Vanishing cycles and their algebraic computation (I)

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The $\mu = \mu$ theorem

- $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ a germ of holom. funct. isol. sing., $F =$ Milnor fibre.
- THEOREM (Milnor):

$$\dim_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^{n+1},0}/(\partial f) =: \boxed{\mu_{\mathrm{alg}} = \mu_{\mathrm{top}}} := \dim_{\mathbb{C}} H^n(F, \mathbb{C})$$

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- Various proofs:
  - Use index of $\mathrm{grad}\, f$ on the Milnor sphere.
  - Deform the function $\longrightarrow \mu_{\mathrm{alg}}$ nondeg. critical pts
  - etc.
- Brieskorn: algebraic formula for the monodromy.
- Important tool: the ***Brieskorn lattice***.

Objective of the lectures

To extend these results to the case

$$f : X \longrightarrow \mathbb{C}$$

X ***smooth quasi-projective*** and f ***regular on*** X.

Local systems in dim. one

- Δ: disc with coord. t, $\Delta^* := \Delta \smallsetminus \{0\}$, $\widetilde{\Delta}^*$: univ. cov.

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- $\mathscr{L}$: local system on Δ^*, $\mathscr{L} \longleftrightarrow (L, \mathrm{T})$,
  L: finite dim. vect. space, T: autom. of L

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- Answer:
  - $p : \widetilde{\Delta}^* \longrightarrow \Delta^*$: univ. covering, $\Rightarrow p^{-1}\mathscr{L}$ trivial
  - $L = \Gamma(\widetilde{\Delta}^*, p^{-1}\mathscr{L})$, T induced by deck-transf.

$$\boxed{L = i_0^{-1} j_* p_* p^{-1} \mathscr{L}}$$

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- $j^{-1}\mathscr{F} = \mathscr{L}$, $\quad i_0^{-1} j_* \mathscr{L} = \ker \left[(\mathrm{T} - \mathrm{Id}) : L \longrightarrow L \right]$ given by the adjunction

$$i_0^{-1} j_* \mathscr{L} = i_0^{-1} \mathscr{F} \longrightarrow i_0^{-1} (j \circ p)_* (j \circ p)^{-1} \mathscr{F}.$$

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$$i_0^{-1} j_* \mathscr{L} = i_0^{-1}\mathscr{F} \longrightarrow i_0^{-1}(j \circ p)_*(j \circ p)^{-1}\mathscr{F}.$$

- $\mathscr{F}$ ***any*** constr. bounded complex on $(\Delta, 0)$.

$$\boxed{\psi_t\mathscr{F} := i_0^{-1} R(j \circ p)_*(j \circ p)^{-1}\mathscr{F}} \tag{1.1.1}$$

$$\boxed{i_0^{-1}\mathscr{F} \longrightarrow \psi_t\mathscr{F} \xrightarrow{\text{can}} \phi_t\mathscr{F} \xrightarrow{+1}}$$

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$\longrightarrow$ Long exact sequence of $\mathbb{C}$-vect. spaces with autom. $\mathbf{T}$

$$\to \underset{\circlearrowleft\, \mathrm{Id}}{\mathscr{H}^j(i_0^{-1}\mathscr{F})} \to \underset{\circlearrowleft\, \mathbf{T}}{\mathscr{H}^j \psi_t \mathscr{F}} \xrightarrow{\mathscr{H}^j \text{can}} \underset{\circlearrowleft\, \mathbf{T}}{\mathscr{H}^j \phi_t \mathscr{F}} \to \underset{\circlearrowleft\, \mathrm{Id}}{\mathscr{H}^{j+1}(i_0^{-1}\mathscr{F})} \to$$

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- If $\mathscr{F} = j_*\mathscr{L}$, $i_0^{-1}\mathscr{F} \simeq \ker(\mathrm{T} - \mathrm{Id}) \Rightarrow$ $\phi_t\mathscr{F} \simeq \mathrm{Im}(\mathrm{T} - \mathrm{Id})$ cohom. in ***degree zero only***.

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- If $\mathscr{F} = Rj_* \mathscr{L}$, $i_0^{-1}\mathscr{F} = \{L \xrightarrow{\mathrm{T}-\mathrm{Id}} L\} \Rightarrow$ $\phi_t \mathscr{F} \overset{\mathrm{can}}{\simeq} \psi_t \mathscr{F}$ cohom. in ***degree zero only***.

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- If $\mathscr{F} = i_{0,*} F$ (skyscrap. sheaf), $\psi_t \mathscr{F} = 0 \Rightarrow$ $\phi_t \mathscr{F} = F[1]$ cohom. in ***degree*** -1 ***only***.

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- Set

$$(1.1.2) \qquad {}^p\psi_t\mathscr{F} = \psi_t\mathscr{F}[-1], \quad {}^p\phi_t\mathscr{F} = \phi_t\mathscr{F}[-1].$$

Then for $\mathscr{F} = j_*\mathscr{L}[\dim\Delta]$, $Rj_*\mathscr{L}[\dim\Delta]$ or $i_{0,*}F$,

$$\boxed{\mathscr{H}^j({}^p\psi_t\mathscr{F}) = 0, \quad \mathscr{H}^j({}^p\phi_t\mathscr{F}) = 0, \quad \text{if } j \neq 0}$$

Perversity in dim. one

- DEFINITION 1.1.3. A constr. complex on $(\Delta, 0)$ is ***perverse*** if
  - $j^{-1}\mathscr{F} = \mathscr{L}[1]$ for some local system $\mathscr{L}$ on Δ^*,
  - $i_0^{-1}\mathscr{F}$ has nonzero cohom. in deg. -1 and 0 at most,
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- THEOREM 1.1.4. A constr. complex $\mathscr{F}$ is perverse iff $\mathscr{H}^j({}^p\psi_t\mathscr{F}) = \mathscr{H}^j({}^p\phi_t\mathscr{F}) = 0$ for $j \neq 0$.

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PROPOSITION 1.1.6. Assume $\mathscr{F}$ perverse on $(\mathbb{A}^1, C)$. Then

- $H^k_c(\overline{\Delta} \setminus I, \mathscr{F}) = 0$ for $k \neq 0$, and

$$\dim H^0_c(\overline{\Delta} \setminus I, \mathscr{F}) = \sum_{c \in C} \dim {}^p\phi_{t-c}\mathscr{F}. \tag{1.1.6*}$$

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- PROOF. Need only consider $\mathscr{F} = i_{c,*}F$ ($c \in C$) and $\mathscr{F} = j_*\mathscr{L}[1]$ ($j : \mathbb{A}^1 \smallsetminus C \hookrightarrow \mathbb{A}^1$).

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- $H^1_c(\overline{\Delta}\smallsetminus I, j_*\mathscr{L}[1]) = H^2_c(\overline{\Delta}\smallsetminus I, j_*\mathscr{L}) = 0$: by duality, non degen. pairing

$$H^0_c(\overline{\Delta}\smallsetminus I, j_*\mathscr{L}^\vee)\otimes H^2_c(\overline{\Delta}\smallsetminus I^c, j_*\mathscr{L}) \longrightarrow H^2_c(\Delta, \mathbb{C}) \simeq \mathbb{C}$$

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- $\beta : \overline{\Delta} \smallsetminus I \hookrightarrow \overline{\Delta}$, then

$$\begin{aligned}\dim H^0_{\mathrm{c}}(\overline{\Delta} \smallsetminus I, j_*\mathscr{L}[1]) &= -\chi(\overline{\Delta}, \beta_!\beta^{-1}j_*\mathscr{L}) \\ &= -\chi(\overline{\Delta} \smallsetminus (I \cup C)) \cdot \mathrm{rk}\,\mathscr{L} - \textstyle\sum_{c\in C} \dim(j_*\mathscr{L})_c \\ &= \#C \cdot \mathrm{rk}\,\mathscr{L} - \textstyle\sum_{c \in C}\big(\mathrm{rk}\,\mathscr{L} - \dim \phi_{t-c}(j_*\mathscr{L})\big)\end{aligned}$$

Nearby and vanishing cycles

- X: cplx manifold, $f : X \longrightarrow \mathbb{C}$ holom. function, $X_0 := f^{-1}(0)$.
- Goal: to glue as a ***cplx of sheaves*** the Milnor fibres of f at each $x \in X_0$.

$$
\begin{array}{ccccccc}
X_0 & \overset{i_0}{\hookrightarrow} & X & \overset{j}{\hookleftarrow} & X^* & \overset{p}{\longleftarrow} & \widetilde{X}^* \\
\downarrow f & & \downarrow f & & \downarrow f & \square & \downarrow \\
\{0\} & \underset{i_0}{\hookrightarrow} & \Delta & \underset{j}{\hookleftarrow} & \Delta^* & \underset{p}{\longleftarrow} & \widetilde{\Delta}^*
\end{array}
$$

Nearby and vanishing cycles for $\mathbb{C}_X$

$$\boxed{\psi_f \mathbb{C}_X := i_0^{-1} R(j \circ p)_* (j \circ p)^{-1} \mathbb{C}_X}$$

$$\mathbb{C}_{X_0} = i_0^{-1} \mathbb{C}_X \longrightarrow \psi_f \mathbb{C}_X$$

$$\boxed{\mathbb{C}_{X_0} = i_0^{-1} \mathbb{C}_X \longrightarrow \psi_f \mathbb{C}_X \xrightarrow{\text{can}} \phi_f \mathbb{C}_X \xrightarrow{+1} .}$$

- $\psi_f \mathbb{C}_X$ supported on X_0,
- $\phi_f \mathbb{C}_X$ supported on $\mathrm{Crit}(f) \cap X_0$,
- Both equipped with monodromy $\mathbf{T}$.

Proper push-forward

THEOREM 1.3.1.

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X' \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f'} \\ & & \Delta \end{array}$$

Assume that π ***proper***. $\mathscr{F}$ bounded cplx. Then

$$\boxed{R\pi_*\psi_f\mathscr{F} \simeq \psi_{f'}R\pi_*\mathscr{F}} \qquad \boxed{R\pi_*\phi_f\mathscr{F} \simeq \phi_{f'}R\pi_*\mathscr{F}}$$

EXAMPLE. If $f : X \longrightarrow \Delta$ ***proper***, then

$$\begin{aligned} Rf_*\psi_f\mathbb{C}_X &\simeq \psi_t Rf_*\mathbb{C}_X, \\ Rf_*\phi_f\mathbb{C}_X &\simeq \phi_t Rf_*\mathbb{C}_X. \end{aligned} \tag{1.3.2}$$

Non proper push-forward

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X' \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f'} \\ & & \Delta \end{array}$$

If π ***not proper***, we may have

$$R\pi_*\psi_f \mathscr{F} \not\simeq \psi_{f'} R\pi_* \mathscr{F}, \quad R\pi_*\phi_f \mathscr{F} \not\simeq \phi_{f'} R\pi_* \mathscr{F}$$

EXAMPLE: $X = B_\varepsilon \subset \mathbb{C}^{n+1}$, $f : X \longrightarrow \mathbb{C}$ isol. sing. at 0, $\pi : U = X \setminus \{0\} \hookrightarrow X$.

- $\phi_{f|U}\mathbb{C}_U = 0 \Rightarrow \pi_!\phi_{f|U}\mathbb{C}_U = 0$.
- $0 \to \pi_!\mathbb{C}_U \to \mathbb{C}_X \to \mathbb{C}_0 \to 0 \Rightarrow$

  $H^n\phi_f\pi_!\mathbb{C}_X \simeq H^n\phi_f\mathbb{C}_X \neq 0 \quad (n \geqslant 1)$

Constructibility

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THEOREM 1.3.3. $\psi_f \mathbb{C}_X$ and $\phi_f \mathbb{C}_X$ have $\mathbb{C}$-constr. cohomol. (*i.e.*, $\exists$ a Whitney stratif. of X_0 s.t. $\mathscr{H}^k \psi_f \mathbb{C}_X$ and $\mathscr{H}^k \phi_f \mathbb{C}_X$ are loc. const. sheaves of f.d. $\mathbb{C}$-vect. spaces on each stratum).

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- COMPLEMENT. Moreover, $\forall x \in X_0$ and $\forall k \in \mathbb{N}$,

$$\mathscr{H}^k(\psi_f \mathbb{C}_X)_x \simeq H^k(F_{\varepsilon,\eta}, \mathbb{C}),$$

  - B_ε: closed ball centered at x,
  - Δ_η: small disc centered at $f(x)$,
  - $f : B_\varepsilon \cap f^{-1}(\Delta_\eta^*) \to (\Delta_\eta^*)$: Milnor-Lê fibr. at x $(0 < \eta \ll \varepsilon \ll 1)$,
  - $\boxed{F_{\varepsilon,\eta} := f^{-1}(\eta) \cap B_\varepsilon}$: Milnor fibre of f at x.

Perversity

THEOREM 1.3.4.

- ${}^p\psi_f {}^p\mathbb{C}_X$ and ${}^p\phi_f {}^p\mathbb{C}_X$ are ***perverse*** (on X_0),
- *i.e.*, $Ri_{0,*} {}^p\psi_f {}^p\mathbb{C}_X$ and $Ri_{0,*} {}^p\phi_f {}^p\mathbb{C}_X$ are ***perverse*** (on X).

COROLLARY 1.3.5. Assume $f : X \to \Delta$ ***proper***. Then $\forall j \in \mathbb{Z}$,

$$({}^p\mathscr{H}^j\, Rf_*)({}^p\psi_f {}^p\mathbb{C}_X) \simeq {}^p\psi_t({}^p\mathscr{H}^j\, Rf_*){}^p\mathbb{C}_X,$$

$$({}^p\mathscr{H}^j\, Rf_*)({}^p\phi_f {}^p\mathbb{C}_X) \simeq {}^p\phi_t({}^p\mathscr{H}^j\, Rf_*){}^p\mathbb{C}_X.$$

Other approaches

$$\psi_f \mathscr{F} := i_0^{-1} Rj_* Rp_* (j \circ p)^{-1} \mathscr{F}$$

Replace with the Alexander complex

$$^A\psi_f \mathscr{F} := i_0^{-1} Rj_* Rp_! (j \circ p)^{-1} \mathscr{F}$$

$\mathscr{L}(\mathrm{T}) = p_! \mathbb{C}_{\widetilde{\Delta}^*}$: local syst. on Δ^* with fibre $\mathbb{C}[\mathrm{T}, \mathrm{T}^{-1}]$ and monodromy $=$ mult. by T.

$$^A\psi_f \mathscr{F} = i_0^{-1} Rj_* (f^{-1} \mathscr{L}(\mathrm{T}) \otimes j^{-1} \mathscr{F})$$

object in $D^{\mathrm{b}}(\mathbb{C}[\mathrm{T}, \mathrm{T}^{-1}])$.

- $\mathscr{F}$ constructible $\Rightarrow {}^A\psi_f \mathscr{F} \simeq {}^p\psi_f \mathscr{F}$.