(with Laurent Clozel) **Number fields with given ramification and self-dual automorphic forms for $GL(2n)$**

(I) Arithmetic problem

S finite set of primes, $\mathbb{Q}_S \subset \overline{\mathbb{Q}}$ max. alg. unramified outside S

Fix $p \in S$, $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \Rightarrow \quad Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow Gal(\mathbb{Q}_S/\mathbb{Q}) = G_S$

Theorem A: if $|S| \geq 2$, this map is injective.

"there are so many n. fields unr. outside S that they generate $\overline{\mathbb{Q}}_p$"

Cor: $|G_S| = |\hat{\mathbb{Z}}|$

Idea to attack thm: Use numb. fields coming from Galois action on ℓ-adic cohom. of unitary Shimura varieties that we control well by work of Kottwitz, Clozel, Harris-Taylor. We are reduced to construct coh. classes $\Leftrightarrow$ aut. forms which have prescribed prop. (analytic pb.)

Rmk:
* Ch 04', theorem holds if $S \ni \ell \neq p$, $\ell \equiv 3\ (4)$ and $\left(\frac{-\ell}{p}\right) = 1$
* it suffices to consider $S = \{\ell, p\}$.
* mention Greenberg, elementary approach?

(II) Aut. forms

Fix ω supercuspidal rep. of $GL_n(\mathbb{Q}_p)$, $\omega^\vee \not\simeq \omega\chi \quad \forall \chi$ unram.

Fix $\ell \neq p$. Denote $\varphi(\omega) =$ Bernstein comp. of $GL_{2n}(\mathbb{Q}_p)$ of $\omega \times \omega^\vee$ $\left\{ I(\chi) = \mathrm{Ind}_{(n,n)}^{GL_{2n}} \omega\chi \times \omega^\vee \chi^{-1} \right\}$

Theorem B: $\exists \Pi$ cusp. aut. form on $GL_{2n}(\mathbb{A}_\mathbb{Q})$,

i) $\Pi^\vee \simeq \Pi$

ii) Π_∞ alg. regular

iii) Π_ℓ Steinberg

iv) $\Pi_p \in \varphi(\omega)$

v) Π unram. outside $\{\ell, p, \infty\}$

Rmks ① Rigidity: $\pi^\vee \simeq \pi$ and control at every place.

$\triangle$ we can't prescribe anything, e.g. we can't produce autodual glob cusp with π_q supercusp. alt. and $\pi_{q'}$ supercusp. symplectic!

(ii) thm B $\Rightarrow$ thm A using HT theorem (ℓ-adic real.$^{\forall m, \forall \omega}$) and some elementary steps.

(iii) In my previous work, I used unitary aut. forms, here $\Rightarrow$ $Gl_m/\mathbb{Q}$.

Our proof will use Arthur's trace form. for $G = Gl_{2n}/\mathbb{Q}$ twisted by. θ

$$\theta(g) = J\,{}^t g^{-1} J^{-1}, \quad \theta^2 = 1, \quad \theta \text{ preserves } T, B, \text{ epinglage.}$$

$$tr(I_\theta R(f)), \quad R = G \circlearrowleft L^2_{cusp}(\mathbb{R}^{*} G(\mathbb{Q}) \backslash G(\mathbb{A}))$$

($\rightarrow$ Need some results that might be contained in Arthur's book at the end) to prove

III Test functions f_∞

We are interested in following rep. of $Gl_{2n}(\mathbb{R})$

$$\left\{ \phi : W_\mathbb{R} \to Gl_{2n}(\mathbb{C}) = \bigoplus_{i=1}^{n} \begin{pmatrix} (z/\bar z)^{p_i} & \\ & (z/\bar z)^{-p_i} \end{pmatrix} \text{ on } W_\mathbb{R} \right\} \longrightarrow \left\{ \phi : W_\mathbb{R} \to Sp_{2n}(\mathbb{C}) \text{ discrete} \right\} \leftrightarrow \left\{ \text{L-paquet of discrete series on } SO^{*}_{2n+1}(\mathbb{R}) \right\}$$

$p_i \in \frac{1}{2} + \mathbb{Z}$, 2 by 2 $\neq$

$\hookrightarrow$ cohom., θ-discrete

$$\{\pi_\lambda\} \longleftarrow \{V_\lambda, \lambda\} \text{ irred. rep. of } SO_{2n+1}(\mathbb{R})$$

Mezo $\exists f_\lambda$ twisted pseudo coeff for π_λ : trace $(A_\theta R(f)) = 0$ if $\rho \not\simeq \pi_\lambda$ tempered, $= 1$ $\rho \simeq \pi_\lambda$.

choice of A_θ: if $\pi^\vee \simeq \pi \Leftrightarrow \pi \simeq \pi \circ \theta$, $\exists A_\theta : \pi \to \pi \circ \theta$, $A_\theta^2 = 1$ determined up to a sign. if π generic, I choose the one which acts by $+1$ on W.f.

(stable conj. classes

Norm map in this context (Waldspurger)

$$\gamma \longrightarrow N\gamma \qquad (\sim \gamma^\theta \gamma)$$

$$\theta\text{-ell} \rightsquigarrow N\gamma \text{ conj. class in } SO(2n+1)(\mathbb{R})$$

<u>Thm C</u> γ θ-simple, $TO_\gamma(f_\lambda) = \begin{cases} \varepsilon(\gamma)\ \mathrm{tr}(N\gamma \mid V_\lambda) & \text{if } \gamma \text{ ellipt.} \\ 0 & \text{else.} \end{cases}$

(for some choice of >0 Haar measure).

Pf. Use results of Bouaziz, Labesse.

(IV) <u>test</u> <u>function at</u> <u>finite primes</u>

- unit of spherical Hecke algebra outside ℓ, p.
- At ℓ, f_ℓ = twisted pseudo coeff. of St_ℓ (via Kottwitz method, EP functions)
- <u>At p</u>. Recall $C_p(\omega)$

<u>positive</u> <u>pseudocoefficients</u> f_p : $\begin{cases} \mathrm{tr}(A_\theta\, \rho(f_p)) = 0 & \rho \notin C_p(\omega) \\ \mathrm{tr}(A_\theta\, I(x)(f_p)) > 0 & I(x) \text{ unitary} \end{cases}$

<u>Thm D</u>. Fix f_p as above (it exists)

(i) $\gamma \longrightarrow STO_\gamma(f_p)$ (on st reg. el.)
is not $\equiv 0$

(ii) $TO_{\gamma_0}(f_p) = STO_{\gamma_0}(f_p) > 0$

$\left\{ \begin{array}{l} \gamma_0 \text{ is the unique } \theta\text{-conj. class with Norm } 1. \\ \ \shortparallel \\ \ 1 \quad . \text{ Its } \theta\text{-centralizer is } Sp_{2n},\ \theta\text{-conj } \gamma_0 = \text{st } \theta\text{-conj } \gamma_0 \\ \qquad\qquad H^1(\mathbb{Q}_p, Sp_{2n}) = 0. \end{array} \right\}$

<u>Rmk</u>: In agreement with results of Shahidi when f_p is a coeff. of a selfdual supercuspidal π_p. $(f_p(1) \neq 0)$
$TO_{\gamma_0}(f_p) \neq 0$ related to symplectic/orthogonal alternative.

V Pf of Thm B

ATF, $f = f_\lambda \otimes f_\ell \otimes f_p \otimes f^{\infty,\ell,p}$ just defined, $\lambda \xrightarrow[\text{off the walls}]{} \infty$

$$\mathrm{tr}(I_\theta R(f)) = \sum_{\gamma \in G(\mathbb{Q})_{ell}} v_\gamma \, \underset{\substack{\text{" (thm C)} \\ \mathrm{tr}(N\gamma | V_\lambda)\, TO_\gamma(f^\infty)}}{TO_\gamma(f)}, \quad \left\{ \begin{matrix} \text{the sum is} \\ \text{finite indp.} \\ \text{of } \lambda \end{matrix} \right\}.$$

Prop: G is a compact connected Lie group $(T, X^*(T), \lambda; V_\lambda)$

$\gamma \in G$ non central $\Rightarrow \dfrac{\mathrm{tr}(\gamma | V_\lambda)}{\dim V_\lambda} \xrightarrow[\lambda \to \infty]{} 0$

⊕ properties of γ_0 (unique class $N\gamma = 1$) $\Rightarrow \dfrac{\mathrm{tr}(I_\theta R(f))}{\dim V_\lambda} \longrightarrow v_{\gamma_0} TO_{\gamma_0}(f^\infty)$.

$\neq 0$ by Thm D (and some work at ℓ)

Rmk: Serre GL_2, ES formula.

VI Pf Thm D (ii). $(TO_{\gamma_0}(f_p) \neq 0)$ local statement but global pf

Step I fix λ_0 (e.g. 0)

Using ST.F (KS) show that some π_0 exists as in Thm B but maybe ramified outside $\{\infty, \ell, p\}$.
(uses thm D (i)) at say S'

Step II positivity argument, W.H. normalisation.
define ≥ 0 pseudo coeff. fns $\pi_{0S'}$ say $f_{S'}$, and apply

ATF:

$$\sum_{\gamma \in G(\mathbb{Q})_{ell}} v_\gamma \, TO_\gamma(f) \geq 0 \quad \forall \lambda.$$
$$> 0 \quad , \lambda = \lambda_0.$$

Prop. G compact Lie group, $\gamma_1 = 1, \gamma_2 \dots \gamma_n$ 2 by 2 $\neq$ conj. classes in G
$\lambda_1 \dots \lambda_n \in \mathbb{C}$ complex numbers.

if $\left| \sum_{i=1}^n \lambda_i \, \mathrm{tr}(\rho(\gamma_i)) \geq 0 \ \forall \rho \in \hat{G}, > 0 \ \rho = \rho_0 \right. \Longrightarrow \lambda_1 > 0$.