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K # field $v \in M_K$ $\rightsquigarrow$ $\mathbb{C}_v$ = Completion Alg. closure of K_v

Thm of Benedetto | K a # field $\varphi \in K[T]$ of degree $d \geq 2$

s = # places of K s.t. φ has not potential good reduction into $\mathbb{C}_v$.

$$\text{Card Preper}(\varphi, K) \leq C(N)(s \log s + 1)$$

$$N = [K : \mathbb{Q}].$$

$\rightsquigarrow$ not particularly difficult but the proof is very clever and technical.

To give a flavor of it only consider the quadratic case.

obs. $s \geq$ # archimedean places.

Thm N fixed $\exists\, C(N) > 0$ For any # field $[K : \mathbb{Q}] \leq N$

for any $c \in K$ $\varphi_c(T) = T^2 + c$

$$\text{Card Preper}(\varphi_c, K) \leq C^s$$

s = # bad reductions

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Analysis of the geometry of $f_c(z) = z^2+c$ in $\mathbb{C}_p$, $v \in \mathcal{M}_{\mathbb{Q}} \cup \{\infty\}$

$$\mathrm{Fix} = \{c_+, c_-\} \qquad c_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-4c}\right).$$

Given $\mathbb{C}_p$ $\quad p \geq 3$

$|c|_p \leq 1 \Rightarrow$ good reduction $\quad K(f) = \overline{B}(0,1)$.

$|c|_p > 1 \Rightarrow$ bad reduction

Lemma $\quad K(f) \subseteq \overline{B}(c_+,1) \sqcup \overline{B}(c_-,1)$.

proof $\quad |c_+| = |c_-| = |c|^{1/2} > 1$

$$f(T+c_+) = T^2 + 2c_+T + c_+^2 + c$$
$$= T^2 + 2c_+T + c_+$$
$$\Rightarrow f\,\overline{B}(c_+,1) = \overline{B}(c_+, \max\{1, |2c_+|, |c_+|\})$$
$$= \overline{B}(c_+, |c|^{1/2})$$

idem for $f\,\overline{B}(c_-,1) = \overline{B}(c_-, |c|^{1/2})$

obs. $\overline{B} = \overline{B}(c_-, |c|^{1/2}) = \overline{B}(c_+, |c|^{1/2})$ since
$$|c_+ - c_-| = |\sqrt{1-4c}| = |c|^{1/2}$$

Now $\quad f^{-1}(\overline{B}) \supseteq \overline{B}(c_+,1) \sqcup \overline{B}(c_-,1)$

since any pt in $\overline{B}$ has only 2 preimages

we get equality $\quad$ ///

Given $\mathbb{C}_2$ $\quad$ same analysis proves

$|c|_2 \leq 4 \Rightarrow$ potential good reduction $\quad K(f) = \overline{B}(c_+,1)$.

$|c|_2 > 4 \Rightarrow K(f) \subseteq \overline{B}(c_+,1) \sqcup \overline{B}(c_-,1)$

Case $\mathbb{C}$

Lemma either $|c| \leq 5$ and $K(P) \subseteq \mathbb{D}\left(\frac{c_+ + c_-}{2}, 3\right)$

or $|c| \geq 5$ $K(P) \subseteq \mathbb{D}(c_+, 2) \sqcup \mathbb{D}(c_-, 2)$

proof.

choose $w \in \mathbb{D}(c_+, 2)$ we prove $P^{-1}(w) \subseteq \mathbb{D}(c_+,2) \cup \mathbb{D}(c_-,2)$

$$\left| \begin{array}{l} z^2 + c = w \\ c_+^2 + c = c_+ \end{array} \right. \qquad \begin{array}{l} |(z - c_+)(z + c_+)| = |w - c_+| \leq 2 \\ c_+ + c_- = 1 \end{array}$$

either $|z - c_+| \leq 2$

or $|z + c_+| \leq 1$ and $|z - c_-| \leq |z + c_+| + 1 \leq 2$

$\Rightarrow$ $P^{-1}(\mathbb{D}(c_+,2) \cup \mathbb{D}(c_-,2)) \subseteq \mathbb{D}(c_+,2) \cup \mathbb{D}(c_-,2)$.

$|c_+ - c_-| = |\sqrt{1-4c}| \geq 4$.

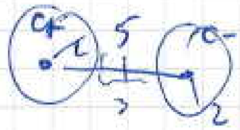
$\Leftrightarrow$ $|1 - 4c| \geq 16$.

$\Leftrightarrow$ $|c - \frac{1}{4}| \geq 4$ $\Rightarrow$

if $|c| \geq 5$ we are in case ②

if $|c| \leq 5$ we have

$|c_+ - c_-| \leq \sqrt{21} \leq 5$



$K(P) \subseteq \mathbb{D}\left(\frac{c_+ + c_-}{2}, \frac{5}{2}\right)$.

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The proof $x_1 \dots x_N$ preperiodic points for P_c

$\mathcal{G} = \{ P$ has potential good reduction at $v \}$

$B_{na} = \{ P$ has bad reduction and $v \notin M_{K,\infty} \}$

$M_K = M_{K,\infty} \sqcup B_{na} \sqcup \mathcal{G}$.

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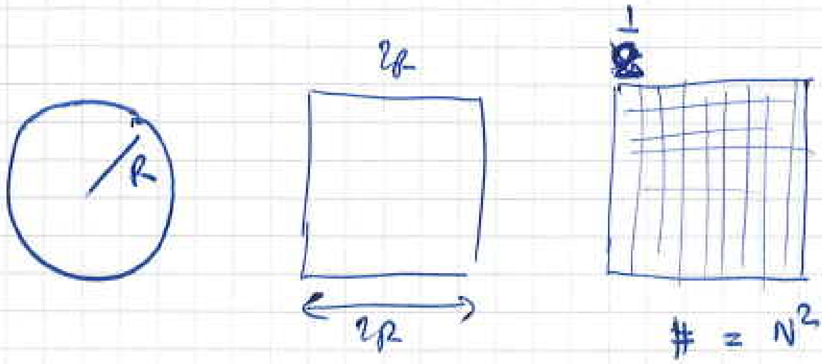
$v \in \mathcal{G}$ $\quad K_v = B_v \quad$ diam $(B_v) = 1$

$v \in \mathcal{B}_{na}$ $\quad K_v \subseteq B_v^+ \sqcup B_v^- \quad$ diam $(B_v^{\pm}) \le 1$.

$v \in M_{K,\infty}$ we can cover K_v by A disks $B_v^{(i)}$ of diam 1



$\# = N^2 \simeq 8R^2$. $\quad \frac{N}{2} = 2R$.

$$A \le \max\{\underset{72}{8\cdot 9}, \underset{64}{2\cdot 32}\} = 144.$$

Suppose

$$n \geq 144^{s_\infty}\, 2^{n s_{na}} \qquad \begin{aligned} s_\infty &= \# M_{K,\infty} \\ s_{na} &= \# \mathcal{B} \\ s &= s_\infty + s_{na}. \end{aligned}$$

$\Rightarrow$ Pigeonhole principle

get $x_i \neq x_j$ s.t. at each place v

x_i & x_j fall into the same disk

of diameter ≤ 1 at a finite place

diameter $\le \sqrt{\frac{1}{2}} < 1$ at an archimedean place

$$1 = \prod |x_i - x_j|^{n_v} \leq \prod_{M_{K,\infty}} |x_i - x_j|^{n} < 1$$

Absurd

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