Cours 7

lunedì 29 marzo 2021 13:53

$\mathbb{K}$ any field car$(\mathbb{K}) = 0$, $\mathbb{K}^{alg} = \mathbb{K}$. $f : \mathbb{P}^N_{\mathbb{K}} \circlearrowleft$ endomorphism, is given by $N+1$ homogeneous polynomials $P_0 \dots P_N$ of degree $\deg(P_j) = d =: \deg(f)$

$f[x_0 : \dots : x_N] = [P_0(x) : \dots : P_N(x)]$ $\bigcap_{j=0}^{N} P_j^{-1}(0) = 0$ ($\Rightarrow \gcd(P_j) = 1$).

local degree at a $\mathbb{K}$-rational point. $p \in \mathbb{P}^N(\mathbb{K})$, $p = [p_0 : \dots : p_N]$

$q = f(p) \in \mathbb{P}^N(\mathbb{K})$, $q = [q_0 : \dots : q_N]$ $\rightsquigarrow \deg_f(p) \in \mathbb{N}^*$

"multiplicity of the equation $f = q$ at p".

Without loss of Generality: may assume $p_0 = 1$, $q_0 = 1$ (up to action of Aut$(\mathbb{P}^N_{\mathbb{K}})$).

Local coordinates at p: $y = (y_1 \dots y_N) \rightsquigarrow [1 : p_1 + y_1 : \dots : p_N + y_N]$

" " " q : $z = (z_1 \dots z_N) \rightsquigarrow [1 : q_1 + z_1 : \dots : q_N + z_N]$

We express f in the coordinates y, z:

$f[1 : p_1 + y_1 : \dots : p_N + y_N] = [1 : q_1 + z_1 : \dots : q_N + z_N] = [P_0(1, \underline{p} + y) : \dots : P_N(1, \underline{p} + y)] =$

$$= \left[1 : \frac{P_1(\dots)}{P_0(\dots)} : \dots : \frac{P_N(\dots)}{P_0(\dots)} \right].$$

Obs: $P_0(1, \underline{p} + y)$ does not vanish at $y = 0$. $\Rightarrow$ f is invertible in $\mathbb{K}[\![y]\!]$.

$z = f(y) = (\phi_1(y), \dots, \phi_N(y))$ $\phi_j \in \mathbb{K}[\![y]\!]$.

Claim (Nullstellensatz) The ideal generated by (ϕ_j) in $\mathbb{K}[\![y]\!]$ is $\mathfrak{M}$-primary.

$\mathfrak{M}$ = maximal ideal $= \langle y_1, \dots, y_N \rangle$.

$$\deg_f(p) = \dim_{\mathbb{K}} \left(\mathbb{K}[\![y]\!] / (\phi_j) \right).$$

1) Thm $f : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ endomorphism, $\deg f = d \geq 1$.

For all $q \in \mathbb{P}^N(\mathbb{K})$, $\sum_{f(p) = q} \deg_f(p) = d^N$ (Bezout)

2) If $g : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ is another endomorphism, then $\forall q \in \mathbb{P}^N(\mathbb{K})$,

$\deg_{g \circ f}(q) = \deg_f(q) \cdot \deg_g(g(q))$

3) If $g: \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ is another endomorphism, [illegible]

$$\deg_{f\circ g}(q) = \deg_g(q) \cdot \deg_f(g(q))$$

Remarks: By 1+2) $\deg(f\circ g) = \deg f \cdot \deg g$.

- $\deg f = 1 \iff f \in \mathrm{Aut}(\mathbb{P}^{N}_{\mathbb{K}}) \cong PGL_{N+1}(\mathbb{K})$
- if $f \in PGL_{N+1}(\mathbb{K})$, $g: \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$, $\deg(f^{-1}\circ g \circ f) = \deg g$.

If $\mathbb{K} = \mathbb{K}^{alg}$, $\mathrm{char}\,\mathbb{K} = 0$ ($\mathbb{K} = \overline{\mathbb{Q}}, \mathbb{C}_p, \mathbb{C}\dots$), $f: \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ endo, $\deg f = d \geq 2$

The number of periodic points of period n is finite and we have:

$$\sum_{f^n(p)=p} \mu(f^n, p) = \frac{d^{n(N+1)} - 1}{d^n - 1}, \quad \text{where } \mu(f,p) = \text{“}\deg_{f-id}(p)\text{”}$$

we will define it properly.

- $\mu(f^n, p) \in \mathbb{N}$, $>0 \iff f^n(p) = p$.

To define $\mu(f,p)$ we look at the local expansion of $f - id$ near p.

$p = [1:0\dots:0]$ $f(p) = p$. Local coordinates: $y = [1: y_1 : \dots : y_N]$

$f(y) = (\phi_1(y), \dots, \phi_N(y))$. $\mu(f,p) := \dim_{\mathbb{K}} \dfrac{\mathbb{K}[[y]]}{(\phi_1 - y_1, \dots, \phi_N - y_N)}$

Proof: - We prove first that $\{f^n(p) = p\}$ is finite.

- By Nullstellensatz, this implies that the ideal $(\phi_1 - y_1, \dots, \phi_N - y_N)$ is $\mathfrak{M}$-primary $\Rightarrow$ $\mu(f,p)$ is well defined.
- Apply Bezout's theorem ($n=1$):

$$(S) \begin{cases} P_0(x) = x_0 t^{d-1} \\ \vdots \\ P_N(x) = x_N t^{d-1} \end{cases}$$

system of equations in $\mathbb{P}^{N+1}_{\mathbb{K}} = \{[t:x]\}$.

One checks that (S) is finite in $\mathbb{P}^{N+1}_{\mathbb{K}}$.

Observe that $(S) \cap \{[0:x]\} = \emptyset$

since f is an endomorphism ($\bigcap_j P_j^{-1}(0) = 0$).

Since (S) is algebraic and does not intersect $\{t=0\}$, it must be finite.

- $(S) \xrightarrow{\mapsto} \{f(p) = p\}$ Rem: $\mapsto$ is surjective

$(S) \subset \mathbb{P}^{N+1}(\mathbb{K})$, $\{f(p)=p\} \subset \mathbb{P}^{N}(\mathbb{K})$

Take $q \in \{f(p) = p\}$, write $q = [x_0 : \dots : x_N]$

$(S) \longrightarrow \{f(p)=p\}$ Claim: Θ is surjective

$\cap \qquad\qquad \cap$

$\mathbb{P}^{N+1}(\mathbb{K}) \quad \mathbb{P}^N(\mathbb{K})$

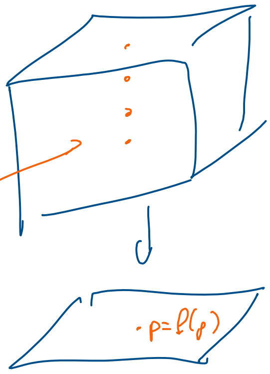
$[t:x] \mapsto [x]$

Take $q \in \{f(p)=p\}$, write $q=[x_0:\dots:x_N]$ s. that $[1:x] \in (S)$.

Take $[t:x] \in \Theta^{-1}(x)$. $P_i(x) = x_i t^{d-1} \Rightarrow t^{d-1}=1$.

$\Rightarrow \# \Theta^{-1}(x) = d-1$.

with multiplicity: $d-1$



Bezout applied to (S):

$$\sum_{[t:x]\in(S)} \text{local mult}_{(S)}(t,y) = d^{N+1}$$ ← product of the degrees of equations of (S)

Fact: for any $[x] \in \{f=\mathrm{id}\}$, for any $[t:x] \in (S)$: $\mathrm{mult}_{(S)}[t:x] = \mu(f[x])$

$$\sum_{f(p)=p} (d-1)\,\mu(f,p) = d^{N+1} - 1$$

⚠ another solution $(S) = \begin{cases} P_0 = x_0 t^{d-1} \\ \vdots \\ P_N = x_N t^{d-1} \end{cases}$ $\#(S) = \prod \text{degrees} = d^{N+1}$

There is a trivial solution $[1:0\dots0]$, of multiplicity 1

because $(x_0 \dots x_N) \neq (0 \dots 0)$, $t^{d-1} = 1 + (d-1)(t-1) + o(t-1)$

Formula for composition: to prove it expand in formal power series.

Observe that: $[1:x]$, $[\zeta:x]$, $\zeta^{d-1}=1 \Rightarrow \mathrm{mult}_{(S)}([1:x]) = \mathrm{mult}_{(S)}[\zeta:x]$

because there is an automorphism of $\mathbb{P}^{N+1}_{\mathbb{K}}$ that preserves the system and maps $[1:x]$ to $[\zeta:x]$ $\quad ([t:x] \mapsto [\zeta t:x])$

Theorem (Shub-Sullivan) Let $f=(f_1 \dots f_N) \in \mathbb{K}[[y]]^N$, $y=(y_1,\dots,y_N)$ $f_i(0)=0$ such that the ideal $\mathfrak{I} := (f_1,\dots f_N)$ is $\mathfrak{M}$-primary.

The sequence $\mu(f^n,p)$ is bounded.

Corollary: $f: \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ of degree $d \geq 2$, $\mathbb{K}=\mathbb{K}^{alg}$. Then $\mathrm{Card}\,\mathrm{Fix}(f^n) \to \infty$.

Proof: same argument as in 1D: $\sum \mu(f^n - \mathrm{id}) \sim d^{nN} \to +\infty$. □

Remarks: 1) this was proved for C^1-maps.

2) There is a recent paper dealing with the analytic case

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2) There is a recent paper dealing with the analytic case

Example: $f=(f_1 \dots f_N)$ $\quad \mathfrak{a}:=\langle f_1, \dots f_N\rangle$ supposed $\mathfrak{M}$-primary.
When $\mu(f,0)=1$? $\quad f-\mathrm{id}=(d_0f-\mathrm{id})+O(2)$.
$\mu(f,0)=1 \Leftrightarrow d_0f-\mathrm{id}\in GL_N(\mathbb{K}) \Leftrightarrow 1$ is NOT an eigenvalue of d_0f.

Proof (SHUB–SULLIVAN). Case $N=1$, when $\mu(f,0)\geq 2$:
$f(y)=y+ay^k+\dots \quad k\geq 2 \quad a\neq 0 \rightsquigarrow f^n(y)=y+any^k+\dots \Rightarrow \mu(f^n,0)=k \ \forall n$.
Lemma: Write $f=df+O(y^2)$. If $\sum_{j=0}^{n-1} df^j\in GL_N$, $\left(\text{this is } \approx \frac{df^n-\mathrm{id}}{df-\mathrm{id}}\right)$.
then $\mu(f^n,0)=\mu(f,0)$
(Lemma $\Rightarrow$ Theorem) may suppose df is upper triangular: $\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_N\end{pmatrix}$

$df=(a_1y_1+a_{12}y_2+\dots a_{1N}y_N,\ a_2y_2+a_{23}y_3+\dots a_{2N}y_N, \dots, a_Ny_N)$

$\mathrm{Spec}\left(\sum_0^{n-1} df^j\right)=\left(\sum_{j=0}^{n-1}a_1^j, \dots, \sum_{j=0}^{n-1}a_N^j\right)$. Hence $\sum_{j=0}^{n-1} df^j\notin GL_N \Leftrightarrow \exists i, \sum_{j=0}^{n-1}a_i^j=0$. $\quad \left(= \begin{cases} \frac{a_i^n-1}{a_i-1} & a_i\neq 1 \\ n & a_i=1.\end{cases}\right)$
$\Leftrightarrow \exists i \quad a_i$ is a n-root of 1, $a_i\neq 1$: $\quad a_i\in U_n\setminus\{1\}$.
$\Leftrightarrow n=mk$ and a_i is a primitive m-root of 1, $m\geq 2$.
Take n and set $\nu=\mathrm{lcm}\{m\geq 2, m|n, \exists i, a_i \text{ is a primitive } m\text{-root of } 1\}$
Apply the lemma to f^ν and $\frac{n}{\nu}$ iterations: $\mu(f^n,0)=\mu(f^\nu,0)$. $\square$

Proof of Lemma (as a purely formal computation)
$f=df+F_1 \qquad F_1\in(\mathfrak{M}^2\cdot\mathbb{K}[[y]])^N \qquad df\in \mathrm{Mat}_N(\mathbb{K})$.
$f^n=df^n+F_n \qquad F_n\in(\mathfrak{M}^2\cdot\mathbb{K}[[y]])^N$
$\mathrm{Id}-f^n=$ "$(\mathrm{Id}-f)\circ\sum_{j=0}^{n-1}f^j$" (it would work if f is linear)
$\mathrm{Id}-f^n=\underbrace{\mathrm{Id}-df^n}-\Theta_n, \qquad \Theta_n:=f^n-df^n. \quad (\Theta_1=f-df)$
$=(\mathrm{Id}+\dots+df^{n-1})(\mathrm{Id}-df)-\Theta_n$.
(*) $=\left(\sum_{j=0}^{n-1}df^j\right)(\mathrm{Id}-f)+\left(\sum_{j=0}^{n-1}df^j\right)(\underbrace{f-df}_{\Theta_1})-\Theta_n$. Notice that $\Theta_n=\sum_{j=0}^{n-1}(df)^{n-1-j}\Theta_1\circ f^j$.

Then. $\left(\sum_{j=0}^{n-1} df^j\right)\Theta_1-\Theta_n=\sum_{j=0}^{n-1} df^{n-1-j}(\Theta_1-\Theta_1\circ f^j)$.

Then: $\left(\sum_{j=0}^{n-1} df^j\right)\theta_1 - \theta_n = \sum_{j=0}^{n-1} df^{n-1-j}(\theta_1 - \theta_1\circ f^j)$.

Observation: write $\mathfrak{I}$ = ideal generated by the components of $(f - id)$

$\mu(f,0) = \dim \mathbb{K}[[y]]/\mathfrak{I}$ Fact: $\theta_1 - \theta_1 \circ f^j \in \mathfrak{I}\cdot\mathfrak{M}$. (*)

Proof of the Lemma: write $\mathfrak{I}_n = \langle f^n - id\rangle$ (ideal generated by the components ...)

let $g_n = \sum_{j=0}^{n-1} df^j \in GL_N$.

$\mathrm{Mult}(\mathfrak{I}_n) = \dim_{\mathbb{K}} \mathbb{K}[[y]]/\mathfrak{I}_n = \dim_{\mathbb{K}} \mathbb{K}[[y]]/g_n^{-1}\cdot\mathfrak{I}_n = \mathrm{mult}(\underbrace{g_n^{-1}\cdot\mathfrak{I}_n}_{=:\,\hat{\mathfrak{I}}_n})$

$\hat{\mathfrak{I}}_n = \langle g_n^{-1}(id - f^n)\rangle \overset{(*)}{=} \langle \underbrace{id - f}_{\mathfrak{I}} + g_n^{-1}\cdot\sum_{j=0}^{n-1} df^{n-1-j}\underbrace{(\theta_1 - \theta_1 f^j)}_{\in\,\mathfrak{I}\mathfrak{M}\ (*)}\rangle$

$\Rightarrow \mathrm{mult}(\hat{\mathfrak{I}}_n) = \mathrm{mult}(\mathfrak{I}_n)$

$\hat{\mathfrak{I}}_n$ is generated by $\hat{g}_1, \dots, \hat{g}_N$, with:

$$\begin{cases} \hat{g}_1 = y_1 - f_1 + \mathfrak{I}\mathfrak{M} \\ \vdots \\ \hat{g}_N = y_N - f_N + \mathfrak{I}\cdot\mathfrak{M}\end{cases}$$

$\hat{f}_1 := y_1 - f_1$, ..., $\hat{f}_N := y_N - f_N$. $\mathfrak{I} = \langle \hat{f}_1, \dots, \hat{f}_N\rangle$

$\hat{g}_i = \hat{f}_i + \sum_{j=1}^{N} a_{ij}\hat{f}_j$ $a_{ij} \in \mathfrak{M}$. $\Rightarrow$ $\hat{g}_i = (\mathrm{unit})\,\hat{f}_i + \sum_{j\neq i} a_{ij}\hat{f}_j$

Claim: $\langle \hat{g}_i\rangle = \langle f\rangle \Rightarrow \mathrm{mult}(\hat{\mathfrak{I}}_i) = \mathrm{mult}(\mathfrak{I}_i)$

$$i\neq j \quad \begin{pmatrix} 1 & & a_{ij}\in\mathfrak{M} \\ & \ddots & \\ a_{ij} & & 1\end{pmatrix}\begin{pmatrix}\hat{f}_1\\ \vdots \\ \hat{f}_N\end{pmatrix} = \begin{bmatrix}\hat{g}_1 \\ \vdots \\ \hat{g}_N\end{bmatrix}$$

invertible (its determinant is invertible in $\mathbb{K}[[y]]$) $\Rightarrow \langle g_i\rangle = \langle f_i\rangle$

(In Shub–Sullivan: they use a topological argument)

Proof of $\theta_1 - \theta_1\circ f^j \in \mathfrak{I}\cdot\mathfrak{M}$,

$\mathfrak{I} = \langle f - id\rangle$ $\theta_1 = f - df. = \sum a_I y^I \in \mathfrak{M}^2$.

m: ... $y^I - y^I$ ... $\mathfrak{I}\,m$

$\mathfrak{J} = \langle f - \mathrm{id} \rangle \qquad O_1 = f - df. = \langle \partial_I y^I \in \mathfrak{M}^k$.

Claim: $y^I - y^I \circ f \in \mathfrak{J} \mathfrak{M}$,

Case $N = 2$. $f = (f_1, f_2)$, $y_1^{e_1} \cdot y_2^{e_2} - f_1^{e_1} f_2^{e_2}$

$e_1 = e_2 = 1$: $y_1 y_2 - f_1 f_2 = y_1 y_2 - (f_1 - y_1 + y_1) f_2 \overset{\text{mod } \mathfrak{J}\cdot\mathfrak{M}}{=} y_1 y_2 - y_1 ((f_2 - y_2) + y_2) \overset{\text{mod } \mathfrak{J}\cdot\mathfrak{M}}{\equiv} 0$.

General case is similar (exercise). $\square$

Concluding remarks

$f : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ endo of degree $d \geq 2$.

- ⚠ in general $\sum_{f^n(x) = x} (\mu(f^n, x) - 1)$ might be unbounded.

Ex: $f(y_1, y_2) = (y_1^2, y_2 + y_2^2) : \mathbb{A}^2 \to \mathbb{A}^2$ extends to an endo of $d = 2$ to $\mathbb{P}^2$.

For ξ, $\xi^{2^n} = \xi$, the $(\xi, 0)$ has multiplicity $\mu(f^n, (\xi, 0)) = 2$.

Thm (Briend–Duval, Fornaess–Sibony)

$f : \mathbb{P}^N_{\mathbb{K}} \circlearrowleft$ endo, $d \geq 2$. $\mathrm{Card}\left(\begin{array}{l}\text{periodic points of period } n \\ \text{so that } \mu(f^n, \cdot) = 1\end{array}\right) \sim \frac{1}{d^{nN}}$.

In other words: $\sum_{f^n(x) = x} (\mu(f^n, x) - 1) = o(d^{nN})$

Problem: estimate $\sum_{f^n(x) = x} (\mu(f^n, x) - 1)$.