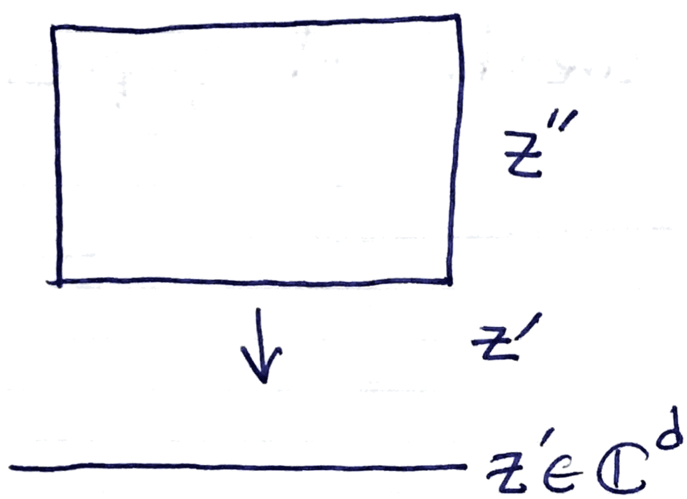
Lecture 15: Tuesday, March 3

Remark: From last time:

Local Parametrization theorem:

$\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n,0)}$ prime ideal

$A = V(\mathfrak{a})$



$$A \subseteq \{|z''| \le C|z'|\}$$

$$|z_j| \le C \max |z_\ell|, \quad \ell = 1, \dots, d$$

$$d+1 \le j \le n$$

$\rightsquigarrow j = n$ $\quad \hat{P}$ = minimal poly. of z_n / $\mathcal{M}_d$

$$= z_n^q + \sum_0^{q-1} a_j(z')\, z_n^{q-j}$$

$$|a_j| = \mathcal{O}(|z'|^j)$$

P_j = minimal polynomial of z_j over $\mathcal{M}_d$

$$= z_j^{q_j} + \sum a_{i,j}(z')\, z_j^{q_j - i}$$

$$|a_{ij}(z^j)| = \mathcal{O}(|z'|^i)$$

$\Rightarrow$ solutions of $(P_j = 0)$

$= \mathcal{O}(|z'|)$.

Cartan Coherence Theorem

M is a complex manifold $A \subseteq M$ analytic subset.

$$\mathcal{I}_A = \{ \mathcal{I}_{A,x} \}$$

$$\mathcal{I}_{A,x} = \{ f \in \mathcal{O}_{M,x},\ f|_A = 0 \}$$

defines a <u>coherent</u> ideal sheaf

<u>First reductions</u>: Can always assume that $M = \Delta^n$ polydisk $\ni 0$.

$$\begin{array}{c}\Delta^n \\ \cup\!| \\ A\end{array}$$

<u>Observation</u>: One only needs to prove that $\mathcal{I}_A$ is of finite type. Indeed,

$$\mathcal{O}^q_{\Delta^n} \xrightarrow{\exists} \mathcal{O}^p_{\Delta^n} \to \mathcal{I}_A \subseteq \mathcal{O}_{\Delta^n}$$

So, Oka's coherence theorem already tells us that the second condition is automatically satisfied.

Look for p and surjective morphism $\mathcal{O}^{p}_{\Delta^n} \to \mathcal{I}_A \to 0$

- $\mathcal{a}$ and $\mathcal{B}$ are two coherent ideal sheaves. Then

$$\mathcal{a} \cap \mathcal{B} = \{\mathcal{a}_x \cap \mathcal{B}_x\}_x \text{ is also coherent.}$$

Obs: $\mathcal{a} + \mathcal{B}$ coherent, $\mathcal{a} \cdot \mathcal{B}$ coherent and $(\mathcal{a} : \mathcal{B}) = \{f \mid f\mathcal{B} \subseteq \mathcal{a}\}$ $\subseteq \mathcal{O}_M$

are all coherent! (proof: using Oka's theorem)

See book by Gunning. (Vol II)

Proof: Reduce Δ^n

$$f_1, f_2, \dots, f_\mu, g_1, g_2, \dots, g_\nu \in \mathcal{O}(\Delta^n)$$

$$\mathcal{a}_x = \langle f_{1,x}, f_{2,x}, \dots, f_{\mu,x} \rangle$$

$$\mathcal{B}_x = \langle g_{1,x}, g_{2,x}, \dots, g_{\nu,x} \rangle \quad \forall x \in \Delta^n$$

$h \in \mathcal{Q}_x \cap \mathcal{B}_x$

$$\Theta : \mathcal{O}_{\Delta^n}^{\mu+\nu} \longrightarrow \mathcal{O}_{\Delta^n}$$

$$(\phi_1, \dots, \phi_{\mu+\nu}) \longmapsto \sum_1^{\mu} \phi_j f_j - \sum_{\mu+1}^{\mu+\nu} \phi_j g_{j-\mu}$$

By Oka's Theorem, $\ker \Theta$ is of finite type.

$h \in \mathcal{Q}_x \cap \mathcal{B}_x$

$h = \sum \phi_i f_i = \sum \phi_j g_{j-\mu}$

$\exists \phi \in \ker(\Theta)$ $\square$

Proof of Cartan's Theorem

Decompose A into irreducible components.

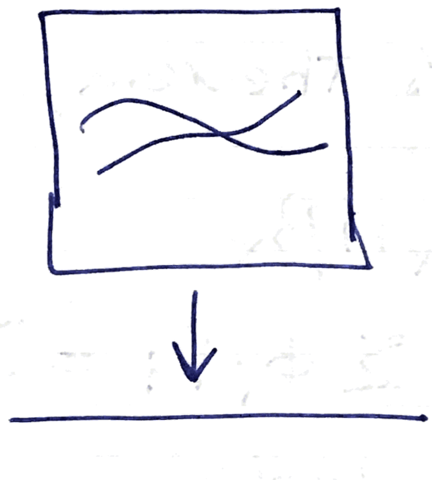
$A = A_1 \cup \dots \cup A_N$.

$\mathcal{I}_A = \bigcap \mathcal{I}_{A_i}$ A_i = <u>irreducible</u>.

We may assume that A is locally irreducible (at 0).

Recall: A is irreducible $\Leftrightarrow$ $\mathcal{I}_A$ is prime.
(primary decomposition of ideals in $\mathcal{O}_{(\mathbb{C}^n,0)}$).

May apply the local parametrization theorem



$$\pi : (z', z'') = z'$$
$$z' \in \mathbb{C}^d, \quad z'' \in \mathbb{C}^{n-d}$$

$\pi : A \longrightarrow \Delta^d$ ramified cover of deg $= q$.

- $\hat{P}$, $\delta(z') = \mathrm{disc}(\hat{P})$.

π is unramified over $\Delta^d \setminus S$
$S = (\delta = 0)$ "bad locus".

$A_0 = A \setminus \pi^{-1}(S)$ is a submanifold of dimension d.

$$\langle \hat{P}(z', z_n) \quad , \quad \delta(z') z_j - B_j(z', z_n) \rangle$$

belongs to $\mathfrak{a}$. In fact,

$$\mathcal{B} := \langle \hat{P}(z', z_n), \delta(z') z_j - B_j(z', z_n) \rangle$$

we proved $\delta^N \tilde{\mathcal{I}}_A \subseteq \mathcal{B} \subseteq \mathcal{I}_A$

for some N.

<u>Obs</u>: If $x \in A_0$, $x = (z', z'')$, $\delta(z') \neq 0$.

then $\mathfrak{a}_x = \mathcal{B}_x$

- Choose generators $f_1, f_2, \ldots, f_\mu$ of $\mathfrak{a}_0 \subseteq \mathcal{O}_{(\mathbb{C}^n, 0)}$

Define

$$\tilde{\mathfrak{a}} = \mathcal{B} + \langle f_1, \ldots, f_\mu \rangle$$

<u>Claim</u>: $\tilde{\mathfrak{a}}_x = \mathcal{I}_{A,x}$ for all x.

See the book by Gunning / Grauert-Remmert

Demailly / Hörmander $\rightsquigarrow$ follow Cartan.

(uses perturbation of coordinates)

- $\tilde{\mathfrak{a}}_0 = \mathcal{I}_{A,0}$ prime.

- $(\tilde{\mathfrak{a}} : \delta) =$ finite type $= \langle g_1, g_2, \ldots, g_\mu \rangle$

<u>Claim</u>: $(\tilde{\mathfrak{a}} : \delta)_x = \tilde{\mathfrak{a}}_x \quad \forall x$

$g_{i,0} \cdot \delta_0 \in \tilde{\mathfrak{a}}_0 \Rightarrow \delta$ does <u>not</u> vanish identically at A_0

$\delta|_A \not\equiv 0 \Rightarrow g_{i,0} \in \tilde{\mathfrak{a}}_0$, so $g_i|_A \equiv 0$.

$f_x \in \mathcal{I}_{A,x}$. Want to show that $f_x \in \tilde{\mathfrak{a}}_x$.

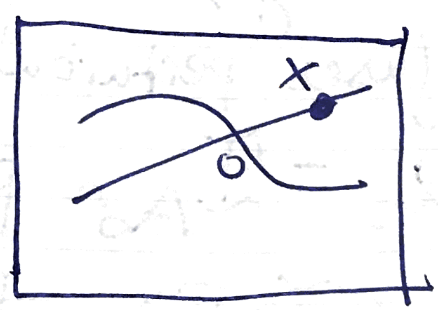
It is clear that $\tilde{\mathfrak{a}} \subseteq \mathcal{I}_A$, and now we are showing the reverse inclusion.

$(\tilde{\mathfrak{a}} : f)$ finite type

Pick $V \ni x$ neighborhood of x such that $f \in \mathcal{O}(V_x)$.

On V, $(\tilde{\mathfrak{a}} : f)$ is finite type.

$= \langle h_1, \dots, h_v \rangle$



(*) $\bigcap \{h_i^{-1}(0)\} \subseteq \{\delta = 0\} \cap V$

Applying Nullstellensatz, we get $\exists r$ such that $\delta_x^r \in (\tilde{\mathfrak{a}}, f)_x$

$$\delta_x^r \cdot f_x \in \tilde{\mathfrak{a}}_x \Rightarrow \delta_x^{r-1} f_x \in \tilde{\mathfrak{a}}_x \underset{\text{induction}}{\Longrightarrow} f_x \in \tilde{\mathfrak{a}}_x.$$

$\square$

Further Coherence Theorems

$\mathbb{C}$-manifold

① $\mathcal{J} \subseteq \mathcal{O}_M$ ideal sheaf

$\mathrm{rad}(\mathcal{J}) = \{\mathrm{rad}(\mathcal{J})_x\}$ is coherent.

② $A \subseteq M$ analytic subset.

$\mathcal{O}_A = \mathcal{O}_M / \underset{\sim}{\mathcal{I}}_A$ is coherent.

③ A is locally irreducible near any of its points.

$\mathcal{O}_{A,x}$ is domain

$\mathrm{Frac}(\mathcal{O}_{A,x}) =: \mathcal{M}_{\mathcal{O}_{A,x}}$

$\widehat{\mathcal{O}}_{A,x}$ also coherent.

$$\mathcal{O}_{A,x} \subseteq \underbrace{\widehat{\mathcal{O}}_{A,x}}_{\text{integral closure}} \subseteq \mathcal{M}_{A,x}$$

$\widehat{\mathcal{O}}_{A,x}$ = normalization sheaf

(4) Direct Image Theorem

$f : X \longrightarrow Y$ holomorphic

X, Y are complex spaces.
($\mathbb{C}$-manifolds)

f = proper (meaning, preimage of a compact set is compact).

$\mathcal{F}$ = coherent sheaf on X.

$$(f_* \mathcal{F})_x = \lim_{U \ni x} \mathcal{F}(f^{-1}(U))$$

$$R^i f_* (\mathcal{F}) = \lim_{V \ni x} H^i(\mathcal{F}(f^{-1}(V)))$$

The theorem states that
$(R^i f_*)(\mathcal{F})$ are coherent $\forall i \geq 0$.

<u>Consequence</u> : $\mathcal{I} = \mathcal{I}_A$ A analytic

$\Rightarrow f_* \mathcal{I}_A$ coherent

$\Rightarrow$ Co-Supp $(f_* \mathcal{I}_A)$ is analytic $\longrightarrow \mathrm{Supp}(\mathcal{O}_M / f_* \mathcal{I}_A)$

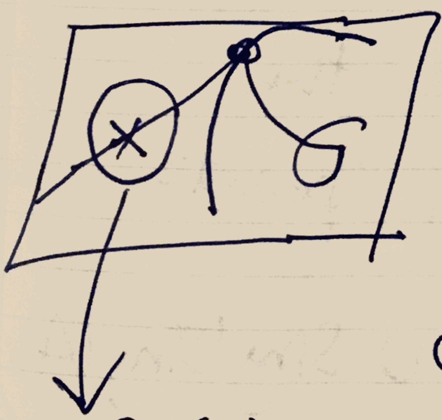
$= f(A)$ is analytic

which is already a very deep theorem

<u>Thm</u>: A analytic $\subseteq \Omega \subseteq \mathbb{C}^n$, $n \geq 1$

$f : \Omega \setminus A \to \mathbb{C}$ holomorphic.

- If f is bounded, then f extends (as a hol. function)
- If $\mathrm{codim}(A) \geq 2$, then f extends (as a hol. function)

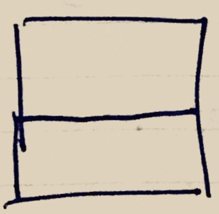


$x \in \mathrm{Reg}(A)$

$\mathrm{Reg}(A) = A \setminus \mathrm{Sing}(A)$

$\mathrm{Sing}(A)$ is analytic

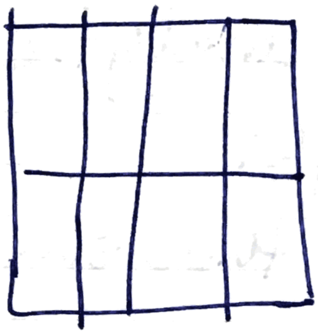
$\mathrm{codim}\, \mathrm{Sing}(A) \geq \mathrm{codim}(A) + 1$



Reduce the situation where A is linear.

$A = \{z'' = 0\}$ $\qquad (z', z'')$

$$\begin{array}{cc} \swarrow & \downarrow \\ z_1, \dots, z_d & z_{d+1}, \dots, z_n \end{array}$$



Fix z'. Look at

$z' \times \Delta^{n-d}$

has dimension ≥ 2

Apply Hartog's $\Rightarrow$ f extends to all the slices

Finally, you keep going (now restrict to $\mathrm{Sing}(A)$, and take the its Regular and singular points

$$\mathrm{Sing}(A) = \mathrm{Reg}(\mathrm{Sing}(A)) \cup \mathrm{Sing}(\mathrm{Sing}(A))$$

$\Rightarrow$ Stratification.

Lecture 16

Thursday, March 5

§3. Subharmonic and plurisubharmonic functions

$\downarrow$ defined on $\mathbb{C}$

$\hookrightarrow$ p.s.h.

$\downarrow$ defined on $\mathbb{C}^n$

psh function $\xleftrightarrow{\text{analogs}}$ convex functions

$\downarrow$

several $\mathbb{C}$

convex geometry

C. Kiselman "psh functions and potential theory in several $\mathbb{C}$-variables"

F. Riesz. 1924 Subharmonic.

P. Lelong 1942

T. Oka 1942

$\Omega \subseteq \mathbb{C}$, open $\quad z = x + iy$

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z \partial \bar z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Laplacian

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar z}$$

(sometimes constant is $\frac{4}{\pi}$, etc.)

Def: $f : \Omega \to \mathbb{R}$ $\mathcal{C}^2$

f is harmonic if $\Delta f = 0$

Ex: If $h : \Omega \to \mathbb{C}$ is holomorphic, then $f = \mathrm{Re}(h)$ is harmonic

$$2f = h + \bar h$$

$$2\partial\bar\partial f = 2\,\partial(\underbrace{\bar\partial h}_{=0}) + 2\,\bar\partial(\underbrace{\partial \bar h}_{=0})$$

obs: harmonic functions are stable by sum, multiplication by $\lambda \in \mathbb{C}$.
if f is harmonic and h holomorphic, then $f \circ h$ is harmonic.

$\bar{\partial}(f \circ h) = (\bar{\partial} f) \circ h \cdot h'$

If we apply ∂ to this equation, we obtain:

$$\Delta (f \circ h) = 0.$$

<u>Thm 1</u>: $h : \mathbb{D}(0,1) \to \mathbb{R}$ harmonic.

Then ① $\exists$ holomorphic f s.t. $h = \mathrm{Re}(f)$

② for any $0 < r < 1$, we have

$$h(0) = \oint_{\cancel{|z|<} \; |\xi| = r} h(\xi)\, d\xi = \iint_{|z| \le r} h(z)\, d\mathrm{Leb}(z)$$

<u>Proof</u>: ① Assume $\Delta h = 0$

$\Leftrightarrow \bar{\partial}(\partial h) = 0$

$\Leftrightarrow \partial h = f(z)\, dz \Rightarrow f$ is holomophic.

$$F(z) = \int_{[0,z]} f(\xi)\, d\xi = \int_{t=0}^{1} f(tz)\, z\, dt$$

Since f is holomorphic $\Rightarrow F$ is holom.

And $F'(z) = f(z)$.

$$v = Re(F) = \frac{1}{2}(F + \overline{F})$$

$$\frac{1}{2}(u_x - i u_y) = F'(z) = f(z)$$

$$= \frac{1}{2}(h_x - i h_y)$$

$$\Rightarrow (u - h) \equiv \text{const.}$$

2) $f(z) = \sum_{n \geq 0} a_n z^n$

$$f(0) = a_0 = \oint_{|\zeta| = r} \left(\sum a_n \zeta^n \right) d\zeta$$

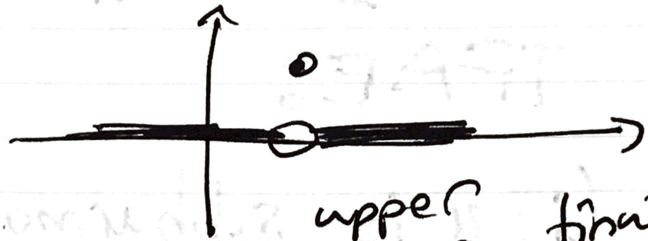
$$0 = \int_{|\zeta| = r} \zeta^n \, d\zeta \quad \text{for } n \geq 1 \quad ✓$$

Subharmonic Functions

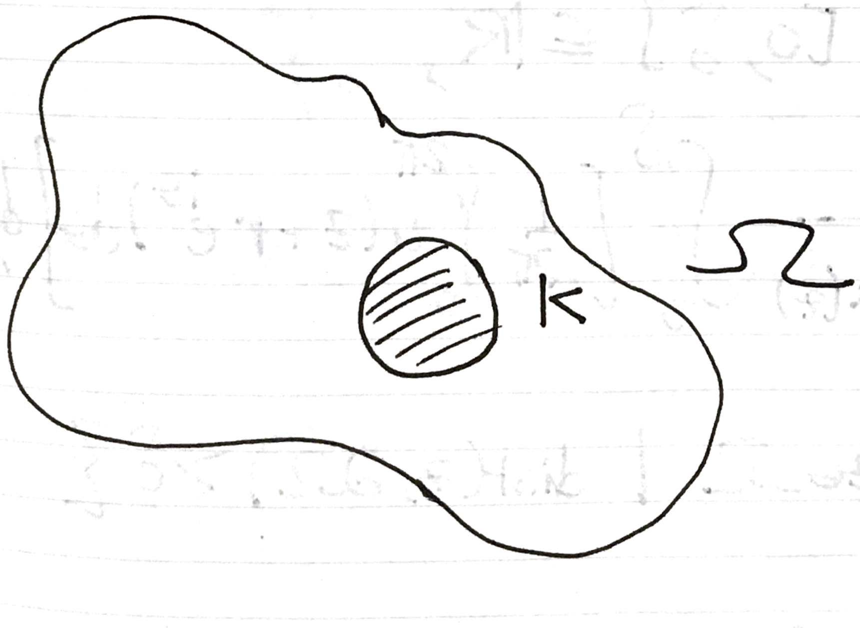
Def: $\mu : \underset{\underset{\mathbb{C}}{\cap\!\shortmid}}{\Omega} \to [-\infty, \infty)$ is subharmonic

if 1) μ is upper semi-continuous

saying that u is upper semicontinuous means that $\{u<c\}$ is open for all c.



(2) $\forall K$ compact inside Ω, for every continuous $h: K \longrightarrow \mathbb{R}$ such that $h|_{\mathrm{Int}(K)}$ is harmonic, and $h \geq u$ on ∂K, then $h \geq u$ on K.



Theorem 2 : $u : \Omega \to [-\infty, \infty)$

~~Xd.~~ u usc (upper-semicontinuous).

TFAE:

(i) u is subharmonic.

(ii) for any closed disk $\overline{D} \subseteq \Omega$
~~XXX~~ for any polynomial $g(z) = \sum a_n z^n$
such that $u \leq \mathrm{Re}(g)$ on ∂D, then
$u \leq \mathrm{Re}(g)$ on D.

(iii) For any $\delta > 0$, for any positive ~~number~~ measure μ on $[0, \delta] \subseteq \mathbb{R}$,

$$u(z) \leq \frac{1}{\int_0^\delta d\mu(z)} \int_0^\delta \left[\frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta\right] d\mu(r)$$

on $\Omega_\delta = \{z \in \Omega \mid \mathrm{dist}(z, \partial\Omega) > \delta\}$.

Proof sketch:

(i) $\Rightarrow$ (ii) clear.

(ii) $\Rightarrow$ (iii)

- u is usc. This implies that u is locally bounded from above.

For given z, $\{u < u(z)+1\}$ open $\ni z$

- for any compact set $K \subseteq \Omega$, $\sup_K u$ is attained, hence finite.

$\Rightarrow \int u$ is well-defined as an element of $[-\infty, \infty)$.

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z+re^{i\theta})\, d\theta$$

"approximate $u|_{\partial D(z,r)}$ by $\mathrm{Re}(P_n)$".

$$u(z) \leq \mathrm{Re}(P_n)(z) = \oint_{|\zeta|=r} \mathrm{Re}(P_n)\, d\zeta$$

Subharmonic by (ii)

$$\sim \oint_{|\zeta|=r} u\, d\sigma_\zeta$$

$$u_n(z) = \sup_{w \in K} \{ u(w) - n|z-w| \}$$

- u_n is continuous, in fact, n-Lipshitz
- $u_{n+1} \leq u_n$
- $z \in K$, $u_n(z) \searrow u(z)$ $\quad D = \mathbb{D}(z,r)$

$$u_n(z) \geq u(z)$$

So, u_n is C^0, and so we get $\exists P_n(z) = \sum a_n z^n$ such that

- $\sup_{\partial D} |u_n - Re(P_n)| \leq \frac{1}{n}$

Stone-Weierstrass theorem

- $Re(P_n) \geq u$ on ∂D

$u(z) \leq Re(P_n)(z)$ at z

so by assumption (a):

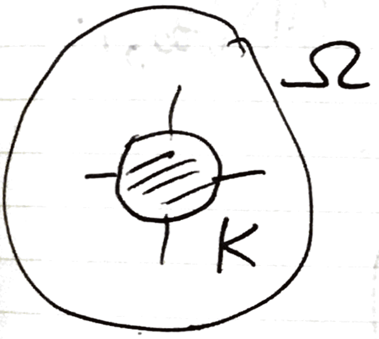
$$u(z) \leq Re(P_n) = \frac{1}{2\pi} \int_{|\zeta|=r} Re(P_n)(\zeta) d\zeta$$

$$= \frac{1}{2\pi} \int_{|\zeta|=r} u_n(\zeta) d\zeta + O(\tfrac{1}{n})$$

letting $n \to \infty$, and using the monotone convergence theorem, we obtain that:

$$u(z) \leq \frac{1}{2\pi} \int_{|\zeta|=r} \mu \, d\zeta$$

(iii) $\Longrightarrow$ (i)



$h \in \mathcal{C}^0(K)$

$h|_{\mathrm{Int}(K)}$ = harmonic

$h \geq u$ on ∂K

$\sup_K (u-h)$ attained $= M > 0$.

(by contradiction)

$F = \{(u-h) = M\}$ set where supremum is attained.

Note that $\cancel{F \subset\subset K}$

$F \subset\subset \mathrm{Int}(K)$ by assumption.

closed

Take $z_0 \in F$ with minimal distance to ∂K.

Let $r = \operatorname{dist}(z_0, \partial K)$, $D(z_0, r) \subseteq K$

$$\oint_{|\zeta - z_0| = r} (u-h) \underset{(iii)}{\geq} (u-h)(z_0) = M > 0$$

$D = \mathbb{D}(z_0, r)$

$\exists$ arc of positive length on $\partial D(z_0, r)$ such that $(u-h)|_A \leq M - \varepsilon$.

$$\oint_{\partial D(z_0, r)} (u-h) = \frac{1}{2\pi r}\left(\int_{\partial D \setminus A} + \int_A\right)$$

$$\leq \frac{1}{2\pi r} \cdot \left(M \cdot \text{length}(\partial D \setminus A) + (M - \varepsilon) \cdot \text{length}(A)\right)$$

$$< M \quad \text{contradiction} \quad \square$$

Convention : $u \equiv -\infty$ is not subharmonic.

$SH(\Omega) = \{u : \Omega \to [-\infty, +\infty) \text{ subharmonic}\}$.

$\to$ Theorem 2 implies

subharmonicity is a local property!

Basic Properties: of subharmonic functions

Maximum Principle : Suppose that u is subharmonic, i.e. $u \in SH(\Omega)$.

If $z_0 \in \Omega$ is a local maximum for u, then u is constant, in a neighborhood of z_0.

[Proof. Apply sub mean value inequality]

① $u, v \in SH(\Omega)$, $u+v, \lambda u, \max\{u, v\} \in SH(\Omega)$

$\lambda > 0$

② $(u_i)_{i \in I}$ any family of subharmonic functions

$u = \sup_{I \ni i} u_i$. If $u < \infty$, and u is u.s.c, then $u \in SH(\Omega)$.

[If the assumption ($u < \infty$ & u is u.s.c.)
then take $u^* =$ use regularization of
$\sup u_i$.]

Comment (for reference)

(3) $u_n \in SH(\Omega)$, $u_n \downarrow u$ (pointwise)
if $u \not\equiv -\infty$, then $u \in SH(\Omega)$.

(4) $\varphi : \mathbb{R} \to \mathbb{R}$ convex & increasing
If $u \in SH(\Omega) \Rightarrow \varphi \circ u \in SH(\Omega)$

Generalization : $\chi : \mathbb{R}^n \to \mathbb{R}$
Assume χ is convex, non-decreasing
in each variable.

$u_1, u_2, \ldots, u_n \in SH(\Omega) \Rightarrow \chi(u_1, \ldots, u_n) \in SH(\Omega)$

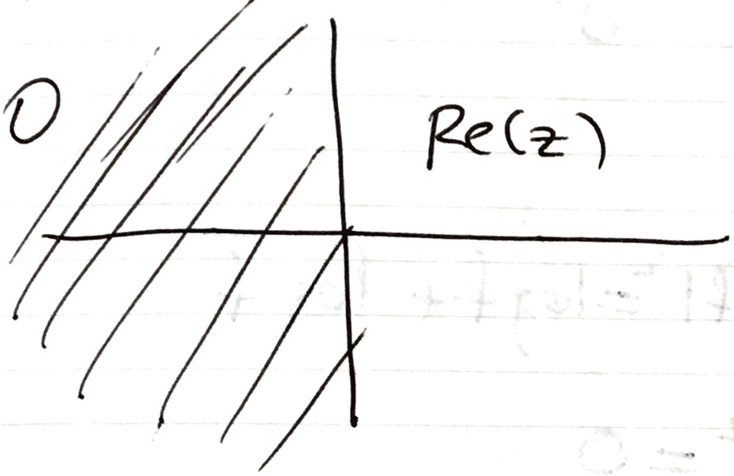
Proof: (of (4)) $\displaystyle \varphi(x) = \sup_{\substack{a>0 \\ ax+b \leq \varphi}} \{ax+b\}$

$\displaystyle \varphi(u) = \sup_{a>0} \{au+b\}$ is subharmonic by (2)

Example: $f \in \mathcal{O}(\Omega)$

- $\max\{Re(f), 0\} \in SH(\Omega)$

So, if $f(z) = z$, $\Omega = \mathbb{C}$, we get



so it is constant on big open subset, but not globally constant.

$\text{Log}|f| \in SH(\Omega)$ if $f \not\equiv 0$

$|f|^\alpha \in SH(\Omega)$ $\forall \alpha > 0$

$\max\{\text{Log}|f_i|\}$, $f_1, \dots, f_n \in \mathcal{O}(\Omega)$

apply $\varphi(x) = e^{\alpha x}$ increasing + convex

$\log|f| \in SH(\Omega)$

$$\log|f| \leq Re(P) \text{ on } \partial D$$

$$\Leftrightarrow \text{Log}|f e^{-P}| \leq 0 \text{ on } \partial D$$

$$\Leftrightarrow |f e^{-P}| \leq 1 \text{ on } \partial D$$

$$\overset{\text{Max-principle}}{\Leftrightarrow} |f e^{-P}| \leq 1 \text{ on } D.$$

extended real line.

Obs: $\text{Log}|f|$ is C^∞ from $\Omega \to [-\infty, +\infty]$

If $f^{-1}(0) \neq \emptyset \Rightarrow \log|f|^{-1}(-\infty) \neq \emptyset$.

If $f^{-1}(0) = \emptyset \Rightarrow \log|f|$ is harmonic

Easier proof: $\log|f|^2 = \log f + \log \bar{f}$

$$\partial\bar{\partial} \log|f|^2 = 0.$$

"want to show that $u \in SH(\Omega)$ $\Longleftrightarrow \Delta u \geq 0$".

Theorem 3: If $u \in SH(\Omega)$, then for any compact subset $K \subseteq \Omega$,

$\int_K u > -\infty$, in other words,

we have $u \in L^1(K)$,

$u \in L^1_{loc}$.

Consequence: $\{u = -\infty\}$ has zero Lebesgue measure.

Warning!

It can be the case

$$\overline{\{\mu = -\infty\}} = \Omega.$$

"Idea: Take $\overline{\{z_n\}} = \Omega$

$$u(z) = \sum_{n \geq 0} \underbrace{\alpha_n}_{\geq 0} \operatorname{Log}|z - z_n|"$$

for suitable weights α_i.

Proof: If $u(z) > -\infty$,

$$\iint_{|u-z|<1} \mu(u)\, d\mathrm{Leb}(u) \geq u(z) > -\infty$$

$E = \{ z \in \Omega_1,\ u$ integrable in one ~~closed~~ open ~~all~~ disks containing z and included in $\Omega \}$.

E is clearly open.

Claim: E is also ~~open~~ closed.

- $\Omega \setminus E$ is open: indeed, $z \in \Omega \setminus E$

if $z_n \in E \longrightarrow z$

$\Rightarrow \exists z_n' \longrightarrow z, \quad u(z_n') > -\infty$

$$u(z_n') \leq \iint_{D(z_n', \varepsilon)} u \, d\mathrm{leb}.$$