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On square roots of class C^m of nonnegative functions of one variable

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Abstract. We investigate the regularity of functions g such that $g^2 = f$, where f is a given nonnegative function of one variable. Assuming that f is of class C^{2m} ($m > 1$) and vanishes together with its derivatives up to order $2m - 4$ at all its local minimum points, one can find a g of class C^m. Under the same assumption on the minimum points, if f is of class C^{2m+2} then g can be chosen such that it admits a derivative of order $m + 1$ everywhere. Counterexamples show that these results are sharp.

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Introduction

In this paper we study the regularity of functions g of one variable whose square is a given nonnegative function f.

For a function f of class C^2, first results are due to G. Glaeser [6] who proved that $f^{1/2}$ is of class C^1 if the second derivative of f vanishes at the zeros of f, and to T. Mandai [8] who proved that one can always choose g of class C^1. More recently in [1] (and later in [7]), for functions f of class C^4, it was proved that one can find g of class C^1 and twice differentiable at every point.

F. Broglia and the authors proved in [3] that this result is sharp in the sense that it is not possible to have in general a greater regularity for g. They also showed that if f is of class C^4 and vanishes at all its (local) minimum points, one can always find g of class C^2 and that the result is sharp. Later, in [4] it was proved that for f of class C^6 vanishing at all its minimum points one can find g of class C^2 and three times differentiable at every point.

In this paper we generalize these results. First we prove that for f of class C^{2m}, $m = 1, 2, \dots, \infty$, vanishing at its (local) minimum points together with all its derivatives up to order $(2m - 4)$ one can find g of class C^m (Theorem 2.2). If the derivatives vanish only up to order $2m - 6$ at all the minimum points, the other assumptions being unchanged, g can be chosen m times differentiable at every point (Theorem 3.1, where m is replaced by $m + 1$).

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Counterexamples are given to show that these assumptions cannot be relaxed and that the regularity of g cannot be improved in general.

1. Precised square roots

In this paper, f will always be a nonnegative function of one real variable whose regularity will be precised below. Our results being of local character, we may and will assume that the support of f is contained in [0, 1].

Definition 1.1. Assuming f of class C^{2m}, $m = 1, 2, \dots, \infty$, we say that g is a *square root of f precised up to order m*, if g is a continuous function satisfying $g^2 = f$ and if, for any (finite) integer $k \le m$ and for any point x_0 which is a zero of f of order exactly $2k$, the function $x \mapsto (x - x_0)^k g(x)$ keeps a constant sign near x_0.

It is clear that g cannot be m times differentiable at every point if this condition is not fulfilled.

It is easy to show the existence of square roots precised up to order m and even to describe all of them. Let us consider the closed set

$$G = \{x \in \mathbb{R} \mid f(x) = 0, f'(x) = 0, \dots, f^{(2m)}(x) = 0\}, \tag{1.1}$$

with the convention that all derivatives vanish if $m = \infty$. Its complement is a union of disjoint intervals J_ν. In J_ν, the zeros of f are isolated and of finite order $\le 2m$. For a square root precised up to order m, one should have $|g| = f^{1/2}$ and the restriction of g to J_ν should be one of two well defined functions $+g_\nu$ and $-g_\nu$ thanks to the condition on the change of sign. There is a bijection between the set of families (ϵ_ν) with $\epsilon_\nu = \pm 1$ and the set of square roots precised up to order m: one has just to set $g(x) = \epsilon_\nu g_\nu(x)$ for $x \in J_\nu$ and $g(x) = 0$ for $x \in G$.

A *modulus of continuity* is a continuous, positive, increasing and concave function defined on an interval $[0, t_0]$ and vanishing at 0. Any continuous function φ defined on a compact set K has a modulus of continuity, *i.e.* a function ω as above such that for every t_1, t_2 with $|t_2 - t_1| < t_0$, one has $|\varphi(t_2) - \varphi(t_1)| < \omega(|t_2 - t_1|)$. One says that $\varphi \in C^\omega(K)$. If $\varphi \in C^k(K)$ and if ω is a modulus of continuity of $\varphi^{(k)}$, one says that $\varphi \in C^{k,\omega}(K)$.

We now state two lemmas taken almost literally from [2, Lemme 4.1, Lemme 4.2 and Corollaire 4.3]. Note that in the rest of this section m will not be allowed to take the value ∞.

Lemma 1.2. *Let $\varphi \in C^{2m}(J)$ be nonnegative, where J is a closed interval contained in $[-1, 1]$, and let $M = \sup \left|\varphi^{(k)}(x)\right|$ for $0 \le k \le 2m$ and $x \in J$. Assume that for some $j \in \{0, \dots, m\}$, the inequality $\varphi^{(2j)}(x) \ge \gamma > 0$ holds for $x \in J$ and that φ has a zero of order $2j$ at some point $\xi \in J$.*

Let us define H and ψ in J by

$$\varphi(x) = (x - \xi)^{2j} H(x), \quad \psi(x) = (x - \xi)^j H(x)^{1/2}.$$

Then, $H \in C^{2m-2j}(J)$ and $\psi \in C^{2m-j}(J)$. Moreover, there exists C_1, depending only on m, such that

$$\left|\psi^{(k)}(x)\right| \le C_1 \gamma^{\frac{1}{2}-k} M^k, \quad k = 1, \dots, 2m - j. \tag{1.2}$$

Lemma 1.3. *Let φ be a nonnegative function of one variable, defined and of class C^{2m} in the interval $[-1, 1]$ such that $\left|\varphi^{(2m)}(t)\right| \le 1$ for $|t| \le 1$ and that* $\max_{0 \le j \le m-1} \varphi^{(2j)}(0) = 1$.

(i) *There exists a universal positive constant C_0, such that*

$$\left|\varphi^{(k)}(t)\right| \le C_0, \quad \text{for } |t| \le 1 \text{ and } 0 \le k \le 2m. \tag{1.3}$$

(ii) *There exist universal positive constants a_j and r_j, $j = 0, \dots, m-1$, such that one of the following cases occurs*:
   (a) *One has $\varphi(0) \ge a_0$ and then $\varphi(t) \ge a_0/2$ for $|t| \le r_0$.*
   (b) *For some $j \in \{1, \dots, m-1\}$ one has $\varphi^{2j}(t) \ge a_j$ for $|t| \le r_j$ and φ has a local minimum in $[-r_j, r_j]$.*

In the following proposition, G is defined by (1.1) and $d(x, G)$ denotes the distance of x from G. When $G = \emptyset$, (a) and (b) are always true and condition (1.4) disappears.

Proposition 1.4. *Assuming that f is of class C^{2m}, the three following properties are equivalent.*

(a) *There exists $g \in C^m$ such that $g^2 = f$.*
(b) *Any function g which is a square root of f precised up to order m belongs to C^m.*
(c) *There exists a modulus of continuity ω such that*

$$\left|\frac{d^k}{dx^k} f^{1/2}(x)\right| \le d(x, G)^{m-k} \omega(d(x, G)), \tag{1.4}$$

for any x such that $f(x) \neq 0$ and any $k \in \{0, \dots, m\}$.

Proof. It is clear that (b)$\Rightarrow$(a): as said above, precised square roots do exist. Under assumption (a), g and its derivatives up to order m should vanish on G. If ω is a modulus of continuity of $g^{(m)}$ one gets $\left|g^{(m)}(x)\right| \le \omega(d(x, G))$. Successive integrations prove that the derivatives $g^{(k)}$ are bounded by the right hand side of (1.4). These derivatives being equal, up to the sign, to those of $f^{1/2}$ when f does not vanish, $(a) \Rightarrow (c)$ is proved.

Let us assume (c) and consider any connected component J_ν of the complement of G. Near each zero of f in J_ν, which is of order exactly $2j$ for some $j \in \{1, \dots, m\}$, the precised square root g_ν is given (up to the sign) by Lemma 1.2

and so it is of class C^m. Moreover, the estimate (1.4) extends by continuity to the points $x \in J_\nu$ where f vanishes and one has

$$\left|g_\nu^{(k)}(x)\right| \le d(x, G)^{m-k}\omega(d(x, G))$$

for $x \in J_\nu$ and $k \in \{0, \dots, m\}$.

If we define g equal to $\epsilon_\nu g_\nu$ in J_ν and to 0 in G, it remains to prove the existence and the continuity of the derivatives of g at any point $x_0 \in G$. By induction, the estimates above prove, for $k = 0, \dots, m-1$, that $g^{k+1}(x_0)$ exists and is equal to 0 and that $g^{k+1}(x) \to 0$ for $x \to x_0$. The proof is complete. □

Corollary 1.5. *Let f be a nonnegative C^∞ function of one variable such that for any m there exists a function g_m of class C^m with $g_m^2 = f$. Then there exists g of class C^∞ such that $g^2 = f$.*

Actually, if g is any square root of f precised up to order ∞, it is precised up to order m for any m and thus of class C^m for any m by the proposition above.

2. Continuously differentiable square roots

We start with an auxiliary result which contains the main argument. The function $f \in C^{2m}$, $m \ge 2$, and the set $G \ne \emptyset$ are as above, and Γ is a closed subset of G. We will use this lemma for $p = 0$, in which case Γ can be disregarded, and for $p = 1$.

Lemma 2.1. *Assume that $m \ne \infty$ and f and all its derivatives up to order $2m - 4$ (included) vanish at all its local minimum points. Assume moreover that there exist a modulus of continuity α and constants $C > 0$ and $p \ge 0$ such that*

$$\left|f^{(2m)}(x)\right| \le C d(x, \Gamma)^{2p}\alpha(d(x, G)). \tag{2.1}$$

Then, there exists a constant $\bar{C}$ such that

$$\left|\frac{d^k}{dx^k} f^{1/2}(x)\right| \le \bar{C} d(x, \Gamma)^p d(x, G)^{m-k}\alpha(d(x, G))^{1/2} \tag{2.2}$$

for any x such that $f(x) \ne 0$ and any $k \in \{0, \dots, m\}$.

Proof. Let J be any connected component of the complement of G and for $x \in J$, let $\widehat{x}$ be (one of) the nearest endpoint(s) of J. The distance between x and $\widehat{x}$ is thus equal to $d(x, G)$ and we remark that, for y between x and $\widehat{x}$, we have $d(y, \Gamma) \le 2d(x, \Gamma)$. Integrating $2m - k$ times the estimate for $f^{(2m)}$ between $\widehat{x}$ and x we get

$$|f^{(k)}(x)| \le C' d(x, \Gamma)^{2p} d(x, G)^{2m-k}\alpha(d(x, G))$$

for $k = 0, \dots, 2m$, the constant C' being independent of J.

Next, for x in J such that $f(x) \neq 0$, we define as in [2],

$$\rho(x) = \max_{0 \le k \le m-1} \left\{ \left[\frac{f_+^{(2k)}(x)}{C' d(x,\Gamma)^{2p} \alpha(d(x,G))} \right]^{\frac{1}{2m-2k}} \right\}.$$

One has thus $\rho(x) \le d(x, G)$ and

$$|f^{(k)}(x)| \le C' d(x,\Gamma)^{2p} \alpha(d(x,G)) \rho(x)^{2m-k}$$

for $k = 0, \dots, 2m$. The auxiliary function

$$\varphi(t) = \frac{f(x + t\rho(x))}{C' d(x,\Gamma)^{2p} \alpha(d(x,G)) \rho(x)^{2m}}$$

is defined in $[-1, 1]$ and satisfies the assumptions of Lemma 1.3. Two cases should be considered.

1. — One has $\varphi(0) \ge a_0$ and then $\varphi(t) \ge a_0/2$ for $|t| \le r_0$ while the derivatives of φ are uniformly bounded by C_0. Thus, there exists an universal constant C'' such that $\left| \frac{d^k}{dx^k} \varphi^{1/2}(t) \right| \le C''$ in this interval. We have thus, by the change of variable $t \mapsto x + t\rho(x)$,

$$\left| \frac{d^k}{dx^k} f^{1/2}(x) \right| \le C'' d(x,\Gamma)^p \rho(x)^{m-k} \alpha(d(x,G))^{1/2}$$

which implies (2.2).

2. — We are in case (b) of Lemma 1.3: all the derivatives of φ are bounded by C_0 and for some $j \in \{1, \dots, m-1\}$ one has $\varphi^{2j}(t) \ge a_j$ for $|t| \le r_j$ and φ has a local minimum at some point $\xi \in [-r_j, r_j]$. Our assumptions imply that $\varphi^{2k}(\xi)$ vanishes for $k \in \{0, \dots, m-2\}$ so j is necessarily equal to $m-1$. We can thus set $\varphi(t) = (t-\xi)^{2m-2} H(t)$ and $\psi(t) = (t-\xi)^{m-1} H(t)^{1/2}$ as in Lemma 1.2. There is a universal constant C''' (computed from C_0 and a_{m-1}) such that $\left| \frac{d^k}{dx^k} \psi(t) \right| \le C'''$ for $|t| \le r_{m-1}$. In particular, for $t = 0$, these derivatives coincide up to the sign with those of $\varphi^{1/2}$. The change of variable $t \mapsto x + t\rho(x)$ gives again the estimates (2.2) on the derivatives of $f^{1/2}(x)$. The proof is complete. □

Theorem 2.2. *Let f be a nonnegative function of one variable of class C^{2m} with $m \ge 2$ such that, at all its minimum points, f and its derivatives up to the order $(2m-4)$ vanish. Then any square root of f precised up to order m is of class C^m.*

Proof. The result is evident if G is empty and we can thus assume $G \neq \emptyset$. If α is a modulus of continuity of $f^{(2m)}$, we have $\left| f^{2m}(x) \right| \le \alpha(d(x,G))$ which is the assumption (2.1) for $p = 0$. By the preceding lemma, we have the estimates

$$\left| \frac{d^k}{dx^k} f^{1/2}(x) \right| \le \bar{C} d(x,G)^{m-k} \alpha(d(x,G))^{1/2}$$

when $f(x) \neq 0$. By Proposition 1.4, this implies that all the square roots precised up to order m are of class C^m. The case $m = \infty$ follows now from Corollary 1.5. □

Remark 2.3. It is certainly not necessary to assume that f vanishes at all its minimum points. For instance, we could also allow nonzero minima at points $\bar{x}_i$, $i \in \mathbb{N}$, provided that the values $f(\bar{x}_i)$ be not "too small". With the notations of Lemma 2.1, it suffices to have $f(\bar{x}_i) \geq C\alpha(d(\bar{x}_i, G))\rho(\bar{x}_i)^{2m}$ for some uniform positive constant C.

It is clear that the assumption $f \in C^{2m}$ of Theorem 2.2 cannot be weakened to $f \in C^{2m-1,1}$ (take $f(t) = t^{2m} + \frac{1}{2}t^{2m-1}|t|$). The two following counterexamples show that in the general case no stronger regularity is possible (Theorem 2.4) and that the vanishing of $2m - 4$ derivatives cannot be replaced by the vanishing of $2m - 6$ derivatives (Theorem 2.5).

Theorem 2.4. *For any given modulus of continuity ω there is a nonnegative function f of class C^∞ on $\mathbb{R}$ such that, at all its minimum points, f and all its derivatives up to the $(2m-4)$-th one vanish, but there is no function g of class $C^{m,\omega}$ such that $g^2 = f$.*

Proof. Let $\chi \in C^\infty(\mathbb{R})$ be the even function with support in $[-2, 2]$ defined by $\chi(t) = 1$ for $t \in [0, 1]$ and by $\chi(t) = \exp\left\{\frac{1}{(t-2)e^{1/(t-1)}}\right\}$ for $t \in (1, 2)$. We note that the logarithm of χ is a concave function on $(1, 2)$. For every $(a, b) \in [0, 1] \times [0, 1]$, $(a, b) \neq (0, 0)$, and every $m \geq 1$ the function $t \mapsto \log(at^{2m} + bt^{2m-2})$ is concave on $(0, +\infty)$ and thus the function

$$t \mapsto \chi^2(t)(at^{2m} + bt^{2m-2})$$

has only one local maximum point and no local minimum points in $(1, 2)$, for its logarithmic derivative vanishes exactly once. Set

$$\begin{aligned} \rho_n &= \frac{1}{n^2}, & t_n &= 2\rho_n + \sum_{j=n+1}^{\infty} 5\rho_j, \\ I_n &= [t_n - 2\rho_n, t_n + 2\rho_n], & \alpha_n &= \frac{1}{2^n} \end{aligned} \tag{2.3}$$

and

$$\varepsilon_n = \omega^{-1}(\alpha_n), \qquad \beta_n = \alpha_n \varepsilon_n^2.$$

Notice that the I_n's are closed and disjoint and that, for $n \geq 4$, one has

$$\varepsilon_n \leq \alpha_n \leq \rho_n\,. \tag{2.4}$$

Define

$$f = \sum_{n=4}^{\infty} \chi^2\Big(\frac{t - t_n}{\rho_n}\Big)(\alpha_n(t - t_n)^{2m} + \beta_n(t - t_n)^{2m-2}).$$

Clearly, f is of class C^∞: this is obvious at every point except perhaps at the origin, but for small $t \in I_n$ and a suitable positive constant C_k one has that

$$|f^{(k)}(t)| \leq C_k \rho_n^{2m-2-k} \alpha_n$$

that converges to 0 as t goes to 0 (which implies that n goes to infinity). Moreover, f takes the value 0 at all its local minimum points, which are the points t_n and the points between I_n and I_{n+1}.

We argue by contradiction and look for functions g of class $C^{m,\omega}$ such that $g^2 = f$; but any such g must be of the form

$$g = \sum_{n=1}^{\infty} \sigma_n \chi\Big(\frac{t-t_n}{\rho_n}\Big)(t-t_n)^{m-1}\sqrt{\beta_n + \alpha_n (t-t_n)^2} \tag{2.5}$$

for some choice of the signs $\sigma_n = \pm 1$. In order to evaluate $g^{(m)}$, let us calculate first $\big(\sqrt{\beta_n + \alpha_n(t-t_n)^2}\big)^{(h)}$ for $h = 1, \ldots, m$. To this end, we will use Faà di Bruno's formula (see [5]), with $F(x) = x^{1/2}$ and $\psi(t)$ given by $\psi(t) = \beta + \alpha t^2$:

$$(F \circ \psi)^{(h)} = \sum_{j=1}^{h} (F^{(j)} \circ \psi) \sum_{p(h,j)} h! \prod_{i=1}^{h} \frac{(\psi^{(i)})^{\mu_i}}{(\mu_i!)(i!)^{\mu_i}},$$

where:

$$p(h,j) = \Big\{ (\mu_1, \ldots, \mu_h) : \mu_i \geq 0\,, \sum_{i=1}^{h} \mu_i = j\,, \sum_{i=1}^{h} i\mu_i = h \Big\}.$$

Now obviously we have:

$$F^{(j)}(x) = \big(x^{1/2}\big)^{(j)} = 2^{-j}\,(2j-3)!!\,(-1)^{j+1}\,x^{1/2-j}\,,$$

where, for n odd, $n!! = 1 \cdot 3 \cdots n$ and, for n even, $n!! = 2 \cdot 4 \cdots n$. Moreover, in our case, the only nonzero terms are those with $i = 1$ or $i = 2$ and $\mu_1 = 2j - h$, $\mu_2 = h - j$, with $\big[\frac{h+1}{2}\big] \leq j \leq h$. So we have:

$$\begin{aligned} &\Big(\sqrt{\beta + \alpha t^2)}\Big)^{(h)} \\ &= \sum_{j=\left[\frac{h+1}{2}\right]}^{h} \frac{h!\,2^{j-h}\,(2j-3)!!\,(-1)^{j+1}\,(\beta+\alpha t^2)^{1/2-j}\alpha^j\,t^{2j-h}}{(2j-h)!\,(h-j)!}\,. \end{aligned} \tag{2.6}$$

We calculate now $g^{(m)}(t)$ for $t \in \tilde{I}_n := [t_n - \rho_n, t_n + \rho_n]$, with g given by (2.5).

We note that on $\tilde{I}_n$ one has $g(t) = \sigma_n (t-t_n)^{m-1}\sqrt{\beta_n + \alpha_n(t-t_n)^2}$, and so, for $t \in \tilde{I}_n$:

$$g^{(m)}(t) = \sigma_n \sum_{h=1}^{m} \frac{(m)!}{h!(m-h)!}(t-t_n)^{h-1}\frac{(m-1)!}{(h-1)!}\Big(\sqrt{\beta_n + \alpha_n(t-t_n)^2}\Big)^{(h)}\,. \tag{2.7}$$

Now, set $t'_n = t_n + \lambda\,\varepsilon_n$, with λ to be chosen later, $1/2 \le \lambda \le 1$, so that, thanks to (2.4), $t'_n \in \tilde{I}_n$. Taking (2.6) and (2.7) into account, we have:

$$g^{(m)}(t'_n) = \sigma_n \alpha_n^{1/2} \sum_{h=1}^{m} \frac{(m)!}{h!(m-h)!} \frac{(m-1)!}{(h-1)!}$$
$$\times \sum_{j=\left[\frac{h+1}{2}\right]}^{h} \frac{h!2^{j-h}(2j-3)!!(-1)^{j+1}\lambda^{2j-1}(1+\lambda^2)^{\frac{1}{2}-j}}{(2j-h)!(h-j)!} = \sigma_n \alpha_n^{1/2} \mathcal{K}_m(\lambda).$$

Since $\mathcal{K}_m(\lambda)$ is a nonzero polynomial of degree $2m-1$ in $\dfrac{\lambda}{(1+\lambda^2)^{1/2}}$, we can choose a value λ_0, $1/2 \le \lambda_0 \le 1$, in such a way that $\mathcal{K}_m(\lambda_0) \neq 0$. But now since $g^{(m)}(t_n) = 0$ we have that

$$\frac{|g^{(m)}(t_n + \lambda_0\,\varepsilon_n) - g^{(m)}(t_n)|}{\omega(\lambda_0\,\varepsilon_n)} = \frac{|g^{(m)}(t_n + \lambda_0\,\varepsilon_n)|}{\omega(\lambda_0\,\varepsilon_n)} = \frac{\alpha_n^{1/2}\,|\mathcal{K}_m(\lambda_0)|}{\omega(\lambda_0\varepsilon_n)}$$
$$\ge \frac{\alpha_n^{1/2}\,|\mathcal{K}_m(\lambda_0)|}{\omega(\varepsilon_n)} = \frac{|\mathcal{K}_m(\lambda_0)|}{\alpha_n^{1/2}}$$

that goes to infinity as $n \to \infty$. □

Theorem 2.5. *There is a nonnegative function f of class C^∞ on $\mathbb{R}$ such that, at all its minimum points, f and all its derivatives up to the $(2m-6)$-th one vanish, but there is no function g of class C^m such that $g^2 = f$.*

Proof. Let χ be a function of class C^∞ as in Theorem 2.4 and define ρ_n, t_n, I_n and α_n as in (2.3); define also

$$\varepsilon_n = \alpha_n, \qquad \beta_n = \alpha_n \varepsilon_n^2$$

and

$$f = \sum_{n=4}^{\infty} \chi^2\Big(\frac{t-t_n}{\rho_n}\Big)\Big(\alpha_n(t-t_n)^{2m-2} + \beta_n(t-t_n)^{2m-4}\Big).$$

The function f is obviously of class C^∞ and satisfies our hypotheses. Again, any function g of class C^{m-1} such that $g^2 = f$ is of the form

$$g = \sum_{n=1}^{\infty} \sigma_n \chi\Big(\frac{t-t_n}{\rho_n}\Big)(t-t_n)^{m-2}\sqrt{\beta_n + \alpha_n(t-t_n)^2}$$

for some choice of the signs $\sigma_n = \pm 1$.

Now, set $t'_n = t_n + \lambda\,\varepsilon_n$, with $1/2 \le \lambda \le 1$: thanks to (2.4), $t'_n \in \tilde{I}_n$. Taking (2.6) and (2.7) into account we have again that

$$g^{(m)}(t'_n) = \sigma_n \frac{\alpha_n^{1/2}}{\varepsilon_n} \sum_{h=2}^{m} \frac{(m)!}{h!(m-h)!} \frac{(m-2)!}{(h-2)!} \times \sum_{j=\left[\frac{h+1}{2}\right]}^{h} \frac{h!2^{j-h}(2j-3)!!(-1)^{j+1}\lambda^{2j-2}(1+\lambda^2)^{\frac{1}{2}-j}}{(2j-h)!(h-j)!} = \sigma_n \frac{1}{\alpha_n^{1/2}} \mathcal{H}_m(\lambda)$$

where $\mathcal{H}_m$ is a polynomial function in $\dfrac{\lambda}{(1+\lambda^2)^{1/2}}$; for some good choice of λ, then, this expression goes to infinity as above. □

3. Differentiable square roots

Theorem 3.1. *Let f be a nonnegative function of one variable of class C^{2m+2} ($2 \le m \le \infty$) such that, at all its minimum points, f and all its derivatives up to the order* $(2m-4)$ *vanish. Then any square root g of f which is precised up to order $m+1$ is of class C^m and its derivative of order $m+1$ exists everywhere.*

Proof. Since f is also a function of class C^{2m} and g is in particular precised up to order m we already know that g is of class C^m.

Let us consider the following closed set

$$\Gamma = \{x \in \mathbb{R} \mid f(x) = 0,\, f'(x) = 0, \ldots, f^{(2m+2)}(x) = 0\}. \tag{3.1}$$

If it is empty, the set G is made of isolated points where $f^{(2m+2)}(x) \neq 0$ and, thanks to the condition on the signs, g is of class C^{m+1}. So, we may assume $\Gamma \neq \emptyset$ and thus, for the same reason, g is of class C^{m+1} outside Γ. What remains to prove is that $g^{(m)}$ is differentiable at each point of Γ.

The function Φ defined by $\Phi(x) = d(x,\Gamma)^{-2} f^{(2m)}(x)$ outside Γ and by $\Phi(x) = 0$ in Γ is continuous and vanishes on G. If α is a modulus of continuity of Φ, one has thus

$$\left| f^{(2m)}(x) \right| \le d(x,\Gamma)^2 \alpha(d(x,G)), \tag{3.2}$$

which is the assumption (2.1) of Lemma 2.1 with $p = 1$. Thanks to this lemma, we get

$$\left| g^{(m)}(x) \right| = \left| \tfrac{d^m}{dx^m} f^{1/2}(x) \right| \le \bar{C} d(x,\Gamma) \alpha(d(x,G))^{1/2}$$

for x such that $f(x) \neq 0$ and $k \in \{0,\ldots,m\}$. By continuity, the estimate of $g^{(m)}(x)$ is also valid for the isolated zeros of f, and it is trivial for $x \in \Gamma$. For $x_0 \in \Gamma$ one has thus $\left| g^{(m)}(x) - g^{(m)}(x_0) \right| / |x - x_0| \le C\alpha(d(x,G))^{1/2}$ which converges to 0 for $x \to x_0$. This proves that $g^{m+1}(x_0)$ exists and is equal to 0, which ends the proof. □

Remark 3.2. We have already proved that, under the assumptions of the theorem, g is not of class C^{m+1} in general (Theorem 2.5 with m replaced by $m+1$). Counterexamples analogous to those given above show that the hypotheses cannot be relaxed.

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