Evolution equations and microlocal analysis

Jean-Michel Bony

Introduction

Given a self-adjoint unbounded operator A on $L^2(\mathbb{R}^n)$, we know from abstract functional analysis that there exists a one parameter group of unitary operators $P_t = e^{itA}$. Is it possible to say more on the structure of P_t when A is a differential or pseudo-differential operator with symbol $a(x, \xi)$?

The case of hyperbolic equations, i.e. when A is a (classical) first order pseudo-differential operator, is well known : P_t is a (classical) Fourier integral operator, associated to a quite natural canonical transformation : the flow at time t of the Hamilton vector field of a. This category of operators has two important properties : 1. There is a good symbolic calculus ; 2. The conjugate of a (classical) pseudo-differential operator by a Fourier integral operator is itself a pseudo-differential operator.

The last property should be substantially modified if one wants to consider more general operators A, for instance the harmonic oscillator $H = \frac{1}{2}(-d^2/dx^2 + x^2)$. It is well known that e^{itH}, for $t = \pi/2$, is the Fourier transformation (up to a scalar factor). Then, if we consider a classical pseudo-differential operator B whose symbol satisfy $\left|\partial_\xi^\alpha \partial_x^\beta b(x, \xi)\right| \leq C_{\alpha\beta}(1+|\xi|)^{-\alpha}$, its conjugate will have a symbol $c(x, \xi) = b(\xi, -x)$ satisfying $\left|\partial_\xi^\alpha \partial_x^\beta c(x, \xi)\right| \leq C_{\alpha\beta}(1+|x|)^{-\beta}$. Such operators may still be called pseudo-differential, but their calculus is highly non classical. Moreover, if we consider e^{itH} for different values of t, it turns out that one should introduce different kinds of non classical calculus.

The good framework for this is the Weyl-Hörmander calculus (see [**Hö1**, section 18.6]) which we recall in section 3. To any (good) Riemannian metric g on the phase space $\mathbb{R}^n \times (\mathbb{R}^n)^*$ are associated classes of symbols and operators (let us call them g-pseudo-differential), and a good symbolic calculus. Our program is the following :

• Given a metric g_0 and a canonical transformation F, calling g_1 the direct image of g_0 by F, we want to define a class of operators denoted by $\mathrm{FIO}(F, g_0, g_1)$ such that the two following properties are satisfied : 1. There is a good symbolic calculus for the composition of these operators ; 2. The conjugate of a g_0-pseudo-differential operator by an invertible element of $\mathrm{FIO}(F, g_0, g_1)$ is a g_1-pseudo-differential operator.

• Given an operator A with symbol a and a metric g_0, is it true that one can find canonical transformations F_t and metrics g_t such that e^{itA} belongs to $\mathrm{FIO}(F_t, g_0, g_t)$?

The present paper is devoted to the first part of this program. We shall see that, under reasonable conditions on F, the class of Fourier integral operators is well defined and has the expected properties. We shall say just a few words in subsection 6.4 about the second part of the program which cannot be treated at the same level of generality. It is easy to determine F_t, g_t, a necessary expression of e^{itA}, and a list of conditions which should be satisfied for having $e^{itA} \in \mathrm{FIO}(F, g_0, g_t)$. However, these conditions are effective only if the assumptions on a give a good control at infinity of the flow of the Hamilton vector field of a. Subsequent papers will be devoted to such sufficient conditions on a.

Let us describe more precisely the content of the paper. In section 1, we recall elementary facts on the Weyl quantization. In section 2 we give some details on metaplectic operators. In particular, we describe precisely their symbols. Metaplectic operators are actually typical examples of Fourier integral operator associated to an *affine* canonical transformation. They will play the role of local model for the general case.

In section 3, we recall some well known facts about the Weyl-Hörmander calculus, including a characterization of pseudo-differential operators in terms of commutators. This allows us to give in section 4 a simple definition of $\mathrm{FIO}(F, g_0, g_1)$ founded on "commutators twisted by F". This almost algebraic definition gives easily properties on composition and adjoints of Fourier integral operators, but cannot guarantee that these classes are non trivial, and in particular contain almost invertible operators.

In section 5, we enter more in the technique of pseudo-differential calculus, with the notion of confined symbol of [**B-L**]. Such a symbol a_Y is concentrated in a ball (for the metric) centered at Y, and its (modified) Fourier transform α_Y is concentrated in a much smaller ball. This is a rather subtle point : the operator $\alpha_Y^w(x, D)$ cannot be considered as concentrated in such a small ball (this would be a violation of the uncertainty principle) but the symbol is ; moreover, it is only in these small balls that F can be identified (up to controllable errors) with its affine tangent map. This allows to give in section 6 a constructive definition of operators in $\mathrm{FIO}(F, g_0, g_1)$: they can be written as integrals $\int U_Y \circ \alpha_Y^w(x, D)\, dY$ where U_Y is a metaplectic operator associated to the affine tangent map of F at Y. The definition of the principal symbol of a Fourier integral operator is now surprisingly simple, it is a section of a very natural line bundle, and the multiplicative property is valid. A sketch of the main proofs, founded on stationary phase arguments, is given in section 7.

1. Weyl Quantization

The *phase space* $\mathcal{X} = \mathbb{R}^n \times (\mathbb{R}^n)^*$ will be equipped with its *symplectic form* σ defined by

$$\sigma(X, Y) = \langle \xi, y \rangle - \langle \eta, x \rangle \, ; \qquad X = (x, \xi),\ Y = (y, \eta).$$

We shall reserve the word *operator* for the elements $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$, the space of linear continuous applications from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into its dual.

Given a temperate distribution a on $\mathcal{X}$, its Weyl quantization is the operator $a^w(x,D)$ (or a^w for short) defined by

$$a^w(x,D)u(x) = \iint e^{i(x-y)\cdot\xi}\, a\left(\tfrac{x+y}{2},\xi\right) u(y)\, \tfrac{dy d\xi}{(2\pi)^n}. \tag{1}$$

The map $a \mapsto a^w$ is an isomorphism of $\mathcal{S}'(\mathcal{X})$ onto $\mathcal{L}(\mathcal{S}(\mathbb{R}^n),\mathcal{S}'(\mathbb{R}^n))$. One says that a is the *symbol* of the operator a^w.

The operator a^w belongs to $\mathcal{L}(\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n))$ if and only if $a \in \mathcal{S}(\mathcal{X})$. We shall say that the operator a^w is *composable* (or that a is a *composable symbol*) if a^w maps $\mathcal{S}(\mathbb{R}^n)$ into itself and $\mathcal{S}'(\mathbb{R}^n)$ into itself. Given two symbols a and b in $\mathcal{S}'(\mathcal{X})$, one of which is composable, their product $a\#b$ is defined by

$$(a\#b)^w = a^w \circ b^w.$$

There is no simple characterization of composable symbols, but one has the following sufficient conditions. We shall see later that pseudo-differential operators and Fourier integral operators are composable.

Proposition 1.1. *A symbol $a \in \mathcal{S}'(\mathcal{X})$ is composable when it satisfies one of the following conditions :*

(i) *The support of a is compact.*

(ii) *The distribution a is a C^∞ function and there exist constants M and C_k such that $\left|\partial^k a(X)\right| \le C_k(1+|X|)^M$ for any partial derivative of order k.*

Remark. The standard quantization $a^{\mathrm{st}}(x,D)$ of a is less symmetric. It is defined by a formula analogous to (1), where $a\left(\frac{x+y}{2},\xi\right)$ is replaced by $a(x,\xi)$. One of the advantages of the Weyl quantization is that the symbol of the formal adjoint of a^w is the complex conjugate $\overline{a}$. In particular a^w is formally self-adjoint if and only if a is real-valued. Another advantage is the symplectic invariance which will play a crucial role below.

Phase symmetries. — Given a point $Y = (y,\eta)$ and δ_Y its Dirac measure, the phase symmetry $\Sigma_Y = (\pi^n\delta_Y)^w$ is given by

$$\Sigma_Y u(x) = e^{2i\langle x-y,\eta\rangle} u(2y-x) \qquad \widehat{\Sigma_Y u}(\xi) = e^{-2i\langle y,\xi-\eta\rangle}\widehat{u}(2\eta-\xi)$$

This formula characterizes the Weyl quantization : from $a(X) = \int a(Y)\delta_Y(X)\,dY$, one deduces $a^w = \int a(Y)\Sigma_Y \frac{dY}{\pi^n}$.

Phase translations. — For $R = (r,\rho)$, the operator τ_R whose symbol is $e^{-i\sigma(\cdot,R)}$ is given by

$$\tau_R u(x) = e^{i\langle x-r/2,\rho\rangle} u(x-r) \qquad \widehat{\tau_R u}(\xi) = e^{-i\langle r,\xi-\rho/2\rangle}\widehat{u}(\xi-\rho)$$

This formula characterizes also the Weyl quantization. For a linear form $l(X) = \sigma(X,R)$, the standard or Weyl quantization $l^w(x,D)$ are the same. The above formula says that $(e^{il})^w$ is nothing but $e^{i(l^w)}$ in the operator theoretical sense. Using the Fourier transformation (see below), one can write any symbol as $a(X) = \int \widehat{a}(R)e^{-2i\sigma(X,R)}\,\frac{dR}{\pi^n}$ and thus

$$a^w = \int \widehat{a}(R)\,\tau_{2R}\,\tfrac{dR}{\pi^n} \tag{2}$$

One has $\Sigma_Y\Sigma_Z = e^{-2i\sigma(Y,Z)}\tau_{2(Y-Z)}$.

Formula for the composition of symbols. — Given a, b and c in $\mathcal{S}(\mathcal{X})$, one has the following expressions

$$a\#b(X) = \iint e^{-2i\sigma(X-S,X-T)} a(S)b(T) \tfrac{dS\,dT}{\pi^{2n}} \tag{3}$$

$$a\#c\#b(X) = \iint e^{-2i\sigma(X-S,X-T)} a(S)b(T)c(S+T-X) \tfrac{dS\,dT}{\pi^{2n}}. \tag{4}$$

The left hand side is well defined when all factors are tempered distributions and when, except perhaps one, they are composable. Usually, the right hand side can then be defined in a weak sense and one can prove by approximation that the equalities are still valid.

The formula (3) could also be written, at least formally, as

$$a\#b(X) = \exp\Big(\tfrac{1}{2i}\sigma(\partial_Y, \partial_Z)\Big) a(Y)b(Z)_{|Y=Z=X},$$

where $\sigma(\partial_Y, \partial_Z)$ is the différential operator on $\mathcal{X}\times\mathcal{X}$ whose expression, in any set of symplectic coordinates, is $\sum \frac{\partial^2}{\partial\eta_j\partial z_j} - \sum \frac{\partial^2}{\partial y_j\partial\zeta_j}$.

This will lead to an asymptotic formula below, but when a (or b) is a polynomial of degree N, that is when a^w is a differential operator with polynomial coefficients, the formula is exact and finite. One has

$$a\#b(X) = \sum_{k=0}^{N} \tfrac{1}{k!} \left(\tfrac{1}{2i}\sigma(\partial_Y, \partial_Z)\right)^k a(Y)b(Z)_{|Y=Z=X}, \tag{5}$$

the first terms being

$$a\#b(X) = a(X)b(X) + \tfrac{1}{2i}\{a\,,\,b\}(X) + \dots$$

where $\{a\,,\,b\}$ is the *Poisson bracket* : $\sum(\frac{\partial a}{\partial\xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial\xi_j})$.

In particular, when $l(X) = \sigma(X,T)$ is a linear function, one has

$$l\#a = la + \tfrac{1}{2i}\{l\,,\,a\} = la + \tfrac{1}{2i}\partial_T a \tag{6}$$

where $\partial_T a$ or $\langle T, \partial_X\rangle\, a(X)$ is the directional derivative

$$\partial_T a(X) = \langle T, \partial_X\rangle\, a(X) = \langle da(X), T\rangle = \lim \varepsilon^{-1}(a(X+\varepsilon T) - a(X)).$$

Fourier transformation. — The phase space can be canonically identified with its dual by the symplectic form. This allows to define the Fourier transform on $\mathcal{X}$ itself. For $a \in \mathcal{S}'(\mathcal{X})$ we set

$$\mathcal{F}a = \widehat{a} = (\pi^n\delta)\#a$$

$$\widehat{a}(X) = \int e^{-2i\sigma(Y,X)} a(Y) \tfrac{dY}{\pi^n}$$

One has $\mathcal{F}\widehat{a} = a$ and $(\pi^n\delta)\#a\#(\pi^n\delta) = \check{a}$, with $\check{a}(X) = a(-X)$.

Twisted convolution. — It is defined by

$$a \circledast b(X) = a\#(\pi^n\delta)\#b(X) = \int e^{-2i\sigma(X,Y)} a(Y)b(X-Y) \tfrac{dY}{\pi^n}.$$

One has $\mathcal{F}(a\#b) = (\pi^n\delta)\#a\#(\pi^n\delta)\#(\pi^n\delta)\#b = \widehat{a} \circledast \widehat{b}$.

Trace. — For $a \in \mathcal{S}(\mathcal{X})$ the operator a^w is of trace class and

$$\mathrm{tr}(a^w) = 2^n \int a(X)\, \tfrac{dX}{\pi^n}. \tag{7}$$

When a or b belongs to $\mathcal{S}(\mathcal{X})$, the other one belonging to $\mathcal{S}'(\mathcal{X})$, one has

$$\mathrm{tr}(a^w \circ b^w) = 2^n \int a\#b(X)\, \tfrac{dX}{\pi^n} = 2^n \int a(S)b(S)\, \tfrac{dS}{\pi^n} \tag{8}$$

2. Metaplectic operators

The relations between the metaplectic group and the Weyl quantization, expressed by the theorem of Segal, are well known (see [**Hö1**]). This section contains no new result, except perhaps the explicit form of the symbols of metaplectic operators which will be useful later. One can find in [**Hö2**] an explicit description of the distribution kernels of metaplectic operators.

The (linear) *symplectic group* $\mathrm{Sp}(n)$ is the group of linear maps M from $\mathcal{X}$ onto itself which preserve the symplectic form : $\sigma(MX, MY) = \sigma(X, Y)$. The *affine symplectic group* $\mathrm{ASp}(n)$ is the set of applications $X \mapsto \chi(X) = MX + R$, with $M \in \mathrm{Sp}(n)$ and $R \in \mathcal{X}$.

THEOREM 2.1 (Segal). *Given any $\chi \in \mathrm{ASp}(n)$, there exists an operator $V = k^w$, uniquely determined up to the multiplication by a complex factor, such that*

$$l\#k = k\#(l \circ \chi)$$

for any affine function l on $\mathcal{X}$.

Proving the uniqueness reduces to prove that k is necessarily a constant when χ is the identity. In this case, one should have $\{l\,,\,k\} = 0$ for any linear form l, which says that all derivatives of k should vanish.

We already know the phase translations τ_R and symmetries Σ_Y which satisfy the property above when χ is the translation of vector R or the symmetry with respect to Y. Let us now assume that both χ and l are linear.

We shall try distributions of the form $k(X) = e^{iB(X)}\delta_{\mathcal{W}}(X)$ where $\mathcal{W}$ is a linear subspace of $\mathcal{X}$, where $\delta_{\mathcal{W}}$ is a "Lebesgue measure" on $\mathcal{W}$, that is a positive measure supported by $\mathcal{W}$ and invariant by its translations, and where $B(X) = b(X, X)$ is a real valued quadratic form on $\mathcal{W}$.

Given $l(X) = \sigma(X, T)$ and $l'(X) = \sigma(X, T')$, one sees, using (6), that the equality $l\#k = k\#l'$ is equivalent to

$$\sigma(X, T)k(X) + \tfrac{1}{2i}\partial_T k(X) = \sigma(X, T')k(X) - \tfrac{1}{2i}\partial_{T'} k(X),$$

which is in turn equivalent to the following two conditions

$$T + T' \in \mathcal{W}$$
$$\forall X \in \mathcal{W}, \qquad \sigma(X, T{+}T') - b(X, T{+}T') = 2\sigma(X, T).$$

If the bilinear form $\sigma{-}b$ is non degenerate on $\mathcal{W}$, given $T \in \mathcal{X}$, the last formula determines a unique $T{+}T' \in \mathcal{W}$, and thus a unique $T' \in \mathcal{X}$. It is not difficult to check that the map $T \mapsto T'$ is symplectic. The theorem is then a direct consequence of the following result of linear algebra whose proof is elementary.

PROPOSITION 2.2 (generalized Cayley transformation). *For $M \in \mathrm{Sp}(n)$ let*

$$\mathcal{W} = \mathrm{range}(I + M)$$

and, for $T_1, T_2 \in \mathcal{W}$,

$$b(T_1, T_2) = \sigma(T_1, (I{-}M)Z_2) \ \textit{where}\ (I{+}M)Z_2 = T_2.$$

(i) *The above definition of* b *does not depend on the choice of* Z_2. *The bilinear form* b *is symmetric and the bilinear forms* $b \pm \sigma$ *are non degenerate on* $\mathcal{W}$.

(ii) *The map* $M \mapsto (\mathcal{W}, b)$ *is bijective from* $\mathrm{Sp}(n)$ *onto the set of couples made of a linear subspace* $\mathcal{W}$ *of* $\mathcal{X}$ *and of a symmetric bilinear form* b *on* $\mathcal{W}$ *such that* $b \pm \sigma$ *is non degenerate on* $\mathcal{W}$.

Remark. When -1 is not an eigenvalue of M, then b is defined on $\mathcal{X}$ itself by $b(T_1, T_2) = \sigma(T_1, CT_2)$, with $C = (I{+}M)^{-1}(I{-}M)$, which is the symplectic version of the Cayley transform.

If we resume to the general case of an affine symplectic map written

$$\chi(X) = Y' + M(X{-}Y),$$

one sees that the $V = k^w$ associated to χ by the theorem of Segal are defined by

$$k(X) = \mathrm{C}^{\mathrm{st}} e^{-2i\sigma\left(X-\frac{Y'}{2}, X-\frac{Y}{2}\right)} e^{iB\left(X-\frac{Y+Y'}{2}\right)} \delta_{\mathcal{W}}\left(X-\tfrac{Y+Y'}{2}\right)$$

where $(\mathcal{W}, b)$ is the generalized Cayley transform of M and $B(X) = b(X, X)$. One has just to compute $\delta_{Y'/2} \# \widetilde{k} \# \delta_{Y/2}$, with $\widetilde{k}$ associated to the linear map, using formula (4). When -1 is not an eigenvalue, k is a function given by the following expression

$$\begin{gathered} k(X) = C^{\mathrm{st}} e^{-2i\sigma\left(X-\frac{Y'}{2}, X-\frac{Y}{2}\right)} e^{i\sigma\left((X-\frac{Y+Y'}{2}), C(X-\frac{Y+Y'}{2})\right)} \\ C = (I{-}M)(I{+}M)^{-1}. \end{gathered} \tag{9}$$

Theorem 2.3. *Let* $V = k^w$ *associated to* χ *as above*

(i) V *is composable.*

(ii) V *is proportional to a unitary operator on* L^2 : *one has* $\overline{k} \# k = \mathrm{C}^{\mathrm{st}}$.

(iii) *For any* $a \in \mathcal{S}'(\mathcal{X})$, *one has*

$$a \# k = k \# (a \circ \chi). \tag{10}$$

These operators V *will be called* generalized (affine) metaplectic operators, *and those* V *which are unitary are called* (affine) metaplectic operators.

For proving (i), it is not difficult to see that, for $u \in \mathcal{S}(\mathbb{R}^n)$, one has $k^w u \in L^\infty$. If l is a linear form, one has then $l(x, D) k^w u = k^w l'(x, D) u \in L^\infty$. By induction, $P(x, D) k^w u \in L^\infty$ for any differential operator P with polynomial coefficients and thus $k^w u \in \mathcal{S}(\mathbb{R}^n)$.

It is clear that $\overline{k}^w$ is a generalized metaplectic operator associated to χ^{-1} and thus one has $l \# (\overline{k} \# k) = (\overline{k} \# k) \# l$ for all l, which proves that $\overline{k} \# k$ is a constant.

The formula (10), which is valid when a is a linear form l, is also valid for $a = e^{il}$. This follows from the fact that the phase translation a^w is then e^{il^w} as an operator. Using (2), this extends by linearity to any symbol a.

The affine metaplectic group. — $\mathrm{AMp}(n)$ is the set of affine metaplectic operators. One has the following exact sequence of groups

$$1 \to S^1 \to \mathrm{AMp}(n) \xrightarrow{\varpi} \mathrm{ASp}(n) \to 1$$

where the projection ϖ associates to V the corresponding χ. One has an analogous exact sequence, replacing $\mathrm{AMp}(n)$ by the group of non-zero generalized metaplectic operators and S^1 by $\mathbb{C}^*$.

The line bundle $\widetilde{\mathrm{AMp}}(n)$ *over* $\mathrm{ASp}(n)$. — It is the set of couples (V, χ) where V is any generalized metaplectic operator (including 0) associated to χ. The projection

$(V,\chi) \mapsto \chi$ will be denoted by $\widetilde{\varpi}$. This line bundle will play an essential role for defining the principal symbol of a Fourier integral operator.

The Lie algebra of AMp(n). — It can be identified with the space of real valued second order polynomials on $\mathcal{X}$, equipped with the Poisson bracket as Lie bracket, in the following way. Let us consider C^1 maps $t \mapsto U_t$ from a neighbourhood of 0 in $\mathbb{R}$ into AMp(n) such that $U_0 = I$. For t small, -1 is not an eigenvalue of the symplectic matrix associated to U_t and the symbol of U_t is thus $e^{ip_t(X)}$, where p_t is a polynomial of degree at most 2, whose coefficients of degree 1 and 2 are real. One has $\frac{d}{dt}U_t|_{t=0} = iq^w$, with $q = \frac{d}{dt}p_t|_{t=0}$. The coefficients of the polynomial q are real, the operator q^w being self-adjoint.

We shall use the following consequences.

Proposition 2.4. (i) *If $t \mapsto U_t$ is a C^1 map from an interval into* AMp(n)*, then one has $\frac{d}{dt}U_t = ip_t^wU_t = iU_tq_t^w$, where p_t and q_t are continuous families of real second order polynomials.*
(ii) *Conversely, given a continuous family p_t of real second order polynomials, the unique solution of $\frac{d}{dt}U_t = ip_t^wU_t$; $U_0 = I$ satisfy $U_t \in$* AMp(n)*.*

The linear case. — There is no proper subgroup of AMp(n) such that its projection is equal to ASp(n) : such a subgroup should contain elements whose projections are translations, it should contain their commutators and thus any constant α with $|\alpha| = 1$.

We can consider $\mathrm{Mp}_\infty(n) = \{V \in \mathrm{AMp}(n) | \, \varpi(V) \in \mathrm{Sp}(n)\}$ and we have also the exact sequence $1 \to S^1 \to \mathrm{Mp}_\infty(n) \xrightarrow{\varpi} \mathrm{Sp}(n) \to 1$, but $\mathrm{Mp}_\infty(n)$ has proper subgroups whose projection is Sp(n). We denote by $\mathrm{Mp}_2(n)$ the smallest of these subgroups, usually called the (linear) metaplectic group, which is a connected 2-sheets covering of Sp(n) : the sequence $1 \to \{\pm 1\} \to \mathrm{Mp}_2(n) \xrightarrow{\varpi} \mathrm{Sp}(n) \to 1$ is exact.

The Lie algebra of $\mathrm{Mp}_\infty(n)$ is (identified to) the set of real *even* polynomials of degree 2 and the Lie algebra of $\mathrm{Mp}_2(n)$ is the set of real *homogeneous* polynomials of degree 2. The Proposition 2.4 is thus valid, replacing AMp(n) by $\mathrm{Mp}_2(n)$ and polynomials by homogeneous polynomials.

3. Pseudo-differential calculus

There are actually many different calculus. Each of which is associated to a (convenient) Riemannian metric g on the phase space. Classes of symbols and of operators will depend on g. This section contains a description of the Weyl-Hörmander calculus, with two simplifying assumptions. We refer to [**Hö1**] and to [**B-L**] for the proofs.

3.1. Admissible metrics. A Riemannian metric g on $\mathcal{X}$ will be identified with a C^∞ map $Y \mapsto g_Y$ from $\mathcal{X}$ into the set of positive definite quadratic forms on $\mathcal{X}$. The ball centered at Y of radius r is defined as

$$B_{Y,r} = \left\{ X \,\middle|\, g_Y(X-Y) \le r^2 \right\}.$$

It is possible to diagonalize simultaneously a positive definite quadratic form and the symplectic form. Given Y, one can choose symplectic coordinates (still denoted by (x,ξ)), such that g_Y has the following form

$$g_Y(dx,d\xi)=\sum_1^n \frac{dx_j^2}{a_j^2}+\sum_1^n \frac{d\xi_j^2}{\alpha_j^2}. \tag{11}$$

The a_j and α_j, which depend on Y are not themselves invariant, but the products $a_j\alpha_j$ do not depend on the choice of the symplectic coordinates.

We shall say that g is *admissible* if it satisfies the conditions A1 to A5 below.

A1. Simplifying assumption. — The products $a_j\alpha_j$ above are equal, their common value being denoted by $\lambda^2(Y)$. Therefore, one can choose symplectic coordinates such that $g_Y=\lambda(Y)^{-1}(dx^2+d\xi^2)$. One sees easily that

$$|\sigma(S,T)|\le \lambda(Y)g_Y(S)^{1/2}g_Y(T)^{1/2}.$$

A2. Fundamental assumption. — $\forall Y,\quad \lambda(Y)\ge 1.$
This means that the unit balls of the metric are sufficiently large. Otherwise, trying to localize in such balls would violate the *uncertainty principle.*

A3. Slowness. — There exists a constant $C>0$ such that

$$g_Y(Y-Z)\le C^{-1}\Longrightarrow \Big(g_Y(T)/g_Z(T)\Big)^{\pm 1}\le C$$

uniformly in Y,Z,T.

As a consequence, the ratio between $g_Y(Y-Z)^{1/2}$ and the geodesic distance between Y and Z is bounded from above and from below as far as Y and Z belong to a ball $B_{X,r}$ (or to a geodesic ball) of radius sufficiently small. In such balls, the triangle inequality is "valid up to a constant" for the quantities $g_Y(Y-Z)^{1/2}$.

A4. Temperance. — There exist constants C and N such that

$$\forall Y,\forall Z,\forall T,\quad \Big(g_Y(T)/g_Z(T)\Big)^{\pm 1}\le C\Big(1+\lambda(Y)^2g_Y(Y-Z)\Big)^N$$

A5. Geodesic temperance. — Let us denote by $D(Y,Z)$ the geodesic distance *for the Riemannian metric* $\lambda(Y)^2g_Y(\cdot)$. Then

$$\forall Y,\forall Z,\forall T,\quad \Big(g_Y(T)/g_Z(T)\Big)^{\pm 1}\le C\Big(1+D(Y,Z)\Big)^N$$

Remark. Most part of the theory remains valid without the assumptions A1 and A5, but the definitions and the proofs are more complicated. For instance, one should define $\lambda(Y)$ as the minimum of the products $a_j\alpha_j$ in (11), one should introduce in A4 the metric g_Y^σ which is equal to $\sum \alpha_j^2dx_j^2+a_j^2d\xi_j^2$ in these coordinates, and the volume of the unit ball $B_{Y,1}$ should be taken into account in the integrals. In our simplified case, one has $g_Y^\sigma=\lambda(Y)^2g_Y$ and the volume of $B_{Y,1}$ is $C^{\mathrm{st}}\lambda(Y)^n$.

The pseudo-differential calculus can be developped in full generality without the assumption A5. However, under this assumption, there exists a very useful characterization of pseudo-differential operators in terms of commutators and the definition of Fourier integral operators becomes much simpler. Moreover, this extra assumption is not so strong : we know no example of a metric satisfying A1 to A4 and not A5.

Example. Classical examples of metrics satisfying A1 to A5 are the following

$$g_Y(dx, d\xi) = \frac{dx^2 + d\xi^2}{\langle Y\rangle^{2\rho}} \quad \text{for } 0 \le \rho \le 1$$

$$g_Y(dx, d\xi) = \langle \eta\rangle^{2\delta} dx^2 + \frac{d\eta^2}{\langle \eta\rangle^{2\rho}} \quad \text{for } \delta \le \rho \le 1 \text{ and } \delta < 1,$$

where $\langle Y\rangle = (1 + |Y|^2)^{1/2}$ and $\langle \eta\rangle = (1 + |\eta|^2)^{1/2}$. One has $\lambda(Y) = \langle Y\rangle^{2\rho}$ in the first case, and $\lambda(Y) = \langle \eta\rangle^{\rho-\delta}$ in the second one.

3.2. Symbols and operators. In what follows, g will be an admissible metric.

A positive function M on $\mathcal{X}$ will be called a *weight* (or a g-weight if necessary) if it satisfies the following slowness and temperance assumptions, for convenient constants C' and N'

$$g_Y(Y-Z) \le {C'}^{-1} \Longrightarrow \Big(M(Y)/M(Z)\Big)^{\pm 1} \le C'$$

$$\Big(M(Y)/M(Z)\Big)^{\pm 1} \le C'\Big(1 + \lambda(Y)^2 g_Y(Y-Z)\Big)^{N'}.$$

The set of g-weights is stable by sum, product, and by $M \mapsto M^s$ for $s \in \mathbb{R}$. The function λ is an important example of g-weight.

Definition 3.1. *For any weight M, the class of symbols $S(M, g)$ is the set of C^∞ functions on $\mathcal{X}$ such that, for $k \ge 0$, one has*

$$|\partial_{T_1} \dots \partial_{T_k} a(X)| \le C_k M(X) \quad \textit{for } g_X(T_j) \le 1.$$

Taking as semi-norms $\|a\|_{k\,;S(M,g)}$ the best constants C_k above, it is a Frechet space.

The properties below are classical (see [**Hö1**] or [**B-L**]), but they could not be proved without (an equivalent of) the procedures of localization which will be developped in section 5.

Theorem 3.2. (i) *Symbols a belonging to $S(M, g)$ are composable. The corresponding operators a^w will be called pseudo-differential operators of weight M. The space of these operators, equipped with the semi-norms of $S(M, g)$, will be denoted by $\Psi(M, g)$.*

(ii) *If a belongs to $S(1, g)$, then a^w is bounded on L^2.*

(iii) *The map $(a_1, a_2) \mapsto a_1 \# a_2$ is continuous from $S(M_1, g) \times S(M_2, g)$ into $S(M_1M_2, g)$.*

Asymptotic development. — For $a_j \in S(M_j, g)$, $j = 1, 2$, the formula (5) becomes an asymptotic one, the "gain" of the symbolic calculus being the function λ :

$$a_1 \# a_2(X) = \sum_{k=0}^{N-1} \tfrac{1}{k!} \left(\tfrac{1}{2i}\sigma(\partial_Y, \partial_Z)\right)^k a_1(Y)a_2(Z)_{|Y=Z=X} + R_N(X) \qquad (12)$$

where the k^{th} term in the sum belongs to $S(M_1M_2\lambda^{-k}, g)$ and R_N belongs to $S(M_1M_2\lambda^{-N}, g)$.

For $A = a^w \in \Psi(M, g)$, one says that b is *a principal symbol of A at weight M* (the weight should be at least implicit) if $a - b \in S(M\lambda^{-1}, g)$. The formula above says in particular that a_1a_2 is a principal symbol (at weight M_1M_2) of $a_1^w \circ a_2^w$ and that $i^{-1}\{a_1\,,\,a_2\}$ is a principal symbol (at weight $M_1M_2\lambda^{-1}$) of the commutator $[a_1^w\,,\,a_2^w]=a_1^wa_2^w-a_2^wa_1^w$.

The following result (see [**B-C**]) will be useful.

Theorem 3.3. *For any weight M, there exists an invertible pseudo-differential operator of weight M, that is $A\in\Psi(M,g)$ and $A'\in\Psi(M^{-1},g)$ such that $AA'=A'A=I$.*

3.3. Characterization.

Definition 3.4. *We shall denote by $S^+(g)$ the space of C^∞ functions b defined on $\mathcal{X}$ such that one has*

$$|\partial_{T_1}\ldots\partial_{T_k}b(X)|\leq C_k\lambda(X)\quad \textit{for } g_X(T_j)\leq 1 \textit{ and } k\geq 1.$$

These symbols are composable. The space of operators b^w will be denoted by $\Psi^+(g)$.

This space looks like $S(\lambda,g)$ except that no estimation is required for the values of b itself. The difference is quite apparent when $\lambda=1$: in this case, elements of $\Psi(\lambda,g)$ are bounded on L^2 while elements of $\Psi^+(g)$ are not in general.

However, for computing a commutator $[b^w\,,\,a^w]$, the formula (12) can still be used, the products ab which are not controlled disappear, and the remaining terms (including the remainder) depend only on the derivatives of b. It is thus not surprising to get the same result as for $b\in S(\lambda,g)$. The converse is an important result.

Theorem 3.5. (i) *For $B\in\Psi^+(g)$ and $A\in\Psi(M,g)$, one has*

$$\operatorname{ad}B\cdot A=[B\,,\,A]\in\Psi(M,g).$$

In particular, it is bounded on L^2 for $M=1$.

(ii) *Conversely, let A be an operator which is bounded on L^2 as well as its iterated commutators*

$$\operatorname{ad}B_1\ldots\operatorname{ad}B_k\cdot A\quad \textit{for } B_j\in\Psi^+(g).$$

Then $A\in\Psi(1,g)$.

We refer to [**Bo1**] for the proof. Assumption A5 is crucial for this characterization, which can be understood as follows. Functions $b\in S^+(g)$ are Lipschitz continuous for the metric λ^2g. Thus, their variation between Y and Z can reach only the λ^2g-geodesic distance $D(Y,Z)$ between these two points. It turns out that one can extract from the commutation relations decay estimates in $(1+D(Y,Z))^{-N}$ but no more, which allows to control the ratio g_Y/g_Z only under assumption A5.

There is another characterization (see [**B-C**]), in terms of localized commutators, which is not so easy to handle but which does not require A5. It should be used for defining the Fourier integral operators in the general case.

4. Fourier integral operators

These operators will be associated to what we shall call an *admissible triple* (F,g_0,g_1), which means that the five following conditions are satisfied.

B1 . — g_0 is an admissible metric on $\mathcal{X}$.

B2. — F is a symplectomorphism (or canonical transformation) from $\mathcal{X}$ onto itself, that is a diffeomorphism of $\mathcal{X}$ which respects the symplectic 2-form : $F_*\sigma=\sigma$. It is equivalent to say that, for any Y, the differential $F'(Y)$ belongs to $\mathrm{Sp}(n)$. We shall denote by $\chi_Y \in \mathrm{ASp}(n)$ the affine tangent map at Y, defined by

$$\chi_Y(X) = F(Y) + F'(Y) \cdot (X - Y). \tag{13}$$

B3. — g_1 is equal to the direct image $F_* g_0$. It is the Riemannian metric on $\mathcal{X}$ defined by

$$g_{1;F(Y)}(T) = g_{0;Y}(F'(Y)^{-1} \cdot T).$$

B4. — The metric g_1 is admissible.

B5. — The derivatives of the vector valued function F satisfy the following uniform bounds :

$$g_{1;F(X)}\Big(\partial_{T_1} \dots \partial_{T_k} F(X)\Big) \le C_k \quad \text{for } g_{0;X}(T_j) \le 1 \tag{14}$$

Since F is an isometry from $(\mathcal{X}, g_0)$ onto $(\mathcal{X}, g_1)$, this condition is obviously satisfied for $k = 1$. It is also valid for F^{-1} from which one deduces that the triple (F^{-1}, g_1, g_0) satisfies also the conditions above.

The fundamental functions λ respectively associated to g_0 and g_1 will be denoted by λ_0 and λ_1. The quadratic forms $g_{0;Y}$ and $g_{1;F(Y)}$ being symplectically equivalent, one has $\lambda_1(F(Y)) = \lambda_0(Y)$.

Remark 4.1. It will be sometimes useful to choose adapted symplectic coordinates at Y_0 and at $F(Y_0)$, which is equivalent to look at $\widetilde{F} = \Phi \circ F \circ \Psi$ where Φ and Ψ belong to $\mathrm{ASp}(n)$. Let $\rho^2 = \lambda_0(Y_0)$ and let $\widetilde{B}$ the euclidean ball of radius ρ centered at 0. We can choose Ψ such it maps $\widetilde{B}$ onto the g_0-unit ball centered at Y_0, we can choose Φ such it maps the g_1-unit ball centered at $F(Y_0)$ onto $\widetilde{B}$ and moreover such that $\widetilde{F}'(0) = I$. The estimates (14) and the slowness imply that one has, with constants independant of Y_0 :

$$\widetilde{F} \text{ maps } \widetilde{B} \text{ into } C^{\mathrm{st}}\widetilde{B}$$

$$\Big|\partial_x^\alpha \partial_\xi^\beta \widetilde{F}(X)\Big| \le C_{\alpha\beta}\, \rho^{1-|\alpha|-|\beta|} \quad \text{for } X \in \widetilde{B}.$$

It is not difficult to see that the estimates B5 are precisely what is needed for proving the following proposition.

Proposition 4.2. *Under assumptions B1 to B5, one has*
(i) *M is a g_1-weight if and only if $M \circ F$ is a g_0-weight.*
(ii) *$a \in S(M, g_1)$ if and only if $a \circ F \in S(M \circ F, g_0)$.*
(iii) *$b \in S^+(g_1)$ if and only if $b \circ F \in S^+(g_0)$.*

Twisted commutators. — If we come back to the heuristic ideas alluded to in the introduction, a Fourier integral operator P associated to F should be such that $b^w \circ P$ is approximately equal to $P \circ (b \circ F)^w$. We are thus led to introduce the following K_b (denoted $K_b^{[F]}$ if necessary)

$$K_b \cdot P = b^w \circ P - P \circ (b \circ F)^w$$

which reduces to $\mathrm{ad}\, b$ when F is the identity.

The following definition is thus an extension, for general F, of the property that characterizes pseudo-differential operators.

Definition 4.3. *An operator P is a Fourier integral operator associated to the admissible triple (F, g_0, g_1) if it is bounded on L^2 and if the iterated twisted commutators*

$$K_{b_1} \dots K_{b_p} \cdot P \qquad \textit{for } b_j \in S^+(g_1)$$

are also bounded on L^2. The space of these operators is denoted by $\mathrm{FIO}(F, g_0, g_1)$.

Remark. To be more precise, these operators should be called Fourier integral operators of weight 1. We shall extend the definition (see remark 4.7 below) to more general weights.

As noticed above, one has $\mathrm{FIO}(I, g, g) = \Psi(1, g)$.

This abstract definition allows to prove easily and almost algebraically the expected formal properties on composition of Fourier integral operators. A more concrete definition will be necessary for proving that these classes of operators are sufficiently large.

Theorem 4.4. *Let (F, g_0, g_1) and (G, g_1, g_2) be admissible triples. Then the triples (F^{-1}, g_1, g_0) and $(G \circ F, g_0, g_2)$ are also admissible.*

(i) *If $P \in \mathrm{FIO}(F, g_0, g_1)$, its adjoint P^* belongs to $\mathrm{FIO}(F^{-1}, g_1, g_0)$.*

(ii) *If $P \in \mathrm{FIO}(F, g_0, g_1)$ and $Q \in \mathrm{FIO}(G, g_1, g_2)$, then $Q \circ P \in \mathrm{FIO}(G \circ F, g_0, g_2)$.*

In the first case, let $b_1 \in S^+(g_1)$ and $b_0 = b_1 \circ F \in S^+(g_0)$. Then, the adjoint of $\overline{b_1}^w P - P \overline{b_0}^w$ which is equal to $-(b_0^w P^* - P^* b_1^w)$ is bounded on L^2, which means that the twisted commutators $K_{b_0}^{[F^{-1}]} \cdot P^*$ are bounded on L^2 for any $b_0 \in S^+(g_0)$. The extension to iterated twisted commutators is immediate.

In the second case, let $b_2 \in S^+(g_2)$, $b_1 = b_2 \circ G$ and $b_0 = b_2 \circ G \circ F$. Then one has

$$K_{b_2}^{[G \circ F]} \cdot (Q \circ P) = \left(K_{b_2}^{[G]} \cdot Q \right) P + Q \left(K_{b_1}^{[F]} \cdot P \right)$$

which proves that the left hand side is bounded on L^2. The case of iterated twisted commutators follows by induction.

Proposition 4.5. *Let M_1 be a g_1-weight and let $M_0 = M_1 \circ F$. Let $A = a^w \in \Psi(M_0, g_0)$ let $B = b^w \in \Psi(M_1^{-1}, g_1)$ and let $P \in \mathrm{FIO}(F, g_0, g_1)$. Then $BPA \in \mathrm{FIO}(F, g_0, g_1)$.*

The proof in the general case requires arguments similar to those of [**Bo1**] and we shall just give the proof when $\lambda_1^{-N} \le M_1 \le \lambda_1^N$ for some N. Let us first consider the case $1 \le M_1 \le \lambda_1$. Then $a \in S^+(g_0)$ and $\tilde{a} = a \circ F^{-1} \in S^+(g_1)$. We have then $PA = \widetilde{A} P + R$, with $\widetilde{A} = \tilde{a}^w \in S(M_1, g_1)$ and $R = K_{\tilde{a}} P \in \mathrm{FIO}(F, g_0, g_1)$. Thus $BPA = (B\widetilde{A})P + BR$ and, by Theorem 4.4, these two products belong to $\mathrm{FIO}(F, g_0, g_1)$.

The case $\lambda_1^{-1} \le M_1 \le 1$ is analogous, writing $BP = P\widetilde{B} + R$, with $\widetilde{B} \in \Psi(M_0^{-1}, g_0)$ and $R \in \mathrm{FIO}(F, g_0, g_1)$.

It is easy to see that any weight M such that $\lambda_1^{-N} \le M \le \lambda_1^N$ can be written as a finite product of weights entering in one of the two cases above. It suffices now to prove that if the proposition is valid for weights M_1' and M_1'' then it is valid for $M_1 = M_1' M_1''$.

With evident notations for M_0, M_0' and M_0'', let $A \in \Psi(M_0, g_0)$. The Theorem 3.3 allows to write $A = A'A''$ with $A' \in \Psi(M_0', g)$ and $A'' \in \Psi(M_0'', g)$. In the same way, any $B \in \Psi(M_1^{-1}, g_1)$ can be written $B = B''B'$ with $B' \in \Psi(M_1'^{-1}, g_1)$

and $B'' \in \Psi(M_1''^{-1}, g_1)$. Then $B'PA'$ and $B''(B'PA')A''$ belong to $\mathrm{FIO}(F, g_0, g_1)$. The proof is complete.

The following corollary uses the "Sobolev spaces" $H(M, g)$ defined in [**B-C**]. Their definition "à la Littlewood-Paley" is a little bit simplified thanks to the assumption A5 :

$$u \in H(M,g) \iff \int M(Y)^2 \, \|\Phi_Y^w u\|_{L^2}^2 \; \lambda(Y)^{-n} \, dY < \infty$$

where (Φ_Y) is the partition of unity defined in subsection 5.2 below.

These spaces can also be characterized as follows. According to Theorem 3.3, there exists an invertible element $A \in \Psi(M, g)$, and one has $u \in H(M, g)$ if and only if $Au \in L^2$. Using the last proposition, one gets the following.

Corollary 4.6. *Let M_1 be a g_1-weight and let $M_0 = M_1 \circ F$. Then, any element $P \in \mathrm{FIO}(F, g_0, g_1)$ maps $H(M_0, g_0)$ into $H(M_1, g_1)$.*

Remark 4.7. Operators in $\mathrm{FIO}(F, g_0, g_1)$ are "of weight 1" and it is now easy to define more general classes. The best is to define their weight as a function μ on the graph of F whose value at point $(Y, F(Y))$ is the common value $M_1(F(Y)) = M_0(Y)$. One can then define $\mathrm{FIO}(F, \mu, g_0, g_1)$ as the set of products PA, or the set of products BP, or the set of products $B'PA'$, with $P \in \mathrm{FIO}(F, g_0, g_1)$ and $A \in \Psi(M_0, g_0)$, or $B \in \Psi(M_1, g_1)$, or $A' \in \Psi(M_0', g_0)$ and $B' \in \Psi(M_1'', g_1')$ with $M_1 = M_1' M_1''$. The fact that all these definitions are equivalent is an immediate consequence of the proposition above.

The reader will state and prove easily properties on adjoint and composition of such operators, and on their action in Sobolev spaces.

5. Localization

In this section g will be an admissible metric. In order to construct partitions of unity related to g, the first idea is to consider the subspace of $S(1, g)$ consisting of symbols supported in the ball $B_{Y,r}$. However, this space is not stable by composition and the good substitute is the following space of confined symbols introduced in [**B-L**]. We refer to [**Bo1**] for the equivalence of the semi-norms below.

5.1. Confinement.

Definition 5.1. *The space $\mathrm{Conf}(g, Y, r)$ is the Schwartz space $\mathcal{S}(\mathcal{X})$ equipped with the following sequence of semi-norms*

$$\|a\|'_{k\,;\mathrm{Conf}(g,Y,r)} = \sup_{l,T_j} \left\| \partial_{T_1} \ldots \partial_{T_l} a(X) \left(1+\lambda(Y)^2 g_Y(X-U_{Y,r})\right)^{k/2} \right\|_{L^\infty(dX)} \quad \text{for } l \le k,\ g_Y(T_j) \le 1$$

or with the (uniformly) equivalent family

$$\|a\|_{k\,;\mathrm{Conf}(g,Y,r)} = \left\| a(X) \left(1+\lambda(Y)^2 g_Y(X-B_{Y,r})\right)^{k/2} \right\|_{L^\infty(dX)} + \left\| \widehat{a}(P) \left(1+\lambda(Y)^2 g_Y(P)\right)^{k/2} \right\|_{L^1(dP)}. \quad (15)$$

Here, $g_Y(X-A)$ (resp. $g_Y(A-B)$) denotes the infimum of $g_Y(X-X')$ (resp. $g_Y(X'-X'')$) for $X' \in A$ (and $X'' \in B$).

A family $(a_Y)_{Y\in\mathcal{X}}$ is *uniformly confined in* $B_{Y,r}$ if $\|a_Y\|_{k\,;\mathrm{Conf}(g,Y,r)}$ is bounded by a constant depending on k but not on Y. A typical example is a family (a_Y) of functions whose support is contained in $B_{Y,r}$ and which is bounded in $S(1,g)$.

In a uniformly confined family, the symbols a_Y are controlled in L^∞ (their L^1 norm is bounded by $C^{\mathrm{st}}\lambda(Y)^n$), with a fast decay outside the ball $B_{Y,r}$. Their Fourier transform are controlled in L^1 (their L^∞ norm is bounded by $C^{\mathrm{st}}\lambda(Y)^n$), but are much more concentrated : they are small outside a g_Y-ball of radius $C^{\mathrm{st}}\lambda(Y)^{-1}$ centered at 0.

The most important result is the following theorem on composition of confined symbols. It uses the function

$$\Delta_r(Y,Z) = 1 + \lambda(Y)^2 g_Y(B_{Y,r}-B_{Z,r}) + \lambda(Z)^2 g_Z(B_{Y,r}-B_{Z,r})\ , \tag{16}$$

measuring the "distance" between Y and Z.

Theorem 5.2. *For r sufficiently small, if (a_Y) and (b_Y) are uniformly confined in $B_{Y,r}$ then, for any N, the family $\Delta_r(Y,Z)^N(a_Y\# b_Z)$ is uniformly confined in $B_{Y,r}$ and in $B_{Z,r}$.*

More precisely, given k and N, there exist C and l which do not depend on a, b, Y, Z such that

$$\|a\#b\|_{k\,;\mathrm{Conf}(g,Y,r)} + \|a\#b\|_{k\,;\mathrm{Conf}(g,Z,r)} \\ \leq C\,\|a\|_{l\,;\mathrm{Conf}(g,Y,r)}\,\|b\|_{l\,;\mathrm{Conf}(g,Z,r)}\,\Delta_r(Y,Z)^{-N}\ . \tag{17}$$

It is important to notice that one gains no decay ($\Delta_r = 1$) when the balls centered at Y and Z intersect. On the other hand, when $g_Y(X-Y)^{1/2} \geq C^{\mathrm{st}} r$, one gains (any power of) both λ and $g_Y(X-Y)$.

5.2. Partitions of unity (first kind). It is easy to construct partitions of unity made of (uniformly) confined symbols. Let f be a nonnegative even smooth function on $\mathbb{R}$ vanishing outside $[-1,1]$ and set $\Theta_Y(X) = f\left(g_Y(X-Y)^{1/2}/r\right)$. Then $I(X) = \int \Theta_Y(X)\lambda(Y)^{-n}\,dY$ belongs to $S(1,g)$ and is bounded from below. Thus one has

$$1 = \int \Phi_Y(X)\lambda(Y)^{-n}\,dY \quad \text{with } \Phi_Y(X) = I(X)^{-1}\Theta_Y(X)$$

and the family (Φ_Y) is uniformly confined (and actually supported) in $B_{Y,r}$.

Such a partition of unity can be used for regularizing the metric g itself and the weights M, replacing them by

$$\widetilde{g}_Y(T) = \int g_Z(T)\Phi_Z(Y)\lambda(Z)^{-n}\,dZ\ ;\ \widetilde{M}(Y) = \int M(Z)\Phi_Z(Y)\lambda(Z)^{-n}\,dZ.$$

The ratios $(\widetilde{g}/g)^{\pm1}$ and $(\widetilde{M}/M)^{\pm1}$ being bounded, it is clear that nothing is changed (except the constants) in the definitions above. So we shall assume in what follows that g and the weights have been regularized, which imply that

$$|\langle T_1,\partial_Y\rangle\ldots\langle T_k,\partial_Y\rangle\, g_Y(T_0)| \leq C_k \text{ for } g_Y(T_j)\leq 1$$
$$|\langle T_1,\partial_Y\rangle\ldots\langle T_k,\partial_Y\rangle\,\langle T_{k+1},\partial_X\rangle\ldots\langle T_{k+l},\partial_X\rangle\,\Phi_Y(X)| \leq C_{k,l} \text{ for } g_Y(T_j)\leq 1.$$

In the same way, a regularized weight M satisfies $M\in S(M,g)$.

The main use of such partitions of unity is to prove the Theorem 3.2 above (see [**B-L**]) where the two following arguments are crucial.

1. Any symbol $a \in S(M,g)$ can be written as a superposition of confined symbols :

$$a = \int M(Y)\, a_Y\, \lambda(Y)^{-n} dY \quad \text{with } (a_Y) \text{ uniformly confined.}$$

One has just to set $a_Y(X) = M(Y)^{-1} a(X) \Phi_Y(X)$.
2. The operators a_Y^w are "almost orthogonal" : one has

$$\|a_Y^w \circ a_Z^w\|_{\mathcal{L}(L^2)} \le C_N \Delta_r(Y,Z)^{-N}$$

as a consequence of (17).

5.3. Partitions of unity (second kind). We shall say that (a_Y) is a *g-regularly confined family* if the families $\langle T_{1Y}, \partial_Y\rangle \ldots \langle T_{kY}, \partial_Y\rangle\, a_Y$ are uniformly confined for any choice of vectors T_{jY} such that $g_Y(T_{jY}) \le 1$.

Let us consider $\alpha_Y = \pi^n \delta_Y \# a_Y$. Up to translations which maintain the functions centered at Y, it is just a Fourier transformation :

$$\text{for} \qquad \widetilde{\alpha}_Y(Z) = \alpha_Y(Y+Z), \quad \text{one has} \quad \widetilde{\alpha}_Y(Z) = \int e^{-2i\sigma(T-Y,Z)} a_Y(T)\, \tfrac{dT}{\pi^n},$$

which means that $\widetilde{\alpha}_Y$ is the Fourier transform of $a_Y(Y-\cdot)$. The following estimates are then consequence of (15).

Proposition 5.3. *Let (a_Y) be a g-regularly confined family and let α_Y and $\widetilde{\alpha}_Y$ defined as above. Then*

$$\Big\| \langle T_1, \partial_Y\rangle \ldots \langle T_k, \partial_Y\rangle\, \widetilde{\alpha}_Y(Z)\, \big(1+\lambda(Y)^2 g_Y(Z)\big)^N \Big\|_{L^1(dZ)} \le C_{k,N}, \tag{18}$$

for $g_Y(T_j) \le 1$.

Let us consider again the g-regularly confined family Θ_Y constructed above, adding the condition that $f(t) = 1$ near 0 and thus that $\Theta_Y(X) = 1$ near Y. Let (ψ_Y) be the family defined by

$$\psi_Y(X) = \widetilde{\psi}_Y(X-Y) = \pi^n \delta_Y \# \Theta_Y(X). \tag{19}$$

The estimates (18) are valid for the functions $\widetilde{\psi}_Y$ and one has moreover

$$\int \psi_Y(X)\, \tfrac{dX}{\pi^n} = 1.$$

Remark. The functions $\alpha_Y(X)$ do not belong uniformly to $S(1,g)$: one has just the following estimates

$$|\alpha_Y(X)| \le C_N \lambda(Y)^n \big(1+\lambda(Y)^2 g_Y(X-Y)\big)^{-N}$$

and moreover, one looses a factor $\lambda(Y)$ for each derivative $\langle T, \partial X\rangle$ with $g_Y(T) \le 1$.

The functions α_Y are much more concentrated than confined symbols. They are small outside a g_Y-ball of radius $C^{\mathrm{st}}\lambda(Y)^{-1}$. This may look as a violation of the uncertainty principle but, if the symbols are more concentrated, the corresponding operators are not. The operator α_Y^w is equal to $\Sigma_Y \circ a_Y^w$, that is an operator confined in $B_{Y,r}$ followed by a phase symmetry with respect to Y. While the usual product $\alpha_Y \alpha_Z$ is small as far as $\lambda(Y)^2 g_Y(Y-Z)$ is large, one has only

$$\|\alpha_Y^w \circ \alpha_Z^w\|_{\mathcal{L}(L^2)} = \|\Sigma_Y \circ a_Y^w \circ a_Z(2Z-\cdot)^w \circ \Sigma_Z\|_{\mathcal{L}(L^2)} \le C_N \Delta(Y,Z)^{-N}, \tag{20}$$

which imply no smallness for $g_Y(Y-Z) \le C^{\mathrm{st}} r$.

Theorem 5.4. *Let M be a (regularized) g-weight.*

(i) *Let (a_Y) be a g-regularly confined family and $\alpha_Y = \pi^n \delta_Y \# a_Y$. Then the operator*

$$B = \int M(Y) \alpha_Y^w \, \tfrac{dY}{\pi^n} \tag{21}$$

belongs to $\Psi(M,g)$ and $X \mapsto M(X) a_X(X)$ is a principal symbol of B.

(ii) *In particular, for $a \in S(M,g)$, the operator $\int a(Y)\psi_Y^w \, dY/\pi^n$ is a pseudo-differential operator of weight M, having a as principal symbol. Thus*

$$\int \psi_Y(X) \tfrac{dY}{\pi^n} = 1 + r(X) \quad \text{with } r \in S(\lambda^{-1}, g).$$

(iii) *Conversely, if $A = a^w \in \Psi(M,g)$ then*

$$Y \mapsto 2^{-n} \operatorname{tr}(A \circ \psi_Y^w) = \int a(S) \psi_Y(S) \tfrac{dS}{\pi^n}$$

is a principal symbol of A.

In order to simplify the notations, we give the proof for $M = 1$. The symbol of B is $b(X) = \int \alpha_Y(X)\, dY/\pi^n$ and one has

$$b(X) = \int \widetilde{\alpha}_{X-Z}(Z) \tfrac{dZ}{\pi^n} = \int \widetilde{\alpha}_X(Z) \tfrac{dZ}{\pi^n} + \int \left\{ \widetilde{\alpha}_{X-Z}(Z) - \widetilde{\alpha}_X(Z) \right\} \tfrac{dZ}{\pi^n}$$

The first integral is equal to $a_X(X)$. In the second one, for $g_X(Z)^{1/2} \geq Cr$ and thus $g_{X-Z}(Z)^{1/2} \geq C'r$, the integral of both terms are $O(\lambda(Z)^{-N})$ for all N. In the remaining integral, using (18) the curly bracket is bounded by $\left(\frac{g_X(Z)}{1+\lambda(Z)^2 g_X(Z)} \right)^{1/2} h(Z)$ with $h \in L^1$. We have proved that $b(X) = a_X(X) + c(X)$ with $|c(X)| \leq \lambda(X)^{-1}$.

For proving that $c \in S(\lambda^{-1}, g)$, one has to prove the same estimate for the derivatives ; $\langle T, \partial X \rangle\, b$ is the integral of $\langle T, \partial X \rangle\, \widetilde{\alpha}_{X-Z}(Z)$ which, for $g_X(T) \leq 1$, is again bounded by (18) and the proof goes along the same way.

The part (ii) is just a particular case, setting $a_Y = a(Y)\Theta_Y$ and $\alpha_Y = a(Y)\psi_Y$.

Remark. The family (ψ_Y) is only an approximate partition of unity. It would be easy to transform it into an exact partition.

One should pay attention to the difference between the two kinds of partitions of unity. The operators Φ_Y^w are almost orthogonal with respect to their element of integration $dY/\lambda(Y)^n$; one has $\int a(Y)\Phi_Y \lambda(Y)^{-n} dY \in S(1,g)$ when a belongs just to L^∞ ; one can use Cotlar lemma in integral form (see [**B-L**]) for proving that the corresponding operator is bounded on L^2. On the other hand, a is not a principal symbol of this operator, even when it belongs to $S(1,g)$, the variation of a in $B_{Y,r}$ being bounded by r but not by $1/\lambda(Y)$.

The situation is quite different for $B = \int a(Y)\psi_Y^w dY/\pi^n$. The ψ_Y^w are not almost orthogonal with respect to their element of integration. If a belongs just to L^∞, the symbol b of this operator is bounded but its derivatives are not and b^w is not bounded on L^2. The assumption $a \in S(1,g)$ is crucial for having $b \in S(1,g)$. This is perhaps more evident if, instead of the change of variable used in the proof of Theorem 5.4, one uses

$$\langle T, \partial X \rangle\, \psi_Y(X) = - \langle T, \partial Y \rangle\, \psi_Y(X) + \beta_Y(X)\,; \quad (\beta_Y)\ g\text{-regularly confined}$$

and integrations by parts in Y for the derivatives of $b(X) = \int a(Y)\psi_Y(X) dY/\pi^n$.

In some sense, (21) should be considered as an oscillatory integral which looks like an absoluty convergent integral modulo $\Psi(\lambda^{-1}, g)$. The fact that a is a principal symbol is due to the concentration of ψ_Y (as a function, recall that ψ_Y^w is not more concentrated than Φ_Y^w).

6. The symbols of Fourier integral operators

6.1. The fiber bundle $\widetilde{\Gamma}$ and its sections. Let us consider again an admissible triple (F, g_0, g_1) as in section 4, the affine tangent map χ_Y being defined by (13).

There is a natural *line bundle* $\widetilde{\Gamma}$ over the graph Γ of F which is the pull back of $\widetilde{\mathrm{AMp}}(n)$ by the application $(Y, F(Y)) \mapsto \chi_Y$.

$$\begin{array}{ccc} \widetilde{\Gamma} & \longrightarrow & \widetilde{\mathrm{AMp}}(n) \\ \Big\downarrow \widetilde{\varpi} & & \Big\downarrow \widetilde{\varpi} \\ \Gamma & \longrightarrow & \mathrm{ASp}(n) \end{array}$$

$\widetilde{\Gamma}$ is made of the triples $(Y, F(Y), V)$ such that V is an extended metaplectic operator (including 0) associated to χ_Y. We still denote by $\widetilde{\varpi}$ the projection $\widetilde{\Gamma} \to \Gamma$. A section of $\widetilde{\Gamma}$ is thus given by $(Y, F(Y)) \mapsto (Y, F(Y), V_Y)$ where either $V_Y = 0$ or V_Y is invertible with $\varpi(V_Y) = \chi_Y$. We shall refer to "the section V_Y" for short.

Given a unitary section V_Y of $\widetilde{\Gamma}$, we know (Proposition 2.4) that we have $\langle T, \partial_Y \rangle V_Y = V_Y \# q_{Y,T}^w$, where q is a real second order polynomial. Changing the section would change only the constant term of $q_{Y,T}$.

We know no way for choosing "better" sections of $\varpi : \mathrm{AMp}(n) \to \mathrm{ASp}(n)$, but there are special sections of $\widetilde{\Gamma}$ which will play an important role.

Theorem 6.1 (horizontal sections). *Given an admissible triple, there exists a unitary section U_Y, unique up to a constant factor, such that*

$$\langle T, \partial_Y \rangle U_Y = i U_Y \# q_{Y,T}^w \quad \textit{with } q_{Y,T}(Y) = 0.$$

Moreover, the first derivatives of $q_{Y,T}$ vanishes at Y. Such sections will be called horizontal.

Set $Y' = F(Y)$ and $M_Y = F'(Y)$. The affine tangent map χ_Y is the composition of three maps : the symmetry of center $Y/2$, the linear map M_Y and the symmetry of center $Y'/2$. Let $k_{0,Y}^w \in \mathrm{Mp}_2(n)$ associated to M_Y and depending smoothly on Y. This is possible because $\mathbb{R}^{2n}$ is contractile, and $k_{0,Y}$ is uniquely determined up to the sign. We know that $\langle T, \partial_Y \rangle k_{0,Y} = i k_{0,Y} \# p_{Y,T}^w$, where $p_{Y,T}$ is a real polynomial, homogeneous of degree 2.

Let us consider the section $k_Y^w = \Sigma_{Y'/2} \circ k_{0,Y}^w \circ \Sigma_{Y/2}$. It is not difficult to compute the derivatives of $k_Y = \pi^{2n} \delta_{Y'/2} \# k_{0,Y} \# \delta_{Y/2}$. One has $\langle T, \partial_Y \rangle \delta(X - Y/2) = -\frac{1}{2} \langle T, \partial_X \rangle \delta(X - Y/2) = -i\sigma(X - Y/2, T) \# \delta(X - Y/2)$ and

$$\begin{aligned} \{\langle T, \partial_Y \rangle \delta(X - Y'/2)\} \# k_{0,Y} &= i\, \delta(X - Y'/2) \# \sigma(X - Y'/2, M_Y T) \# k_{0,Y} \\ &= i\, \delta(X - Y'/2) \# k_{0,Y} \# \sigma(M_Y X - Y'/2, M_Y T). \end{aligned}$$

Summing up, we get

$$\langle T,\partial_Y\rangle\, k_Y = i\pi^{2n}\,\delta_{Y'/2}\# k_{0,Y}\# r_{Y,T}\#\delta_{Y/2}$$

with
$$\begin{aligned} r_{Y,T}(X) &= p_{Y,T}(X) - \sigma(X-Y/2,T) + \sigma(M_Y X - Y'/2, M_Y T) \\ &= p_{Y,T}(X) + \tfrac{1}{2}\left\{\sigma(Y,T) - \sigma(F(Y),F'(Y)\cdot T)\right\}. \end{aligned}$$

If we introduce the 1-form $\alpha = \sigma(Y,dY) = \sum \eta_j dy_j - y_j d\eta_j$, the curly bracket is equal to $\langle \alpha - F^*\alpha, T\rangle$. One has $d(\alpha - F^*\alpha) = d\alpha - F^*(d\alpha) = 0$ for $d\alpha/2$ is the symplectic 2-form which is preserved by F. Thus, there exists a smooth function $H(Y)$ globally defined on $\mathbb{R}^{2n}$ such that $dH = \frac{1}{2}(\alpha - F^*\alpha)$. One has

$$\langle T,\partial_Y\rangle \left(e^{-iH(Y)}k_Y\right) = \left(e^{-iH(Y)}k_Y\right)\# p_{Y,T}(Y-\cdot).$$

Setting $U_Y = e^{-iH(Y)}k_Y^w$, the proof is complete.

Definition 6.2. *Given an admissible triple, one says that a section V_Y of $\widetilde{\Gamma}$ is of class $S(1)$ (or belongs to $S(1,\widetilde{\Gamma})$) if one can write $V_Y = f(Y)U_Y$ with U_Y horizontal and $f \in S(1,g_0)$.*

If (V_Y) and (W_Z) are sections of class $S(1)$ for admissible triples (F,g_0,g_1) and (G,g_1,g_2) respectively, then $Y \mapsto W_{F(Y)} \circ V_Y$ is a $S(1)$ section for the triple $(G\circ F, g_0, g_2)$ and $Z \mapsto V^*_{F^{-1}(Z)}$ is a $S(1)$ section for the triple (F^{-1}, g_1, g_0).

It suffices to consider the case of horizontal sections. One has

$$\langle T,\partial_Y\rangle\, U_Y = iU_Y \circ {q_{Y,T}}^w = i{q'_{Y,T}}^w \circ U_Y$$

where $q'_{Y,T} = q_{Y,T}\circ \chi_Y^{-1}$ is a second order polynomial vanishing at $F(Y)$. Thus $\langle T,\partial_Y\rangle\, U_Y^* = -iU_Y^* \# {q'_{Y,T}}^w$ which express that $F(Y) \mapsto U_Y^*$ is a horizontal section for (F^{-1}, g_1, g_0).

In the same way, if V and W are horizontal, one has with simplified notations

$$\begin{aligned} \langle T,\partial_Y\rangle\, W_{F(Y)}\circ V_Y &= iW_{F(Y)}\circ \widetilde{q}^w \circ V_Y + iW_{F(Y)}\circ V_Y \circ q \\ &= iW_{F(Y)}\circ V_Y \circ (q + \widetilde{q}\circ \chi_Y^{-1})^w, \end{aligned}$$

where $\widetilde{q}$ vanishes at $F(Y)$ and where q (and thus $q + \widetilde{q}\circ\chi_Y^{-1}$) vanish at Y.

Remark. Given a (regularized) weight μ defined on Γ (see remark (4.7)) the space $S(\mu;\widetilde{\Gamma})$ of sections of weight μ is just the space of sections V such that $\mu^{-1}V$ belongs to $S(1;\widetilde{\Gamma})$.

6.2. Main results. We state below the main results of the theory, the sketch of their proofs will be given in the next section

Theorem 6.3 (Existence of many FIOs). *Let (F,g_0,g_1) an admissible triple, V_Y a $S(1)$ section of $\widetilde{\Gamma}$, a_Y a g_0-regularly confined family and $\alpha_Y = \pi^n \delta_Y \# a_Y$. Then the operator*

$$P = \int V_Y \circ \alpha_Y^w \,\tfrac{dY}{\pi^n} \tag{22}$$

belongs to $\mathrm{FIO}(F,g_0,g_1)$.

Theorem 6.4 (Principal symbol). *Let P défined by (22) and set*

$$b(Y) = \int \alpha_Y(X)\tfrac{dX}{\pi^n};\quad c(Y) = 2^{-n}\,\mathrm{tr}\,(P\circ \psi_Y^w \circ V_Y^*)\,, \tag{23}$$

where ψ_Y *is an approximate partition of unity defined by* (19)*. Then* $c \in S(1, g_0)$ *and* $b - c \in S(\lambda_0^{-1}, g_0)$*.*

For the next theorem, we introduce another admissible triple (G, g_1, g_2) and consider (H, g_0, g_2) with $H = G \circ F$. We shall denote by $\Gamma_F, \ldots, \widetilde{\Gamma}_H$ the corresponding graphs and bundles.

Theorem 6.5 (Product law). *Let* V_Y *[resp.* W_Z*] a* $S(1)$ *section of* $\widetilde{\Gamma}_F$ *[resp.* $\widetilde{\Gamma}_G$*], let* a_Y *[resp.* b_Z*] a* g_0*- [resp.* g_1*-] regularly confined family, let* $\alpha_Y = \pi^n \delta_Y \# a_Y$ *and* $\beta_Z = \pi^n \delta_Z \# a_Z$*.*

Let P *defined by* (22) *and* $Q = \int W_Z \beta_Z^w \frac{dZ}{\pi^n}$*. Then one has*

$$Q \circ P = \int W_{F(Y)} \circ V_Y \circ \gamma_Y^w \frac{dY}{\pi^n}$$

with $Y \mapsto \delta_Y \# \gamma_Y$ g_0*-regularly confined and*

$$\Big(\int \gamma_Y(X) \tfrac{dX}{\pi^n} \Big) \equiv \Big(\int \alpha_Y(X) \tfrac{dX}{\pi^n} \Big) \Big(\int \beta_{F(Y)}(X) \tfrac{dX}{\pi^n} \Big) \quad (\text{mod } S(\lambda_0^{-1}, g_0))$$

6.3. Conséquences. We can now define the principal symbol of a Fourier integral operator when it can be written as (22) : it is an element of $S(1; \widetilde{\Gamma})$ given by $(Y, F(Y)) \mapsto b(Y) V_Y$ (or $c(Y) V_Y$) with the notations of (23). This definition is quite natural : up to lower order terms, the principal symbol of P at Y is the best approximation of P at this point by a generalized metaplectic operator.

This principal symbol is uniquely determined, up to an element of $S(\lambda^{-1}; \widetilde{\Gamma})$ and depend neither on the decomposition (22) (c does not depend on it) nor on the choice of the partition ψ_Y (b does not depend on it). Moreover, any section in $S(1; \widetilde{\Gamma})$ is a principal symbol of an operator of the form (22).

When the triple is (I, g_0, g_0), we recover the theory of the principal symbol of pseudo-differential operators, a function a on $\mathcal{X}$ being identified to the section $Y \to a(Y) I$.

It is now easy to find $P \in \mathrm{FIO}(F, g_0, g_1)$ and $Q \in \mathrm{FIO}(F^{-1}, g_1, g_0)$ of the form (22) whose principal symbol are inverse. We know by Theorem 6.5 that $Q \circ P$ is a pseudo-differential operator whose principal symbol is 1, that is $Q \circ P = I + R$ with $R \in \Psi(\lambda^{-1}, g_0)$.

Let now T a general element of $\mathrm{FIO}(F, g_0, g_1)$. One has $T = (TQ)P - TR$ and $TQ \in \Psi(1, g_0)$ while $TR \in \mathrm{FIO}(F, \lambda^{-1}, g_0, g_1)$. We know by Theorem 5.4 that $TQ = A + S$ with $A = \int \alpha_Y^w \frac{dY}{\pi^n}$ and $S \in \Psi(\lambda^{-1}, g_0)$. Then $T = AP + (SP - TR)$ and we have proved the following.

Theorem 6.6. (i) *Any* $T \in \mathrm{FIO}(F, g_0, g_1)$ *can be written* $T = T_0 + T'$ *where* T_0 *admits a decomposition* (22) *and* $T' \in \mathrm{FIO}(F, \lambda^{-1}, g_0, g_1)$*. The principal symbols of* T *are defined as those of* T_0*. One of them is given by*

$$(Y, F(Y)) \mapsto c(Y) V_Y \quad \textit{with} \quad c(Y) = 2^n \operatorname{tr}(T \circ \psi_Y^w \circ V_Y^*) .$$

(ii) *In the geometric situation of Theorem 6.5, if* $(Y, F(Y)) \mapsto V_Y$ *is a principal symbol of* $S \in \mathrm{FIO}(F, g_0, g_1)$ *and if* $(Z, G(Z)) \mapsto W_Z$ *is a principal symbol of* $T \in \mathrm{FIO}(G, g_1, g_2)$*, then* $(Y, H(Y)) \mapsto W_{F(Y)} \circ V_Y$ *is a principal symbol of* $T \circ S$*.*

Remark. Actually, one can write $T = T_0 + T'$ with $T' \in \mathrm{FIO}(F, \lambda^{-\infty}, g_0, g_1)$ (that is belonging to $\mathrm{FIO}(F, \lambda^{-N}, g_0, g_1)$ for all N). It is just a matter of symbolic calculus of pseudo-differential operators.

If there exists an exactly invertible Fourier integral operator in $\mathrm{FIO}(F, g_0, g_1)$ of the form (22), one can take $T' = 0$ and any element of $\mathrm{FIO}(F, g_0, g_1)$ can be decomposed as in (22). One has just to construct an exact partition of unity (ψ_Y). This will be the situation if F is the flow at time t of the hamiltonian field H_a and if we succeed in constructing e^{ita^w} as indicated below.

The reader will have no difficulty for defining the principal symbols of elements of $\mathrm{FIO}(F, \mu, g_0, g_1)$, which are elements of $S(\mu; \widetilde{\Gamma})$ uniquely determined modulo elements of $S(\mu\lambda^{-1}; \widetilde{\Gamma})$.

6.4. Towards evolution equations. Let us come back to the situation alluded to in the introduction. Given a real-valued C^∞ symbol a defined on $\mathcal{X}$, we would like to define the e^{ita^w} as Fourier integral operators. We shall just show how to proceed and what are the conditions which should be satisfied in order to succeed. Most of these conditions are expressed in terms of the hamiltonian flow of a, and cannot be simply expressed in terms of a itself.

Let us recall that the Hamilton vector field H_a is defined by $\langle da, T\rangle = \sigma(H_a, T)$, and that $H_a = (\partial a/\partial\xi_j; -\partial a/\partial x_j)$ in any set of symplectic coordinates. We shall denote by F_t the flow of H_a, that is $t \mapsto X(t) = F_t(X_0)$ is the unique solution of $\frac{d}{dt}X(t) = H_a(X(t))$; $X(0) = X_0$. We shall assume that the flow is global, and F_t is for any t a symplectomorphism of $\mathcal{X}$.

Given an admissible metric g_0 on $\mathcal{X}$, we define $g_t = F_{t*}g_0$. One has to assume that (F_t, g_0, g_t) is an admissible triple, which is not so easy to check. Of course, the Riemannian and symplectic structures are preserved by F_t, but the affine structure is not ; proving that g_t is tempered, which express a relation between Riemannian and affine structures, requires a good knowledge of the flow at infinity.

We want to write

$$e^{ita^w} = \int k_{tY}^w \alpha_{tY}^w \tfrac{dY}{\pi^n},$$

where k_{tY}^w is a metaplectic operator associated to the affine symplectic map χ_{tY} tangent to F_t at Y. We shall write as follows the second order Taylor expansion of a at $F_t(Y)$:

$$a(X) = b_{tY}(X) + r_{tY}(X),\ b_{tY} \text{ polynomial of d}^\circ 2,\ r_{tY}(X) = o(|X - F_t(Y)|^2)$$

It is not difficult to check that one can take k_{tY} solution of $\frac{d}{dt}k_{tY} = ib_{tY} \# k_{tY}$. The equation for the α_{tY} is then

$$\frac{d}{dt}\alpha_{tY} = i\widetilde{r_{tY}} \# \alpha_{tY} \quad \text{with } \widetilde{r_{tY}} = r_{tY} \circ \chi_{tY}$$

with a given partition of unity as initial condition : $\int \alpha_{0Y} \frac{dY}{\pi^n} = 1$. Setting $a_{tY} = \delta_Y \# \alpha_{tY}$, the equation for the a_{tY} is analogous, $\widetilde{r_{tY}}(X)$ being replaced by $\widetilde{r_{tY}}(2Y - X)$.

We choose the a_{0Y} such that it is a regularly confined family, and our initial problem is reduced to the proof that a_{tY} is regularly confined, which is a matter of uniform estimates (expressed in terms of the quadratic form g_{0Y}) for the $\widetilde{r_{tY}}$.

The whole construction starts from a^w defined on $\mathcal{S}$ and $\mathcal{S}'$, and not from a realization of it as a self-adjoint operator on L^2. Actually, when the construction works, it proves also that a^w defined on $\mathcal{S}$ is essentially self-adjoint and that its weak and strong extensions coïncide.

7. Sketch of the proofs

Let us consider the operator

$$P = \int V_Y \circ \alpha_Y^w \, \tfrac{dY}{\pi^n} \;, \quad \alpha_Y = \pi^n \delta_Y \# a_Y, \tag{24}$$

where we can assume that $V_Y = k_Y^w$ is a horizontal section of $\widetilde{\Gamma}$. We have to prove that P is a Fourier integral operator as defined in section 4. We shall first prove that $K_b \cdot P$, for $b \in S^+(g_1)$, can be also written as $\int k_Y^w \beta_Y^w \, dY$, the β_Y having the same properties as the α_Y. It will remain to prove that P is bounded on L^2.

For both problems, the answer would be easy if we could gain powers of λ, that is if $\lambda(Y)^N a_Y$ were uniformly confined for any N. On the other hand, as far as the g_Y-distance between Y and Z is $\geq Cr$, in the estimations of the semi-norms of $\alpha_Y \circ \alpha_Z$ one gains not only any power of the distance, but also any power of $\lambda(Y)$. As a rather easy consequence, we can assume when necessary that the problem is localized in a ball $B_{Y_0,Cr}$, that is $a_Y = 0$ for $Y \notin B_{Y_0,Cr}$, as far as the estimates are independent on Y_0. For the same reason, we can always assume that a symbol confined in $B_{Y,r}$ is actually supported in $B_{Y,Cr}$.

7.1. Twisted commutators. — One has

$$K_b \cdot P = b^w P - P(b\circ F)^w = \int k_Y^w \left((b\circ\chi_Y)^w \delta_Y^w a_Y^w - \delta_Y^w a_Y^w (b\circ F)^w\right) dY \tag{25}$$

Up to a term which is written as (24), we can commute $(b\circ F)^w$ and a_Y. However, one has $f^w \delta_Y^w = \delta_Y^w f(2Y - \cdot)^w$ and (25) is equivalent (i.e. modulo integrals of the form (24)) to

$$\int k_Y^w \delta_Y^w c_Y^w a_Y^w \, dY$$

where c_Y is the difference between $(b\circ F)$ and the symmetrized of $(b\circ\chi_Y)$. We have $c_Y(Y) = 0$ and, up to equivalence, we can assume that c_Y is supported in $B_{Y,Cr}$. One has then not only $c_Y \in S^+(g_0)$ but $c_Y \in S(\lambda, g_0)$ (uniformly).

Using Taylor expansion, we can write $c_Y(X) = \sum_1^{2n} \sigma(X-Y, T_{jY}) \# \gamma_{jY}(X) + \gamma_{0Y}$ with $\gamma_{jY} \in S(1, g_0)$ (and actually uniformly confined) and $g_Y(T_{jY}) \leq 1$. Up to equivalence, we can neglect γ_{0Y} and, dropping the index, we are led to consider terms like

$$k_Y \# \delta_Y \# \sigma(X-Y, T) \# a'_Y$$

where a'_Y stands for $\gamma_{jY} \# a_Y$. In view of $\delta_Y \# \sigma(X-Y, T) = \frac{-1}{2i} \langle T, \partial X \rangle \, \delta_Y = \frac{1}{2i} \langle T, \partial Y \rangle \, \delta_Y$, we have to study

$$\theta_Y = k_Y \# (\langle T, \partial Y \rangle \, \delta_Y) \# a'_Y.$$

It is time to use the fact that k_Y^w is a horizontal section. One has $\langle T, \partial_Y \rangle \, k_Y = i k_Y \# q_(\cdot - Y)^w$ where q is a homogeneous quadratic polynomial. Moreover (it is perhaps easier to check it using the special coordinates of the remark 4.1) one has $|q(Z)| \leq C\lambda(Y) g_0(Z)$. Introducing again a cut off in $B_{Y,Cr}$, one can write $q(X-Y) \simeq \sum \sigma(X-Y, T_j) \# \varphi_j(X)$ with $c_j \in S(1, g_0)$ (and actually confined). The uniform norm of c_j is bounded by $C^{\mathrm{st}} r$ and we can choose r such that it is small.

At that step, it is useful to assume that we are localized in some ball $B_{Y_0,Cr}$ and that (T_j) is a basis in which g_{Y_0} is diagonal. Summing up, we have

$$\langle T_j, \partial_Y\rangle \{k_Y \# \delta_Y\} = k_Y \# \sum_1^{2n} \langle T_l, \partial_Y\rangle\, \delta_Y \# (\delta_{jl} + \varphi_{jl}) + \text{ error}$$

where δ_{jl} is the Kronecker symbol and $\varphi_{jl} \in S(1, g_0)$ has small uniform norm. The error coming from the cut off, after composition with a'_Y will decay as any power of λ^{-1} and can be neglected. Now, the symbolic calculus allows to invert the above matrix of elements of $S(1, g)$, up to elements of $S(\lambda^{-\infty}, g)$, and we can write

$$k_Y \# \langle T_j, \partial_Y\rangle\, \delta_Y \simeq \sum \left(\langle T_l, \partial_Y\rangle \{k_Y \# \delta_Y\}\right) \# \theta_{jl}$$

with $\theta_{jl} \in S(1, g_0)$. Thus, our expression of $K_b \cdot P$ can be written as a sum of terms of the following type

$$\int \left(\langle T, \partial_Y\rangle \{k_Y \# \delta_Y\}\right) \# \theta_Y \# a_Y \, dY,$$

and an integration by parts will convert it into a term similar to the right hand side of (24).

Localization in a euclidean ball. — For proving the boundedness in L^2, we can assume that P is localized in a ball $B_{Y_0,Cr}$. Up to composition with fixed metaplectic operators (see remark 4.1) we can then assume that $Y_0 = F(Y_0) = 0$, that $F'(Y_0) = I$ and that the metrics g_0 and g_1 are equal to $g = (dx^2 + d\xi^2)/\rho^2$ with $\rho^2 = \lambda_0(Y_0)$. In what follows, we shall call $a \mapsto \partial_T a$ a *e-derivative* if $|T| \leq 1$ and a *g-derivative* if $|T| \leq \rho$, that is if $g(T) \leq 1$.

We are thus reduced to prove the boundedness of

$$Q = \int_{|Y| \leq r\rho} k_Y^w \, \alpha_Y^w \, \tfrac{dY}{\pi^n},$$

the symbols α_Y being 0 outside the ball.

We know by the proof of Theorem 6.1 that the horizontal section which is the identity for $Y = 0$ has the following form

$$k_Y = e^{-iH(Y)} \exp\left\{-2i\sigma\left(X - \tfrac{F(Y)}{2}, X - \tfrac{Y}{2}\right) + iB_Y\left(X - \tfrac{Y+F(Y)}{2}\right)\right\}. \tag{26}$$

The e-derivatives of order p of F are $O(\rho^{1-p})$. From $B_Y(W) = \sigma(W, C_Y W)$ and $C_Y = (I + F'(Y))^{-1}(I - F'(Y))$, where $F'(Y)$ belongs to a small neighbourhood of I, we deduce that the e-derivatives of order p of C_Y are $O(\rho^{-p})$. Moreover, C_Y and the first derivatives of $F(Y) - Y$ vanishes for $Y = 0$.

The proof of Theorem 6.1 says also that

$$\begin{aligned} 2\langle T, \partial Y\rangle\, H(Y) &= \sigma(Y, T) - \sigma(F(Y), F'(Y) \cdot T) \\ &= \sigma(Y, T) - \sigma\left(Y + O(|Y|^2), T + O(|Y|)T\right) \end{aligned}$$

Thus, the partial derivatives of order p of H are $O(\rho^{2-p})$. Moreover, H and its derivatives up to order 2 vanish at $Y = 0$.

The operators $k_Y^w \circ \alpha_Y^w$ are uniformly bounded in L^2, but they are not almost orthogonal, and the volume of the ball is $\sim \rho^{2n}$. The gain of this power of ρ will result from a stationary phase argument.

7.2. Computation of Q^*Q. — We have

$$\begin{aligned}\overline{q}\#q &= \iint \overline{k_Z}\#\delta_{F(Z)}\#\overline{\widetilde{a}_Z}\#\widetilde{a}_Y\#\delta_{F(Y)}\#k_Y\,\tfrac{dY\,dZ}{\pi^{2n}}\\ &= \iint \overline{k_Z}\#L_{YZ}\#k_Y\,\tfrac{dY\,dZ}{\pi^{2n}}.\end{aligned} \tag{27}$$

The function $\widetilde{a}_Y = a_Y \circ \chi_Y^{-1}$ is confined in a ball centered at $F(Y)$ and can be considered as confined in a ball $B_{0,C'r}$ defined by $|Z| \leq C'r\rho$. An easy computation shows that

$$H_{YZ}(X) = e^{-2i\sigma(X-F(Z),X-F(Y))}l_{YZ}(X) \tag{28}$$

where the functions l_{YZ}, and their g-derivatives are uniformly confined in the ball $B_{0,C'r}$.

It is actually possible to modify slightly (27) and (28) as follows : the function $k_Y(X)$ is replaced by the (usual) product $c_Y(X)k_Y(X)$ with c_Y uniformly confined in $B_{0,C'r}$, the function l_{YZ} is modified but keeps the same properties. This comes from two arguments. The first one, which is just a matter of symbolic calculus, says that $a_Y = a'_Y\#a''_Y + \varepsilon_Y$ where a'_Y, a''_Y and $\lambda(Y)^N\varepsilon_Y$ for any N are uniformly confined in slightly larger balls. Neglecting the term coming from ε_Y, one can then replace k_Y by $a''_Y\#k_Y$ in (27) and the function L_{YZ}, constructed with a'_Y instead of a_Y, keeps the same properties.

The second argument says that, when k^w is a metaplectic operator associated to an affine transformation close to the identity, and when a'' is confined in $B_{0,r}$, one has $a''\#k = ck$, the symbol c being confined, with a control of its semi norms, in $B_{0,Cr}$. When $k(X) = e^{i\sigma(X,CX)}$ (the case of phase translations is easy), the value of c is given by

$$c(X) = \int e^{-2i\sigma(X-Y,H)}e^{-2i\sigma(CX,H)}e^{i\sigma(H,CH)}a''(Y)\,\tfrac{dY\,dH}{\pi^{2n}}$$

which correspond to a Gauss transform followed by a change of variable. The main argument is similar to that of [**Hö1**, Lemma 7.6.4].

Applying the formula (4) to the modification of (27) we get

$$\overline{k_Z}\#L_{YZ}\#k_Y = \iint e^{-2i\sigma(X-S,X-T)}\overline{c_Z(S)k_Z(S)}c_Y(T)k_Y(T)\; L_{YZ}(S{+}T{-}X)\,\tfrac{dS\,dT}{\pi^{2n}}$$

$$\overline{q}\#q(X) = \iiiint e^{i\Phi(X,Y,Z,S,T)}f(X,Y,Z,S,T)\,dY\,dZ\,dS\,dT,$$

and

$$\begin{aligned}\Phi(X,Y,Z,S,T) &= H(Z) - H(Y) - 2\sigma(X{-}S,X{-}T)\\ &\quad + B_Z(S - \tfrac{Z+F(Z)}{2}) + \sigma\left(S{-}\tfrac{F(Z)}{2}, S{-}\tfrac{Z}{2}\right) - B_Y(T - \tfrac{Y+F(Y)}{2})\\ &\quad - \sigma\left(S{-}\tfrac{F(Y)}{2}, S{-}\tfrac{Y}{2}\right) - 2\sigma(S{+}T{-}X{-}F(Z), S{+}T{-}X{-}F(Y)).\end{aligned}$$

The function f is uniformly bounded as well as its g-derivatives. Moreover, it has the decay of a confined symbol with respect to the variables X, Y, Z, S, T and we can as well assume that it is supported in a ball $B_{0,Cr}$ for all these variables.

7.3. Boundedness of Q in L^2. — We shall prove that Q^*Q (which depend of course on the initial Y_0) belongs to $\Psi(1,g)$ with uniform semi-norms. The change of variable $(X, Y, Z, S, T) = \rho(X', Y', Z', S', T')$ in the formula above leads to an integral

$$I(X') = \int e^{i\rho^2\Phi'(X',Y',Z',S',T')} f'(X', Y', Z', S', T')\rho^{8n}\, dY'\, dZ'\, dS'\, dT'$$

and we have to prove that $I(X')$ is bounded together with all its e-derivatives. The function f' has its support in a fixed ball of euclidean radius Cr and all its e-derivatives are uniformly bounded.

The phase Φ' has the same expression as Φ, except that F is replaced by $\widetilde{F}(Z') = \rho^{-1}F(\rho Z)$, and so on. The estimates given above on the derivatives of F and H imply that the e-derivatives of Φ' are uniformly bounded.

We use now the stationary phase method as expressed in [**Hö1**, th. 7.7.5]. If the hessian of Φ' with respect to (X', Y', S', T') is (uniformly) non degenerate at the origin, and if r had been choosen sufficiently small, then all the e-derivatives of I are uniformly bounded. One should remark that $H(Y)$ and $B_Y\left(\frac{Y+F(Y)}{2}\right)$ are actually cubic terms at 0 and that the same is true for $\sigma\left(S-\frac{F(Y)}{2}, S-\frac{Y}{2}\right) = \sigma\left(S-\frac{F(Y)}{2}, \frac{F(Y)-Y}{2}\right)$.

For $X = 0$, one has actually

$$\Phi'(0, Y', Z', S', T') = -2\sigma(S', T') - 2\sigma(S'+T'-Z', S'+T'-Y') + o(|Y'|^2 + |Z'|^2 + |S'|^2 + |T'|^2)$$

and the quadratic form in $\mathbb{R}^{8n}$ (which would be exactly the phase if Q were a pseudo-differential operator) is non degenerate. This proves that I is bounded with all its e-derivatives and thus that $\overline{q}\#q$ is bounded with all its g-derivatives. The operator $Q^*Q \in \Psi(1, g)$ is bounded on L^2, which ends the proof.

We do not give here the proofs of theorems 6.4 and 6.5 which are simpler and introduce no new idea.

References

[Bo1] Bony, Jean-Michel. Caractérisations des opérateurs pseudo-différentiels. Séminaire Équations aux Dérivées Partielles, 1996–1997, Exp. No. XXIII, 17 pp., École Polytech.

[B-C] Bony, Jean-Michel; Chemin, Jean-Yves. Espaces fonctionnels associés au calcul de Weyl-Hörmander. Bull. Soc. Math. France 122 (1994), no. 1, 77–118.

[B-L] Bony, Jean-Michel; Lerner, Nicolas. Quantification asymptotique et microlocalisations d'ordre supérieur. Ann. Sci. École Norm. Sup. (4) 22 (1989), no. 3, 377–433.

[Hö1] Hörmander, Lars. The analysis of linear partial differential operators. Springer-Verlag, Berlin, 1990.

[Hö2] Hörmander, Lars. Symplectic classification of quadratic forms, and general Mehler formulas. Math. Z. 219 (1995), no. 3, 413–449.

École polytechnique, Centre de Mathématiques, 91128 Palaiseau cedex, France
E-mail address: bony@math.polytechnique.fr