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INTRODUCTION TO PURE NON-COMMUTATIVE HODGE STRUCTURES

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INTRODUCTION

These note intend to introduce and give examples of pure non-commutative Hodge structures, also denoted by pure nc. Hodge structures, as introduced by Katzarkov, Kontsevich and Pantev in [**KKP08**]. A pure nc. Hodge structure is a small refinement of the notion of a trTERP structure, as introduced previously by Claus Hertling [**Her03**]. The refinement only consists in taking care of the $\mathbb{R}$-structure (or the $\mathbb{Q}$-structure) on the Stokes data of the trTERP structure. So, both notions are very similar.

When forgetting the $\mathbb{Q}$- or $\mathbb{R}$-structure, a pure nc. Hodge structure is nothing but the data of a connection on a trivial vector bundle on $\mathbb{P}^1$, having a pole of order at most two at the origin and at infinity, and no other pole. The notion of $\mathbb{Q}$- or $\mathbb{R}$-structure is more delicate, as it involves the Birkhoff-Riemann-Hilbert correspondence for connections with irregular singularities.

In various geometrical situations, such a trivial bundle is not given, and only the data of the connection near the origin is present. The main question is then to exhibit such an extension of the connection to a neighbourhood of infinity.

The main technique used to produce geometrical example is the Fourier-Laplace transformation. Starting from a usual polarized variation of pure Hodge structure on a punctured affine complex line, one can define by Fourier-Laplace transformation such a connection in the neighbourhood of the origin. The theorem which is explained in Lecture 3 explains the way it works. Note that it uses the polarization in an essential way.

As an application of this technique, we explain in Lecture 4 how a pair of Stokes matrices satisfying a positivity property can give rise to a pure nc. Hodge structure. This result was conjectured by Claus Hertling and Christian Sevenheck in [**HS07**], and proved in [**HS11**].

There are other applications, as the existence of a tt* structure on the germ of Frobenius manifold attached to a convenient and non-degenerate Laurent polynomial, that I will not treat in these notes, since variations of polarized pure nc. Hodge structures are not considered here.

Lecture 1 gives the necessary tools to understand Stokes data, and Lecture 2 explains how to construct a pure nc. Hodge structure from data near the origin of $\mathbb{P}^1$ only.

I refer to [**Sab11**] for a more complete survey on these notions.

LECTURE 1

CONNECTIONS WITH A POLE OF ORDER TWO

1.1. Germs of meromorphic connections

Let z be a coordinate on the complex line. We will consider six kinds of objects, written similarly $(\mathscr{H}, \nabla)$:

(1) $\mathscr{H}$ is a free $\mathbb{C}\{z\}$-module of finite rank μ and ∇ is a meromorphic connection $\nabla_{\partial_z} : \mathscr{H} \to \mathbb{C}(\{z\}) \otimes \mathscr{H}$.

(2) $\mathscr{H}$ is a free $\mathscr{O}_{\mathbb{C}}$-module of finite rank μ and ∇ is a meromorphic connection $\nabla_{\partial_z} : \mathscr{H} \to \mathscr{O}_{\mathbb{C}}(*0) \otimes \mathscr{H}$ with a possible pole at $z = 0$ and no other pole.

(3) $\mathscr{H}$ is a free $\mathscr{O}^{\mathrm{an}}_{\mathbb{P}^1}(*\infty)$-module of finite rank μ and ∇ is a meromorphic connection having a possible pole at the origin, a regular singularity at infinity, and no other pole.

(4) $\mathscr{H}$ is a free $\mathscr{O}^{\mathrm{alg}}_{\mathbb{P}^1}(*\infty)$-module of finite rank μ and ∇ is a meromorphic connection having a possible pole at the origin, a regular singularity at infinity, and no other pole.

(5) $\mathscr{H}$ is a free $\mathscr{O}_{\mathbb{A}^1}$-module of finite rank μ and ∇ is a rational connection $\nabla_{\partial_z} : \mathscr{H} \to \mathscr{O}_{\mathbb{A}^1}(*0) \otimes \mathscr{H}$ with a possible pole at $z = 0$, no other pole at finite distance, and a regular singularity at infinity.

(6) $\mathscr{H}$ is a free $\mathbb{C}[z]$-module of finite rank μ and ∇ is a rational connection $\nabla_{\partial_z} : \mathscr{H} \to \mathbb{C}[z, z^{-1}] \otimes \mathscr{H}$ having a regular singularity at infinity.

All five categories of objects are known to be equivalent:

- We pass from (1) to (2) by extending the local system from a small punctured disc around the origin to a local system on $\mathbb{C}^*$.
- We pass from (2) to (3) by Deligne's meromorphic extension at infinity on $\mathbb{P}^1$.
- We pass from (3) to (4) by a GAGA theorem.
- We pass from (4) to (5) by restricting to $\mathbb{A}^1$ (in the Zariski topology).
- We pass from (5) to (6) by taking global sections.
- We pass from (6) to (1) by tensoring with $\mathbb{C}\{z\}$ over $\mathbb{C}[z]$.

Note however that going from (1) to (6) is a very transcendental operation, since it necessitates computing solutions of the differential equation near the origin and extending them on the whole complex line. It consists then in finding, inside $\mathscr{H}$ as in (1), a free $\mathbb{C}[z]$-module which generates it over $\mathbb{C}[z]$, on which the connection has a regular singularity at infinity and no other pole than the origin.

1.2. Numbers

Let us consider $(\mathscr{H},\nabla)$ as living on $\mathbb{P}^1$ as in (3) or (4). We will assume that the eigenvalues of the monodromy on $\mathbb{C}^*$ have absolute value equal to one, in order to simplify the explanation. Deligne's construction furnishes, for each $\gamma\in\mathbb{R}$ a locally free $\mathscr{O}_{\mathbb{P}^1}$-module $V^\gamma\mathscr{H}\subset\mathscr{H}$ so that it coincides with $\mathscr{H}$ on $\mathbb{P}^1\smallsetminus\{\infty\}$ and the connection ∇ has a logarithmic pole at infinity, with residue having eigenvalues in $[\gamma,\gamma+1)$. This defines a decreasing filtration. Each $V^\gamma\mathscr{H}$ has a Birkhoff-Grothendieck decomposition $V^\gamma\mathscr{H}\simeq\mathscr{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathscr{O}_{\mathbb{P}^1}(a_\mu)$, $a_1\geqslant\cdots\geqslant a_\mu$. Set $v_\gamma=\#\{i\mid a_i\geqslant 0\}$ and $\nu_\gamma=v_\gamma-v_{>\gamma}\geqslant 0$. We can express these numbers a little differently. We have a natural morphism

$$\mathscr{O}_{\mathbb{P}^1}\otimes_{\mathbb{C}}\Gamma(\mathbb{P}^1,V^\gamma\mathscr{H})\longrightarrow V^\gamma\mathscr{H}$$

whose image is denoted by $\mathscr{V}^\gamma$. This is a subbundle of $V^\gamma\mathscr{H}$ in the sense that $V^\gamma\mathscr{H}/\mathscr{V}^\gamma$ is also a locally free sheaf of $\mathscr{O}_{\mathbb{P}^1}$-modules; more precisely, fixing a Birkhoff-Grothendieck decomposition as above, we have $\mathscr{V}^\gamma=\bigoplus_{i|a_i\geqslant 0}\mathscr{O}_{\mathbb{P}^1}(a_i)$ (indeed, for any line bundle $\mathscr{O}_{\mathbb{P}^1}(k)$, $\mathscr{O}_{\mathbb{P}^1}\otimes_{\mathbb{C}}\Gamma(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(k))\to\mathscr{O}_{\mathbb{P}^1}(k)$ is onto if $k\geqslant 0$ and 0 if $k<0$) so $\mathscr{V}^\gamma$ is a direct summand of $V^\gamma\mathscr{H}$ of rank v_γ. Restricting to $\mathbb{C}$, we get a decreasing filtration $\mathscr{V}^\bullet$ of $\mathscr{H}$ indexed by $\mathbb{R}$. The graded pieces $\mathrm{gr}^\gamma_{\mathscr{V}}\mathscr{H}:=\mathscr{V}^\gamma/\mathscr{V}^{>\gamma}$ are locally free $\mathscr{O}_{\mathbb{C}}$-modules (being isomorphic to the kernel of $\mathscr{H}/\mathscr{V}^{>\gamma}\to\mathscr{H}/\mathscr{V}^\gamma$), and $\nu_\gamma=\mathrm{rk}\,\mathrm{gr}^\gamma_{\mathscr{V}}\mathscr{H}$.

The set of pairs $(-\gamma,\nu_\gamma)$ is called the *spectrum at infinity* of the meromorphic connection.

Example. $f:U\to\mathbb{A}^1$ a tame regular function on a smooth quasi-projective variety U with $\dim U=n$. Set $\mathscr{H}=\Omega^n(U)[z]/(z\mathrm{d}+\mathrm{d}f)\Omega^{n-1}(U)[z]$ and ∇_{∂_z} in induced by the action of ∂_z-f/z^2. The first component of the spectrum at infinity is known to be contained in $[0,n]\cap\mathbb{Q}$ and the spectrum is symmetric with respect to $n/2$.

***Exercise* (The Harder-Narasimhan filtration).** Let us fix $\gamma\in\mathbb{R}$. Denote by $F^pV^\gamma\mathscr{H}$ the Harder-Narasimhan filtration of $V^\gamma\mathscr{H}$, i.e., set $i_p=\max\{i\mid a_i\geqslant p\}$. Then $F^pV^\gamma\mathscr{H}:=\mathscr{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathscr{O}_{\mathbb{P}^1}(a_{i_p})$.

(1) Show that $F^0V^\gamma\mathscr{H}=\mathscr{V}^\gamma$.

(2) Show that $F^pV^\gamma\mathscr{H}=\mathscr{V}^{\gamma+p}$.

***Exercise* (Mochizuki).** Assume that ∇ has a pole of order $\leqslant 2$ at the origin. Then the meromorphic connection $\nabla:V^\gamma\mathscr{H}\to V^\gamma\mathscr{H}\otimes\Omega^1_{\mathbb{P}^1}(2\{0\}+1\{\infty\})$ sends $F^pV^\gamma\mathscr{H}$ into $F^{p-1}V^\gamma\mathscr{H}\otimes\Omega^1_{\mathbb{P}^1}(2\{0\}+1\{\infty\})$. [*Hint*: Use the Birkhoff-Grothendieck decomposition and the standard differential on each term to write $\nabla=\mathrm{d}+A$ and prove the result for the $\mathscr{O}$*-linear* morphism $A:V^\gamma\mathscr{H}\to V^\gamma\mathscr{H}\otimes\Omega^1_{\mathbb{P}^1}(2\{0\}+1\{\infty\})$.]

In other words, away from the pole, the Harder-Narasimhan filtration satisfies the Griffiths transversality property with respect to the connection.

1.3. Meromorphic connections with a pole of order $\leqslant 2$

From now on, we assume that the connection has a pole of order $\leqslant 2$. The Levelt-Turrittin theorem gives a normal form for the formalized connection, i.e., up to a formal meromorphic base change, after a possible ramification of z. We assume here that no ramification is needed for the connections we consider. In other words we make the following assumption (*nr. exponential type*).

***Assumption* (Exponential type with no ramification).** *There exists a finite subset* $C\subset\mathbb{C}$ *and for each* $c\in C$ *a germ* $(\mathscr{H}_c,\nabla_c)$ *with possible double pole but with* regular singularity*, such that*

$$\mathbb{C}[\![z]\!]\otimes_{\mathbb{C}\{z\}}(\mathscr{H},\nabla)\simeq\bigoplus_{c\in C}\Big[\mathbb{C}[\![z]\!]\otimes_{\mathbb{C}\{z\}}(\mathscr{H}_c,\nabla_c+c\,\mathrm{Id}\;dz/z^2)\Big].$$

If one accepts base changes with poles, then this is equivalent to asking that there exists a germ $(\mathscr{H}_c,\nabla_c)$ *with a* simple pole *such that*

$$\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}\{z\}}(\mathscr{H},\nabla)\simeq\bigoplus_{c\in C}\Big[\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}\{z\}}(\mathscr{H}_c,\nabla_c+c\,\mathrm{Id}\;dz/z^2)\Big].$$

We will write

$$(\mathscr{G}_c,\nabla_c):=\mathbb{C}(\{z\})\otimes_{\mathbb{C}\{z\}}(\mathscr{H}_c,\nabla_c)\quad\text{and}\quad(\widehat{\mathscr{G}}_c,\widehat{\nabla}_c)=\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}(\{z\})}(\mathscr{G}_c,\nabla_c).$$

We can understand this property from a global point of view. Set $\tau=z^{-1}$ and, with respect to the description (6) regard $\mathscr{H}$ as a $\mathbb{C}[\tau]$-module with a connection. By inverse Laplace transformation, it defines a module over the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$, i.e., t acts as $-\nabla_{\partial_\tau}$ and ∂_t acts as τ.

Exercise. $(\mathscr{H},\nabla)$ has nr. exponential type if and only if, when $\mathscr{H}$ is regarded as a $\mathbb{C}[t]\langle\partial_t\rangle$-module, it has only regular singularities, at finite distance and at $t=\infty$.

Examples.

(1) Let us write the connection as $\nabla=\mathrm{d}+A\,\mathrm{d}z$ with

$$A=A_{-2}z^{-2}+A_{-1}z^{-1}+\cdots:\mathscr{H}\longrightarrow z^{-2}\mathscr{H}.$$

Then A_{-2} is well defined as an $\mathscr{O}$-linear endomorphism of $\mathscr{H}$. If A_{-2} is semi-simple, then $(\mathscr{H},\nabla)$ has nr. exponential type.

(2) On the other hand, consider the connection with matrix

$$A(z)\mathrm{d}z:=P(z)\Big(\frac{Y}{z}+\mathrm{Id}\Big)P(z)^{-1}\cdot\frac{\mathrm{d}z}{z},$$

with

$$Y=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},\quad P(z)=\mathrm{Id}+zZ,\quad Z=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

Then the matrix $z^2A(z)=P(Y+z\,\mathrm{Id})P^{-1}$ has characteristic polynomial equal to $\chi(\lambda)=(\lambda-z)^3$, but one can check that it needs ramification (this contradicts a criterion given [**KKP08**, Rem. 2.13]).

1.4. Stokes filtration

The Stokes data (Betti data) only depend on the meromorphic bundle $(\mathscr{H}[z^{-1}],\nabla)$ that we denote from now on by $(\mathscr{G},\nabla)$.

Let $\mathscr{L}$ be the local system $\ker\nabla$ on $\mathbb{C}^*$. It has rank μ. We regard it as a local system on S^1 with coordinate $e^{i\theta}$, if $\theta=\arg z$.

***Theorem* (Deligne, Magrange).** *The category of $(\mathscr{G}, \nabla)$ of nr. exponential type is equivalent to the category of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ of nr. exponential type.*

Let $\mathscr{A}^{\mathrm{mod}}$ be the sheaf of germs on S^1 of holomorphic functions with moderate growth on a sector of $\mathbb{C}^*$, and similarly $\mathscr{A}^{\mathrm{rd}}$ with rapid decay. Asymptotic analysis of horizontal sections of ∇ shows that $\nabla_{\partial_z} : \mathscr{A}^{\mathrm{mod}} \otimes \mathscr{G} \to \mathscr{A}^{\mathrm{mod}} \otimes \mathscr{G}$ is onto, as well as $\nabla_{\partial_z} : \mathscr{A}^{\mathrm{rd}} \otimes \mathscr{G} \to \mathscr{A}^{\mathrm{rd}} \otimes \mathscr{G}$. Their kernels form nested subsheaves $\mathscr{L}_{<0} \subset \mathscr{L}_{\leqslant 0}$ of $\mathscr{L}$. Moreover, $\mathscr{L}_{\leqslant 0}/\mathscr{L}_{<0}$ is a local system on S^1 whose monodromy is that of $(\mathscr{G}_0, \nabla_0)$ in the Levelt-Turrittin decomposition of $(\mathscr{G}, \nabla)$.

For each $c \in \mathbb{C}$ we can define a pair $(\mathscr{L}_{<c}, \mathscr{L}_{\leqslant c})$ of subsheaves of $\mathscr{L}$ by considering $\nabla - c \operatorname{Id} \mathrm{d}z/z^2$ acting on $\mathscr{A}^{\mathrm{mod}} \otimes \mathscr{G}$ resp. $\mathscr{A}^{\mathrm{rd}} \otimes \mathscr{G}$.

This family $(\mathscr{L}, ((\mathscr{L}_{<c}, \mathscr{L}_{\leqslant c})_{c \in \mathbb{C}})$ has the following properties, that can be taken as the definition of a *Stokes-filtered local system of nr. exponential type*.

(1) (Grading condition) For each $c \in \mathbb{C}$, $\mathrm{gr}_c \mathscr{L} := \mathscr{L}_{\leqslant c}/\mathscr{L}_{<c}$ is a local system which is zero except for c in a finite subset $C \subset \mathbb{C}$, and $\sum_c \operatorname{rk} \mathrm{gr}_c \mathscr{L} = \mu$.

(2) (Filtration condition) For each $\theta \in S^1$ and each $c \in \mathbb{C}$, $\mathscr{L}_{<c,\theta} = \sum_{c' <_\theta c} \mathscr{L}_{\leqslant c', \theta}$, where $c' <_\theta c$ means

$$c \neq c' \quad \text{and} \quad \arg(c - c') \in \theta + (\pi/2, 3\pi/2) \mod 2\pi.$$

(This is a partial order on $\mathbb{C}$: c' is not comparable to c at θ if and only if $(c - c')e^{-i\theta}$ is purely imaginary.)

One can define the notion of a Stokes-filtered local system of nr. exponential type on any base field, and this leads for example to the notion of a $\mathbb{Q}$-structure on $(\mathscr{G}, \nabla)$: this is by definition a Stokes-filtered local system of nr. exponential type defined over $\mathbb{Q}$, whose tensorization by $\mathbb{C}$ is isomorphic to that attached to $(\mathscr{G}, \nabla)$.

1.5. Stokes data

The previous description of the Stokes filtration is independent of any choice. On the other hand, the description with Stokes data below depends on some choices. Let C be a non-empty finite subset of $\mathbb{C}$. We say that $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is *generic* with respect to C if the set C is totally ordered with respect to $\leqslant_\theta$. Once θ_o generic with respect to C is chosen, there is a unique numbering $\{c_1, \dots, c_n\}$ of the set C in strictly increasing order. We will set $\theta'_o = \theta_o + \pi$. Note that the order is exactly reversed at θ'_o, so that $-C$ is numbered as $\{-c_1, \dots, -c_n\}$ by θ'_o.

The category of Stokes data of type (C, θ_o) (defined over a base field, say $\mathbb{Q}$ or $\mathbb{C}$) has objects consisting of two families of vector spaces $(L_{c,1}, L_{c,2})_{c \in C}$ and a diagram of morphisms

$$\bigoplus_{i=1}^n L_{c_i,1} = L_1 \overset{S}{\underset{S'}{\rightrightarrows}} L_2 = \bigoplus_{i=1}^n L_{c_i,2}$$

such that

(1) $S = (S_{ij})_{i,j=1,\dots,n}$ is block-upper triangular, i.e., $S_{ij} : L_{c_i,1} \to L_{c_j,2}$ is zero unless $i \leqslant j$, and S_{ii} is invertible (so $\dim L_{c_i,1} = \dim L_{c_i,2}$, and S itself is invertible),

(2) $S' = (S'_{ij})_{i,j=1,\dots,n}$ is block-lower triangular, i.e., $S'_{ij} : L_{c_i,1} \to L_{c_j,2}$ is zero unless $i \geqslant j$, and S'_{ii} is invertible (so S' itself is invertible).

A morphism of Stokes data of type (C, θ_o) consists of morphisms of vector spaces $\lambda_{c,\ell} : L_{c,\ell} \to L'_{c,\ell}$, $c \in C$, $\ell = 1, 2$, which are compatible with the corresponding diagrams as above. This allows one to classify Stokes data of type (C, θ_o) up to isomorphism. The monodromy T_1 on L_1 is defined by $\mathrm{T}_1 = S^{-1}S'$. *Grading* the Stokes data means replacing (S, S') with their block diagonal parts. There is a natural notion of tensor product in the category of Stokes data of type (C, θ_o), and a duality from Stokes data of type (C, θ_o) to Stokes data of type $(-C, \theta_o)$.

Fixing bases in the spaces $L_{c,\ell}$, $c \in C$, $\ell = 1, 2$, allows one to present Stokes data by matrices (Σ, Σ') where $\Sigma = (\Sigma_{ij})_{i,j=1,\dots,n}$ (resp. $\Sigma' = (\Sigma'_{ij})_{i,j=1,\dots,n}$) is block-lower (resp. -upper) triangular and each Σ_{ii} (resp. Σ'_{ii}) is invertible. The matrix $\Sigma_{ii}^{-1}\Sigma'_{ii}$ is the matrix of monodromy of $L_{c_i,1}$, while $\Sigma^{-1}\Sigma'$ is that of the monodromy of L_1.

Given θ_o generic with respect to C, there is an equivalence (depending on θ_o) between the category of Stokes filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ defined over the base field with jumping indices in C and that of Stokes data of type (C, θ_o) defined over the base field, which is compatible with grading, duality and tensor product (cf. e.g., [**HS11**, §2]).

One can give another description as follows, by emphasizing the monodromy T. Stokes data of type (C, θ_o) consist of a graded vector space $L = \bigoplus_{i=1}^n L_{c_i}$ endowed with an automorphism T such that, for each k, the endomorphism $\mathrm{T}_{k,k} : L_{c_k} \to L_{c_k}$ and the endomorphism $\mathrm{T}_{\leqslant k,\leqslant k} : \bigoplus_{i=1}^k L_{c_i} \to \bigoplus_{i=1}^k L_{c_i}$ are invertible. *Grading* the Stokes data consists in replacing T with its block-diagonal part. Note that a morphism φ of Stokes data is a graded morphism $\varphi = \bigoplus_k \varphi_k$, with $\varphi_k : L_k \to L_k$, which commutes with T. In particular, φ_k commutes with $T_{k,k}$.

Let us finally remark that the Stokes data attached to a meromorphic connection of nr. exponential type depend on the choice of the generic θ_o. Changing θ_o changes the Stokes data attached to the *same* $(\mathscr{G}, \nabla)$.

1.6. Isomonodromy deformations

Let X be a complex manifold and let $\mathscr{G}$ be a meromorphic bundle on $X \times \mathbb{C}$ with pole along the divisor $X \times \{0\}$. An *integrable* meromorphic connection on $\mathscr{G}$ has *good* nr. exponential type its restriction to each slice $\{x_o\} \times \mathbb{C}$ has nr. exponential type and if $C := \bigsqcup_{x \in X} C_x \subset X \times \mathbb{C}$ is a *non-ramified covering* over X. We also say that $(\mathscr{G}, \nabla)$ is a good meromorphic flat bundle with nr. exponential type along $X \times \{0\}$.

Theorem. *Assume that X is 1-connected. Then the restriction functor $(\mathscr{G}, \nabla) \mapsto (\mathscr{G}_x, \nabla_x)$ is an equivalence of categories.*

As a consequence, for any connected X, given any base point $x^o \in X$ and a non-ramified covering $C \subset X \times (\mathbb{C}^n \smallsetminus \text{diagonals})$ of X, there is an equivalence between the category of germs along $X \times \{0\}$ of meromorphic connections $(\mathscr{G}, \nabla)$ of nr. exponential type C and representations

$$\pi_1(X, x^o) \longrightarrow \mathrm{Aut}(\mathscr{G}_{|x_o}, \nabla_{|x_o}).$$

LECTURE 2

PURE NON-COMMUTATIVE HODGE STRUCTURES

2.1. Pure nc. Hodge structures from the operator point of view

Let H be a finite-dimensional complex vector space. A polarized pure complex Hodge structure of weight $w \in \mathbb{Z}$ consists of the data of a positive definite Hermitian form h on H and an h-orthogonal decomposition $H = \bigoplus_p H^{p,w-p}$. The role of the weight is to fix the grading. A $\mathbb{Q}$-Betti structure is a $\mathbb{Q}$-subspace $H_{\mathbb{Q}}$ which generates H over $\mathbb{C}$.

Equivalently, a polarized pure complex Hodge structure of weight w consists of the data $(H, h, \mathscr{Q}, w)$ where $\mathscr{Q}$ is an h-self-adjoint endomorphism of H with half-integral eigenvalues: $\mathscr{Q} = (p - w/2)\,\mathrm{Id}$ on $H^{p,w-p}$. The Weil operator C can be written as $C = \exp \pi i \mathscr{Q}$, and C^2 is the monodromy of the connection $\nabla = \mathrm{d} - \mathscr{Q}\mathrm{d}z/z$ on the trivial bundle $H \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}^1}$. A $\mathbb{Q}$-Betti structure induces such a structure on the local system $\ker \nabla$.

A *polarized pure nc. Hodge structure of weight* w consists of data $(H, h, \mathscr{Q}, \mathscr{U}, w)$, where $\mathscr{Q}$ is an h-self-adjoint endomorphism of H (no restriction on the eigenvalues), and $\mathscr{U}$ is any endomorphism of H. It is useful to introduce the connection on the trivial holomorphic bundle $H \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}^1}$:

$$(*) \qquad \nabla = \mathrm{d} + \Big(z^{-1}\mathscr{U} - (\mathscr{Q} + (w/2)\,\mathrm{Id}) - z\mathscr{U}^\dagger\Big)\frac{\mathrm{d}z}{z},$$

where $\mathscr{U}^\dagger$ is the h-adjoint of $\mathscr{U}$. A $\mathbb{Q}$-Betti structure is a $\mathbb{Q}$-structure on the local system $\ker \nabla$, plus a $\mathbb{Q}$-structure on the Stokes data of ∇ at $z = 0$ and $z = \infty$. I.e., the Stokes-filtered local system attached to ∇ should be defined over $\mathbb{Q}$.

However, in many interesting examples, such a structure is not given in this way, and one has to extract it from a set of data defined in the neighbourhood of $z = 0$ only.

2.2. Gluing of vector bundles

We start with $(\mathscr{H}, \nabla)$ as in §1.1, and we denote by $(\mathscr{L}, \mathscr{L}_\bullet)$ the Stokes-filtered local system associated to $(\mathscr{G}, \nabla)$. It will be convenient and equivalent to regard $\mathscr{L}$ as a local system on $S^1 = \{|z| = 1\}$, since the Stokes structure only depends on $z/|z|$.

Gluing with a real structure. — Assume $\mathscr{L}$ defined over $\mathbb{R}$ (or $\mathbb{Q}$), i.e., $\mathscr{L} \simeq \overline{\mathscr{L}}$.

- Set $\gamma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$, $z \mapsto 1/\overline{z}$. Notice $\gamma_{|S^1} = \mathrm{Id}$.
- Glue $\mathscr{H}$ with $\gamma^*\overline{\mathscr{H}}$ to get $\widetilde{\mathscr{H}}$ (holomorphic vector bundle on $\mathbb{P}^1$):

$$\mathscr{H}_{|S^1} = \mathscr{O}_{|S^1} \otimes \mathscr{L} \simeq \mathscr{O}_{|S^1} \otimes \overline{\mathscr{L}} = (\gamma^*\overline{\mathscr{H}})_{|S^1}$$

- Since the gluing is ∇-flat, it is compatible with the connections ∇ and $\gamma^*\overline{\nabla}$ and produces a connection $\widetilde{\nabla}$ on $\widetilde{\mathscr{H}}$ with a pole at 0 and ∞ only.

Gluing with a sesquilinear pairing. — Let ι be the involution $z \mapsto -z$. Assume we are given an isomorphism $\mathcal{C} : \mathscr{L}^\vee \simeq \iota^{-1}\overline{\mathscr{L}}$. We call it a nondegenerate ι-sesquilinear pairing on $\mathscr{L}$, since we can regard it as a pairing

$$\mathcal{C} : \mathscr{L} \otimes \iota^{-1}\overline{\mathscr{L}} \longrightarrow \mathbb{C}_{S^1}.$$

- Set $\sigma := \gamma \circ \iota : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$, $z \mapsto -1/\overline{z}$.
- Glue $\mathscr{H}^\vee$ with $\sigma^*\overline{\mathscr{H}}$ to get $\widehat{\mathscr{H}}$ (holomorphic vector bundle on $\mathbb{P}^1$):

$$\mathscr{H}^\vee_{|S^1} = \mathscr{O}_{|S^1} \otimes \mathscr{L}^\vee \overset{\mathcal{C}}{\simeq} \mathscr{O}_{|S^1} \otimes \iota^{-1}\overline{\mathscr{L}} = (\sigma^*\overline{\mathscr{H}})_{|S^1}.$$

Since this gluing is ∇-flat, it is compatible with the connections ∇ and $\sigma^*\overline{\nabla}$ and produces a connection $\widetilde{\nabla}$ on $\widehat{\mathscr{H}}$ with a pole at 0 and ∞ only..

Assume moreover that $\mathcal{C} : \mathscr{L}^\vee \simeq \iota^{-1}\overline{\mathscr{L}}$ is ι-Hermitian. Then the construction produces a natural σ-Hermitian isomorphism $\mathscr{S} : \widehat{\mathscr{H}^\vee} \to \sigma^*\overline{\widehat{\mathscr{H}}}$ and therefore, if we set $H := \Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})$, there is an isomorphism

$$h = \Gamma(\mathbb{P}^1, \mathscr{S}) : \Gamma(\mathbb{P}^1, \widehat{\mathscr{H}^\vee}) \xrightarrow{\sim} \overline{\Gamma(\mathbb{P}^1, \sigma^*\widehat{\mathscr{H}})} = \overline{\Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})} = \overline{H}.$$

If $\widehat{\mathscr{H}}$ is trivial, then $\Gamma(\mathbb{P}^1, \widehat{\mathscr{H}^\vee}) = H^\vee$, and H comes equipped with a non-degenerate Hermitian for h.

Comparison. — Assume

- we are given a real structure $\mathscr{L} \simeq \overline{\mathscr{L}}$ on $\mathscr{L}$,
- we are given a non-degenenerate pairing $\mathcal{Q}_{\mathbb{R}} : \mathscr{L}_{\mathbb{R}} \otimes \iota^{-1}\mathscr{L}_{\mathbb{R}} \to \mathbb{R}$, and a non-degenenerate $\mathscr{O}$-bilinear pairing $\mathcal{Q} : (\mathscr{H}, \nabla) \otimes \iota^*(\mathscr{H}, \nabla) \to (z^{-w}\mathscr{O}, \mathrm{d})$ for some $w \in \mathbb{Z}$, so that $\mathcal{Q}_{\mathbb{C}}$ corresponds to the restriction of $\mathcal{Q}$ to $(\mathscr{G}, \nabla)$ by the Riemann-Hilbert corresopndence.

Then, on the one hand, from the real structure on $\mathscr{L}$ we get $\widetilde{\mathscr{H}}$. On the other hand, $\mathcal{Q}$ defines a non-degenerate ι-sesquilinear pairing $\mathcal{C}$, hence $\widehat{\mathscr{H}}$.

***Lemma* (C. Hertling).** *Under these assumptions, $\mathcal{Q}$ induces an isomorphism*

$$\widetilde{\mathscr{H}} \xrightarrow{\sim} \iota^*\widehat{\mathscr{H}} \otimes \mathscr{O}_{\mathbb{P}^1}(w).$$

Consequence. $\widetilde{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}(w)^{\mathrm{rk}\,\mathscr{H}} \iff \widehat{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}^{\mathrm{rk}\,\mathscr{H}}$.

In such a case, if the connection ∇ has a pole of order $\leqslant 2$, it takes the form $(*)$.

2.3. Pure non-commutative Hodge structures

2.3.1. Pure Hodge structures from the twistor point of view. — Let z be a new variable. Then the decreasing filtration $F^{\bullet}H$ defined by $F^pH = \bigoplus_{p' \geqslant p} H^{p', w-p'}$ allows one to define a free $\mathbb{C}[z]$-module $\mathscr{H} = \bigoplus_p F^pHz^{-p}$, which satisfies

$$\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} \mathscr{H} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} H.$$

It is equipped with a connection ∇ (induced by the differential d) which has a pole of order one at the origin. The local system $\ker \nabla$ on $\mathbb{C}^* = \{z \neq 0\}$ is trivial (monodromy equal to identity) with fibre H, and it has a rational constant sub local system with fibre $H_{\mathbb{Q}}$.

Let $\gamma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$ be as in §2.2. Then $\gamma^*\overline{\mathscr{H}} = \sum_q \overline{F^qH}z^q$ is a $\mathbb{C}[z^{-1}]$-free module, and $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} \gamma^*\overline{\mathscr{H}} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \overline{H} \simeq \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} H$, due to the real structure. One can then glue the bundles $\mathscr{H}$ and $\gamma^*\overline{\mathscr{H}}$ into a holomorphic bundle $\widetilde{\mathscr{H}}$ on $\mathbb{P}^1$. The opposedness (or bi-grading) property is then equivalent to the property that $\widetilde{\mathscr{H}}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(w)^{\dim H}$.

We will follow this approach for defining a pure nc. $\mathbb{Q}$-Hodge structure.

2.3.2. Definition of a polarized pure nc. $\mathbb{Q}$-Hodge structure

Definition. Data:

- $(\mathscr{H}, \nabla)$ having a pole of order two with no ramification,
- $(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})$ a Stokes-filtered $\mathbb{Q}$-local system on S^1,
- a pairing $\mathcal{Q}_{\mathrm{B}} : (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \otimes \iota^{-1}(\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}) \longrightarrow (\mathbb{Q}_{S^1}, \mathbb{Q}_{S^1,\bullet})$

($(\mathbb{Q}_{S^1}, \mathbb{Q}_{S^1,\bullet})$: trivial Stokes filtration on $\mathbb{Q}_{S^1}$).

We say that $\big((\mathscr{H}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \mathcal{Q}_{\mathrm{B}}\big)$ is a *polarized pure nc. $\mathbb{Q}$-Hodge structure of weight w* if it satisfies the following properties:

(1) $\mathcal{Q}_{\mathrm{B}}$ is non-degenerate $(-1)^w$-ι-symmetric (in particular, it induces a non-degenerate $(-1)^w$-ι-symmetric pairing on each local system $\mathrm{gr}_c\mathscr{L}_{\mathbb{Q}}$).

(2) The $(-1)^w$-ι-symmetric pairing $\mathcal{Q}$ that $\mathcal{Q}_{\mathrm{B}}$ induces on $(\mathscr{G}, \nabla) = \mathscr{O}(*0) \otimes (\mathscr{H}, \nabla)$ through the RH correspondence, which takes values in $\mathscr{O}_{\mathbb{C}}(*0)$ satisfies:

$$\mathcal{Q}\big(\mathscr{H} \otimes \iota^*\mathscr{H}\big) \subset z^{-w}\mathscr{O}_{\mathbb{C}}$$

and is non-degenerate as such.

(3) Letting $\mathcal{C}$ be the ι-sesquilinear pairing associated to $i^{-w}\mathcal{Q}$ (hence $\mathcal{C}$ is a nondegenerate ι-Hermitian pairing), then

(a) $\widehat{\mathscr{H}}$ is trivial (i.e., $\widetilde{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}(w)^d$, opposedness),

(b) $h := \Gamma(\mathbb{P}^1, \mathscr{S})$ is positive definite on $H := \Gamma(\mathbb{P}^1, \widehat{\mathscr{H}})$ (polarisation).

2.3.3. Relation with the operator definition. — Let $\big((\mathscr{H}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \mathcal{Q}_{\mathrm{B}}\big)$ be a polarized pure nc. $\mathbb{Q}$-Hodge structure of weight w. Since $\widehat{\mathscr{H}}$ is trivial, we can choose a global basis of $\widehat{\mathscr{H}}$ and express the connection ∇ in this frame. Since ∇ has a pole of order two at zero and infinity, and no other pole, its matrix takes the form $(*)$ in this basis. One then checks that $\mathscr{Q}$ is self-adjoint and $\mathscr{U}^\dagger$ is the h-adjoint of $\mathscr{U}$.

2.3.4. Irregular Hodge numbers. — Let $\big((\mathscr{H},\nabla),(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet}),\mathcal{Q}_{\mathrm{B}}\big)$ be a polarized pure nc. $\mathbb{Q}$-Hodge structure of weight w. For each $\alpha\in[0,1)$, the *irregular Hodge numbers* $h^{p,w-p}(\alpha)$ are the numbers

$$h^{p,w-p}(\alpha):=\operatorname{rk}\operatorname{gr}^p_F V^{-\alpha}\mathscr{H},$$

where $F^\bullet V^{-\alpha}\mathscr{H}$ is the restriction to $\mathscr{H}$ of the Harder-Narasimhan filtration of $V^{-\alpha}\mathscr{H}$ (cf. §1.2).

2.4. A few words about variations of polarized pure nc. $\mathbb{Q}$-Hodge structures

Let X be a complex manifold. A *variation of polarized pure nc.* $\mathbb{Q}$*-Hodge structure of weight* w parametrized by X consists of the following data:

- $(\mathscr{H},\nabla)$ a holomorphic bundle on $X\times\mathbb{C}$ with a flat meromorphic connection ∇ having poles of Poincaré rank one along $\{z=0\}$ (i.e., $z\nabla$ is logarithmic) and no other pole,
- A Stokes-filtered $\mathbb{Q}$-local system $(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})$ on $X\times S^1$,
- a pairing

$$\mathcal{Q}_{\mathrm{B}}:(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})\otimes\iota^{-1}(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})\longrightarrow(\mathbb{Q}_{X\times S^1},\mathbb{Q}_{X\times S^1,\bullet})$$

($(\mathbb{Q}_{X\times S^1},\mathbb{Q}_{X\times S^1,\bullet})$: trivial Stokes filtration on $\mathbb{Q}_{X\times S^1}$),
subject to the condition that, for each $x\in X$, the restriction to $\{x\}\times\mathbb{C}$ is a polarized pure nc. $\mathbb{Q}$-Hodge structure of weight w.

Hodge structures	Hodge structures (Nc)	Nc. Hodge structures
Filtered vect. sp. $(H,F^\bullet H)$	$(\mathscr{H},\nabla)=\oplus_p(F^pHz^{-p},\mathrm{d})$ free $\mathbb{C}[z]$-mod + connect.	$\mathscr{H}$ free $\mathscr{O}_{\mathbb{A}^1}$-mod., ∇: connect., $0=$ only pole, ord. $\leqslant 2$, n.r.
H	$\mathscr{L}=\ker\nabla$ on $\mathscr{H}_{\vert S^1}$	idem
$H_{\mathbb{Q}}$	$\mathscr{L}_{\mathbb{Q}}$ (cst $\mathbb{Q}$-loc. syst. on S^1)	$\mathscr{L}_{\mathbb{Q},\bullet}$ (Stokes-filt. $\mathbb{Q}$-loc. syst.)
$F^pH\cap\overline{F^{w-p+1}H}=0\ \forall\,p$	$\widetilde{\mathscr{H}}\simeq\mathscr{O}_{\mathbb{P}^1}(w)^{\operatorname{rk}\mathscr{H}}$	idem
Q: $(-1)^w$-sym. non-deg. $\mathbb{Q}$-bilin. form.	$\mathcal{Q}:\mathscr{L}_{\mathbb{Q}}\otimes\iota^{-1}\mathscr{L}_{\mathbb{Q}}\to\mathbb{Q}$ $(-1)^w$-ι-sym. non-deg.	$\mathcal{Q}:\mathscr{L}_{\mathbb{Q},\bullet}\otimes\iota^{-1}\mathscr{L}_{\mathbb{Q},\bullet}\to\mathbb{Q}_\bullet$ idem
s.t. $Q(H^{p,q},H^{p',q'})=0$ for $p'\neq w-p$	$(\mathscr{H},\nabla)\otimes\iota^*(\mathscr{H},\nabla)\to(z^{-w}\mathscr{O},\mathrm{d})$ non-deg.	idem idem
$Q\rightsquigarrow h$ Herm. form	$\mathcal{Q}\rightsquigarrow\mathcal{C}$: ι-Herm. on $\mathscr{L}$	idem
H	$\widehat{\mathscr{H}}$ trivial, $H:=\Gamma(\mathbb{P}^1,\widehat{\mathscr{H}})$	idem
h def. >0	h def. >0 on $\Gamma(\mathbb{P}^1,\widehat{\mathscr{H}})\simeq H$	idem

TABLE 1. Comparison table

LECTURE 3

PURE NC. $\mathbb{Q}$-HODGE STRUCTURE THROUGH FOURIER-LAPLACE

3.1. Producing a pure nc. $\mathbb{Q}$-Hodge structure by Fourier-Laplace transformation

Let $C \subset \mathbb{A}^1$ be a finite set of points on the complex affine line with coordinate t. Let $(\mathcal{V}_{\mathbb{Q}}, F^\bullet V, \nabla, Q_{\mathrm{B}})$ be a variation of polarized pure Hodge structure of weight $w \in \mathbb{Z}$ on $X := \mathbb{A}^1 \smallsetminus C$. Namely,

- (V, ∇) is a holomorphic vector bundle with connection on X,
- $F^\bullet V$ is a finite decreasing filtration of V by holomorphic sub-bundles satisfying the Griffiths transversality property: $\nabla F^p V \subset F^{p-1} V \otimes_{\mathscr{O}_X} \Omega^1_X$,
- $\mathcal{V}_{\mathbb{Q}}$ is a $\mathbb{Q}$-local system on X with $\mathcal{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = V^\nabla$,
- $Q_{\mathrm{B}} : \mathcal{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{V}_{\mathbb{Q}} \to \mathbb{Q}$ is a nondegenerate $(-1)^w$-symmetric pairing,

all these data being such that the restriction at each $x \in X$ is a polarized pure Hodge structure of weight w. We denote by Q the nondegenerate flat pairing

$$Q : (V, \nabla) \otimes (V, \nabla) \longrightarrow (\mathscr{O}_X, d)$$

that we get from Q_{B} through the canonical isomorphism $\mathscr{O}_X \otimes_{\mathbb{Q}} \mathcal{V}_{\mathbb{Q}} = V$. The associated nondegenerate sesquilinear pairing is denoted by

$$k : (V, \nabla) \otimes_{\mathbb{C}} \overline{(V, \nabla)} \longrightarrow \mathscr{C}^\infty_X,$$

which can also be obtained from $k_{\mathrm{B}} : \mathcal{V} \otimes_{\mathbb{C}} \overline{\mathcal{V}} \to \mathbb{C}$ similarly. It is $(-1)^w$-Hermitian and $i^{-w} k$ induces a flat Hermitian pairing on the C^∞-bundle $(\mathscr{C}^\infty_X \otimes_{\mathscr{O}_X} V, \nabla + \overline{\partial})$. We can regard $(V, \nabla, F^\bullet V, i^{-w} k)$ as a variation of polarized complex pure Hodge structure of weight 0.

Theorem. *Let $(\mathcal{V}_{\mathbb{Q}}, F^\bullet V, \nabla, Q_{\mathrm{B}})$ be a variation of polarized pure $\mathbb{Q}$-Hodge structure of weight $w \in \mathbb{Z}$ on $X := \mathbb{A}^1 \smallsetminus C$. Then its Fourier-Laplace transform*

$${}^F(\mathcal{V}_{\mathbb{Q}}, F^\bullet V, \nabla, Q_{\mathrm{B}}) = ((\mathscr{H}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), -{}^F(j_* Q_{\mathrm{B}}))$$

is a polarized pure nc. $\mathbb{Q}$-Hodge structure of weight $w+1$.

3.2. The Betti side

We denote by $j : \mathbb{A}^1 \smallsetminus C \hookrightarrow \mathbb{A}^1$ the inclusion.

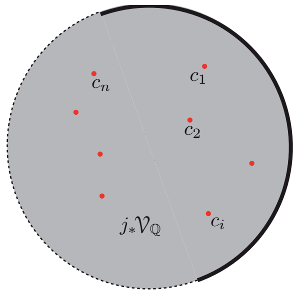
- Fix $z_o \in S^1$. Define Φ_{z_o} as the family of closed sets $S \subset \mathbb{A}^1$ such that

$$\overline{S} \cap \{(\infty, e^{i\theta}) \mid \mathrm{Re}(e^{i\theta}/z_o) \geqslant 0\} = \varnothing \quad \text{in } \mathbb{A}^1 \cup S^1_\infty.$$

We consider the inclusions (α is open and β is closed)

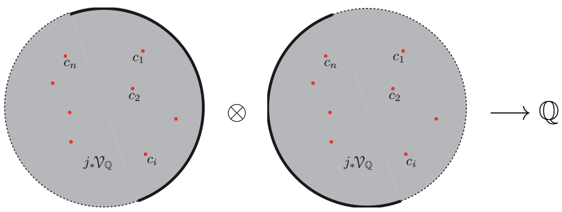
$$\mathbb{A}^1 \xhookrightarrow{\alpha} \mathbb{A}^1 \cup \{(\infty, e^{i\theta}) \mid \mathrm{Re}(e^{i\theta}/z_o) < 0\} \xhookrightarrow{\beta} \mathbb{A}^1 \cup S^1_\infty =: \widetilde{\mathbb{P}}^1.$$

- $(\mathscr{L}_\mathbb{Q})_{z_o} = H^1_{\Phi_{z_o}}(\mathbb{A}^1, j_*\mathcal{V}_\mathbb{Q}) = H^1(\widetilde{\mathbb{P}}^1, \beta_!\alpha_*j_*\mathscr{V}_\mathbb{Q})$:



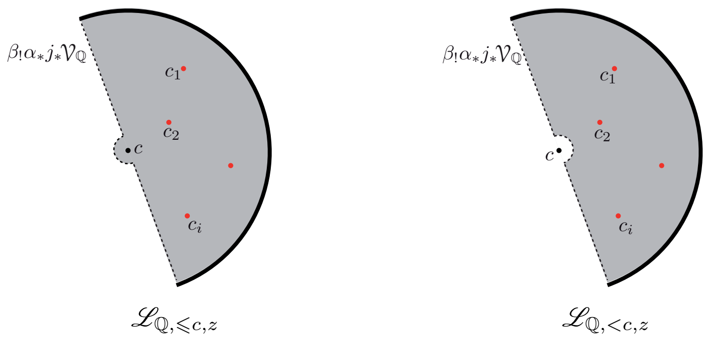
- $({}^F(j_*Q_\mathrm{B}))_{z_o}$ is the cup product followed by Q_B (note: $\Phi_{z_o} \cap \Phi_{-z_o} =$ family of compact sets in $\mathbb{A}^1$):

$$H^1_{\Phi_{z_o}}(\mathbb{A}^1, j_*\mathcal{V}_\mathbb{Q}) \otimes H^1_{\Phi_{-z_o}}(\mathbb{A}^1, j_*\mathcal{V}_\mathbb{Q}) \longrightarrow H^2_\mathrm{c}(\mathbb{A}^1, \mathbb{Q}) \simeq \mathbb{Q},$$



Note that here appears the involution ι.

- For $c \in \mathbb{C}$, we define $\mathscr{L}_{\mathbb{Q},\leqslant c,z}$ and $\mathscr{L}_{\mathbb{Q},<c,z}$ by using a picture similar to the above one:



3.3. De Rham side

3.3.1. De Rham side for the variation of pure Hodge structure. — The bundle (V, ∇) can be extended in a unique way as a free $\mathscr{O}_{\mathbb{P}^1}(*C \cup \{\infty\})$-module with a connection ∇ having a regular singularity at $C \cup \{\infty\}$ (Deligne's meromorphic extension). Taking global sections on $\mathbb{P}^1$ produces a left module $\widetilde{M}$ on the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$. The minimal extension (along C) of $\widetilde{M}$ is the unique submodule M of $\widetilde{M}$ which coincides with $\widetilde{M}$ after tensoring both by $\mathbb{C}(t)$, and which has no quotient submodule supported in C (it is characterized by the property that $\mathrm{DR}^{\mathrm{an}} M = j_*\mathscr{V}$).

3.3.2. How to get $(\mathscr{G},\nabla)$: Laplace transform of M. — Set

$$G=\mathbb{C}[t]\langle\partial_t,\partial_t^{-1}\rangle\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}M,$$

and define the action of $\mathbb{C}[z,z^{-1}]\langle\partial_z\rangle$ on G as follows: $z\cdot m=\partial_t^{-1}m$, $z^{-1}\cdot m=\partial_t m$, and $z^2\partial_z m=tm$. One can show that G is a free $\mathbb{C}[z,z^{-1}]$-module, and the action of ∂_z is that of a connection (i.e., satisfies Leibniz rule). Its analytization as a free $\mathscr{O}(*0)$-module with connection is denoted by $(\mathscr{G},\nabla)$.

Theorem. *$\mathbb{C}\otimes_{\mathbb{Q}}(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})$ is the Stokes-filtered local system attached to $(\mathscr{G},\nabla)$.*

3.3.3. A direct definition of $\mathcal{C}$: the Fourier transformation. — The pairing k extends first (due to the regularity of the connection) as a pairing

$$\widetilde{k}:\widetilde{M}\otimes_{\mathbb{C}}\overline{\widetilde{M}}\longrightarrow\mathscr{S}'(\mathbb{A}^1\smallsetminus C),$$

where $\mathscr{S}'(\mathbb{A}^1)$ denotes the Schwartz space of temperate distributions on $\mathbb{A}^1=\mathbb{R}^2$, and

$$\mathscr{S}'(\mathbb{A}^1\smallsetminus C):=\mathbb{C}\big[t,\textstyle\prod_{c\in C}(t-c)^{-1}\big]\otimes_{\mathbb{C}[t]}\mathscr{S}'(\mathbb{A}^1).$$

Then one shows that, when restricted to $M\otimes_{\mathbb{C}}\overline{M}$, $\widetilde{k}$ takes values in $\mathscr{S}'(\mathbb{A}^1)$, and we denote it by k.

Set $z'=z^{-1}$ (it corresponds to ∂_t in the Laplace correspondence above). The Fourier transformation $\mathrm{F}_t:\mathscr{S}'(\mathbb{A}^1_t)\to\mathscr{S}'(\mathbb{A}^1_{z'})$ with kernel $\exp(\overline{tz'}-tz')\frac{i}{2\pi}\,dt\wedge d\overline{t}$ is an isomorphism from the Schwartz space $\mathscr{S}'(\mathbb{A}^1_t)$ considered as a $\mathbb{C}[t]\langle\partial_t\rangle\otimes_{\mathbb{C}}\mathbb{C}[\overline{t}]\langle\partial_{\overline{t}}\rangle$-module, to $\mathscr{S}'(\mathbb{A}^1_{z'})$ considered as a $\mathbb{C}[z']\langle\partial_{z'}\rangle\otimes_{\mathbb{C}}\mathbb{C}[\overline{z}']\langle\partial_{\overline{z}'}\rangle$-module.

Composing k with F_t and restricting to $\mathbb{C}^*$ produces a sesquilinear pairing ${}^F k:(\mathscr{G},\nabla)\otimes\iota^*\overline{(\mathscr{G},\nabla)}\to(\mathscr{C}^\infty_{\mathbb{C}^*},\mathrm{d})$, whose horizontal part restricted to S^1 defines a pairing $\mathcal{C}:\mathscr{L}\otimes\iota^{-1}\overline{\mathscr{L}}\to\mathbb{C}_{S^1}$ as in §2.2.

The pairing ${}^F k$ restricts to horizontal sections of $(\mathscr{G},\nabla)$ to produce a Betti ι-sesquilinear pairing $({}^F k)_{\mathrm{B}}$ on $\mathscr{L}$. It is defined only over $\mathbb{C}^*$. On the other hand, in a way similar to the definition of ${}^F(j_*Q_{\mathrm{B}})$, there is a topological Laplace transform ${}^F(j_*k_{\mathrm{B}})$, which is compatible with the Stokes filtration. In fact, ${}^F(j_*k_{\mathrm{B}})$ is the ι-sesquilinear pairing associated with the ι-bilinear pairing ${}^F(j_*Q_{\mathrm{B}})$ and the real structure on $\mathscr{L}$. The comparison between both is given by:

Theorem. *Over $\mathbb{C}^*$ we have $({}^F k)_{\mathrm{B}}=\frac{i}{2\pi}{}^F(j_*k_{\mathrm{B}})$.* □

Remark. The change of weight from w to $w+1$ in the theorem follows from the $i/2\pi$ in this formula.

3.4. Hodge side: $(\mathscr{H},\nabla)$ as the Brieskorn lattice of the filtration $F^\bullet M$

The Hodge filtration $F^\bullet V$ extends, according to a procedure due to M. Saito and relying on Schmid's theory of limits of variations of polarized pure Hodge structures, to a good filtration $F^\bullet M$ of M as a $\mathbb{C}[t]\langle\partial_t\rangle$-module.

We denote by $\widehat{\mathrm{loc}}:M\to G$ the natural morphism (the kernel and cokernel of which are isomorphic to powers of $\mathbb{C}[t]$ with its natural structure of left $\mathbb{C}[t]\langle\partial_t\rangle$-module).

For any lattice L of M, i.e., a $\mathbb{C}[t]$-submodule of finite type such that $M = \mathbb{C}[\partial_t] \cdot L$, we define the associated Brieskorn lattice as

$$G_0^{(L)} = \sum_{j\geqslant 0} \partial_t^{-j} \widehat{\mathrm{loc}}(L).$$

This is a $\mathbb{C}[\partial_t^{-1}]$-submodule of G. Moreover, because of the relation $[t, \partial_t^{-1}] = (\partial_t^{-1})^2$, it is naturally equipped with an action of $\mathbb{C}[t]$. If M has a regular singularity at infinity, then $G_0^{(L)}$ has finite type over $\mathbb{C}[\partial_t^{-1}]$. We have $G = \mathbb{C}[\partial_t] \cdot G_0^{(L)}$.

Let us now consider a filtered $\mathbb{C}[t]\langle\partial_t\rangle$-module. Let $p_0 \in \mathbb{Z}$. We say that $F^\bullet M$ *is generated by* $F^{p_0}M$ if, for any $\ell \geqslant 0$, we have $F^{p_0-\ell}M = F^{p_0}M + \cdots + \partial_t^\ell F^{p_0}M$. Then $F^{p_0}M$ is a lattice of M. Moreover, the $\mathbb{C}[\partial_t^{-1}]$-module $\partial_t^{p_0} G_0^{(F^{p_0})}$ does not depend on the choice of the index p_0, provided that the generating assumption is satisfied. We thus define the *Brieskorn lattice of the filtration* $F^\bullet M$ as

$$G_0^{(F)} = \partial_t^{p_0} G_0^{(F^{p_0})} \quad \text{for some (or any) index } p_0 \text{ of generation.}$$

If we also set $z = \partial_t^{-1}$, then one can show that $G_0^{(F)}$ is a free $\mathbb{C}[z]$-module which satisfies $G = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} G_0^{(F)}$ and which is stable by the action of $z^2\partial_z := t$. Its analytization $\mathscr{H}$ is a free $\mathscr{O}$-module on which the connection has a pole of order $\leqslant 2$.

LECTURE 4

STOKES MATRICES AND PURE NC. HODGE STRUCTURES

4.1. Deligne-Malgrange lattices

We explain here an example of pairs $(\mathscr{H}, \nabla)$ of nr. exponential type. It can be obtained as the Brieskorn lattice of a natural filtration of M, namely the filtration by Deligne lattices.

Let $(\mathscr{G}, \nabla)$ be a meromorphic connection of nr. exponential type (cf. §1.3). The functor which associates to any lattice $\mathscr{H}$ of $\mathscr{G}$ (i.e., a $\mathbb{C}\{z\}$-free submodule such that $\mathbb{C}(\{z\}) \otimes_{\mathbb{C}\{z\}} \mathscr{H} = \mathscr{G}$) its formalization $\mathbb{C}[\![z]\!] \otimes_{\mathbb{C}\{z\}} \mathscr{H}$ is an equivalence between the full subcategory of lattices of $\mathscr{G}$ and that of lattices of $\widehat{\mathscr{G}}$ (cf. [**Mal96**]). In particular, let us consider for each $c \in C$ and $a \in \mathbb{R}$ the Deligne lattices $\widehat{\mathscr{G}}_c{}^a$ (resp. $\widehat{\mathscr{G}}_c{}^{>a}$) of the regular connection $(\widehat{\mathscr{G}}_c, \widehat{\nabla}_c)$ considered in §1.3, characterized by the property that the connection $\widehat{\nabla}_c$ on $\widehat{\mathscr{G}}_c^a$ has a simple pole and the real parts of the eigenvalues of its residue belong to $[a, a+1)$ (resp. to $(a, a+1]$). Clearly, $\widehat{\mathscr{G}}_c{}^{a+1} = z\widehat{\mathscr{G}}_c{}^a$ and $\widehat{\mathscr{G}}_c{}^{>a+1} = z\widehat{\mathscr{G}}_c{}^{>a}$.

According to the previous equivalence, there exist unique lattices of $\mathscr{G}$, denoted by $\mathrm{DM}^a(\mathscr{G}, \nabla)$ (resp. $\mathrm{DM}^{>a}(\mathscr{G}, \nabla)$, which induce, by formalization, the decomposed lattice $\bigoplus_c(\mathscr{E}^{-c/z} \otimes \widehat{\mathscr{G}}_c{}^a)$ (resp. $\bigoplus_c(\mathscr{E}^{-c/z} \otimes \widehat{\mathscr{G}}_c{}^{>a})$). They are called the Deligne-Malgrange lattices of $(\mathscr{G}, \nabla)$. We regard them as defining a decreasing filtration of $\mathscr{G}$.

Lemma. *Any morphism $(\mathscr{G}, \nabla) \to (\mathscr{G}', \nabla')$ of meromorphic connections of nr. exponential type is strictly compatible with the filtration by Deligne-Malgrange lattices.*

Sketch of proof. The associated formal morphism is block-diagonal with respect to the decomposition of §1.3, and each diagonal block induces a morphism between the corresponding regular parts, which is known to be strict with respect to the filtration by the Deligne lattices. □

The behaviour by duality below is proved similarly by reducing to the regular singularity case (cf. e.g., [**Sab02**, §III.1.b] or [**Sab06**, Lem. 3.2]).

Lemma. *Let $(\mathscr{G}, \nabla)$ be as above and let $(\mathscr{G}, \nabla)^\vee$ be the dual meromorphic connection. Then there are canonical isomorphisms*

$$[\mathrm{DM}^a(\mathscr{G}, \nabla)]^\vee \simeq \mathrm{DM}^{>-a-1}[(\mathscr{G}, \nabla)^\vee],$$
$$[\mathrm{DM}^{>a}(\mathscr{G}, \nabla)]^\vee \simeq \mathrm{DM}^{-a-1}[(\mathscr{G}, \nabla)^\vee]. \qquad \square$$

We will use this lemma as follows.

Corollary. *Let $(\mathscr{G},\nabla)$ be of nr. exponential type, with associated Stokes structure $(\mathscr{L},\mathscr{L}_\bullet)$. Let*

$$\mathcal{Q}_{\mathrm{B}}:(\mathscr{L},\mathscr{L}_\bullet)\otimes_{\mathbb{C}}\iota^{-1}(\mathscr{L},\mathscr{L}_\bullet)\longrightarrow\mathbb{C}$$

be a nondegenerate bilinear pairing. Let

$$\mathcal{Q}:(\mathscr{G},\nabla)\otimes_{\mathbb{C}(\{z\})}\iota^*(\mathscr{G},\nabla)\longrightarrow(\mathbb{C}(\{z\}),\mathrm{d})$$

be the nondegenerate pairing corresponding to $\mathcal{Q}_{\mathrm{B}}$ via the Riemann-Hilbert correspondence. Then, for each $a\in\mathbb{R}$, $\mathcal{Q}$ extends in a unique way as a nondegenerate pairing

$$\mathrm{DM}^a(\mathscr{G},\nabla)\otimes_{\mathbb{C}\{z\}}\iota^*\,\mathrm{DM}^{>-a-1}(\mathscr{G},\nabla)\longrightarrow(\mathbb{C}\{z\},\mathrm{d}).\qquad\square$$

Corollary. *With the previous assumptions, assume moreover that a is an integer. Then,*

(1) *if none of the monodromies of the $\widehat{\mathscr{G}_c}$ has 1 as an eigenvalue, then $\mathrm{DM}^a(\mathscr{G},\nabla)=\mathrm{DM}^{>a}(\mathscr{G},\nabla)$ for each integer a, and $\mathcal{Q}$ induces a nondegenerate pairing*

$$\mathrm{DM}^a(\mathscr{G},\nabla)\otimes_{\mathbb{C}\{z\}}\iota^*\,\mathrm{DM}^a(\mathscr{G},\nabla)\longrightarrow(z^{2a+1}\mathbb{C}\{z\},\mathrm{d}),$$

(2) *if none of the monodromies of the $\widehat{\mathscr{G}_c}$ has -1 as an eigenvalue, then $\mathrm{DM}^{a-1/2}(\mathscr{G},\nabla)=\mathrm{DM}^{>a-1/2}(\mathscr{G},\nabla)$ for each integer a, and $\mathcal{Q}$ induces a nondegenerate pairing*

$$\mathrm{DM}^{a-1/2}(\mathscr{G},\nabla)\otimes_{\mathbb{C}\{z\}}\iota^*\,\mathrm{DM}^{a-1/2}(\mathscr{G},\nabla)\longrightarrow(z^{2a}\mathbb{C}\{z\},\mathrm{d}).\qquad\square$$

4.2. Pure nc. Hodge structures from Deligne-Malgrange lattices

Let $C\subset\mathbb{C}$ be a finite set, let $\theta_o\in\mathbb{R}/2\pi\mathbb{Z}$ be generic with respect to C (cf. §1.3) defining thus a numbering $\{c_1,\dots,c_n\}$ of the set C in strictly increasing order, and let Σ be a *block-lower triangular invertible square matrix* of size d with entries in $\mathbb{Q}\subset\mathbb{R}$, the blocks being indexed by C ordered by θ_o. Under some assumptions on Σ, we will associate to these data and to each integer w a connection with a pole of order two $(\mathscr{H},\nabla)$ with $\mathbb{Q}$-Betti structure $(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})$ and a nondegenerate $(-1)^w$-ι-symmetric nondegenerate $\mathbb{Q}$-pairing $\mathcal{Q}$ as in §2.3.2, giving rise in particular to a TERP$(-w)$-structure. We will denote these data by $\mathrm{ncH}(C,\theta_o,\Sigma,w)$.

The matrix Σ determines Stokes data $((L_{c,1},L_{c,2}),S,S')$ of type (C,θ_o) (cf. §1.5) by setting $L_1=L_2=\mathbb{Q}^d$, and $L_{c,j}$ ($c\in C$, $j=1,2$) correspond to the blocks of Σ, which defines a linear morphism $S:L_1\to L_2$, and we define S' as the linear morphism attached to $\Sigma':=(-1)^w\cdot{}^t\Sigma$. These Stokes data in turn correspond to a Stokes filtered local system $(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})$. The underlying local system $\mathscr{L}_{\mathbb{Q}}$ is completely determined by the $\mathbb{Q}$-vector space $L_1=\mathbb{Q}^d$ together with (the conjugacy class of) its monodromy, whose matrix is $(-1)^w\Sigma^{-1}\cdot{}^t\Sigma$. On the other hand, each diagonal block Σ_{c_i} of Σ gives rise to an invertible matrix $(-1)^w\Sigma_{c_i}^{-1}\cdot{}^t\Sigma_{c_i}$, which represents the monodromy of the meromorphic connection corresponding to $\widehat{\mathscr{G}_{c_i}}$ in the decomposition of §1.3.

A nondegenerate ι-pairing $\mathcal{Q}_{\mathrm{B}}$ on $(\mathscr{L}_{\mathbb{Q}},\mathscr{L}_{\mathbb{Q},\bullet})$ with values in $\mathbb{Q}$ is determined by a pair of nondegenerate pairings $\mathcal{Q}_{12}:L_{1,\mathbb{Q}}\otimes L_{2,\mathbb{Q}}\to\mathbb{Q}$ and $\mathcal{Q}_{21}:L_{2,\mathbb{Q}}\otimes L_{1,\mathbb{Q}}\to\mathbb{Q}$ which satisfy $\mathcal{Q}_{21}(x_2,x_1)=\mathcal{Q}_{12}(S^{-1}x_2,S'x_1)$, and the $(-1)^w$-ι-symmetry amounts to $\mathcal{Q}_{21}(x_2,x_1)=(-1)^w\mathcal{Q}_{12}(x_1,x_2)$ (cf. [**HS11**, (3.3) & (3.4)]). In the fixed bases of L_1 and L_2, we define $\mathcal{Q}_{21}(x_2,x_1)={}^tx_2\cdot x_1$, so that $\mathcal{Q}_{12}(x_1,x_2)={}^tx_1{}^t\Sigma\cdot\Sigma'^{-1}x_2$; the

$(-1)^w$-symmetry follows from $\Sigma' = (-1)^w \cdot {}^t\Sigma$. From the Riemann-Hilbert correspondence we finally obtain a nondegenerate $(-1)^w$-ι-symmetric pairing

$$\mathcal{Q}_{\mathrm{B}} : ((\mathscr{G}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})) \otimes \iota^*((\mathscr{G}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet})) \longrightarrow (\mathbb{C}(\{z\}), d, \mathbb{Q}_{S^1}).$$

We will set (using the notation of §4.1, and stressing upon the fact that the construction of $(\mathscr{G}, \nabla)$ and $(\mathscr{L}, \mathscr{L}_\bullet)$ above depends on the parity of w):

$$\mathrm{ncH}(C, \theta_o, \Sigma, w) = (\mathrm{DM}^{-(w+1)/2}(\mathscr{G}, \nabla), (\mathscr{L}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q},\bullet}), \mathcal{Q}_{\mathrm{B}}).$$

Let us note that the last corollary of §4.1 reads as follows:

Corollary. *Let (C, θ_o, Σ, w) be as above. Assume that $\ker(\Sigma_{c_i} + {}^t\Sigma_{c_i}) = 0$ for all i. Then $\mathrm{DM}^{>-(w+1)/2} = \mathrm{DM}^{-(w+1)/2}$ and the pairing $\mathcal{Q}_{\mathrm{B}}$ above induces a nondegenerate $(-1)^w$-ι-symmetric pairing, also denoted by $\mathcal{Q}_{\mathrm{B}}$:*

$$\mathrm{DM}^{-(w+1)/2}(\mathscr{G}, \nabla) \otimes_{\mathbb{C}\{z\}} \iota^* \mathrm{DM}^{-(w+1)/2}(\mathscr{G}, \nabla) \longrightarrow (z^{-w}\mathbb{C}\{z\}, \mathrm{d}).$$

***Theorem* (cf. [HS11]).** *Let (C, θ_o, Σ, w) be as above. We moreover assume the following:*

1. *for each $c \in C$, the diagonal block Σ_c of Σ satisfies $\ker(\Sigma_c + {}^t\Sigma_c) = 0$,*
2. *the quadratic form $\Sigma + {}^t\Sigma$ is positive semi-definite.*

Then $\mathrm{ncH}(C, \theta_o, \Sigma, w)$ is a polarized pure nc. $\mathbb{Q}$-Hodge structure of weight w.

Sketch of proof. Although the data are clearly defined, a direct proof starting from these data is not known. The problem is to relate the positivity property of the Stokes matrix with the existence of a metric of a certain kind, and to prove that $\widetilde{\mathscr{H}} \simeq \mathscr{O}_{\mathbb{P}^1}(w)^d$. The way to construct $\widetilde{\mathscr{H}}$ by the gluing procedure of §2.2 is difficult to relate with the Deligne-Malgrange lattices.

The proof exhibits $\mathrm{ncH}(C, \theta_o, \Sigma, w)$ as obtained by Laplace transform from a unitary flat bundle on $\mathbb{C} \smallsetminus C$, regarded as a variation of polarized pure Hodge structure of type $(0, 0)$. More precisely, the $\mathbb{C}[t]\langle\partial_t\rangle$-module M is the regular holonomic module obtained as the intermediate (or middle) extension of the corresponding flat bundle from $\mathbb{C} \smallsetminus C$ to $\mathbb{C} = \mathbb{A}^1$, and the lattice L considered in §3.4 is the Deligne lattice of M which is equal to $V^{>-1}$ at each $c \in C$. □

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