathbf{P}^1}$.

Proof of theorem 1.3.8. — Assume first that there exists a point in $\mathbf{P}^1$ taken as the point at infinity, such that one has $\mathcal{M} = j_+ j^* \mathcal{M}$ where $j : U = \mathbf{P}^1 - \{\infty\} \hookrightarrow \mathbf{P}^1$ denotes the inclusion and $j^*\mathcal{M}$ denotes the restriction of $\mathcal{M}$ to this affine open set U. Because U is affine one has

$$H^i(\mathbf{P}^1, \Omega^k \otimes \mathcal{M}) = H^i(U, \Omega^k \otimes j^*\mathcal{M}) = 0$$

for $i \neq 0$ and $k = 0, 1$ and

$$H^0(\mathbf{P}^1, \Omega^k \otimes \mathcal{M}) = H^0(U, \Omega^k \otimes j^*\mathcal{M}) = \Omega^k(U) \otimes \mathcal{M}(U).$$

Moreover one has an exact sequence

$$\begin{aligned} 0 \longrightarrow \boldsymbol{H}^0(\mathbf{P}^1, \Omega^\bullet \otimes \mathcal{M}) \longrightarrow H^0(\mathbf{P}^1, \Omega^0 \otimes \mathcal{M}) \longrightarrow& \\ \longrightarrow H^0(\mathbf{P}^1, \Omega^1 \otimes \mathcal{M}) \longrightarrow \boldsymbol{H}^1(\mathbf{P}^1, \Omega^\bullet \otimes \mathcal{M}) \longrightarrow \cdots& \end{aligned}$$

and consequently $\boldsymbol{H}^\star(\mathbf{P}^1, \Omega^\bullet \otimes \mathcal{M})$ is equal to the cohomology of the algebraic de Rham complex over U. One may then apply theorem 1.3.6

In general one uses the exact sequence of localization

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow j_+ j^* \mathcal{M} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{K}$ and $\mathcal{C}$ have support at ∞. One may apply the previous result to $j_+ j^*\mathcal{M}$ and also to $\mathcal{K}$ and $\mathcal{C}$ (by changing the point at infinity) and one obtains the theorem using long exact sequences of hypercohomology.

2. Algebraic/analytic comparison theorems (Abstract)

This section has not been written up. I shall only give the abstract of the results explained during the lecture. The reader will find useful references in the bibliographical guide.

2.1. GAGA

One introduces the sheaf $\mathcal{D}^{an}_{\mathbf{P}^1}$ of differential operators on $\mathbf{P}^1$ with analytic coefficients and proves the coherence of $\mathcal{D}^{an}_{\mathbf{P}^1}$. If $\mathcal{M}^{an}$ is a coherent $\mathcal{D}^{an}_{\mathbf{P}^1}$-module, GAGA theorem asserts that there exists a coherent $\mathcal{D}_{\mathbf{P}^1}$-module $\mathcal{M}$ such that $\mathcal{M}^{an} = \mathcal{D}^{an}_{\mathbf{P}^1} \otimes_{\mathcal{D}} \mathcal{M}$. The proof is based on the existence of a lattice in $\mathcal{M}^{an}$, *i.e* a coherent $\mathcal{O}^{an}_{\mathbf{P}^1}$-module which generates $\mathcal{M}^{an}$ at each point of $\mathbf{P}^1$ (one then may apply the classical GAGA theorem). This result is in fact local and has to be proved only for holonomic modules. One then uses the formal structure of such modules at a singular point to get a formal lattice which will be used to construct the convergent lattice.

One proves also that $\mathcal{M}$ is holonomic if and only if $\mathcal{M}^{an}$ is so.

2.2. Analytic de Rham complex and local comparison

When $\mathcal{M}^{an}$ is a $\mathcal{D}^{an}_{\mathbf{P}^1}$-module one may define the *analytic de Rham complex* $\mathrm{DR}^{an}(\mathcal{M}^{an})$. When $\mathcal{M}^{an}$ is holonomic, this complex has constructible cohomology. It is in some sense dual to the complex of solutions of $\mathcal{M}^{an}$ in $\mathcal{O}^{an}$ introduced in II–1.3.2.

Assume that $\mathcal{M}^{an}$ is holonomic. Let V be a small neighborhood of a singular point $x \in \mathbf{P}^1$ of $\mathcal{M}^{an}$ and let $j : V^* = V - \{x\} \hookrightarrow V$ be the open inclusion. If $\mathcal{M}^{an}$ is a meromorphic connexion at x (*i.e* its germ at x is a localized $\mathcal{D}^{an}_x$-module), then the *local comparison theorem* asserts that if $\mathcal{M}_{an}$ is *regular* at x, one has

$$\mathrm{DR}^{an}(\mathcal{M}^{an})_x = \Big[\boldsymbol{R} j_* j^{-1} \mathrm{DR}^{an}(\mathcal{M}^{an})\Big]_x .$$

2.3. Global comparison theorem

Let $\mathcal{M}$ be a $\mathcal{D}_U$-module, where U is an affine open set of $\mathbf{P}^1$. The *global comparison theorem* asserts that if $(j_+\mathcal{M})^{an}$ is *regular* (at each point of $\mathbf{P}^1$), the cohomology of the algebraic de Rham complex $\Omega^\bullet \otimes \mathcal{M}$ is equal to the hypercohomology

$$\boldsymbol{H}^*(U, \mathrm{DR}^{an}(\mathcal{M}^{an})).$$

This result can be deduced from the local comparison theorem and standard homological arguments.

2.4. Global index

When $\mathcal{M}$ is a holonomic $\mathcal{D}_U$-module but perhaps not regular, the difference between Euler characteristics of $\Omega^\bullet(U) \otimes \mathcal{M}$ and $\boldsymbol{H}^*(U, \mathrm{DR}^{an}(\mathcal{M}^{an}))$ can be computed in term of local irregularities. This gives a *global index theorem*.

Bibliographical guide

1. Chapter I

The results of section 1 may easily be generalized to many variables. The reader may consult the book [9], from where are taken the proofs in this section. The definition of Gevrey conditions may be found in [20].

Division theorems are important in analytic geometry. They allow to analyse the structure of ideals. For instance the notion of a division basis of an ideal was used at the beginning of the century for convergent or formal power series in many variables (it is also called Gröbner basis). The fact that this notion may be used also in the case of differential operators comes from the fact that the graded ring $\mathrm{gr}^F\mathcal{D}$ for the F-filtration is a ring of polynomials with coefficients in a ring of power series. The results of section 2 are developed in [16] (*see* also [30] for analogous results in many variables).

The notion of holonomy (introduced in section 3) is not difficult in dimension 1. For generalizations, the reader may consult [9]. The microlocal approach to characteristic varieties is developed in the book by M. Kashiwara [12] as well as in the second part of the book of F. Pham [14]. Bernstein's approach to holonomy may be found in his paper [17], and also in the paper by F. Ehlers in [10].

Section 4 makes the link between the classical notion of a meromorphic connection and the notion of a localized $\mathcal{D}$-module (following [33]). Classically, a meromorphic connection consists of a vector bundle with a connection admitting poles. In the case of many variables, it is necessary to consider also coherent sheaves. The notion which is independent of such a vector bundle (or sheaf) is obtained by working over the ring of functions localized on poles of the connection (in §4 it is the ring $\mathbf{C}[x, x^{-1}]$). The main theorem is theorem 4.2.3. It has been generalized in many variables by Kashiwara [31]. We have insisted on the V-filtration because it is important for the theory of moderate nearby and vanishing cycles. The proof of proposition 4.3.3 given here is taken from [33] where it is attributed to N. Katz.

It is easy to define the Newton polygon for an operator (even in many variables). One important point (which was singled out by Y. Laurent in [32]) is to define it for $\mathcal{D}$-modules. It is easier to define first the set of slopes of this polygon. That is what is done in §5.1. Such a notion appears usually when one wants to compare many filtrations on the same object.

The structure theorem of formal regular meromorphic connections (subsection 5.2) is essentially taken from [7] and [19]. The quite simple proof of theorem 5.4.7 (*via* theorem 5.3.1) is taken from [33] (see now [35]). It is a small simplification of the proof in [24]. Other proofs are given in [26], [23], [25], [21] (see also [36]) and [22].

The notion of (moderate) nearby and vanishing cycles (section 6.1) is important for the classification of regular holonomic $\mathcal{D}$-modules in many variables. For the link with microsolutions the reader may consult [16] as well as the paper of L. Narvaez in [34]. The approach given here is due to Beilinson, Kashiwara, Malgrange.

2. Chapter II

§1.1 contains classical results. The proof of theorem 1.1.1 is taken from [7]. In §1.2 the various regularity criteria are easy consequences of theorem 1.1.1. Proposition 1.2.2 was stated in [16].

In §1.3, theorem 1.3.10 and its proof are due to Malgrange [19]. It has been generalized recently by Z. Mebkhout to the case of many variables (the positivity statement is replaced by a perversity statement concerning the *irregularity sheaf*).

The local index theorem of §1.4, which is a consequence of theorem 1.3.10, has been generalized by Kashiwara in [12]. In dimension bigger than one, the geometry of the characteristic variety is more complicated and *local Euler obstruction* enters in the formula for $\chi_{\mathrm{alg}}(\mathcal{M})$. The formula for the index given in proposition 1.4.6 is due to Deligne [11].

The presentation given in §2 is due to Malgrange [28], [33] (*see* also [27]). The paragraph on asymptotic expansions is a summary of a chapter in Wasow's book [7]. In that book the proof of theorem 2.3.1 does not clearly separate the formal case and the analytic case. That is why we have followed [27]. However, the proof of theorem 2.4.3 may be found in [7]. Another presentation as well as a generalization to the case power series satisfying Gevrey conditions are given in [29].

3. Chapter III

One should consult [10] for the first section of the chapter and [11], [35] for the second one.

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