

Kneser neighbours and orthogonal Galois representations in dimensions 16 and 24

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(joint work with Jean Lannes)

Let $n \geq 1$ be an integer. Recall that an even unimodular lattice in the standard euclidean space \mathbb{R}^n is a lattice $L \subset \mathbb{R}^n$ of covolume 1 with $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$. Let X_n denote the set of isometry classes of even unimodular lattices in \mathbb{R}^n . As is well-known, X_n is a finite set which is non-empty if and only if $n \equiv 0 \pmod{8}$. For example, the lattice

$$E_n = D_n + \mathbb{Z} \frac{e_1 + \dots + e_n}{2},$$

$\{e_1, \dots, e_n\}$ denoting the canonical basis of \mathbb{R}^n and D_n the sublattice of index 2 in \mathbb{Z}^n whose elements (x_i) satisfy $\sum_i x_i \equiv 0 \pmod{2}$, is even unimodular for $n \equiv 0 \pmod{8}$.

The set X_n has been determined in only three cases. One has $X_8 = \{E_8\}$ (Mordell), $X_{16} = \{E_8 \oplus E_8, E_{16}\}$ (Witt) and Niemeier showed that X_{24} has 24 explicit elements (see [V]). The number of numerical coincidences related to Niemeier's lists is quite extraordinary and makes that list still mysterious. For the other values of n the Minkowski-Siegel-Smith mass formula shows that X_n is huge, perhaps impossible to describe. For instance, X_{32} already has more than $80 \cdot 10^6$ elements ([S]).

Let $L \subset \mathbb{R}^n$ be an even unimodular lattice, and let p be a prime; Kneser defines a p -neighbour of L as an even unimodular lattice $M \subset \mathbb{R}^n$ such that $M \cap L$ has index p in L (hence in M). The relation of being p -neighbours turns X_n into a graph which was shown to be connected by Kneser, providing a theoretical way to compute X_n from the single lattice E_n . This is actually the way Kneser and Niemeier computed X_n for $n \leq 24$, using the prime $p = 2$ and the huge number of symmetries present in those cases.

In this paper, we are interested in giving an explicit formula for the number $N_p(L, M)$ of p -neighbours of L which are isometric to M . Equivalently, it amounts to determining the \mathbb{Z} -linear operator $T_p : \mathbb{Z}[X_n] \rightarrow \mathbb{Z}[X_n]$ defined by $T_p[L] = \sum [N]$, $[-]$ denoting the isometry class of a lattice, the summation being over all the p -neighbours of L .

Before stating our main results, let us mention that the p -neighbours of a given even unimodular lattice L are in canonical bijection with the \mathbb{F}_p -points of the projective quadric C_L over \mathbb{Z} defined by the quadratic form $x \mapsto \frac{x \cdot x}{2}$ on L . The quadric C_L is hyperbolic over \mathbb{F}_p for each prime p , thus L has exactly

$$c_n(p) = |C_L(\mathbb{F}_p)| = 1 + p + p^2 + \dots + p^{n-2} + p^{n/2-1}$$

p -neighbours, where $n = \text{rk}_{\mathbb{Z}} L$. We are thus interested in the partition of the quadric $C_L(\mathbb{F}_p)$ into $|X_n|$ parts (some of them being possibly empty) given by the isometry classes. Of course, $N_p(E_8, E_8) = c_8(p)$ as $X_8 = \{E_8\}$, thus the first interesting case (perhaps known to specialists!) is $n = 16$.

Theorem A : *Let $n = 16$. In the basis $E_8 \oplus E_8, E_{16}$ the matrix of T_p is*

$$c_{16}(p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 + p + p^2 + p^3) \frac{1 + p^{11} - \tau(p)}{691} \begin{bmatrix} -405 & 286 \\ 405 & -286 \end{bmatrix},$$

$\sum_{n \geq 1} \tau(n)q^n = q \prod_{n \geq 1} (1 - q^n)^{24}$ denoting Ramanujan's Δ function.

One first makes two well-known observations. First, the operators T_p commute with each others. Second, they are self-adjoint for the scalar product defined by $\langle [L], [M] \rangle = \delta_{[L],[M]} |O(L)|$, which amounts to saying that for all $L, M \in X_n$ we have

$$(1) \quad N_p(L, M) |O(M)| = N_p(M, L) |O(L)|.$$

Our main question (for any n) is thus equivalent to first finding a basis of $\mathbb{R}[X_n]$ made of eigenvectors common to all of the T_p operators, and then to describing the system of eigenvalues (λ_p) of the (T_p) on each of these eigenvectors. If $n = 16$ it is not difficult to compute T_2 , and this was essentially done by Borchers for $n = 24$ (see [N-V]). In both cases, the eigenvalues of T_2 are distinct integers (this was noticed by Nebe and Venkov [N-V] for $n = 24$). Let us mention the important presence of $(c_n(p))$ as "trivial" system of eigenvalues : formula (1) shows that $\sum_{L \in X_n} [L] |O(L)|^{-1} \in \mathbb{Q}[X_n]$ is an eigenvector for T_p with eigenvalue $c_n(p)$.

Assume now $n = 16$. The non-trivial system of eigenvalues is related to Ramanujan's Δ -function in a non-trivial way. Our proof relies on Siegel theta series

$$\vartheta_g : \mathbb{Z}[X_n] \rightarrow M_{\frac{n}{2}}(\mathrm{Sp}_{2g}(\mathbb{Z})),$$

the latter space being the space of classical Siegel modular forms of weight $n/2$ and genus g . The generalized Eichler commutation relation ([R], [W]) asserts that ϑ_g intertwines T_p with some explicit Hecke operator on the space of Siegel modular forms. By a classical result of Witt, Kneser and Igusa (see [K]),

$$\vartheta_g(E_8 \oplus E_8) = \vartheta_g(E_{16}) \quad \text{if } g \leq 3,$$

whereas $F = \vartheta_4(E_8 \oplus E_8) - \vartheta_4(E_{16})$ does not vanish, thus the T_p -eigenvalue we are looking for is related to the Hecke eigenvalues of $F \in S_8(\mathrm{Sp}_8(\mathbb{Z}))$. A result by Poor and Yuen [P-Y] asserts that the latter space is 1-dimensional (generated by the famous Schottky form!). But another non-trivial member of this space is Ikeda's lift of Ramanujan's Δ function (see [I]), whose Hecke eigenvalues are explicitly given in terms of Δ . By unravelling the precise formulae we obtain Theorem A. Actually, we found a direct proof of the existence of Ikeda's lift of Δ that relies on the triality for the reductive group $\mathrm{PGO}_{E_8}^+$ over \mathbb{Z} and two theta series constructions.

Pre-Theorem* B : *There is an explicit formula as well for T_p if $n = 24$.*

In this case it is more difficult to find the non trivial systems of eigenvalues on $\mathbb{Q}[X_{24}]$. Five of them were actually identified as Ikeda lifts in the work of Nebe and Venkov [N-V], with a particular one due to Borchers-Freitag-Weissauer. We rather rely on Chapter 9 of the book [A] by Arthur ... which is still missing at

the moment; hence the * in the statement above. The relation with automorphic forms comes from the canonical identification $X_n = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / G(\widehat{\mathbb{Z}})$ where G is the \mathbb{Z} -orthogonal group of E_n , so that $\mathbb{C}[X_n]$ is canonically the dual of the space of automorphic forms of G of level 1 and trivial coefficients. The quickest way (although perhaps inappropriate!) to state our results is in terms of Galois representations.

Fix a prime ℓ . Thanks to the works of many authors (including [A]), for any system of eigenvalues $\pi = (\lambda_p)$ of (T_p) on $\overline{\mathbb{Q}}_\ell[X_n]$, there exists a unique continuous, semi-simple representation

$$\rho_{\pi, \ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GO}(n, \overline{\mathbb{Q}}_\ell)$$

which is unramified outside ℓ and such that $\text{Trace}(\rho_{\pi, \ell}(\text{Frob}_p)) = \lambda_p$ for each prime $p \neq \ell$. This Galois representation is furthermore crystalline at ℓ with Hodge-Tate numbers $0, 1, \dots, n-2$, the number $n/2 - 1$ occurring twice.

For example, if $\pi = (c_n(p))$ then $\rho_{\pi, \ell} = (\oplus_{i=0}^{n-2} \omega^i) \oplus \omega^{n/2-1}$, where ω is the ℓ -adic cyclotomic character. Moreover, Theorem A implies that if π is the non-trivial system of eigenvalues when $n = 16$, then $\rho_{\pi, \ell} = \rho_{\Delta, \ell} \otimes (\oplus_{i=0}^3 \omega^i) \oplus \omega^7 \oplus (\oplus_{i=0}^6 \omega^i) \otimes \omega^4$, where $\rho_{\Delta, \ell}$ is Deligne's Galois representation attached to Δ .

Consider now the following collection of orthogonal ℓ -adic Galois representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with similitude factor ω^{22} and distinct Hodge-Tate numbers in $\{0, 2, \dots, 22\}$. For r odd and $1 \leq r \leq 23$, set

$$(2) \quad [r] = (\oplus_{i=0}^{r-1} \omega^i) \otimes \omega^{\frac{23-r}{2}}.$$

For r even, $k \in \{12, 16, 18, 20, 22\}$, $k+r \leq 24$, set

$$(3) \quad \Delta_k[r] = \rho_{\Delta_k, \ell} \otimes (\oplus_{i=0}^{r-1} \omega^i) \otimes \omega^{\frac{24-(k+r)}{2}},$$

where Δ_k is a generator of $S_k(\text{SL}_2(\mathbb{Z}))$. Set

$$(4) \quad \text{Sym}^2 \Delta = \text{Sym}^2 \rho_{\Delta, \ell}.$$

For $(j, k) \in \{(6, 8), (4, 10), (8, 8), (12, 6)\}$ then the space of vector-valued Siegel modular cusp forms of genus 2 (for $\text{Sp}_4(\mathbb{Z})$) and coefficient $\text{Sym}^j \otimes \det^k$ has dimension one (Tsushima, see [VdG]). Let $\Delta_{j,k}$ be a generator and let $\rho_{\Delta_{j,k}, \ell}$ be its associated 4-dimensional ℓ -adic Galois representation (Weissauer); it is symplectic with Hodge-Tate numbers $0, k-2, j+k-1$, and $j+2k-3$. Set

$$(5) \quad \Delta_{j,k}[2] = \rho_{\Delta_{j,k}, \ell} \otimes (1 \oplus \omega) \otimes \omega^{\frac{24-2k-j}{2}}.$$

Fact: *There are exactly 24 representations of dimension 24 which are direct sums of representations in the list (2)–(5) above and whose Hodge-Tate numbers are $0, 1, \dots, 22$ with 11 occurring twice.*

For instance, $\text{Sym}^2 \Delta \oplus \Delta_{20}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [9]$, $\Delta_{4,10}[2] \oplus \Delta_{18}[2] \oplus \Delta[4] \oplus [3] \oplus [1]$, and $\text{Sym}^2 \Delta \oplus \Delta_{6,8}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [5]$ are three of them. A more precise form of pre-Theorem B (still conditionnal to Arthur's results) is then the following.

Theorem* B : *The Galois representations attached to the 24 systems of eigenvalues π occuring in $\mathbb{Q}[X_{24}]$ are exactly the Galois representations above.*

Thanks to this list, the only unknown to determine T_p on $\mathbb{Z}[X_{24}]$ are the Hecke eigenvalues of the four Siegel cusp forms $\Delta_{j,k}$. They have actually been computed by Van der Geer and Faber for all $p \leq 11$, and even for some of them up to $p = 37$: see [VdG]. We checked that their results are consistant with Borcherds's computation of T_2 . Better yet, we directly proved that for any even unimodular root lattice $L \subset \mathbb{R}^{24}$ with Coxeter number $h(L)$ we have

$$N_p(\text{Leech}, L) = 0 \text{ if } p < h(L).$$

In particular, if $L \in \{E_{24}, E_8^3, E_8 \oplus E_{16}, A_{24}^+\}$ and $p \leq 23$ then $N_p(\text{Leech}, L) = 0$. This simple fact allowed us to confirm all the eigenvalues given in the table of Van der Geer and Faber for $p \leq 23$ and to compute¹ T_p for those p . Using the Ramanujan estimates for the Δ_k and $\Delta_{j,k}$, and with the convention $h(\text{Leech}) = 1$, we obtain for instance :

Corollary* : *Assume $p \geq 11$. If L and M are two even unimodular lattices in \mathbb{R}^{24} with Coxeter numbers $h(L) \geq h(M)$, then $N_p(L, M) \geq 1$ if, and only if, $p \geq h(L)/h(M)$.*

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¹See <http://www.math.polytechnique.fr/~chenevier/niemeier/niemeier.html> for tables.