Integral of a monomial over a simplex. User's guide

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This document is a companion for the two Maple programs

Integral of a monomial over a simplex, by means of Brion theorem and iterated Maple Laurent expansions,

Integral of a monomial over a simplex, by means of Brion theorem and iterated geometric series.

It contains two parts. First, we see what these programs do. In the second part, we briefly recall the mathematical background.

I. Commands.

Let $S = (S_1, \ldots, S_{n+1})$ be a set of $n+1$ affinely independant points in $\mathbb{R}^d$. We denote by $\mathfrak{s}$ the convex hull of the set S. It is a n-dimensional simplex in the affine space $< S >$ spanned by the set S. Let $(m_1, \ldots, m_d)$ be a list of d nonegative integers. This program computes the integral over $\mathfrak{s}$ of the monomial $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$, with respect to a specified Lebesgue measure. We consider three cases. Denote by $\mathrm{lin}(S)$ the vector space parallel to $< S >$.

Case 1) The basic computation. The *simplex related* Lebesgue measure is defined by the basis of $\mathrm{lin}(S)$ made of the n edges of $\mathfrak{s}$ at any of its vertices. For instance, at vertex S_1 the basis is $(S_i - S_1, i = 2, \ldots, n+1)$. This Lebesgue measure does not depend on the choice of the vertex. The simplex $\mathfrak{s}$ has volume $\frac{1}{n!}$.

Case 2) In this case we assume that the simplex is rational. The Lebesgue measure is defined by the intersection lattice $\mathrm{lin}(S) \cap \mathbb{Z}^d$. Thus the basic parallelepiped of this lattice has volume 1. In particular, if $\mathfrak{s}$ is full-dimensional, we obtain the standard Lebesgue measure of $\mathbb{R}^d$.

Case 3) The Lebesgue measure on $\mathrm{lin}(S)$ is defined by a positive definite scalar product Q on R^d.

The following commands can be used.

```
>basic_simplex_integral:=proc(S,d,m)
```

Here S is a list of $n+1$ points in $\mathbb{R}^d$, m is a list of d nonegative integers. This command returns the value of the integral

$$\int_{\mathfrak{s}} x_1^{m_1} \dots x_d^{m_d} dm_{\mathfrak{s}}(x)$$

here $dm_{\mathfrak{s}}(x)$ denotes the simplex related Lebesgue measure.

```
>relative_simplex_integral:=proc(S,d,m)
```

Here S is a list of $n+1$ **rational** points in $\mathbb{R}^d$, m is a list of d nonegative integers. This command returns the value of the integral

$$\int_{\mathfrak{s}} x_1^{m_1} \dots x_d^{m_d} dm_{\mathfrak{s}}(x),$$

where $dm_{\mathfrak{s}}(x)$ denotes the Lebesgue measure defined by the intersection lattice $\mathrm{lin}(S) \cap \mathbb{Z}^d$.

```
>simplex_face_integral:=proc(S,d,K,m)
```

Here S is a list of $d+1$ **rational** points in $\mathbb{R}^d$, K is a sublist of $[1 \dots d]$, m is a list of d nonegative integers. The sublist K defines a subset $S(K)$ of S with convex hull $\mathfrak{s}(K)$. This command returns the value of the integral

$$\int_{\mathfrak{s}(K)} x_1^{m_1} \dots x_d^{m_d} dm_{\mathfrak{s}}(x),$$

where $dm_{\mathfrak{s}}(x)$ denotes the Lebesgue measure defined by the intersection lattice $\mathrm{lin}(S(K)) \cap \mathbb{Z}^d$. This particular case is useful for programs which implement Euler-Maclaurin formulas.

```
>euclidean_simplex_integral:=proc(S,Q,d,m)
```

Here S is a list of $n+1$ points in $\mathbb{R}^d$, Q is a positive definite scalar product on R^d, m is a list of d nonegative integers. This command returns the value of the integral

$$\int_{\mathfrak{s}} x_1^{m_1} \dots x_d^{m_d} dm_{\mathfrak{s}}(x),$$

where $dm_{\mathfrak{s}}(x)$ denotes the Lebesgue measure on $\mathrm{lin}(S)$ defined by the scalar product.

II. Mathematical background.

II-1. The method: Brion's formula and iterated Laurent series

Let $\mathfrak{s}$ be a $n-$dimensional simplex in $\mathbb{R}^d$ with rational vertices $s_i, 1 \leq i \leq n+1$. Let $dm_{\mathfrak{s}}$ be the relative Lebesgue measure on $< \mathfrak{s} >$. We want to compute the integral

$$\int_{\mathfrak{s}} x_1^{m_1} \dots x_n^{m_d} dm_{\mathfrak{s}}(x). \tag{1}$$

We start by observing that (1) is equal to the coefficient of $\frac{\xi_1^{m_1} \dots \xi_n^{m_d}}{m_1! \dots m_d!}$ in

$$\int_{\mathfrak{s}} e^{\langle \xi, x \rangle} dm_{\mathfrak{s}}(x)$$

Let $\mathrm{vol}(\mathfrak{s})$ be the corresponding volume of the simplex. Our method is based on the following formula.

$$\int_{\mathfrak{s}} e^{\langle \xi, x \rangle} dm_{\mathfrak{s}}(x) = (-1)^n n! \ \mathrm{vol}(\mathfrak{s}) \sum_{i=1}^{n+1} \frac{e^{\langle \xi, s_i \rangle}}{\prod_{j \neq i} (\langle \xi, s_j - s_i \rangle)}. \tag{2}$$

Formula (2) follows from Brion's theorem on polyhedra. It is also easily obtained by using the standard parametrisation of the simplex. Notice that although each term in this sum has poles, the poles cancel and the sum is a holomorphic function of ξ.

Example. Dimension 1.

$$\int_{s_1}^{s_2} e^{\xi x} dx = \frac{e^{\xi s_2} - e^{\xi s_1}}{\xi} = -(s_2 - s_1) \left(\frac{e^{\xi s_1}}{\xi(s_2 - s_1)} + \frac{e^{\xi s_2}}{\xi(s_1 - s_2)} \right).$$

Example. Let $\mathfrak{s}$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0),(1,0),(0,1)$. In this case, (2) becomes

$$\int_{\mathfrak{s}} e^{\xi_1 x_1+\xi_2 x_2} dx_1 dx_2 = \frac{1}{\xi_1\xi_2} + \frac{e^{\xi_1}}{\xi_1(\xi_1-\xi_2)} + \frac{e^{\xi_2}}{\xi_2(\xi_2-\xi_1)}.$$

Thus we compute (1) as the coefficient of $\frac{\xi_1^{m_1}\dots\xi_n^{m_d}}{m_1!\dots m_d!}$ in the right-hand-side of (2). Our method consists in applying iterated Laurent series expansions to each meromorphic function in the sum (2) with respect to the variables $\xi_1,\dots,\xi_d$, one at a time.

Several methods can be used in order to compute the one dimensional Laurent series. In the program **Integral over simplex with Laurent expansion.mw**, we just use the Maple command `laurent`. In the program **Integral-with-geometric-series.mw**, we do the expansions by hand, being careful to keep as few terms as possible.

II-2. Relative volume

Case 1) The volume with respect to the intersected lattice is computed as follows. For $j=1,\dots,n$, let $C_j = s_{j+1}-s_1$. Thus the vectors $C_j, j=1,\dots,n$ form a basis of the vector space $\mathrm{lin}(S)$. Let C be the (d,n)-matrix with column vectors C_j. Let A be the (n,n) matrix which consists of the first n rows of a rowwise Hermite form of the matrix C. Then the relative volume of the simplex $\mathfrak{s}$ is given by

$$n!\,\mathrm{vol}(\mathfrak{s}) = |\det A|.$$

This follows from the next lemma.

Lemma 1 *Let L be a subspace of $V \cong \mathbb{R}^d$ with $\dim L = n$. Denote by $(e_1,\dots,e_d)$ the standard basis of $\mathbb{R}^d$ and by $(\overline{f_1},\dots,\overline{f_d})$ the restriction of the standard dual basis to L. Let $g_1,\dots,g_n$ be a basis of the lattice generated by $(\overline{f_1},\dots,\overline{f_d})$, and let $(B_1,\dots,B_n)$ be the dual basis of L.*
i) $(B_1,\dots,B_k)$ is a basis of the lattice $L\cap\mathbb{Z}^d$.
ii) Let

$$C_j = \sum_{i=1}^{d} c_{ij} e_i, \quad \text{for } j=1,\dots,n, \tag{3}$$

be a basis of the subspace L. Let A be the (n,n)-matrix made of the first n rows of a rowwise integer reduced Hermite form of the matrix

$$C = (c_{ij})_{1\leq i\leq d, 1\leq j\leq n}.$$

Then A^{-1} is the matrix of a basis $(B_1,\ldots,B_n)$ of $L\cap\mathbb{Z}^d$ with respect to the basis C_j.
iii) The normalized Lebesgue measure dm_L on L defined by the lattice $L\cap\mathbb{Z}^d$ is given by the formula

$$\int_L h\, dm_L = |\det A| \int_{\mathbb{R}^k} h(x_1C_1+\ldots x_nC_n)\, dx_1\ldots dx_n. \tag{4}$$

Proof. A basis $(B_1,\ldots,B_n)$ of L is a basis of the lattice $L\cap\mathbb{Z}^d$ if and only if the dual basis (B_i^*) is a basis of the projected lattice of $L^*\cong L^*/L^\perp$, whence i).

Let $H=a_{ij}$ be a rowwise Hermite form of the matrix C. Thus

$$H = UC$$

where $U\in \mathrm{SL}(d,\mathbb{Z})$ and H is an upper triangular (d,n) matrix. Let $(C_j^*, j = 1\ldots,n)$ be the dual basis of L^*. We have $\overline{f_i}=\sum_{j=1}^n c_{ij}C_j^*$, for $i=1\ldots d$. Thus the first n rows of the matrix H form a basis of the lattice generated by $(\overline{f_1},\ldots,\overline{f_d})$, expressed in the basis C_j^*. This proves ii) and iii) $\square$.