**Claude Sabbah**

# NON-COMMUTATIVE HODGE STRUCTURES

## MAINZ, MARCH 29-31, 2012

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**2000 *Mathematics Subject Classification.*** 14D07, 34M40.

***Key words and phrases.*** Non-commutative Hodge structure, Fourier-Laplace transformation, Brieskorn lattice.

This research was supported by the grant ANR-08-BLAN-0317-01 of the Agence nationale de la recherche.
*March 29, 2012*

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***Abstract.*** These lectures survey recent results on a generalization of the notion of a Hodge structure. The main example is related to the Fourier-Laplace transform of a variation of polarizable Hodge structure on the punctured affine line, like the Gauss-Manin systems of a proper or tame algebraic function on a smooth quasi-projective variety. Variations of non-commutative Hodge structures often occur on the tangent bundle of Frobenius manifolds, giving rise to a tt* geometry. The notes closely follow the article with the same title, to appear in Ann. Institut Fourier (Grenoble), 2011, arXiv: `1107.5890`, and the reader will find precise references and more details therein.

# CONTENTS

# LECTURE 1

# WHY?

**Summary.** Why do non-commutative Hodge structures occur in algebraic geometry? After recalling the notion of a Hodge structure and that of variation of such objects, we give various reasons explaining the needs for a generalization of this notion. We give a simple linear algebra approach to the notion of nc. Hodge structure, but such an object does not occur in this way usually, so another approach, called "twistor", will be introduced in the next lecture, which will also be useful for understanding what a variation of nc. Hodge structure is.

## 1.1. Hodge structures and their variations

**1.1.1. Hodge structures.** Let $H$ be a finite dimensional complex vector space. A *complex Hodge structure* of weight $w \in \mathbb{Z}$ consists of a grading $H = \bigoplus_{p\in\mathbb{Z}} H^{p,w-p}$ (Hodge decomposition). Equivalently, it consists of a semi-simple endomorphism $\mathscr{Q}$ of $H$ with half-integral eigenvalues. The eigenspace of $\mathscr{Q}$ corresponding to the eigenvalue $p - w/2$, $p \in \mathbb{Z}$, is $H^{p,w-p}$. The role of the weight only consists in fixing the grading.

A *real Hodge structure* consists of a complex Hodge structure together with a $\mathbb{R}$-vector space $H_\mathbb{R}$ such that $H = \mathbb{C}\otimes_\mathbb{R} H_\mathbb{R}$, with respect to which $H^{w-p,p} = \overline{H^{p,w-p}}$. Then the matrix of $\mathscr{Q}$ in any basis of $H_\mathbb{R}$ is purely imaginary.

On the other hand, a *polarization* of a complex Hodge structure is a nondegenerate $(-1)^w$-Hermitian pairing $k$ on $H$ such that the Hodge decomposition is $k$-orthogonal and such that the Hermitian form $h$ on $H$ defined by $h_{|H^{p,w-p}} = i^{p-(w-p)}k_{|H^{p,w-p}} = i^{-w}(-1)^p k_{|H^{p,w-p}}$ is positive definite, in other words, defining the Weil operator $C$ by $e^{\pi i\mathscr{Q}}$, $h(u,v) = k(Cu,v)$. For a real Hodge structure, the real polarization $Q$ is then defined as the real part of $k$, and it is $(-1)^w$-symmetric on $H_\mathbb{R}$.

A $\mathbb{Q}$-Hodge structure of weight $w$ consists of a real Hodge structure of weight $w$ together with a $\mathbb{Q}$–structure $H_\mathbb{Q} \subset H_\mathbb{R}$. Important is the polarization, which is a rational bilinear form $S$ on $H_\mathbb{Q}$ which is $(-1)^w$-symmetric, such that $Q = (2\pi)^{-w}S$ is a polarization of the underlying real Hodge structure, that is, if we set $h(x,y) = (2\pi)^{-w}S(Cx,\overline{y})$, the gradation is $h$-orthogonal and $h$ is a positive definite Hermitian form $h$ on $H$.

***Example.*** Let $X$ be a smooth projective complex variety equipped with an ample line bundle. Then the primitive cohomology $H^k_{\mathrm{prim}}(X,\mathbb{Q})$ ($k\leqslant\dim X$) is equipped with a $\mathbb{Q}$-Hodge structure for which the polarization is defined from the Poincaré duality pairing and the cup product by the Chern class of the ample line bundle.

**1.1.2. Variations of Hodge structures and their limits.** Let $V$ be a holomorphic vector bundle on a complex manifold $X$, equipped with a *flat holomorphic connection* $\nabla:V\to\Omega^1_X\otimes V$. Let $\mathscr{V}=\ker\nabla$ the associated locally constant sheaf of $\mathbb{C}$-vector spaces, so that $V\simeq\mathscr{O}_X\otimes_{\mathbb{C}}\mathscr{V}$, and let $(H,\nabla+\overline{\partial})$ the associated $C^\infty$-bundle, so that $V=\ker\overline{\partial}:H\to\mathscr{A}^{(0,1)}_X\otimes H$. A *variation of complex Hodge structure* of weight $w$ consists of a grading $H=\bigoplus_p H^{p,w-p}$ by $C^\infty$ sub-bundles such that, for each $p$,

(1) $F^pH:=\bigoplus_{p'\geqslant p}H^{p,w-p}$ is stable by $\overline{\partial}$ and defines a holomorphic sub-bundle $F^pV\subset V$,

(2) (Griffiths transversality) $\nabla F^pV\subset\Omega^1_X\otimes F^{p-1}V$.

Note that the semi-simple operator $\mathscr{Q}$ on $H$ has constant eigenvalues.

A *variation of real Hodge structure* consists of the supplementary data of a real $C^\infty$ sub-bundle $H_{\mathbb{R}}\subset H$ which is $\nabla+\overline{\partial}$*-horizontal* or equivalently a real sub-local system $\mathscr{V}_{\mathbb{R}}\subset\mathscr{V}$ satisfying $\mathscr{V}=\mathbb{C}\otimes_{\mathbb{R}}\mathscr{V}_{\mathbb{R}}$, such that, at each $x\in X$, $H_{\mathbb{R},x}$ is a real structure on the complex Hodge structure $H_x=\bigoplus_p H_x^{p,w-p}$.

On the other hand, a *polarization* of a variation of complex Hodge structure is a $(\nabla+\overline{\partial})$*-horizontal* non-degenerate $(-1)^w$-Hermitian pairing $k$ on $H$, or equivalently a non-degenerate $(-1)^w$-Hermitian pairing $k$ on $\mathscr{V}$, so that $h(u,v):=k(e^{\pi i\mathscr{Q}}u,v)$ is a polarization at each $x\in X$, that is, $h$ is a Hermitian metric on $H$. For a real Hodge structure, the polarization $Q$ is defined as the real part of $k$ and is $(-1)^w$-symmetric on $\mathscr{V}_{\mathbb{R}}$.

A variation of $\mathbb{Q}$-Hodge structure of weight $w$ consists of a variation of $\mathbb{R}$-Hodge structure of weight $w$ together with a $\mathbb{Q}$-structure $\mathscr{V}_{\mathbb{Q}}\subset\mathscr{V}_{\mathbb{R}}$ and, in case of a polarization, a bilinear form $S$ on $\mathscr{V}_{\mathbb{Q}}$ which induces a polarization on each fibre.

Let $D$ be a divisor with normal crossings in $X$ and set $j:X^*:=X\smallsetminus D\hookrightarrow X$. Let $(V,\nabla,F^\bullet V,k)$ be a variation of polarized complex Hodge structure of weight $w$ on $X^*$.

***Theorem* (Regularity theorem).** *Under these assumptions, the subsheaf $(j_*V)^{\mathrm{lb}}$ of $j_*V$, consisting of local sections whose $h$-norm is locally bounded near $D$, is a locally free sheaf on which the connection $\nabla$ has at most logarithmic poles. Moreover, for each $p$, $j_*F^pV\cap(j_*V)^{\mathrm{lb}}$ (intersection taken in $j_*V$) is a locally free sheaf.*

***Example.*** Let $f:Y\to X$ be a smooth projective morphism. Then $R^kf_*\mathbb{Q}_Y$ is a polarized variation of $\mathbb{Q}$-Hodge structure.

## 1.2. The need for an extension of the notion of a Hodge structure

**1.2.1. Irregular Hodge theory.** Let $f : X \to \mathbb{A}^1$ a regular function on an algebraic manifold. What kind of a structure do we have on the twisted de Rham cohomology $H^*(X, f; \mathbb{C}) := \boldsymbol{H}^*\big((X, (\Omega_X^\bullet, \mathrm{d} + \mathrm{d}f\wedge)\big)$? This is related to the properties of exponential periods, i.e., integrals $\int_\Gamma e^f\omega$, $\omega$ an algebraic form on $X$, $\Gamma$ a locally closed cycle. The $\mathbb{Q}$-structure is understood: $H^*(X, f; \mathbb{Q}) = H^*(X, \mathrm{Re}(f) = +\infty; \mathbb{Q})$ (cf. Lecture 3).

Deligne remarked in 1984 that the classical formula

$$\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$$

suggests that for $f(x) = -x^2 : \mathbb{A}^1 \to \mathbb{A}^1$, $H^*(X, f; \mathbb{C})$ has a Hodge structure of type $(1/2, 1/2)$. In the case where $X$ is a curve, Deligne defines a filtration of the complex $(\Omega_X^\bullet, \mathrm{d} + \mathrm{d}f\wedge)$ indexed by rational numbers, and shows a $E_1$-degeneracy property, looking like the standard Hodge $\Rightarrow$ de Rham degeneacy.

**1.2.2. Fourier transform of a variation of Hodge structure.** Let $(V, \nabla, F^\bullet V, k)$ be a variation of polarized complex (or real, or rational) Hodge structure of weight $w$ on $\mathbb{A}^1 \smallsetminus \{p_1, \dots, p_r\}$ (or more generally on $\mathbb{A}^n \smallsetminus D$). In order to define its Fourier transform, one has first to extend $(V, \nabla)$ as a left regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module on the Weyl algebra with coordinate $t$ (more generally $t = (t_1, \dots, t_n)$). One chooses the *minimal extension* $M$, characterized by the property that $\mathrm{DR}\, M$ is the intermediate extension $\mathrm{IC}(\mathscr{V}_{\mathbb{C}})$.

Add a new variable $\tau$ and consider the cokernel

$${}^F\!M = \mathrm{coker}\left[M[\tau] \xrightarrow{\nabla_{\partial_t} - \tau} M[\tau]\right].$$

This is a $\mathbb{C}[\tau]$-module equipped with a connection $\nabla_{\partial_\tau}$ induced by the action: $\nabla_{\partial_\tau}(\sum_k m_k\tau^k) = \sum_k[(k+1)m_{k+1} - tm_k]\tau^k$. We get in this way a holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module with a regular singularity at $\tau = 0$, an irregular singularity at $\tau = \infty$ and no other singularity. According to the regularity theorem, its restriction to $\mathbb{A}^1_\tau \smallsetminus \{0\}$ cannot underly a variation of polarized Hodge structure. Does it underly a variation of some kind of Hodge structure?

This question was already asked by Katz and Laumon with analogy to the results for the Fourier-Deligne transformation of $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{A}^1_{\mathbb{F}_q}$.

**1.2.3. Hodge structure in Singularity theory (local and global aspects)**

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function with an isolated singularity. Steenbrink has produced a canonical mixed Hodge structure on the space of vanishing cycles $H^{n-1}(F, \mathbb{Q})$ ($F$ = Milnor fibre), which has been expressed by Varchenko in terms of the Brieskorn lattice. However, this mixed Hodge structure does not arise as a limit of a variation of polarized Hodge structure. Is it the limit of a variation of some kind of Hodge structure?

A similar question, suitably modified, can be raised in the case of a regular function with isolated singularities and tame at infinity (cf. Lecture 3).

As an application, when $f$ is the Landau-Ginzburg potential associated to a toric Fano manifold, the nc. Hodge structure constructed with the Brieskorn lattice allows one to construct a tt*-structure on the quantum cohomology of the toric Fano manifold.

**1.2.4. Hodge structures for non-commutative spaces.** The terminology "non-commutative Hodge structure" (which should not be confused with that of non-abelian Hodge theory developed by C. Simpson) has been introduced by Katzarkov-Kontsevich-Pantev to cover the kind of Hodge structure one should expect on the periodic cyclic cohomology of smooth compact non-commutative spaces. The point is that the periodic cyclic cohomology comes naturally equipped, as a $\mathbb{C}(\!(z)\!)$-vector space, with a connection having a pole of order two. There is an analogue of the degeneration of the Hodge $\Rightarrow$ de Rham spectral sequence, but many other properties are still lacking.

## 1.3. N.c. Hodge structures from the operator point of view

A *complex nc. Hodge structure* of weight $w \in \mathbb{Z}$, consists of the data $(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)$, where $\mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}$ are endomorphisms of $H$. When $w$ is fixed, these data form a category, where morphisms are linear morphisms $H \to H'$ commuting with the endomorphisms $\mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}$. For a complex Hodge structure, we have $\mathscr{U} = \mathscr{U}^\dagger = 0$ and $\mathscr{Q}$ is as above. The category of complex nc. Hodge structures of weight $w \in \mathbb{Z}$ is abelian.

***Example.*** Assume $\mathscr{U} = \mathscr{U}^\dagger = 0$ and $\mathscr{Q}$ is semi-simple. One can decompose $H = \bigoplus_{\lambda \in \mathbb{C}^*} H_\lambda$, where $H_\lambda$ is the $\lambda$-eigenspace of $e^{-2\pi i \mathscr{Q}}$. Then each $(H_\lambda, 0, 0, \mathscr{Q}, w)$ is a Hodge structure of weight $w$ and we can regard $(H, 0, 0, \mathscr{Q}, w)$ as a Hodge structure of weight $w$ equipped with a semi-simple automorphism, with eigenvalue $\lambda$ on $H_\lambda$.

In order to understand various operations on complex nc. Hodge structures, we associate to $(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)$ the $\mathbb{C}[z]$-module $\mathscr{H} = \mathbb{C}[z] \otimes_{\mathbb{C}} H$, with the connection

$$(*) \qquad \nabla = \mathrm{d} + \big(z^{-1}\mathscr{U} - (\mathscr{Q} + (w/2)\,\mathrm{Id}) - z\mathscr{U}^\dagger\big)\frac{\mathrm{d}z}{z}.$$

This connection has a (possibly irregular) singularity at $z = 0$ and $z = \infty$, and no other singularity. Duality and tensor product are defined in a natural way, according to the rules for connections. Hence

$$(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)^\vee = (H^\vee, -{}^t\mathscr{U}, -{}^t\mathscr{U}^\dagger, -{}^t\mathscr{Q}, -w),$$

and $(H_1, \mathscr{U}_1, \mathscr{U}_1^\dagger, \mathscr{Q}_1, w_1) \otimes (H_2, \mathscr{U}_2, \mathscr{U}_2^\dagger, \mathscr{Q}_2, w_2)$ has weight $w_1 + w_2$ and the endomorphisms are defined by formulas like $\mathscr{U}_1 \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes \mathscr{U}_2$. The involution $\iota : z \mapsto -z$ induces a functor $\iota^*$, with $\iota^*(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w) = (H, -\mathscr{U}, -\mathscr{U}^\dagger, \mathscr{Q}, w)$.

*Real nc. Hodge structures.* The complex conjugate of the complex nc. Hodge structure is defined as

$$\overline{(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)} := (\overline{H}, \overline{\mathscr{U}}^\dagger, \overline{\mathscr{U}}, -\overline{\mathscr{Q}}, w),$$

where $\overline{H}$ is the $\mathbb{R}$-vector space $H$ together with the conjugate complex structure. A real structure $\kappa$ on $(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)$ is an isomorphism from it to its conjugate, such that $\overline{\kappa} \circ \kappa = \mathrm{Id}$. A real structure consists therefore in giving a real structure $H_{\mathbb{R}}$ on $H$, with respect to which $\mathscr{U}^\dagger = \overline{\mathscr{U}}$ and $\overline{\mathscr{Q}} + \mathscr{Q} = 0$. We denote such a structure by $(H_{\mathbb{R}}, \mathscr{U}, \mathscr{Q}, w)$. Morphisms are $\mathbb{R}$-linear morphisms compatible with $\mathscr{U}$ and $\mathscr{Q}$. Real nc. Hodge structures $(H_{\mathbb{R}}, \mathscr{U}, \mathscr{Q}, w)$ satisfy properties similar to that of complex nc. Hodge structures and we have similar operations defined in a natural way.

*Polarization of a complex nc. Hodge structure.* A *polarization* of $(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)$ is a nondegenerate Hermitian form $h$ on $H$ such that

- $h$ is positive definite,
- $\mathscr{U}^\dagger$ is the $h$-adjoint of $\mathscr{U}$ and $\mathscr{Q}$ is self-adjoint with respect to $h$.

It is useful here to introduce the complex Tate object $\mathbb{T}_{\mathbb{C}}(\ell)$ defined as $(\mathbb{C}, 0, 0, 0, -2\ell)$ for $\ell \in \mathbb{Z}$, corresponding to the Hodge structure $\mathbb{C}^{-\ell,-\ell}$. The Tate twist by $\mathbb{T}_{\mathbb{C}}(\ell)$ is simply denoted by $(\ell)$. The last condition is equivalent to asking that $h$ defines an isomorphism

$$(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w) \xrightarrow{\sim} \iota^* \overline{(H, \mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}, w)}^\vee (-w).$$

The tensor product

$$(H_1, \mathscr{U}_1, \mathscr{U}_1^\dagger, \mathscr{Q}_1, h_1, w_1) \otimes (H_2, \mathscr{U}_2, \mathscr{U}_2^\dagger, \mathscr{Q}_2, h_2, w_2)$$

of polarized complex nc. Hodge structures is defined by the supplementary relation $h = h_1 \otimes h_2$, and is also polarized.

*Polarization of a real nc. Hodge structure and the Betti structure.* Although the notion of a real nc. Hodge structure seems to be defined over $\mathbb{R}$, the real vector space $H_{\mathbb{R}}$ does not contain the whole possible "real" information on the structure, in cases more general than that of a Hodge structure. The Weil operator is not defined in this setting. The formula $C = e^{\pi i \mathscr{Q}}$ for Hodge structures exhibits the Weil operator as a square root of the monodromy of the connection $\mathrm{d} - \mathscr{Q}\mathrm{d}z/z$. This suggests that the monodromy of the connection $\nabla$ defined by $(*)$ should be taken into account in order to properly define the notion of a real nc. Hodge structure, and further, that of a nc. $\mathbb{Q}$-Hodge structure. Even further, if $\nabla$ has an irregular singularity, the Betti real structure is encoded in the Stokes data attached to the connection, not only in the monodromy, together with the notion of a $\mathbb{Q}$-Betti structure.

Here is another drawback of the presentation of a complex nc. Hodge structure as a vector space with endomorphisms: the notion of a variation of such objects is not defined in a holomorphic way, exactly as the spaces $H^{p,w-p}$ do not vary holomorphically in classical Hodge theory. Good variations are characterized by the property of the Hermitian metric to be harmonic, and the endomorphisms $\mathscr{U}, \mathscr{Q}$ satisfy relations encoded in the notion of a *CV structure*. The "twistor" approach of the next lecture will be more adapted to the notion of a variation of nc. $\mathbb{Q}$-Hodge structure.

# LECTURE 2

# WHAT?

**Summary.** What is... a non-commutative Hodge structure? We give the "twistor" definition of a nc. $\mathbb{Q}$-Hodge structure with details. In order to do so, we first recall some known results on bundles with meromorphic connections on a neighbourhood of the origin in the complex line, and we describe gluing procedures to get bundles with meromorphic connections on the Riemann sphere.

## 2.1. Meromorphic connections with a pole of order $\leqslant 2$

**2.1.1. Exponential type with no ramification.** Let $z$ be a coordinate on the complex line. We will consider four kinds of objects, written similarly $(\mathscr{H}, \nabla)$:

(1) $\mathscr{H}$ is a free $\mathbb{C}\{z\}$-module of finite rank and $\nabla$ is a meromorphic connection $\nabla_{\partial_z} : \mathscr{H} \to \mathbb{C}(\{z\}) \otimes \mathscr{H}$ with a pole of order $\leqslant 2$, that is, $\nabla_{\partial_z}(\mathscr{H}) \subset \frac{1}{z^2}\mathscr{H}$.

(2) $\mathscr{H}$ is a free $\mathscr{O}_{\mathbb{C}}$-module of finite rank and $\nabla$ is a meromorphic connection $\nabla_{\partial_z} : \mathscr{H} \to \mathscr{O}_{\mathbb{C}}(2 \cdot 0) \otimes \mathscr{H}$ with a pole of order $\leqslant 2$ at $z = 0$ and no other pole.

(3) $\mathscr{H}$ is a free $\mathscr{O}_{\mathbb{A}^1}$-module of finite rank and $\nabla$ is a meromorphic connection $\nabla_{\partial_z} : \mathscr{H} \to \mathscr{O}_{\mathbb{A}^1}(2 \cdot 0) \otimes \mathscr{H}$ with a pole of order $\leqslant 2$ at $z = 0$, no other pole at finite distance, and a regular singularity at infinity.

(4) $\mathscr{H}$ is a free $\mathbb{C}[z]$-module of finite rank and $\nabla$ is a meromorphic connection $\nabla_{\partial_z} : \mathscr{H} \to \frac{1}{z^2} \otimes \mathscr{H}$, and a regular singularity at infinity.

All four categories of objects are known to be equivalent: by the sheafification functor for the Zariski topology (4) $\mapsto$ (3), the analytization functor (3) $\mapsto$ (2) (this uses Deligne's canonical extension at infinity) and the germ at the origin functor (2) $\mapsto$ (1), and I will often not distinguish between them.

***Assumption* (Exponential type with no ramification).** *There exists a finite subset $C \subset \mathbb{C}$ such that*

$$\mathbb{C}[\![z]\!] \otimes_{\mathbb{C}\{z\}} (\mathscr{H}, \nabla) \simeq \bigoplus_{c \in C} \Big[\mathbb{C}[\![z]\!] \otimes_{\mathbb{C}\{z\}} (\mathscr{H}_c, \nabla_c)\Big],$$

*where $\nabla_c - c\,\mathrm{Id}\; dz/z^2$ has a regular singularity.*

**2.1.2. Stokes structures.** The local system $\mathscr{L}$ attached to $(\mathscr{H},\nabla)_{|\mathbb{C}^*}$ comes equipped with a family of pairs of nested subsheaves $\mathscr{L}_{<c} \subset \mathscr{L}_{\leqslant c}$ for each $c \in \mathbb{C}$, which satisfies the properties below. We will say that $(\mathscr{L},\mathscr{L}_\bullet)$ is a *Stokes-filtered local system.* Note that these properties can be defined for a local system over any field (and can be adapted over a ring), and we will use the notion of Stokes-filtered $\mathbb{Q}$-local system. For a fixed $z \in \mathbb{C}^*$, define a partial order $\leqslant_z$ on $\mathbb{C}$ compatible with addition by setting $c \leqslant_z 0$ iff $c = 0$ or $\mathrm{Re}(c/z) < 0$ (and $c <_z 0$ iff $c \neq 0$ and $\mathrm{Re}(c/z) < 0$). This partial order on $\mathbb{C}$ only depends on $z/|z| \in S^1$. The required properties are as follows.

- For each $z \in \mathbb{C}^*$, the germs $\mathscr{L}_{\leqslant c,z}$ form an exhaustive increasing filtration of $\mathscr{L}_z$, compatible with the order $\leqslant_z$.
- For each $z \in \mathbb{C}^*$, the germ $\mathscr{L}_{<c,z}$ can be recovered as $\sum_{c'<_z c} \mathscr{L}_{\leqslant c',z}$.
- The graded sheaves $\mathscr{L}_{\leqslant c}/\mathscr{L}_{<c}$ are local systems on $\mathbb{C}^*$.
- The rank of $\bigoplus_{c\in\mathbb{C}} \mathscr{L}_{\leqslant c}/\mathscr{L}_{<c}$ is equal to the rank of $\mathscr{L}$, so that both local systems are locally isomorphic, and there is only a finite set $C \subset \mathbb{C}$ of jumping indices.

A $\mathbb{Q}$-*structure* consists of a Stokes-filtered local system $(\mathscr{L}_\mathbb{Q},\mathscr{L}_{\mathbb{Q},\bullet})$ defined over $\mathbb{Q}$ such that $(\mathscr{L},\mathscr{L}_\bullet) = \mathbb{C}\otimes_\mathbb{Q}(\mathscr{L}_\mathbb{Q},\mathscr{L}_{\mathbb{Q},\bullet})$. In particular, the monodromy of $\mathscr{L}$ is defined over $\mathbb{Q}$. On the other hand, one can define a Riemann-Hilbert functor $(\mathscr{G},\nabla) := \mathscr{O}(*0)\otimes_\mathscr{O}(\mathscr{H},\nabla) \mapsto (\mathscr{L},\mathscr{L}_\bullet)$, which is an equivalence of categories compatible with duality and tensor product.

The decomposition in the assumption is unique, and the formalization functor $(\mathscr{G},\nabla) \mapsto \mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}(\{z\})}(\mathscr{G},\nabla)$ corresponds, via the Riemann-Hilbert functor $(\mathscr{G},\nabla) \mapsto (\mathscr{L},\mathscr{L}_\bullet)$ to the Stokes grading functor $(\mathscr{L},\mathscr{L}_\bullet) \mapsto \bigoplus_{c\in\mathbb{C}} \mathscr{L}_{\leqslant c}/\mathscr{L}_{<c}$, so that the local system associated to $(\mathscr{H}_c,\nabla_c)_{|\mathbb{C}^*}$ is $\mathrm{gr}_c\mathscr{L}$. As a consequence, if we define the notion of a $\mathbb{Q}$-structure on $(\mathscr{H},\nabla)$ as a $\mathbb{Q}$-structure on the associated Stokes-filtered local system, such a $\mathbb{Q}$-structure induces a $\mathbb{Q}$-structure on each $(\mathscr{H}_c,\nabla_c)$.

***Remark.*** One can give an equivalent presentation in terms of Stokes matrices.

## 2.2. Gluing of vector bundles

We start with $(\mathscr{H},\nabla)$ as in § 2.1.1, and we denote by $(\mathscr{L},\mathscr{L}_\bullet)$ the Stokes-filtered local system associated to $(\mathscr{G},\nabla)$. It will be convenient and equivalent to regard $\mathscr{L}$ as a local system on $S^1 = \{|z| = 1\}$, since the Stokes structure only depends on $z/|z|$.

*Gluing with a real structure.* Assume $\mathscr{L}$ defined over $\mathbb{R}$ (or $\mathbb{Q}$), i.e., $\mathscr{L} \simeq \overline{\mathscr{L}}$.

- Set $\gamma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$, $z \mapsto 1/\overline{z}$. Notice $\gamma_{|S^1} = \mathrm{Id}$.
- Glue $\mathscr{H}$ with $\gamma^*\overline{\mathscr{H}}$ to get $\widetilde{\mathscr{H}}$ (holomorphic vector bundle on $\mathbb{P}^1$):

$$\mathscr{H}_{|S^1} = \mathscr{O}_{|S^1}\otimes\mathscr{L} \simeq \mathscr{O}_{|S^1}\otimes\overline{\mathscr{L}} = (\gamma^*\overline{\mathscr{H}})_{|S^1}$$

*Gluing with a sesquilinear pairing.* Assume we are given an isomorphism $\mathcal{C} : \mathscr{L}^\vee \simeq \iota^{-1}\overline{\mathscr{L}}$, where $\iota$ is the involution $z \mapsto -z$. We call it a nondegenerate $\iota$-sesquilinear pairing on $\mathscr{L}$, since we can regard it as a pairing

$$\mathcal{C} : \mathscr{L} \otimes \iota^{-1}\overline{\mathscr{L}} \longrightarrow \mathbb{C}_{S^1}.$$

- Set $\sigma := \gamma \circ \iota : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$, $z \mapsto -1/\overline{z}$.
- Glue $\mathscr{H}^\vee$ with $\sigma^*\overline{\mathscr{H}}$ to get $\widehat{\mathscr{H}}$ (holomorphic vector bundle on $\mathbb{P}^1$):

$$\mathscr{H}^\vee_{|S^1} = \mathscr{O}_{|S^1} \otimes \mathscr{L}^\vee \overset{\mathcal{C}}{\simeq} \mathscr{O}_{|S^1} \otimes \iota^{-1}\overline{\mathscr{L}} = (\sigma^*\overline{\mathscr{H}})_{|S^1}.$$

Assume moreover that $\mathcal{C} : \mathscr{L}^\vee \simeq \iota^{-1}\overline{\mathscr{L}}$ is $\iota$-Hermitian. Then the construction produces a natural isomorphism $\mathscr{S} : \widehat{\mathscr{H}}^\vee \to \sigma^*\overline{\widehat{\mathscr{H}}}$ and therefore an isomorphism $\Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})^\vee \xrightarrow{\sim} \Gamma(\mathbb{P}^1, \sigma^*\overline{\widehat{\mathscr{H}}}) = \overline{\Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})}$. If $\widehat{\mathscr{H}}$ is trivializable, then $H := \Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})$ comes equipped with a Hermitian form $h = \Gamma(\mathbb{P}^1, \mathscr{S})$.

*Comparison.* Assume

- we are given a real structure $\mathscr{L} \simeq \overline{\mathscr{L}}$ on $\mathscr{L}$,
- we are given a non-degenenerate pairing $\mathcal{Q}_\mathbb{R} : \mathscr{L}_\mathbb{R} \otimes \iota^{-1}\mathscr{L}_\mathbb{R} \to \mathbb{R}$, and a non-degenenerate $\mathscr{O}$-bilinear pairing $\mathcal{Q} : (\mathscr{H}, \nabla) \otimes \iota^*(\mathscr{H}, \nabla) \to (z^{-w}\mathscr{O}, \mathrm{d})$ for some $w \in \mathbb{Z}$, so that $\mathcal{Q}_\mathbb{C}$ corresponds to the restriction of $\mathcal{Q}$ to $(\mathscr{G}, \nabla)$ by the Riemann-Hilbert corresopndence.

Then, on the one hand, from the real structure on $\mathscr{L}$ we get $\widetilde{\mathscr{H}}$. On the other hand, $\mathcal{Q}$ defines a non-degenerate $\iota$-sesquilinear pairing $\mathcal{C}$, hence $\widehat{\mathscr{H}}$.

***Lemma* (C. Hertling).** *Under these assumptions,* $\mathcal{Q}$ *induces an isomorphism*

$$\widetilde{\mathscr{H}} \xrightarrow{\sim} \iota^*\widehat{\mathscr{H}} \otimes \mathscr{O}_{\mathbb{P}^1}(w).$$

***Consequence*.** $\widetilde{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}(w)^{\operatorname{rk}\mathscr{H}} \iff \widehat{\mathscr{H}}$ *is trivial.*

## 2.3. Non-commutative Hodge structures

**2.3.1. Hodge structures from the twistor point of view.** Let $z$ be a new variable. Then the decreasing filtration $F^\bullet H$ defined by $F^pH = \bigoplus_{p' \geqslant p} H^{p', w-p'}$ allows one to define a free $\mathbb{C}[z]$-module $\mathscr{H} = \bigoplus_p F^pHz^{-p}$, which satisfies $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} \mathscr{H} = \mathbb{C}[z, z^{-1}] \otimes_\mathbb{C} H$. It is equipped with a connection $\nabla$ (induced by the differential $d$) which has a pole of order one on $\mathscr{H}$. The local system $\ker\nabla$ on $\mathbb{C}^* = \{z \neq 0\}$ is trivial (monodromy equal to identity) with fibre $H$, and it has a rational constant sub local system with fibre $H_\mathbb{Q}$.

Let $\gamma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$ be as in § 2.2. Then $\gamma^*\overline{\mathscr{H}} = \sum_q \overline{F^qH}z^q$ is a $\mathbb{C}[z^{-1}]$-free module, and $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} \gamma^*\overline{\mathscr{H}} = \mathbb{C}[z, z^{-1}] \otimes_\mathbb{C} \overline{H} \simeq \mathbb{C}[z, z^{-1}] \otimes_\mathbb{C} H$, due to the real structure. One can then glue the bundles $\mathscr{H}$ and $\gamma^*\overline{\mathscr{H}}$ into a holomorphic bundle $\widetilde{\mathscr{H}}$ on $\mathbb{P}^1$. The opposedness (or gradation) property is then equivalent to the property that $\widetilde{\mathscr{H}}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(w)^{\dim H}$.

We will follow this approach for defining a nc. $\mathbb{Q}$-Hodge structure.

### 2.3.2. Definition of a polarized nc. $\mathbb{Q}$-Hodge structure

***Definition.*** Data:

- $(\mathscr{H}, \nabla)$ having a pole of order two with no ramification,
- $(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})$ a Stokes-filtered $\mathbb{Q}$-local system on $S^1$,
- a pairing

$$\mathcal{Q}_{\mathrm{B}} : (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \otimes \iota^{-1}(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \longrightarrow (\mathbb{Q}_{S^1}, \mathbb{Q}_{S^1,\bullet})$$

($(\mathbb{Q}_{S^1}, \mathbb{Q}_{S^1,\bullet})$: trivial Stokes filtration on $\mathbb{Q}_{S^1}$).

We say that $\big((\mathscr{H}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \mathcal{Q}_{\mathrm{B}}\big)$ is a *polarized nc. $\mathbb{Q}$-Hodge structure of weight $w$* if it satisfies the following properties:

(1) $\mathcal{Q}_{\mathrm{B}}$ is non-degenerate $(-1)^w$-$\iota$-symmetric (in particular, it induces a non-degenerate $(-1)^w$-$\iota$-symmetric pairing on each local system $\mathrm{gr}_c \mathscr{L}_{\mathbb{Q}}$).

(2) The $(-1)^w$-$\iota$-symmetric pairing $\mathcal{Q}$ that $\mathcal{Q}_{\mathrm{B}}$ induces on $(\mathscr{G}, \nabla) = \mathscr{O}(*0) \otimes (\mathscr{H}, \nabla)$ through the RH correspondence, which takes values in $\mathscr{O}_{\mathbb{C}}(*0)$ satisfies:

$$\mathcal{Q}\big(\mathscr{H} \otimes \iota^* \mathscr{H}\big) \subset z^{-w} \mathscr{O}_{\mathbb{C}}$$

and is non-degenerate as such.

(3) Letting $\mathcal{C}$ be the $\iota$-sesquilinear pairing associated to $i^{-w}\mathcal{Q}$ (hence $\mathcal{C}$ is a nondegenerate $\iota$-Hermitian pairing), then

(a) $\widehat{\mathscr{H}}$ is trivial (i.e., $\widetilde{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}(w)^d$, opposedness),

(b) $h := \Gamma(\mathbb{P}^1, \mathscr{S})$ is positive definite on $H := \Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})$ (polarisation).

***Remark.*** In a $\mathscr{O}$-basis of $\mathscr{H}$ induced by a $\mathbb{C}$-basis of $H$, the connection $\nabla$ takes the form $(*)$ given in § 1.3.

## 2.4. A few words about variations of polarized nc. $\mathbb{Q}$-Hodge structures

Let $X$ be a complex manifold. A *variation of polarized nc. $\mathbb{Q}$-Hodge structure of weight $w$* parametrized by $X$ consists of the following data:

- $(\mathscr{H}, \nabla)$ a holomorphic bundle on $X \times \mathbb{C}$ with a flat meromorphic connection $\nabla$ having poles of Poincaré rank one along $\{z = 0\}$ (i.e., $z\nabla$ is logarithmic) and no other pole,
- A Stokes-filtered $\mathbb{Q}$-local system $(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})$ on $X \times S^1$,
- a pairing

$$\mathcal{Q}_{\mathrm{B}} : (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \otimes \iota^{-1}(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \longrightarrow (\mathbb{Q}_{X\times S^1}, \mathbb{Q}_{X\times S^1,\bullet})$$

($(\mathbb{Q}_{X\times S^1}, \mathbb{Q}_{X\times S^1,\bullet})$: trivial Stokes filtration on $\mathbb{Q}_{X\times S^1}$),

subject to the condition that, for each $x \in X$, the restriction to $\{x\} \times \mathbb{C}$ is a polarized nc. $\mathbb{Q}$-Hodge structure of weight $w$.

| Hodge structures | Hodge structures (Nc) | Nc. Hodge structures |
|---|---|---|
| Filtered vect. sp. $(H, F^\bullet H)$ | $(\mathscr{H}, \nabla) = \oplus_p(F^pHz^{-p}, \mathrm{d})$ free $\mathbb{C}[z]$-mod + connect. | $\mathscr{H}$ free $\mathscr{O}_{\mathbb{A}^1}$-mod., $\nabla$: connect., $0$ = only pole, ord. $\leqslant 2$, n.r. |
| $H$ | $\mathscr{L} = \ker\nabla$ on $\mathscr{H}_{\vert S^1}$ | idem |
| $H_{\mathbb{Q}}$ | $\mathscr{L}_{\mathbb{Q}}$ (cst $\mathbb{Q}$-loc. syst. on $S^1$) | $\mathscr{L}_{\mathbb{Q},\bullet}$ (Stokes-filt. $\mathbb{Q}$-loc. syst.) |
| $F^pH \cap \overline{F^{w-p+1}H} = 0\ \forall p$ | $\widetilde{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}(w)^{\mathrm{rk}\,\mathscr{H}}$ | idem |
| $Q$: $(-1)^w$-sym. non-deg. $\mathbb{Q}$-bilin. form. | $\mathcal{Q} : \mathscr{L}_{\mathbb{Q}} \otimes \iota^{-1}\mathscr{L}_{\mathbb{Q}} \to \mathbb{Q}$ $(-1)^w$-$\iota$-sym. non-deg. | $\mathcal{Q} : \mathscr{L}_{\mathbb{Q},\bullet} \otimes \iota^{-1}\mathscr{L}_{\mathbb{Q},\bullet} \to \mathbb{Q}_\bullet$ idem |
| s.t. $Q(H^{p,q}, H^{p',q'}) = 0$ for $p' \neq w-p$ | $(\mathscr{H}, \nabla) \otimes \iota^*(\mathscr{H}, \nabla) \to (z^{-w}\mathscr{O}, \mathrm{d})$ non-deg. | idem idem |
| $Q \rightsquigarrow h$ Herm. form | $\mathcal{Q} \rightsquigarrow \mathcal{C}$: $\iota$-Herm. on $\mathscr{L}$ | idem |
| $H$ | $\widehat{\mathscr{H}}$ trivial, $H := \Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})$ | idem |
| $h$ def. $> 0$ | $h$ def. $> 0$ on $\Gamma(\mathbb{P}^1, \widehat{\mathscr{H}}) \simeq H$ | idem |

TABLE 1. Comparison table

***Example* (The rescaling of a nc. $\mathbb{Q}$-Hodge structure).** C. Hertling has considered the action of $\mathbb{C}^*$ on the category of connections $(\mathscr{H}, \nabla)$ with a pole of order two obtained by rescaling the variable $z$. For $x \in \mathbb{C}^*$, consider the map $\mu_x : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $\mu_x(z) = xz$. The rescaled connection is $\mu_x^*(\mathscr{H}, \nabla)$. Since

$$\mu_x^{-1}(S^1) = \{z \mid |xz| = 1\} \neq S^1 \quad \text{if } |x| \neq 1,$$

we define the pull-back local system $\mu_x^{-1}\mathscr{L}$ by working with local systems on $\mathbb{C}^*$ (recall that the inclusion $S^1 \hookrightarrow \mathbb{C}^*$ induces an equivalence of categories of local systems on the corresponding spaces).

The rescaling acts on the category of objects $(\mathscr{H}, \nabla, \mathscr{L}_{\mathbb{R}})$ (by the same procedure as above), on the category of objects $(\mathscr{H}, \nabla, \mathcal{C})$ since $\iota$ commutes with $\mu_x$, and similarly on the category of objects $(\mathscr{H}, \nabla, \mathscr{L}_{\mathbb{Q}}, \mathcal{Q}_{\mathrm{B}})$ (without paying attention to the nc. Hodge property at the moment). It also acts on Stokes-filtered local systems $(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})$ in a way compatible, by the Riemann-Hilbert correspondence, to the action on the meromorphic bundles $(\mathscr{G}, \nabla)$: the subsheaf $(\mu_x^{-1}\mathscr{L})_{\mathbb{Q},\leqslant c}$ is defined as $\mu_x^{-1}(\mathscr{L}_{\mathbb{Q},\leqslant c/x})$.

If $(\mathscr{H}, \nabla, \mathscr{L}_{\mathbb{R}})$ is pure of weight $w$ (resp. if $(\mathscr{H}, \nabla, \mathcal{C})$ is pure of weight $0$ and polarized, resp. if $(\mathscr{H}, \nabla, \mathscr{L}_{\mathbb{Q}}, \mathcal{Q}_{\mathrm{B}})$ is pure of weight $w$ and polarized) then, *provided* $|x-1|$ *is small enough*, the corresponding rescaled object remains pure (resp. pure and polarized) of the same weight: this follows from the rigidity of trivial bundles on $\mathbb{P}^1$. In this way, we obtain a variation of polarized nc. $\mathbb{Q}$-Hodge structure parametrized by some open neighbourhood of 1 in $\mathbb{C}^*$. On the other hand, this may not remain true for all values of the rescaling parameter $x$.

The subcategory of pure polarized nc. Hodge structures which remain so by rescaling by any $x \in \mathbb{C}^*$ is a global analogue of a nilpotent orbit in the theory of variation of polarized Hodge structures.

# LECTURE 3

# HOW?

**Summary.** How do the nc. Hodge structures are produced in a natural way? The basic construction comes from the Fourier-Laplace transform of a variation of polarized Hodge structure in dimension one. From this construction one deduces nc. Hodge properties of more geometric objects as the exponentially twisted de Rham cohomology.

## 3.1. Producing a nc. $\mathbb{Q}$-Hodge structure by Fourier-Laplace transformation

*Starting point.* Let $C \subset \mathbb{A}^1$ be a finite set of points on the complex affine line with coordinate $t$. Let $(\mathcal{V}_\mathbb{Q}, F^\bullet V, \nabla, Q_\mathrm{B})$ be a variation of polarized Hodge structure of weight $w \in \mathbb{Z}$ on $X := \mathbb{A}^1 \smallsetminus C$. Namely,

- $(V, \nabla)$ is a holomorphic vector bundle with connection on $X$,
- $F^\bullet V$ is a finite decreasing filtration of $V$ by holomorphic sub-bundles satisfying the Griffiths transversality property: $\nabla F^p V \subset F^{p-1} V \otimes_{\mathscr{O}_X} \Omega^1_X$,
- $\mathcal{V}_\mathbb{Q}$ is a $\mathbb{Q}$-local system on $X$ with $\mathcal{V}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = V^\nabla$,
- $Q_\mathrm{B} : \mathcal{V}_\mathbb{Q} \otimes_\mathbb{Q} \mathcal{V}_\mathbb{Q} \to \mathbb{Q}$ is a nondegenerate $(-1)^w$-symmetric pairing,

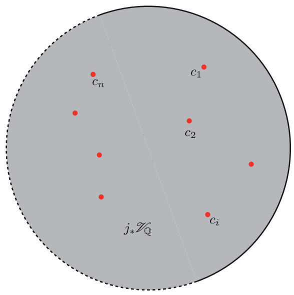
all these data being such that the restriction at each $x \in X$ is a polarized Hodge structure of weight $w$ (cf. § 1.1.2). We denote by $Q$ the nondegenerate flat pairing $(V, \nabla) \otimes (V, \nabla) \to (\mathscr{O}_X, d)$ that we get from $Q_\mathrm{B}$ through the canonical isomorphism $\mathscr{O}_X \otimes_\mathbb{Q} \mathcal{V}_\mathbb{Q} = V$. The associated nondegenerate sesquilinear pairing is denoted by $k : (V, \nabla) \otimes_\mathbb{C} \overline{(V, \nabla)} \to \mathscr{C}^\infty_X$, which can also be obtained from $k_\mathrm{B} : \mathcal{V} \otimes_\mathbb{C} \overline{\mathcal{V}} \to \mathbb{C}$ similarly. It is $(-1)^w$-Hermitian and $i^{-w} k$ induces a flat Hermitian pairing on the $C^\infty$-bundle $(\mathscr{C}^\infty_X \otimes_{\mathscr{O}_X} V, \nabla + \overline{\partial})$. We can regard $(V, \nabla, F^\bullet V, i^{-w} k)$ as a variation of polarized complex Hodge structure, pure of weight 0.

***Theorem.*** *Let $(\mathcal{V}_\mathbb{Q}, F^\bullet V, \nabla, Q_\mathrm{B})$ be a variation of polarized $\mathbb{Q}$-Hodge structure of weight $w \in \mathbb{Z}$ on $X := \mathbb{A}^1 \smallsetminus C$. Then its Fourier-Laplace transform $((\mathscr{H}, \nabla), (\mathscr{L}_\mathbb{Q}, \mathscr{L}_{\mathbb{Q},\bullet}), -\widehat{j_* Q_\mathrm{B}})$ is a pure polarized nc. $\mathbb{Q}$-Hodge structure of weight $w+1$.*

- Fix $z_o \in S^1$. Define $\Phi_{z_o}$ as the family of closed sets $S \subset \mathbb{A}^1$ such that

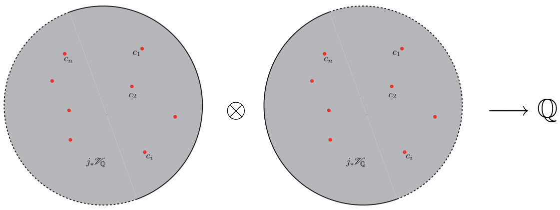
$$\overline{S} \cap \{(\infty, e^{i\theta}) \mid \mathrm{Re}(e^{i\theta}/z_o) \geqslant 0\} = \varnothing \quad \text{in } \mathbb{A}^1 \cup S^1_\infty.$$

- $(\mathscr{L}_{\mathbb{Q}})_{z_o} = H^1_{\Phi_{z_o}}(\mathbb{A}^1, j_*\mathcal{V}_{\mathbb{Q}}) = H^1(\widetilde{\mathbb{P}}^1, \beta_!\alpha_*j_*\mathscr{V}_{\mathbb{Q}})$ (with $\widetilde{\mathbb{P}}^1 = \mathbb{C} \cup S^1_\infty$):



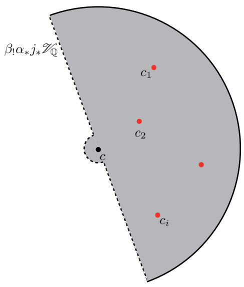

- $(\widehat{j_*Q_{\mathrm{B}}})_{z_o}$ is the cup product followed by $Q_{\mathrm{B}}$ (note: $\Phi_{z_o} \cap \Phi_{-z_o} =$ family of compact sets in $\mathbb{A}^1$):

$$H^1_{\Phi_{z_o}}(\mathbb{A}^1, j_*\mathcal{V}_{\mathbb{Q}}) \otimes H^1_{\Phi_{-z_o}}(\mathbb{A}^1, j_*\mathcal{V}_{\mathbb{Q}}) \longrightarrow H^2_{\mathrm{c}}(\mathbb{A}^1, \mathbb{Q}) \simeq \mathbb{Q},$$

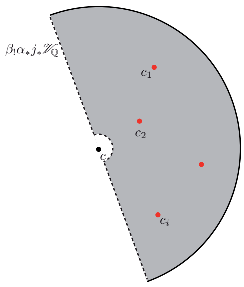



Note that here appears the involution $\iota$.

- For $c \in \mathbb{C}$, we define $\mathscr{L}_{\mathbb{Q},\leqslant c,z}$ and $\mathscr{L}_{\mathbb{Q},<c,z}$ by using a picture similar to the above one:




$\mathscr{L}_{\mathbb{Q},\leqslant c,z}$




$\mathscr{L}_{\mathbb{Q},<c,z}$

*Intermediate step for the de Rham and Hodge side*

*De Rham side*: The bundle $(V, \nabla)$ can be extended in a unique way as a free $\mathscr{O}_{\mathbb{P}^1}(*C \cup \{\infty\})$-module with a connection $\nabla$ having a regular singularity at $C \cup \{\infty\}$ (Deligne's meromorphic extension). Taking global sections on $\mathbb{P}^1$ produces a left module $\widetilde{M}$ on the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$. The minimal extension (along $C$) of $\widetilde{M}$ is the unique submodule $M$ of $\widetilde{M}$ which coincides with $\widetilde{M}$ after tensoring both by $\mathbb{C}(t)$, and which has no quotient submodule supported in $C$ (it is characterized by the property that $\mathrm{DR}^{\mathrm{an}} M = j_*\mathscr{V}$). The pairing $k$ extends first (due to the regularity of the connection) as a pairing $\widetilde{k} : \widetilde{M} \otimes_{\mathbb{C}} \overline{\widetilde{M}} \to \mathscr{S}'(\mathbb{A}^1 \smallsetminus C)$, where

$\mathscr{S}'(\mathbb{A}^1)$ denotes the Schwartz space of temperate distributions on $\mathbb{A}^1 = \mathbb{R}^2$, and $\mathscr{S}'(\mathbb{A}^1 \smallsetminus C) := \mathbb{C}\big[t, \prod_{c\in C}(t-c)^{-1}\big] \otimes_{\mathbb{C}[t]} \mathscr{S}'(\mathbb{A}^1)$. Then one shows that, when restricted to $M \otimes_{\mathbb{C}} \overline{M}$, $\widetilde{k}$ takes values in $\mathscr{S}'(\mathbb{A}^1)$, and we denote it by $k$.

*Hodge side*: The Hodge filtration $F^\bullet V$ extends, according to a procedure due to M. Saito and relying on Schmid's theory of limits of variations of polarized Hodge structures, to a good filtration $F^\bullet M$ of $M$ as a $\mathbb{C}[t]\langle\partial_t\rangle$-module.

*How to get $(\mathscr{G}, \nabla)$: Laplace transform of $M$.* Set $G = \mathbb{C}[t]\langle\partial_t, \partial_t^{-1}\rangle \otimes_{\mathbb{C}[t]\langle\partial_t\rangle} M$, and define the action of $\mathbb{C}[z, z^{-1}]\langle\partial_z\rangle$ on $G$ as follows: $z\cdot m = \partial_t^{-1}m$, $z^{-1}\cdot m = \partial_t m$, and $z^2\partial_z m = tm$. One can show that $G$ is a free $\mathbb{C}[z, z^{-1}]$-module, and the action of $\partial_z$ is that of a connection (i.e., satisfies Leibniz rule). Its analytization as a free $\mathscr{O}(*0)$-module with connection is denoted by $(\mathscr{G}, \nabla)$.

*How to get $(\mathscr{H}, \nabla)$: the Brieskorn lattice of the filtration $F^\bullet M$.* We denote by $\widehat{\mathrm{loc}}$ : $M \to G$ the natural morphism (the kernel and cokernel of which are isomorphic to powers of $\mathbb{C}[t]$ with its natural structure of left $\mathbb{C}[t]\langle\partial_t\rangle$-module). For any lattice $L$ of $M$, i.e., a $\mathbb{C}[t]$-submodule of finite type such that $M = \mathbb{C}[\partial_t]\cdot L$, we define the associated Brieskorn lattice as

$$G_0^{(L)} = \sum_{j\geqslant 0} \partial_t^{-j}\widehat{\mathrm{loc}}(L).$$

This is a $\mathbb{C}[\partial_t^{-1}]$-submodule of $G$. Moreover, because of the relation $[t, \partial_t^{-1}] = (\partial_t^{-1})^2$, it is naturally equipped with an action of $\mathbb{C}[t]$. If $M$ has a regular singularity at infinity, then $G_0^{(L)}$ has finite type over $\mathbb{C}[\partial_t^{-1}]$. We have $G = \mathbb{C}[\partial_t]\cdot G_0^{(L)}$.

Let us now consider a filtered $\mathbb{C}[t]\langle\partial_t\rangle$-module. Let $p_0 \in \mathbb{Z}$. We say that $F^\bullet M$ *is generated by* $F^{p_0}M$ if, for any $\ell \geqslant 0$, we have $F^{p_0-\ell}M = F^{p_0}M + \cdots + \partial_t^\ell F^{p_0}M$. Then $F^{p_0}M$ is a lattice of $M$. Moreover, the $\mathbb{C}[\partial_t^{-1}]$-module $\partial_t^{p_0}G_0^{(F^{p_0})}$ does not depend on the choice of the index $p_0$, provided that the generating assumption is satisfied. We thus define the *Brieskorn lattice of the filtration* $F^\bullet M$ as

$$G_0^{(F)} = \partial_t^{p_0}G_0^{(F^{p_0})} \quad \text{for some (or any) index } p_0 \text{ of generation.}$$

If we also set $z = \partial_t^{-1}$, then one can show that $G_0^{(F)}$ is a free $\mathbb{C}[z]$-module which satisfies $G = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} G_0^{(F)}$ and which is stable by the action of $z^2\partial_z := t$. Its analytization $\mathscr{H}$ is a free $\mathscr{O}$-module on which the connection has a pole of order $\leqslant 2$.

*A direct definition of $\mathcal{C}$: the Fourier transformation.* Set $z' = z^{-1}$ (it corresponds to $\partial_t$ in the Laplace correspondence above). The Fourier transformation $\mathrm{F}_t : \mathscr{S}'(\mathbb{A}^1_t) \to \mathscr{S}'(\mathbb{A}^1_{z'})$ with kernel $\exp(\overline{tz'} - tz')\frac{i}{2\pi}\,dt\wedge d\overline{t}$ is an isomorphism from the Schwartz space $\mathscr{S}'(\mathbb{A}^1_t)$ considered as a $\mathbb{C}[t]\langle\partial_t\rangle \otimes_{\mathbb{C}} \mathbb{C}[\overline{t}]\langle\partial_{\overline{t}}\rangle$-module, to $\mathscr{S}'(\mathbb{A}^1_{z'})$ considered as a $\mathbb{C}[z']\langle\partial_{z'}\rangle \otimes_{\mathbb{C}} \mathbb{C}[\overline{z}']\langle\partial_{\overline{z}'}\rangle$-module.

Composing $k$ with $\mathrm{F}_t$ and restricting to $\mathbb{C}^*$ produces a sesquilinear pairing ${}^F\!k$ : $(\mathscr{G}, \nabla) \otimes \iota^*\overline{(\mathscr{G}, \nabla)} \to (\mathscr{C}^\infty_{\mathbb{C}^*}, \mathrm{d})$, whose horizontal part restricted to $S^1$ defines a pairing $\mathcal{C} : \mathscr{L} \otimes \iota^{-1}\overline{\mathscr{L}} \to \mathbb{C}_{S^1}$ as in § 2.2.

The pairing ${}^F k$ restricts to horizontal sections of $(\mathscr{G}, \nabla)$ to produce a Betti $\iota$-sesquilinear pairing $({}^F k)_{\mathrm{B}}$ on $\mathscr{L}$. It is defined only over $\mathbb{C}^*$. On the other hand, in a way similar to the definition of $\widehat{j_* Q_{\mathrm{B}}}$, there is a topological Laplace transform $\widehat{j_* k_{\mathrm{B}}}$, which is compatible with the Stokes filtration. In fact, $\widehat{j_* k_{\mathrm{B}}}$ is the $\iota$-sesquilinear pairing associated with the $\iota$-bilinear pairing $\widehat{j_* Q_{\mathrm{B}}}$ and the real structure on $\mathscr{L}$. The comparison between both is given by:

***Lemma.*** *Over* $\mathbb{C}^*$ *we have* $({}^F k)_{\mathrm{B}} = \frac{i}{2\pi} \widehat{j_* k_{\mathrm{B}}}$. □

***Remark.*** The change of weight from $w$ to $w+1$ in the theorem follows from the $i/2\pi$ in this formula.

### 3.2. The nc. $\mathbb{Q}$-Hodge structure attached to a tame function

Let $X$ be a complex smooth quasi-projective variety and let $f : X \to \mathbb{A}^1$ be a regular function on it, that we regard as a morphism to the affine line $\mathbb{A}^1$ with coordinate $t$. For each $k \in \mathbb{Z}$, the perverse cohomology sheaf ${}^p\mathscr{H}^k(\boldsymbol{R} f_* \mathbb{Q}_X)$ underlies a mixed Hodge module. The fibre at $z = 1$ of its topological Laplace transform is the $k$-th *exponential cohomology space of* $X$ *with respect to* $f$ (or simply of $(X, f)$).

For the sake of simplicity, we will only consider the case of a cohomologically tame function $f : U \to \mathbb{A}^1$ on a smooth affine complex manifold $U$, for which there is only one non-zero exponential cohomology space. Cohomological tameness implies that there exists a diagram

$$\begin{array}{ccc} U & \overset{\kappa}{\hookrightarrow} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} \\ & & \mathbb{A}^1 \end{array}$$

where $X$ is quasi-projective and $F$ is projective, such that the cone of natural morphism $\kappa_! \mathbb{Q}_U \to \boldsymbol{R}\kappa_* \mathbb{Q}_U$ has no vanishing cycle with respect to $F - c$ for any $c \in \mathbb{C}$.

We will use the perverse shift convention by setting ${}^p\mathbb{Q}_U = \mathbb{Q}[\dim U]$. By Poincaré-Verdier duality, we have a natural pairing

$$Q_{\mathrm{B}} : \boldsymbol{R} f_! {}^p\mathbb{Q}_U \otimes_{\mathbb{Q}} \boldsymbol{R} f_* {}^p\mathbb{Q}_U \longrightarrow \mathbb{Q}_{\mathbb{A}^1}[2].$$

Considering the $\mathbb{Q}$-perverse sheaf $\mathscr{F} = {}^p\mathscr{H}^0(\boldsymbol{R} f_* {}^p\mathbb{Q}_U)$, we therefore get a morphism $\mathbb{D}\mathscr{F} \to \mathscr{F}$, whose kernel and cokernel (in the perverse sense) are constant sheaves up to a shift. Let $(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})$ be the Stokes-filtered local system on $S^1$ deduced from the topological Laplace transform of $\mathscr{F}$. It comes equipped with a nondegenerate pairing

$$\mathfrak{Q}_B := -\widehat{j_* Q_{\mathrm{B}}} : (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \otimes \iota^{-1}(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \longrightarrow \mathbb{Q}_{S^1}.$$

On the other hand, let $G_0$ denote the Brieskorn lattice of $f$. By definition,

$$G_0 = \Omega^{\dim U}(U)[z] \big/ (z\mathrm{d} - \mathrm{d}f\wedge)\Omega^{\dim U - 1}(U)[z],$$

and set $G = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} G_0$, with the action of $\nabla_{\partial_z}$ induced by $\partial_z + f/z^2 = e^{f/z} \circ \partial_z \circ e^{-f/z}$ on $\Omega^{\dim U}(U)[z]$. We also set $G_k = z^{-k} G_0$. For $\ell \in \mathbb{Z}$ we set $\varepsilon(\ell) = (-1)^{\ell(\ell-1)/2}$.

***Theorem.*** *The data* $((G_{\dim U}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \varepsilon(\dim U - 1)\mathcal{Q}_B)$ *is a polarized nc.* $\mathbb{Q}$*-Hodge structure which is pure of weight* $\dim U$.

*Sketch of proof.* We first replace the perverse sheaf $\mathscr{F}$ defined above with $\mathscr{F}_{!*} := {}^p\mathscr{H}^0(\boldsymbol{R}F_* \kappa_{!*} {}^p\mathbb{Q}_U)$, which generically is the local system of intersection cohomology of the fibres of $F$, and we have a corresponding Poincaré-Verdier duality pairing $Q_{\mathrm{B},!*}$, whose topological Laplace transform $-\widehat{j_* Q_{\mathrm{B},!*}}$ coincides with $\mathcal{Q}_{\mathrm{B}}$. By applying M. Saito's results on polarizable Hodge $\mathscr{D}$-modules, together with the theorem of § 3.1, we find that $((G_0^{\mathrm{H}}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \varepsilon(\dim U - 1)\mathcal{Q}_B)$ is a pure polarized nc. $\mathbb{Q}$-Hodge structure of weight $\dim U$, where $G_0^{\mathrm{H}}$ is the Brieskorn lattice of the Hodge filtration of the Hodge module corresponding to $\mathscr{F}_{!*}$. Taking also into account the shift between the standard filtration and M. Saito's Hodge filtration, we have $G_0^{\mathrm{H}} = G_{\dim U}$. □

***Corollary.*** *The data*

$$((G_0, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \varepsilon(\dim U - 1)\mathcal{Q}_{\mathrm{B}})$$

*is a pure polarized nc.* $\mathbb{Q}$*-Hodge structure of weight* $-\dim U$. □