Giorno 4

09-11-2022

exercice 16

$\omega \in \mathbb{C}$ $\quad \mathrm{Im}(\omega) > 0$

$\Lambda = \mathbb{Z} + \mathbb{Z}\omega$ réseau dans $\mathbb{C}$

on considère $S = \mathbb{C}/\Lambda$ est une surface de Riemann compacte.

Thm $\varphi : S - \{0\} \to \mathbb{C}^2$ hol.

et $P(x,y) = y^2 - (x^3 + Ax + B)$ tq

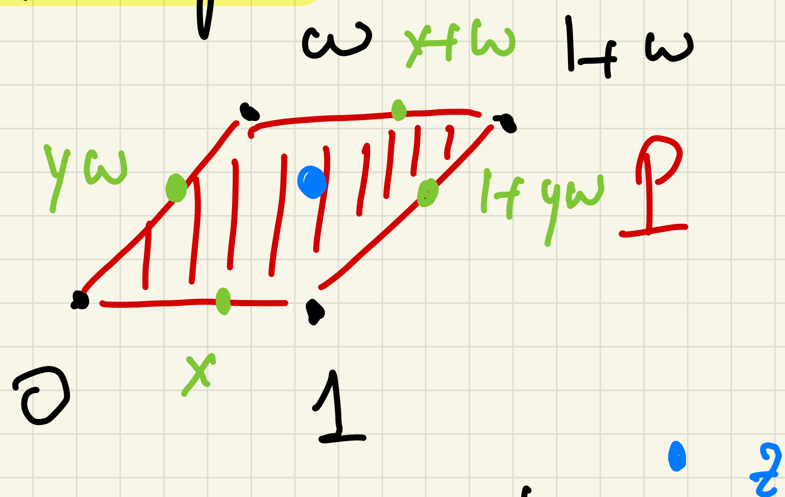
- $\{P = 0\} \cap \left\{\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0\right\} = \emptyset$
- $\varphi : S \setminus \{0\} \to \{P = 0\}$ est un biholomorphisme

pour démontrer le thm, on va étudier les fonctions elliptiques

$f : \mathbb{C} \to \mathbb{C}$ tq $f(z+\lambda) = f(z)$

pour tout $z \in \mathbb{C}$, $\lambda \in \Lambda$

f méromorphe



$z = x + y\omega$ $\quad x, y \notin \mathbb{Z}$

$\exists z' \in \mathrm{Int}(P)$ $\quad z - z' \in \Lambda$.

On a $z \in \mathbb{C} \setminus \Lambda$ $\exists \varepsilon > 0$

$|z - \lambda| \geq \varepsilon |\lambda| \quad \forall \lambda \in \Lambda.$

- Λ est discret. donc

$\Lambda \cap D(z, R)$ est fini

$\{\lambda_1 \dots \lambda_n\}$ $\quad |z - \lambda_i| \geq \varepsilon |\lambda_i|$

$z \notin \Lambda \quad |z - \lambda_i| > 0 \quad \lambda_i \neq 0.$

- $\lambda \in \Lambda \setminus D(z, R)$

$|z - \lambda| = |\lambda| \left| 1 - \frac{z}{\lambda} \right| \geq \frac{1}{2} |\lambda|.$

on choisit R pour que

$|z - \lambda| > R \Rightarrow \left|\frac{z}{\lambda}\right| \leq 1/2$

$$\varphi(z) = \wp^{\varepsilon}(z+v) - \wp^{\varepsilon}(z) \equiv \text{cst}$$

en 0 $\quad \wp^{\varepsilon}(z) = \dfrac{1}{z^2} + \underbrace{\left(\sum_{\lambda \neq 0} \dfrac{1}{(z-\lambda)^2} - \dfrac{1}{\lambda^2}\right)}_{\text{hol}} = g$

$$\wp^{\varepsilon}(z+v) = \frac{1}{(z+v)^2} + \sum_{\lambda \neq 0} \frac{1}{(z+v-\lambda)^2} - \frac{1}{\lambda^2}$$

en 0

$$= \frac{1}{(z+v)^2} + \underbrace{\boxed{\sum_{\substack{\lambda \neq 0 \\ \neq v}} \frac{1}{(z+v-\lambda)^2} - \frac{1}{\lambda^2}}}_{h} + \frac{1}{z^2} - \frac{1}{v^2}$$

$g(0) = 0$

$h(0) = 0$

$$\underset{\displaystyle 0}{\underset{\|}{\wp^{\varepsilon}(z+v) - \wp^{\varepsilon}(z)}} = \cancel{\frac{1}{z^2}} - \frac{1}{v^2} + \frac{1}{(z+v)^2} - h(z) - \cancel{\frac{1}{z^2}} - g(z)$$

d) $f : \mathbb{C} \to \mathbb{C}$ $f(z+\lambda) = f(z)$

$\underline{P} = [0,1] + [0,1]\omega$

$P \cap$ Pôle de $(f) \cup$ Zéro de $(f) \subseteq \mathrm{Int}(P)$

Lemme $\boxed{\sum_{p \in \underline{P}} \mathrm{Res}_p(f) = 0}$

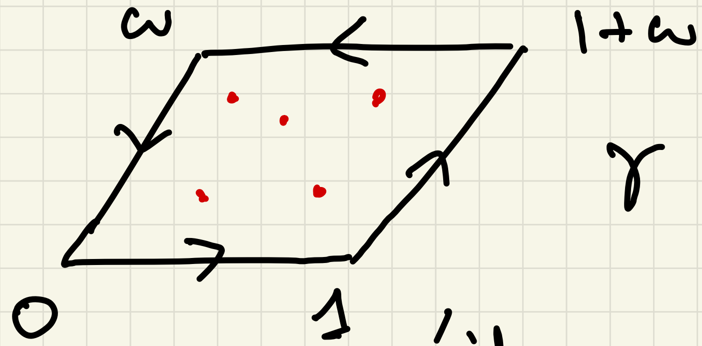
$\{z_i\}$ pôles et zéro de f dans $\underline{P}$

$\mathrm{mult}_{z_i}(f) = m_i$

$\boxed{\sum m_i = 0 \qquad \sum m_i z_i \in \underline{\Lambda}}$

Formule de Cauchy

des résidus



$$\frac{1}{2i\pi}\int_\gamma f\,dz = \overset{\text{résidu}}{\sum \operatorname{res}_{z_i}(f)}$$

$$\| $$

$$\frac{1}{2i\pi}\left[\boxed{\cancel{\int_0^1}} + \cancel{\int_1^{1+\omega}} + \boxed{\cancel{\int_{1+\omega}^{\omega}}} + \cancel{\int_\omega^0}\right]$$

$$\int_0^1 f(z)\,dz = \int_0^1 f(z+\omega)\,dz = \int_\omega^{1+\omega} f(z)\,dz$$

$$= -\int_{1+\omega}^{\omega} f(z)\,dz.$$

$$\Rightarrow \sum \operatorname{res}_{z_i}(f) = 0$$

On travaille avec $g = \frac{f'}{f}$

f non constante $\Rightarrow$ g méromorphe
$\neq 0$ elliptique

pôle de g $\subseteq$ pôle de f $\cup$
zéro de f

$$\sum_{Pol(g) \cap P} \text{res}(g) = 0$$

z_i pôle ou zéro de f.

$$f(z) = a(z - z_i)^{m_i}(1 + h)$$

$a \neq 0 \quad h(z_i) = 0$

$$\operatorname{res}_{z_i}(g) \overset{?}{=}$$

$$g = \frac{f'}{f} = \frac{\cancel{a}\, m_i (z-z_i)^{\cancel{m_i}-1}(1+\tilde{h})}{\cancel{a}(z-\cancel{z_i)^{m_i}}(1+h)}$$

$$h(z_i) = \tilde{h}\,|_{z_i} = 0$$

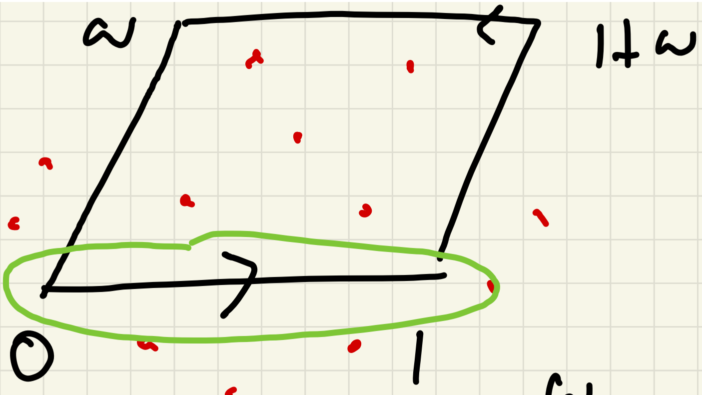
$$\operatorname{res}_{z_i}(g) = m_i$$

$$\Rightarrow \quad \sum m_i = 0$$

$$h(z) = z\,\frac{f'}{f}$$ (pas elliptique!)

$$\sum \operatorname{res}_{z_i}(h) = 2i\pi^{-} \int h(z)\,dz$$

$\{z_i\}$ = pôles et zéros de f ds P



$$h = z\,\frac{f'}{f}$$

$$\int_0^1 h(z)\,dz + \int_{1+\omega}^{\omega} h(z)\,dz$$

$$= \int_0^1 z\,\frac{f'}{f}\,dz - \int_0^1 (z+\omega)\,\frac{f'}{f}\,dz$$

$$= -\int_0^1 \omega\,\frac{f'}{f}\,dz = -\omega \log f \Big|_0^1$$

$\log f$ = défini à $2i\pi\mathbb{Z}$ près

$$f(1) = f(0)$$

$$\Rightarrow \log f(1) - \log f(0) \in 2i\pi\mathbb{Z}$$

$$\int_{\lozenge} h(z)\,dz = \int_0^1 - \int_\omega^{1+\omega} + \int_1^{\omega+1} - \int_0^\omega$$

$$= -\omega \left(\operatorname{Log} f(1) - \log f(0) \right)$$
$$= 2i\pi\, k\, \omega \qquad k \in \mathbb{Z}$$

$$= 2i\pi\, \ell \qquad \ell \in \mathbb{Z}.$$

Conclusion
$$\int_{\lozenge} h(z)\,dz = 2i\pi\, \mu$$

avec $\mu \in \Lambda$.

Thm des résidus pour $h = z \frac{f'}{f}$

$$\frac{1}{2i\pi} \int_{\partial P} h(z)\, dz = \sum \mathrm{res}_{z_i}(h)$$

$$\Cap \qquad\qquad\qquad\qquad \| $$

$$\Lambda \qquad\qquad\qquad\qquad z_i\, m_i$$

$$\sum m_i z_i \in \Lambda$$ ///

e) $g(z) = (\wp')^2 - 4\wp^3 + 60\, G_4\, \wp - 140\, G_6$

$g : \mathbb{C} \to \mathbb{C}$ méromorphe elliptique

on veut montrer que $g \equiv 0$

$$\text{Pôles de } (g) \subseteq \text{Pôles}(\wp) \cup \text{Pôles}(\wp') = \text{Pôles}(\wp) \subseteq \Lambda .$$

Lemme au voisinage de 0

$\boxed{g(z) \text{ est hol. et s'annule en } 0}$

donc g elliptique et hol $\overset{(a)}{\Rightarrow} g \equiv 0$

- $\wp(z) = \dfrac{1}{z^2} + h(z)$ avec $h(0) = 0$

G_2 si h hol.

$$\wp^3(z) = \frac{1}{z^6} + \frac{c_4}{z^4} + \frac{c_2}{z^2} + c_0 + O(z)$$

$$\wp'(z) = \sum_{\lambda \in \Lambda} \frac{-2}{(z+\lambda)^3} = -\frac{2}{z^3} + \sum_{\lambda \neq 0} -\frac{2}{\lambda^3} + O(z)$$

$$= -\frac{2}{z^3} - 2G_3 + O(z)$$

$$(\wp'(z))^2 = \left(\sum \frac{-2}{(z+\lambda)^3} \right)^2 = \left(-\frac{2}{z^3} + \sum_{\lambda \neq 0} \frac{-2}{(z+\lambda)^3} \right)^2$$

$$= \frac{4}{z^6} + \frac{b_4}{z^4} + \frac{b_2}{z^2} + b_0 + O(z)$$

$$g(z) = (\wp'(z))^2 - 4\wp(z)^3 + 60\, G_4\, \wp(z) - 140\, G_6$$

$$\| $$

$$= \cancel{\frac{4}{z^6}} + \frac{b_4}{z^4} + \frac{b_2}{z^2} + b_0 - 4\left(\cancel{\frac{1}{z^6}} + \frac{c_4}{z^4} + \frac{c_2}{z^2} + c_0\right)$$

$$+ 60\, G_4 \left(\frac{1}{z^2}\right) - 140\, G_6$$

$$\begin{cases} b_4 = 4c_4 \\ b_2 = 4c_2 - 60\, G_4 \\ b_0 = 4c_0 + 140\, G_6 . \end{cases}$$

///

$g = 0$

$$\begin{tikzcd}
\mathbb{C}/\Lambda \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathbb{C}^2 \\
\mathbb{C}/\Lambda - \{0\} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

$$z \longmapsto (\wp(z), \wp'(z))$$

φ hol.

$\phi(z+\lambda) = \phi(z)$ pour tout z

cons $\Rightarrow \exists\ \varphi : \mathbb{C}/\Lambda - \{0\} \to \mathbb{C}^2$

(e) $\varphi(\mathbb{C}/\Lambda - \{0\}) \subseteq \{ f(x,y) = 0 \}$

$$\boxed{f = y^2 - 4x^3 + 60\,G_4\,x - 140\,G_6}$$

Prop. $\{P=0\} \cap \{\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0\} = \phi$

Gauss $\Rightarrow$ $\underset{\substack{\parallel \\ C_P}}{\{P=0\}}$ est une surface de Riemann. Aussi

$\pi_1(x,y) = x$
$\pi_2(x,y) = y$ $\quad C_P \to \mathbb{C}$

sont holomorphes.

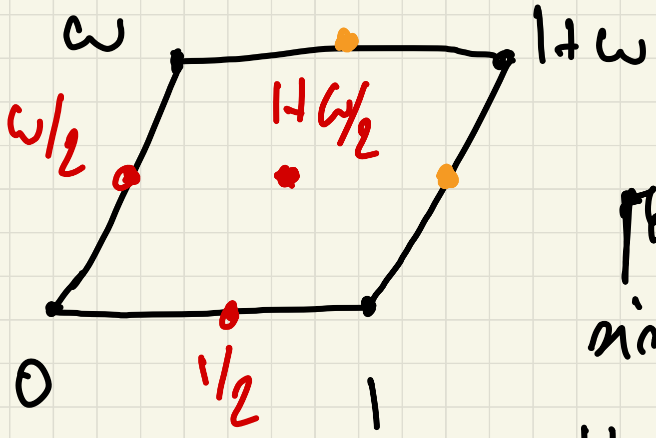
$\varphi : \mathbb{C}/\Lambda - \{0\} \longrightarrow C_P$

est $\boxed{\text{holomorphe}}$ car $\pi_1 \circ \varphi = \wp$

et $\pi_2 \circ \varphi = \wp'$ sont holomorphes.

(f) $e_1 = \wp(1/2)$ $\quad e_2 = \wp(\omega/2)$

$e_3 = \wp\left(\frac{1+\omega}{2}\right)$



$\wp'$ admet 3 racines simples en $1/2$, $\omega/2$, $\frac{1+\omega}{2}$ (modulo Λ)

• $\wp'(z) = -\wp'(-z)$

z_0 tq $2z_0 = 0$ modulo Λ.

$$\{z_0,\ 2z_0 \in \Lambda\} = \Lambda \cup \left\{\frac{1}{2} + \Lambda\right\} \cup \left\{\frac{\omega}{2} + \Lambda\right\} \cup \left\{\frac{1+\omega}{2} + \Lambda\right\}.$$

$\Rightarrow \wp'(z_0) = 0$

$2z_0 \in \Lambda.$ $\qquad 2z_0 = \lambda$

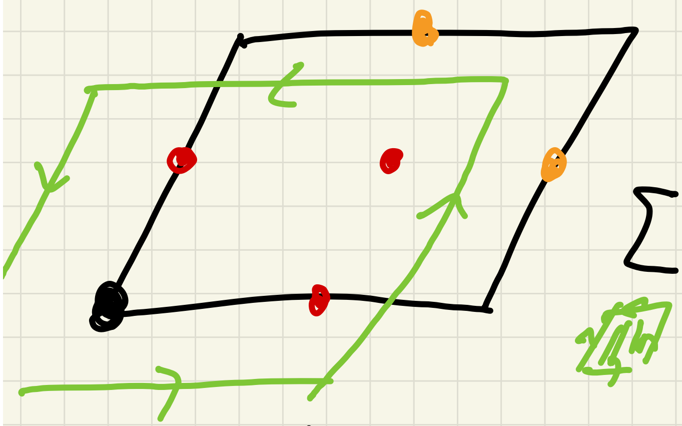
$z_0 = -z_0 + \lambda$

$$\wp'(z_0) = \wp'(-z_0+\lambda) = \wp'(-z_0)$$

$$\| $$

$$-\wp'(-z_0) \qquad \Rightarrow \wp'(z_0) = 0$$

$$\wp'(1/2) = \wp'(\omega/2) = \wp'(1+\omega/2) = 0$$



question (d)

$$\sum m_i = 0$$

$$-\mathrm{ord}_0(\wp') = 3 = \sum_{\wp'(z_i)=0} \mathrm{ord}_{z_i}(\wp')$$

$$\geq$$

$$\mathrm{ord}_{1/2}\wp' + \mathrm{ord}_{\omega/2}\wp' + \mathrm{ord}_{\frac{1+\omega}{2}}\wp'$$

$\Rightarrow \operatorname{ord}_{1/2}(\wp') = \operatorname{ord}_{\omega/2}(\wp') = \operatorname{ord}_{\frac{1+\omega}{2}}(\wp')$

$= 1$

$\wp'$ a 3 zéros simples (mod Λ)

$\rightarrow 1/2,\ \omega/2,\ 1+\omega/2\ .$ ///

$e_1 = \wp(1/2) \qquad e_2 = \wp(\omega/2)$

$e_3 = \wp(1+\omega/2) \qquad \boxed{e_1 \neq e_2 \neq e_3\ .}$

On regarde pôles et zéros de $\wp - e_1$

$\wp - e_1$ pôle d'ordre 2 dans en 0

$\Rightarrow \wp - e_1$ a exactement 2 zéros avec multiplicité (modulo Λ).

$$(\wp - e_1)(1/2) = 0$$

$$\wp'(1/2) = 0$$

$\Rightarrow$ $1/2$ unique zéro d'ordre 2

pour $\wp$ modulo Λ.

$\Rightarrow$ $\wp(\omega/2) \neq 0$ $\quad \wp\left(\frac{1+\omega}{2}\right) \neq 0$

$e_1 \neq e_2 \qquad e_1 \neq e_3$.

pour montrer que $e_2 \neq e_3$ on considère $\wp - e_2$

$$(\wp'(z))^2 = 4(\wp(z) - \alpha_1)(\wp(z) - \alpha_2)(\wp(z) - \alpha_3)$$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ par (e)

$\wp_\Lambda'(1/2) = 0 \quad (\wp'(\omega/2) = 0)$

$\Rightarrow \wp(1/2) = e_1$ solution de

$(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)$

$\Rightarrow e_1 \in \{\alpha_1, \alpha_2, \alpha_3\}$

$\{\alpha_1, \alpha_2, \alpha_3\} \supseteq \{e_1, e_2, e_3\}$

Comme $e_1 \neq e_2 \neq e_3$

$\Rightarrow \{\alpha_1, \alpha_2, \alpha_3\} = \{e_1, e_2, e_3\}$

$$\boxed{(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)}$$

$$P(x,y) = y^2 - 4(x-e_1)(x-e_2)(x-e_3)$$

$$(P=0) \cap \left(\frac{\partial P}{\partial y}=0\right) \cap \left(\frac{\partial P}{\partial x}=0\right) \ni (x,y)$$

$$\frac{\partial P}{\partial y} = 2y \qquad \varphi(x) = 4(x-e_1)(x-e_2)(x-e_3)$$

$y=0$ & $\varphi(x)=0$ & $\varphi'(x)=0$

$\Rightarrow$ pas possible.

(g) $\varphi : \mathbb{C}/\Lambda - \{0\} \longrightarrow \{P=0\}$ est un biholomorphisme.

surjectivité : $(x,y) \in \{\mathcal{E}=0\}$.

on regarde la fonction elliptique $\wp(z)-x$ qui admet un pôle d'ordre 2 en 0 donc admet un zéro z_0 $\quad \wp(z_0)=x$

$$\left(\wp'(z_0)\right)^2 = Q\left(\wp(z_0)\right) = \underset{\displaystyle y^2}{\underset{\parallel}{Q(x)}}$$

$$\Rightarrow \wp'(z_0) = \pm y$$

$$\Rightarrow \phi(-z_0) = (x,y) \text{ ou}$$

$$\phi(z_0) = (x,y).$$

injectivité : $\phi(z_1) = \phi(z_2)$

$\wp - \wp(z_1)$ elliptique pôle ordre 2 en 0

$\wp(z) - \wp(z_1)$ s'annule en z_1 et $-z_1$

si $2z_1 \notin \Lambda$

zéros $\{\wp(z) - \wp(z_1)\} = \{\pm z_1 + \Lambda\}$

par continuité vrai si $2z_1 \in \Lambda$

mais $z_1 \notin \Lambda$

$$\phi(z_1) = \phi(z_2) \iff \begin{array}{l} \wp(z_1) = \wp(z_2) \\ \wp'(z_1) = \wp'(z_2) \end{array}$$

$\Rightarrow$ $z_2 = z_1$ ou $z_2 = -z_1$

$\Rightarrow$ $z_2 = z_1$

$\Rightarrow$ ϕ est injective. ///