1. The eigencurve

Let D be the definite quaternion algebra of discriminant d, N an integer prime to d and p an odd prime such that $(p, Nd) = 1$. Recall that we have defined compact open subgroups $K_1(N,p) = K_1(N) \subset K_1(N) \subset \widehat{\mathcal{O}}^\times$ (for a fixed maximal order $\mathcal{O}$ of D).

Recall that $\mathcal{H}$ is the abstract polynomial ring $\mathbb{Z}[U_p, \{T_\ell\}_{(\ell,Npd)=1}]$. If $k \geq 2$ is an integer, we have a space $S_k^D(N,p) := S_k(K_1(N,p))$ of modular forms, which is an $\mathcal{H}$-module in a natural way. In order to study p-adic congruences we fix once and for all fields embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

(*System of Hecke eigenvalues*) For each $k \geq 2$ and $f \in S_k^D(N,p)$ which is a nonzero common eigenform for all the elements of $\mathcal{H}$, let us define a ring homomorphism $\psi_f : \mathcal{H} \to \mathbb{C}$ by the formula $h(f) = \psi_f(h)f$ for each $h \in \mathcal{H}$. This is the *system of eigenvalues of f (with respect to $\mathcal{H}$)*. We have seen that $\psi_f(\mathcal{H}) \subset \overline{\mathbb{Q}}$ so via $\iota_p\iota_\infty^{-1}$ we may and shall view it with coefficients in $\overline{\mathbb{Q}}_p$. Consider the subset

$$\mathcal{Z} \subset \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}$$

of all the elements of the form (ψ_f, k) as above when f and k vary as above.

Recall that the weight space $\mathcal{W}$ is the p-adic character variety of $\mathbb{Z}_p^\times$ in dimension 1. It is a disjoint union of $p-1$ open unit balls. We consider the embedding $\mathbb{Z} \to \mathcal{W}(\mathbb{Q}_p)$ given by $k \mapsto (x \mapsto x^{k-2})$.

THEOREM 1.1. *There is a reduced rigid analytic space $\mathcal{C}$ over $\mathbb{Q}_p$ equipped with:*

- *A ring homomorphism $\psi : \mathcal{H} \longrightarrow \mathcal{O}(\mathcal{C})$,*
- *An analytic morphism $\kappa : \mathcal{C} \longrightarrow \mathcal{W}$ (the "weight map"),*
- *An accumulation and Zariski-dense subset $Z \subset \mathcal{C}(\overline{\mathbb{Q}}_p)$,*

such that the following conditions are satisfied:

(i) *$\psi(U_p)$ is an invertible function on $\mathcal{C}$ and the morphism $\nu := (\kappa, \psi(U_p)^{-1}) : \mathcal{C} \longrightarrow \mathcal{W} \times \mathbb{G}_m$ is a finite morphism.*

(ii) *For each open affinoid $\mathcal{V} \subset \mathcal{W} \times \mathbb{G}_m$, the natural map*

$$\psi \otimes \nu^* : \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}(\mathcal{V}) \longrightarrow \mathcal{O}(\nu^{-1}(\mathcal{V}))$$

is surjective.

(iii) *The evaluation map $\mathcal{C}(\overline{\mathbb{Q}}_p) \to \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p)$,*

$$x \mapsto (\psi_x, \kappa(x)),\ \ \psi_x := (h \mapsto \psi(h)(x)),$$

induces a bijection $Z \xrightarrow{\sim} \mathcal{Z}$.

The quadruple $(\mathcal{C}, \psi, \kappa, Z)$ is unique with these properties. It has the following extra properties:

(a) *$\mathcal{C}$ is an equidimensional curve.*[1]

[1] That is, $\mathcal{C}$ is admissible covered by affinoids $V \subset \mathcal{C}$ such that $\mathcal{O}(V)$ is equidimensional of dimension 1 (recall it is a noetherian ring). In particular, there is no isolated point.

(b) *("κ is locally finite flat") $\mathcal{C}$ is admissibly covered by open affinoids $\mathcal{U} \subset \mathcal{C}$ such that $\kappa(\mathcal{U})$ is an open affinoid of $\mathcal{W}$ and such that the induced morphism $\kappa : \mathcal{U} \to \kappa(\mathcal{U})$ is a finite flat morphism. Moreover, any $x \in \mathcal{C}$ has a basis of affinoid neighborhoods of this form.*

(c) *If $\mathcal{U}$ is as in (b), then Z is Zariski-dense in $\mathcal{U}$ if and only if each connected component of $\kappa(\mathcal{U})$ meets $\mathbb{Z}$.*

The curve $\mathcal{C}$ above (equipped with ψ, κ and Z), is called the *quaternionic eigencurve of tame level N for D.*

This statement refers to some notions of rigid analytic geometry that we have to explain. In what follows X is a rigid analytic space over $\mathbb{Q}_p$.

(*Closed subsets*) Let $F \subset X$ be a subset. We say that F is a closed (or analytic) subset if there is an admissible covering of X by open affinoids X_i such that $F \cap X_i$ is defined by an ideal I_i in $\mathcal{O}(X_i)$ for each i. In this case there is a unique structure of reduced analytic space on F such that the affinoids $F \cap X_i = \mathrm{Sp}(\mathcal{O}(X_i)/\sqrt{I_i})$ form an admissible covering of F. When X is quasi-Stein, i.e. a countable admissible increasing union of open affinoids X_i with $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ having dense image, it is a result of Kiehl that any closed subset of X is defined by the vanishing of a collection of global analytic functions on X.

(*Zariski-dense subsets*) A subset $Z \subset X$ is said *Zariski-dense* if the only closed subset of X containing Z is X itself. When X is affinoid, it is equivalent to ask that Z is Zariski-dense in $\mathrm{Max}(\mathcal{O}(X))$, or even in $\mathrm{Spec}(\mathcal{O}(X))$ by the Jacobson property. The *Zariski-closure* of Z in X is the intersection of all the closed subsets of X containing Z, it is a closed subset.

(*Accumulation subsets*) A subset Z is an accumulation subset if for any $z \in Z$ and any open affinoid $\mathcal{U} \subset X$ containing z, there is another open affinoid $z \in \mathcal{U}' \subset \mathcal{U}$ such that $Z \cap \mathcal{U}'$ is Zariski-dense in $\mathcal{U}'$.

Example : $\mathbb{Z}$ (or $\mathbb{N}$) is a Zariski-dense accumulation subset of $\mathcal{W}$.

(*Finite morphisms*) A finite morphism $f : X \to Y$ is a morphism of rigid analytic spaces such that for some admissible covering of Y by open affinoids Y_i, then $f^{-1}(Y_i)$ is an open affinoid of X and the associated ring homomorphism $f^* : \mathcal{O}(Y_i) \to \mathcal{O}(f^{-1}(Y_i))$ is finite in the usual sense. Let us mention here that it is a fact due to Tate that if A is an affinoid algebra and B is a finite A-algebra, then B is an affinoid algebra as well.

We now prove the theorem. Note that the statement does not mention any p-adic modular form, although we shall really use them to prove the result! The uniqueness property is here for psychological reasons and shall not be needed nor proved below. Let us prove the existence of $\mathcal{C}$, following Coleman and Mazur. Let

$$F = \mathrm{Fred}_{U_p} \in 1 + T\mathcal{O}(\mathcal{W})\{\{T\}\}$$

be the Fredholm power series of U_p acting on the space of families of p-adic modular forms of level N, as defined at the end of the previous lecture. As already said, we may view it as an analytic function on $\mathcal{W} \times \mathbb{A}^1$, hence on $\mathcal{W} \times \mathbb{G}_m$. Define the associated Fredholm hypersurface

$$Z(F) \subset \mathcal{W} \times \mathbb{G}_m$$

defined by $F = 0$. It defines a closed subset (take $I_i = (F)$ for each i), but also has a natural structure of not necessarily reduced rigid analytic space given by the coherent sheaf of ideals generated by F. By definition of F, an element $(\kappa, \lambda) \in Z(F)(\overline{\mathbb{Q}}_p)$ if and only if there is a p-adic modular form $f \neq 0$ in $S_{\kappa,m}(N) \otimes \overline{\mathbb{Q}}_p$, for $m \geq m(\kappa)$, such that $U_p(f) = \lambda^{-1} f$ (such an f is of finite slope $v(\lambda)$ for U_p). What is essentially missing to get $\mathcal{C}$ is the action of the other operators T_ℓ for $(\ell, Npd) = 1$.

We thus want to factor F and apply Riesz-Coleman theory. Of course, the theory of Newton polygon for power series over general affinoids does not quite work as well as for finite extensions of $\mathbb{Q}_p$, but we have the following positive result due to Coleman and extended by Buzzard

Proposition 1.2. (The special covering of a Fredholm hypersurface) *Let Y be any reduced rigid analytic space and $F \in 1 + T\mathcal{O}(Y)\{\{T\}\}$. Let $\mathcal{I}$ be the set of open affinoids $\mathcal{U} \subset Z(F)$ such that:*

(a) *$\mathcal{V} = \mathrm{pr}_1(\mathcal{U})$ is an open affinoid of Y,*
(b) *$F_{|\mathcal{V} \times \mathbb{G}_m}$ has a factorization of the form GH where $G \in 1 + T\mathcal{O}(\mathcal{V})[T]$ has a unit leading coefficient, and $H \in 1 + T\mathcal{O}(\mathcal{V})\{\{T\}\}$ is prime to G in $\mathcal{O}(\mathcal{V})\{\{T\}\}$,*
(c) *$\mathcal{U}$ is the locus $G = 0$ in $\mathcal{V} \times \mathbb{G}_m$.*

In particular, $\mathcal{O}(\mathcal{U}) = \mathcal{O}(\mathcal{V})[T]/(G(T))$ (so $\mathcal{U}$ is finite flat over $\mathcal{V}$ of degree $\deg(G)$) and $Z(F) \cap (\mathcal{V} \times \mathbb{G}_m)$ is the admissible disjoint union of $\mathcal{U}$ and of the zeros of H.)

Then $\mathcal{I}$ is an admissible covering of $Z(F)$. It is stable by finite intersections, taking connected components, and by pull-back over affinoids : if $\mathcal{U} \in \mathcal{I}$ and $\mathcal{V}' \subset \mathrm{pr}_1(\mathcal{U})$ is an open affinoid, then $\mathrm{pr}_1^{-1}(\mathcal{V}') \cap \mathcal{U} \in \mathcal{I}$.

Applying this to the case above ($Y = \mathcal{W}$), we can now give a construction of $\mathcal{C}$. We shall construct $\mathcal{C}$ as a finite covering of the Fredholm hypersurface $Z(F) \subset \mathcal{W} \times \mathbb{G}_m$. Fix $\mathcal{U} \in \mathcal{I}$, as well as the corresponding factorization

$$F_{|\mathcal{V}} = GH \in \mathcal{O}(\mathcal{V})\{\{T\}\}.$$

Chose some m so that $\mathcal{V} \subset \mathcal{W}_m$, thus U_p is well defined on $S_{m,\mathcal{V}}(N)$ and has Fredholm series $F_{|\mathcal{V}}$. The Riesz-Coleman theory gives a natural decomposition

$$S_{m,\mathcal{V}}(N) = P \oplus Q,$$

where in particular P is projective of finite rank over $\mathcal{O}(\mathcal{V})$, preserved by the action of $\mathcal{H}$, and such that the characteristic polynomial of U_p on P is G^*. Consider the natural $\mathcal{O}(\mathcal{V})$-algebra morphism

$$\mathcal{O}(\mathcal{V}) \otimes_{\mathbb{Z}} \mathcal{H} \to \mathrm{End}_{\mathcal{O}(\mathcal{V})}(P).$$

Its image is a commutative $\mathcal{O}(\mathcal{V})$-algebra of finite type, hence is an affinoid algebra by Tate's result recalled above. Denote by $\mathcal{C}(\mathcal{U})$ the affinoid spectrum of this affinoid algebra. By construction we have :

(a) A ring homomorphism $\psi_{\mathcal{U}} : \mathcal{H} \to \mathcal{O}(\mathcal{C}(\mathcal{U}))$,

(b) A finite flat morphism[2] $\kappa_{\mathcal{U}} : \mathcal{C}(\mathcal{U}) \to \mathcal{V}$,

(c) $\psi_{\mathcal{U}}(U_p)$ is killed by $G^*(T)$ (Cayley-Hamilton), so that $\psi_{\mathcal{U}}(U_p)$ invertible (the constant coefficient of G^* is a unit), and $\nu_{\mathcal{U}} = (\kappa_{\mathcal{U}}, \psi_{\mathcal{U}}^{-1}) : \mathcal{C}(\mathcal{U}) \to \mathcal{U}$ is a finite morphism as well.

One easily checks that these $\mathcal{C}(\mathcal{U})$ together with the $\psi_{\mathcal{U}}$ and the $\nu_{\mathcal{U}}$ glue when $\mathcal{U}$ varies in the admissible covering $\mathcal{I}$: this defines a rigid analytic space called $\mathcal{C}$, admissible covered by the $\mathcal{C}(\mathcal{U})$, $\mathcal{U} \in \mathcal{I}$, plus a ψ and a ν satisfying (i), (ii), (a) and (b) of the statement!

This $\mathcal{C}$ is an equidimensionnal curve by (b) as so is $\mathcal{W}$. It can be shown to be reduced but even if we don't know this we can replace $\mathcal{C}$ constructed here by its nilreduction, which does not affect anything in (i), (ii), (a) and (b) before.

It only remains to define Z and show (iii). Note that (iii) now has a meaning, so we have a map

$$\mathcal{C}(\overline{\mathbb{Q}}_p) \to \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p),$$

$x \mapsto (\psi_x, \kappa(x))$. That map is easily seen to be injective by properties (i) and (ii) (note that any two points of $\mathcal{C}$ belong to a same open affinoid of the form $\nu^{-1}(\mathcal{V})$ s in (ii)). We can characterize its image :

LEMMA 1.3. *The image of* $\mathcal{C}(\overline{\mathbb{Q}}_p) \to \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathcal{W}(\overline{\mathbb{Q}}_p)$ *is exactly the set of elements of the form* (ψ_f, κ) *where* $f \neq 0 \in S_{m,\kappa}(N)$ *is of finite slope and an eigenform for all the Hecke operators and* $h(f) = \psi_f(h)f$ *for all* $h \in \mathcal{H}$.

Indeed, this follows by construction by applying the following lemma to $A = \mathcal{O}(\mathcal{V})$, $B = \mathcal{C}(\mathcal{U})$ and P as above :

LEMMA 1.4. *Let* A *be a noetherian commutative ring,* P *a projective finite type* A*-module and* $B \subset \mathrm{End}_A(P)$ *a commutative* A*-algebra. Then for any maximal ideal* m *of* A*, the natural morphism*

$$B/mB \to \mathrm{End}_{A/m}(P/mP)$$

has a nilpotent kernel.

Proof — By localizing at m we may assume that A is local and $P = A^n$. In this case the result follows from the following remark: if $M \in mM_n(A)$ then $M^n \in mA[M]$ by the Cayley-Hamilton identity. □

Recall that we have defined in the last lecture a $\mathbb{Q}_p \otimes \mathcal{H}$-module of $S_k^D(N,p)$ (actually denoted by $S_k^D(N,p)$ as well) which is a submodule of $S_{0,k}(N)$. As we shall see later, it is entirely made of finite slope elements (see prop 3.2). It follows that we can define Z as the inverse image of $\mathcal{Z}$ by the map given in (iii). However, we still have to show that Z defined this way is Zariski-dense and accumulation. We first show the accumulation property. It is enough to show property (c) of the statement.

Proof of (c): Fix some open affinoid of $\mathcal{C}$ of the form $\mathcal{C}(\mathcal{U})$ such that each connected component of $\kappa(\mathcal{U}) = \mathcal{V}$ meets $\mathbb{Z}$. It follows that for any $C > 0$, the

[2]Note that for any connected open affinoid $\mathcal{V} \subset \mathcal{W}$, then $\mathcal{O}(\mathcal{V})$ is a Dedekind ring. The flatness follows as $\mathcal{O}(\mathcal{C}(\mathcal{U}))$ is certainly torsion free over $\mathcal{O}(\mathcal{V})$.

integers $> C$ in $\mathcal{V}$ are Zariski-dense in $\mathcal{V}$ (why?). As $\mathcal{C}(\mathcal{U}) \to \mathcal{V}$ is finite flat it follows that the $x \in \mathcal{C}(\mathcal{U})$ such that $\kappa(x)$ is an integer $> C$ are Zariski-dense in $\mathcal{C}(\mathcal{U})$ for any real number C. Apply this to

$$C = \mathrm{Max}_{x \in \mathcal{C}(\mathcal{U})} v(\psi(U_p)(x)) + 1,$$

which exists by the maximum modulus principle applied to $\psi(U_p)^{-1}$ on $\mathcal{C}(\mathcal{U})$. By the "small slope forms are classical" result of the previous lecture, it follows that if $x \in \mathcal{C}(\mathcal{U})$ is such that $\kappa(x)$ is an integer $> C$, then any form $f \in S_{0,\kappa(x)}(N)$ such that $\psi_f = \psi_x$ has slope $v(\psi(U_p)(x)) < \kappa(x) - 1$ hence is in $S_k^D(N,p) : x \in Z$.

Let us finally prove that Z is Zariski-dense in $\mathcal{C}$. The proof uses the notion of irreducible components of a rigid analytic space (see Coleman-Mazur as well as the paper by B. Conrad with this title). An irreducible component of X is a closed subset which is irreducible, i.e. not the union of two proper closed subsets, and which is maximal with these properties for the inclusion. It is a fact that any rigid analytic space is set-theoretically the union of its irreducible components. It turns out that any $F \in 1 + T\mathcal{O}(\mathcal{W})\{\{T\}\}$ may be written as a convergent countable product

$$F = \prod_i F_i^{n_i}$$

where the $F_i \in 1 + T\mathcal{O}(W)\{\{T\}\}$ are irreducible (thus $\neq 1$), coprime, and $n_i \in \mathbb{N}$, and that the irreducible components of $Z(F)$ are then the $Z(F_i)$ (nilreduced).

By property (c), it is actually enough to show that any irreducible component W of $\mathcal{C}$ contains a point x such that $\kappa(x) \in \mathbb{Z}$. Indeed, if $\mathcal{U}$ is an affinoid neighborhood of such an x in $\mathcal{C}$ as in (c), then $W \cap \mathcal{U}$ is a union of irreducible components of $\mathcal{U}$ (a general fact), in each of which Z is Zariski-dense by (c). But $W \cap \mathcal{U}$ is an open affinoid of W, hence is Zariski-dense in W (a general fact again), and we are done.

As $\nu : \mathcal{C} \to Z(F)$ is componentwise surjective (this is a finite map between two spaces of the same equidimension 1), it is enough to check that any irreducible component $Z(F_i)$ of $Z(F)$ contains an element (κ, λ) with $\kappa \in \mathbb{Z}$. Write $F_i = 1 + T(\sum_{n \geq 0} a_n T^n)$ and choose j such that $a_j \neq 0$. Then a_j cannot vanish on the whole of $\mathbb{Z}$ as $\mathbb{Z}$ is Zariski-dense in $\mathcal{W}$, thus there is some $k \in \mathbb{Z}$ such that $a_j(k) \neq 0$. But then the power series $F_i(k) = 1 + T\mathbb{Q}_p\{\{T\}\}$ is not constant hence necessarily has a zero ! In other words $k \in \mathrm{pr}_1(Z(F_i))$ and we are done. □

2. The Galois pseudocharacter on the eigencurve and the infinite fern

Let S be the set of primes of $\mathbb{Q}$ dividing $Npd\infty$.

THEOREM 2.1. *There is a unique continuous 2-dimensional pseudo-character $T : G_{\mathbb{Q},S} \to \mathcal{O}(\mathcal{C})$ such that $T(\mathrm{Frob}_\ell) = \psi(T_\ell)$ for each $(\ell, Npd) = 1$.*

To give sense to this statement we need to say which topology we put on the global functions. If X is any reduced rigid analytic space then for any open affinoid $\mathcal{U} \subset X$ we may consider the semi-norm $|f|_{\mathcal{U}} = \sup_{x \in \mathcal{U}} |f(x)|$ on $\mathcal{O}(X)$. This collection of semi-norms defines a topology on $\mathcal{O}(X)$ ("topology of uniform convergence on all open affinoids"), for which it is complete. We set $\mathcal{O}(X)^{\leq 1} = \{f \in \mathcal{O}(X), |f(x)| \leq 1 \ \forall x \in X\}$.

Lemma 2.2. *$\mathcal{O}(\mathcal{C})^{\leq 1}$ is a compact subset of $\mathcal{O}(\mathcal{C})$ and contains $\psi(T_\ell)$ for each $(\ell, Npd) = 1$.*

Let us prove the theorem from this lemma. The idea is to deduce it from the existence of Galois representations attached to the classical quaternionic modular forms (i.e. the $z \in Z$) and from the Zariski-density of Z in $\mathcal{C}$. Note that for each $z = (\psi_f, k) \in Z$ we may consider the pseudo-character $T_z : G_{\mathbb{Q},S} \to \overline{\mathbb{Q}}_p$ defined by $T_z = \mathrm{trace}(\rho_{f,\iota})$. We can thus consider the continuous pseudo-character

$$T : G_{\mathbb{Q},S} \longrightarrow \prod_{z \in Z} \overline{\mathbb{Q}}_p, \ g \mapsto (T_z(g))_{z \in Z},$$

where the right-hand side is equipped with the product topology. Remark that the evatluation map $f \mapsto (f(z))_{z \in Z}$ induces a injection

$$j : \mathcal{O}(\mathcal{C})^{\leq 1} \to \prod_{z \in Z} \overline{\mathbb{Q}}_p$$

(as Z is Zariski-dense) which is a homeomorphism onto its image, as it is continuous and $\mathcal{O}(\mathcal{C})^{\leq 1}$ is compact. It is thus enough to show that $T(G_{\mathbb{Q},S}) \subset \mathrm{Im}(j)$. But

$$T_z(\mathrm{Frob}_\ell) = \psi(T_\ell)(z)$$

so that $T(\mathrm{Frob}_\ell) \in \mathrm{Im}(j)$. We conclude by Cebotarev's theorem as $\mathrm{Im}(j)$ is compact, hence closed. □

We now indicated a way to prove the lemma. That $\psi(T_\ell) \in \mathcal{O}(\mathcal{C})^{\leq 1}$ follows from the fact that T_ℓ has norm ≤ 1 on each $S_{m,\kappa}(N)$ and by lemma 1.3. To check the compacitness of $\mathcal{O}(\mathcal{C})^{\leq 1}$ one may invoke the following lemma, that we leave as an exercise (see e.g. my book with Bellaïche chapter 7).

Lemma 2.3. *Say that a rigid analytic space X is (strictly) nested if X is the admissible increasing union of open affinoids $X_n \subset X_{n+1}$ such that the restriction maps $\mathcal{O}(X_{n+1}) \to \mathcal{O}(X_n)$ are compact $\mathbb{Q}_p$-linear maps.*

- *If X is reduced and nested, then $\mathcal{O}(X)^{\leq 1}$ is a compact subset of $\mathcal{O}(X)$,*
- *$\mathcal{W}$, $\mathbb{G}_m$ and $\mathcal{W} \times \mathbb{G}_m$ are nested,*
- *If $X \to Y$ is a finite morphism with Y nested, then X is nested.*

Corollary 2.4. *Let $\kappa \in \mathcal{W}(L)$ and let $f \neq 0 \in S_{m,\kappa}(N) \otimes_L \overline{\mathbb{Q}}_p$ be a finite slope modular $\mathcal{H}$-eigenform, with eigenvalue a_ℓ under T_ℓ. Then there is a unique semisimple Galois representation $\rho_f : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that $\mathrm{trace}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell$ whenever $(\ell, Npd) = 1$.*

Proof — Indeed, define $T_x : G_{\mathbb{Q},S} \to \overline{\mathbb{Q}}_p$ by evaluating T at the point $x \in \mathcal{C}(\overline{\mathbb{Q}}_p)$ corresponding to f and define ρ_f as the unique semisimple representation whose trace is T_x. □

Let $\mathfrak{X}$ be the p-adic character variety of $G_{\mathbb{Q},S}$ in dimension 2. It is equipped with a universal pseudo-character $T^{\mathrm{univ}} : G_{\mathbb{Q},S} \to \mathcal{O}(\mathfrak{X})$ of dimension 2.

COROLLARY 2.5. *There is a unique morphism $\Phi : \mathcal{C} \to \mathfrak{X}$ such that $T = \Phi^* \circ T^{\mathrm{univ}}$. Its image contains all the $x \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ such that $\rho_x \simeq \rho_{f,\iota}$ where f is a quaternionic eigenform for D of level $K_1(N,p)$.*

Indeed, this follows from the universal property of $\mathcal{W}$ (this property was given for affinoids but it extends at once to any rigid space) and from the theorem above.

DEFINITION 2.6. *The infinite fern of quaternionic modular forms for D of tame level N is the set-theoretic image $\mathcal{F} = \mathrm{Im}(\Phi) \subset \mathfrak{X}$.*

At present, the global geometry of $\mathcal{C}$ is not well understood. Here is a famous open problem raised by Coleman-Mazur :

CONJECTURE 2.7. *$\mathcal{C}$ has only finitely many irreducible components. Equivalently, the power series F has only finitely many distinct prime factors in $1 + T\mathcal{O}(\mathcal{W})\{\{T\}\}$.*

Dispite the concrete nature of F, there is at present no triple (D, N, p) for which this conjecture is known (see however...). It is tempting to conjecture for instance that for $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, $N = 1$ and $p = 3$, then F is a prime.

It is not difficult to show however that F is not a polynomial (exercise) and that it is not divisible by the square of a prime in $1+T\mathcal{O}(\mathcal{W})\{\{T\}\}$. In geometric terms, $Z(F)$ is reduced and of infinite degree over $\mathcal{W}$.

3. Coleman's arcs and the Gouvêa-Mazur theorem

We keep the notations of the previous paragraphs.

3.1. Twin forms. On the (complex) space $S_k^D(N) = S_k(K_1(N))$ we have an action of two Hecke operators T_p and S_p, given by the respective double cosets $\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$ and $\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$, whose respective degrees are $p+1$ and 1.

On the (complex) space $S_k^D(N,p)$, we also have an action of two Hecke operators U_p and S_p, again given by the respective double cosets $I\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}I$ and $I\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}I$, whose respective degrees are now p and 1.

There are two ways to embed $S_k^D(N)$ into $S_k^D(N,p)$. Indeed, remark that the element

$$\delta = uw = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

normalizes I inside $\mathrm{GL}_2(\mathbb{Q}_p)$, thus it induces an automorphism of $S_k^D(N,p)$ by the formula $\delta(f)(x) = f(x\delta)$. In particular, we have a natural $\mathbb{C}$-linear map

$$j : S_k^D(N) \times S_k^D(N) \longrightarrow S_k^D(N,p), (f,g) \mapsto f + \delta(g).$$

If we set $\mathcal{H}' = \mathbb{Z}[\{T_\ell, (\ell, Ndp) = 1\}] \subset \mathcal{H}$, then j is $\mathcal{H}'[S_p]$ equivariant but we would like to understand how $U_p \circ j$ and T_p are related.

We shall obtain a nicer formula if we consider the conjugate $U_p^* := \delta U_p \delta^{-1}$ instead of U_p, which is also the Hecke operator on $S_k^D(N,p)$ associated to $[I \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} I]$.

PROPOSITION 3.2. (a) *The map j is injective unless $k = 2$ in which case its kernel is the 1-dim subspace $(f, -f)$ of constant functions.*

(b) *U_p^* preserves* $\mathrm{Im}(j)$ *and*

$$U_p^* \circ j = j \circ \begin{pmatrix} T_p & pS_p \\ -\mathrm{id} & 0 \end{pmatrix}.$$

(c) *The subspace of $f \in S_k^D(N,p)$ such that both f and $\delta(f)$ are in the kernel of the natural projection $S_k^D(N,p) \to S_k^D(N)$ is a complement of* $\mathrm{Im}(j)$ *stable by $\mathcal{H}'[S_p, U_p]$. On this complement we have $U_p^2 = S_p$.*

Proof — To check (a), remark that if (f,g) is in the kernel of j, then the function f is invariant on the right by the subgroup generated by $\mathrm{GL}_2(\mathbb{Z}_p)$ and $u\mathrm{GL}_2(\mathbb{Z}_p)u^{-1}$, and in particular by $uwu^{-1} = \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix}$. It follows that f is right-invariant by any standard unipotent elements and then by $\mathrm{SL}_2(\mathbb{Q}_p)$. By the strong approximation theorem, it follows that f factors through the norm $N : D_f^\times \to \mathbb{A}_f^\times$, and then that f is constant as $N(D^\times K_1(N)) = \mathbb{Q}_{>0}^* \widehat{\mathbb{Z}}^\times = \mathbb{A}_f^\times$.

Let us check (b). A set of representatives of $I \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} I/I$ is given by the $v_i = \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$ for $i = 0, \ldots, p-1$. A set of representatives of $\mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p)/\mathrm{GL}_2(\mathbb{Z}_p)$ is given by the same elements plus u. This gives the first row. For the second row remark that $p^{-1}v_i\delta = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$.

Part (c) is a similar computation left to the reader. □

The result above can be interpreted as follows. Fix some $f \in S_k^D(N)$ a nonzero common eigenform for $\mathcal{H}'$. Note that we are in level $K_1(N)$ here rather than $K_1(N,p)$. It is a fact (*multiplicity one theorem*) that such an f is automatically an eigenform for T_p and S_p as well. If we write $T_p(f) = a_p f$ and $S_p(f) = \varepsilon p^{k-2}$, we easily check from the definition that ε is necessarily a root of unity, and the same multiplicity one theorem ensures that a_p and ε only depend on the system of $\mathcal{H}'$-eigenvalues of f, and not really on f itself.

Unless f is constant,[3] the proposition above shows that the forms f and $\delta(f)$ generate a 2-dimensional subspace in $S_k^D(N,p)$ stable by U_p and under which U_p has the characteristic polynomial

$$X^2 - a_p X + \varepsilon p^{k-1}.$$

[3]When f is constant, then $k = 2$ and of course $U_p(f) = pf$.

The roots of this polynomial will be referred as *the p-Weil numbers of f*. Actually when f is non-constant, they are truly p-Weil numbers of weight $k-1$ by Deligne (in the usual sense).[4] For each choice α of p-Weil number, there is thus an $f_\alpha \in S_k^D(N,p)$ which is an $\mathcal{H}$-eigenform, with the same $\mathcal{H}'$-system of eigenvalue as f, and such that $U_p(f) = \alpha f$. The forms f_α, f_β obtained this way from the two possible p-Weil numbers are called the *twin forms associated to f*. Note that f_α and f_β correspond to two distinct points in $\mathcal{C}$ in general having the same Galois representation. Moreover, α, β are p-adic integers and that

$$v(\alpha) + v(\beta) = k - 1.$$

Define

$$Z^{\mathrm{old}} \subset Z$$

as the system of eigenvalues of $\mathcal{H}$ obtained this way from all the possible twin forms, and set $Z^{\mathrm{new}} = Z \backslash Z^{\mathrm{old}}$.

COROLLARY 3.3. *Z^{new} is a discrete subset of $\mathcal{C}$.*

Indeed, for $z \in Z^{\mathrm{new}}$, then $\kappa(z)$ is an integer ≥ 2 and $v(\psi(U_p)(z)) = \frac{\kappa(x)-2}{2}$ by the previous proposition part (c). But if $\mathcal{U} \subset \mathcal{C}$ is an open affinoid, then $x \mapsto v(\psi(U_p)(x))$ is bounded on $\mathcal{U}$ as $\psi(U_p)$ is invertible, thus $Z^{\mathrm{new}} \cap \mathcal{U}$ is finite as there are only finitely many systems of $\mathcal{H}$-eigenvalues on the $S_k^D(N)$ when k is bounded.

3.4. Coleman's arcs.

DEFINITION 3.5. *Say that $x \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ is a quaternionic modular point of level N for D, or for short that x is quaternionic modular, if $\rho_x \simeq \rho_{f,\iota}$ for some $\mathcal{H}'$-eigenform in $S_k^D(N)$. We also write x_f for the point associated to such an f.*

Fix $x_f \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ a quaternionic modular point of some weight k. Fix a p-Weil number α of f and consider the point

$$z \in Z^{old} \subset \mathcal{C}(\overline{\mathbb{Q}}_p)$$

associated to f_α. Choose some affinoid neighborhood of $\mathcal{U}$ of z in $\mathcal{C}$ as in property (b) of $\mathcal{C}$. We may choose $\mathcal{U}$ small enough so that $v(\psi(U_p)(x))$ is constant for $x \in \mathcal{U}$, hence and equal to its value $v(\alpha)$ at z. The image

$$C_{f,\alpha} = \Phi(\mathcal{U}) \subset \mathcal{F} \subset \mathfrak{X}$$

is called *a Coleman arc* associated to f_α. By definition, it contains $\Phi(z) = x_f$. It is not really canonical as the neigbourhood $\mathcal{U}$ chosen above was not, but its germ at x would be. In any case, the statements below will not depend on which $\mathcal{U}$ has been chosen to define $C_{f,\alpha}$.

LEMMA 3.6. *$C_{f,\alpha}$ is a locally closed one dimensional subspace of $\mathfrak{X}$ and the map $\Phi : \mathcal{U} \to C_{f,\alpha}$ is finite.*

Proof — This may be seen as follows. First, there is a natural morphism

$$\det : \mathfrak{X} \to \mathcal{W}$$

[4] That is, algebraic numbers whose absolute value is $p^{(k-1)/2}$ in any complex embedding.

obtained as follows. If T_A is an A-point of $\mathfrak{X}(A)$, we may define its determinant $\det(T_A) = \frac{T_A(g)^2 - T_A(g^2)}{2}$. This defines a rigid analytic morphism from $\mathfrak{X}$ to the p-adic character variety of $\mathrm{G}_{\mathbb{Q},S}^{\mathrm{ab}}$ in dimension 1. By the natural restriction morphism $\mathbb{Z}_p^\times \to G_{\mathbb{Q},S}^{\mathrm{ab}}$ at p, we obtain a natural morphism $\det : \mathfrak{X} \to \mathcal{W}$. It turns out that on $\mathcal{C}$ the two natural morphisms $\det \cdot \Phi$ and κ are essentially the same : for each $x \in \mathcal{C}$, $\det(\rho_x)_{|\mathbb{Z}_p^\times}$ is the character $\kappa(x)$ multiplied by the cyclotomic character. Indeed, this is true when $x \in Z$ and follows in general by Zariski-density of Z in $\mathcal{C}$.

Now, set $\mathcal{V} = \kappa(\mathcal{U}) \subset \mathcal{W}$ (an open affinoid) and define $\mathfrak{X}_\mathcal{V}$ as the admissible open subspace of $\mathfrak{X}$ such that $\det(\rho_x)(-1) \in \mathcal{V}$. It follows that the composite morphism

$$\mathcal{U} \xrightarrow{\Phi} \mathfrak{X}_\mathcal{V} \xrightarrow{\det(-1)} \mathcal{V}$$

is the morphism κ. As $\kappa_{|\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is finite flat, it follows that $\Phi : \mathcal{U} \to \mathfrak{X}_\mathcal{V}$ is finite as well, so $C_{f,\alpha} = \Phi(\mathcal{U})$ is closed in $\mathfrak{X}_\mathcal{V}$, and $C_{f,\alpha}$ is finite flat over $\mathcal{V}$ (via the determinant). □

LEMMA 3.7. (i) *The quaternionic modular points are Zariski-dense and accumulation in $C_{f,\alpha}$.*

(ii) *Assume that $v(\alpha), v(\beta)$ are distinct, $\neq \frac{k-2}{2}$ and $< k-1$. Then $C_{f,\alpha} \cap C_{f,\beta}$ contains only finitely many modular points (including x_f).*

Proof — Indeed, the first part follows from the similar property for $\mathcal{U}$ (property (b) and (c) of the eigencurve) and corollary 3.3.

For the second, let $y \in C_{f,\alpha} \cap C_{f,\beta}$ be a quaternionic modular point, so $y = x_{f'}$ for some eigenform $f' \in S_{k'}(N)$ of some weight k'. It follows that f' has a p-Weil number α' of valuation $v(\alpha)$ and another β' of valuation $v(\beta)$ (this is actually a little more subtle to show than it may seem : why?). It follows that they are exactly the two p-Weil numbers of f' so that

$$k' - 1 = v(\alpha') + v(\beta') = v(\alpha) + v(\beta) = k - 1.$$

In particular, $k = k'$: f' has the same weight as f. We conclude the proof as $S_k^D(N)$ is finite dimensional. □

LEMMA 3.8. *If $\Omega \subset \mathfrak{X}$ is an open affinoid containing a quaternionic modular point, then the Zariski-closure of the modular points in Ω has dimension ≥ 2.*

Proof — By choosing the Coleman arcs small enough then Ω will contain the following subset (give a picture).

Although it is intuitively clear that an affinoid of dimension 1 cannot contain such a subset, it is not so trivial to prove it. Here is an argument. Assume by absurdum that the Zariski-closure $X \subset \Omega$ of the modular points has dimension 1. The singular locus of a reduced 1-dim. affinoid being finite, it follows that X contains a modular smooth point $x_f \in X$. Two small enough Coleman's arcs $C_{f,\alpha}$ and $C_{f,\beta}$ are in Ω, hence in X as Z is Zariski-dense in $\mathcal{U}$ (for α and β). As x_f is smooth they both have to contain an affinoid neighborhood of x_f in X. Such an affinoid will contain infinitely many modular points by the accumulation property of Z, which contradicts the previous lemma.

□

Define an essentially modular point in $\mathfrak{X}$ as a point y such that $\rho_y \simeq \rho_x(m)$ where x is modular and $m \in \mathbb{Z}$.

THEOREM 3.9. *If $\Omega \subset \mathfrak{X}$ is an open affinoid containing at least one modular point, then the Zariski-closure of the essentially modular points in Ω has dimension ≥ 3.*

Proof — For this step we need some result of p-adic Hodge theory. A theorem of Sen defines a polynomial

$$P = T^2 + aT + b \in \mathcal{O}(\mathfrak{X})[T]$$

such that for all $x \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, the Sen polynomial of the p-adic Galois representation $\rho_{x|\mathrm{G}_{\mathbb{Q}_p}}$ is $P(x) = T^2 + a(x)T + b(x)$. A result of Faltings-Jordan asserts that if x is modular with $\rho_x \simeq \rho_f$ for $f \in S_k^D(N)$, then $P(x) = T(T + k - 1)$. In particular, each modular point in $\mathfrak{X}$ belongs to the closed subset $b = 0$ ("Sen null space").

On the other hand, there is a natural morphism $\mathcal{W} \times \mathfrak{X} \to \mathfrak{X}$ given on the moduli interpretation by

$$(\kappa_A, T_A) \mapsto (T_A \otimes \kappa_A)(g) = T_A(g)\kappa_A(\chi(g)),$$

where $\chi : G_{\mathbb{Q},S} \to \mathbb{Z}_p^\times$ is the cyclotomic character. If $\kappa \in \mathcal{W}(L)$, on checks that $P(\kappa.x) = P(x - s)$ where $s \in L$ is the derivative at 1 of κ. It follows that the Sen-null space $b = 0$ has everywhere codimension 1 in $\mathfrak{X}$. The theorem follows then from the previous lemma, plus the fact that $\mathbb{Z}$ is Zarski-dense in $\mathcal{W}$. □

COROLLARY 3.10. *If $\bar{\rho}$ is regular and $\mathfrak{X}(\bar{\rho})$ contains a modular point, then the essentially-modular points are Zariski-dense in $\mathfrak{X}(\bar{\rho})$.*

Indeed, if $\bar{\rho}$ is regular then we have seen that $\mathfrak{X}(\bar{\rho})$ is irreducible of dimension 3 (it is even an open unit ball).

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– Mazur "An infinite fern in the deformation space of 2-dim Galois representations",

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– Chenevier "The infinite fern of Galois representations of unitary type".

See also the works of Buzzard, Calegari and others on the p=2 tame level 1 eigencurve, as well as my paper "Quelques courbes de Hecke se plongent dans l'espace de Colmez".