Cours 8

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5. Dynamical heights

Thm B $f : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ endomorphism of degree $d \geq 2$, $\mathbb{K}/\mathbb{Q}$ finite extension
Then the set $\mathrm{Preper}(f, \mathbb{K}) = \{x \in \mathbb{P}^N(\mathbb{K}),\ x \text{ has finite orbit}\}$ is finite

It will follow from the construction of a canonical height
$h_f : \mathbb{P}^N(\mathbb{K}^{alg}) \to \mathbb{R}_+$ s.t.:

(a) $h_f(f(x)) = d \cdot h_f(x)$

(b) Northcott: Fix $\delta, M > 0$. The set $\{x \in \mathbb{P}^N(\mathbb{K}^{alg}),\ h_f(x) \leq M,\ \deg(x) \leq \delta\}$ is finite.

$\mathrm{Gal}(\mathbb{K}^{alg}/\mathbb{K}) \curvearrowright \mathbb{P}^N(\mathbb{K}^{alg})$ $\quad x = [x_0 : \dots : x_N]$ $\quad \sigma \in \mathrm{Gal}(\mathbb{K}^{alg}/\mathbb{K})$
$\sigma(x) = [\sigma(x_0) : \dots : \sigma(x_N)]$. $\deg(x) :=$ Cardinality of the orbit of x under the action of $\mathrm{Gal}(\mathbb{K}^{alg}/\mathbb{K})$.

Fact: if $N = 1$, $\deg(x) = \deg_{\mathbb{K}}\left(\frac{x_1}{x_0}\right)$

Observation: (a) + (b) $\Rightarrow$ thm B: $\mathrm{Preper}(f, \mathbb{K}) \subseteq \{h_f = 0\}$

$$x \Rightarrow h_f(x) = \frac{1}{d^n} \underbrace{h_f(f^n(x))}_{\text{unif. bounded}} \to 0$$

1) Extend naive height from $N = 1$ to higher dimensions.

$\mathbb{K}/\mathbb{Q}$ finite extension $x = [x_0 : \dots : x_N] \in \mathbb{P}^N(\mathbb{K})$, $x_i \in \mathbb{K}$.

$M_{\mathbb{K}} = \{$set of multiplicative norms on $\mathbb{K}$ whose restriction to $\mathbb{Q}$ is $|\cdot|_p$, $p \in \mathcal{P} \cup \{\infty\}\}$

$\log \|x\|_v = \max \log |x_i|_v$

⚠ abuse of notation ($\log \|x\|_v$ depend on the representant of $x \in \mathbb{P}^N(\mathbb{K})$)

$\log \|\lambda x\|_v = \log \|\lambda\|_v + \log \|x\|_v$

Def: Standard (height) of $\mathbb{P}^N(\mathbb{K})$.

$x \in \mathbb{P}^N(\mathbb{K})$, $\quad h_{\mathbb{K}}(x) = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} n_v \cdot \log \|x\|_v \qquad n_v = \underset{\geq 1}{[\mathbb{K}_v : \mathbb{Q}_v]}$

Rem: $h_{\mathbb{K}}(x)$ does not depend on the representative: $\forall \lambda \in \mathbb{K}^*$

$$h_{\mathbb{K}}([\lambda x]) = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \cdot \sum n_v \log \|\lambda x\|_v = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \Big(\underbrace{\sum n_v \log \|\lambda\|_v}_{\overset{\shortparallel}{0} \text{ (product formula}} + \sum n_v \log \|x\|_v \Big) = h_{\mathbb{K}}([x])$$

- $h_{\mathbb{K}}(x) \geq 0$: Take $x = [x_0 : \dots : x_N]$ may assume $x_0 \neq 0$.

$\rightsquigarrow x = [1 : x_1' : \dots : x_N'] \quad x_i' := \frac{x_i}{x_0} \Rightarrow \log \|(1, x_i')\|_v = \max\{0, \log |x_i'|_v\} \geq 0$.

- Want to give a formula for $h_{\mathbb{K}}$ not depending on the extension $\mathbb{K}$.

For each $p \in M_{\mathbb{Q}}$, we fix an embedding $\mathbb{K} \hookrightarrow \mathbb{C}_p$ $(\mathbb{C}_\infty = \mathbb{C})$

$(*)$ $\forall x \in \mathbb{P}^N(\mathbb{K})$, $\quad h_{\mathbb{K}}(x) = \frac{1}{\deg(x)} \sum_{p \in M_{\mathbb{Q}}} \sum_{y \in G \cdot x} \log \|y\|_p \qquad G = \mathrm{Gal}(\mathbb{K}^{alg}/\mathbb{K})$

$\Rightarrow$ - If $x \in \mathbb{K} \cap \mathbb{K}'$, $\mathbb{K}, \mathbb{K}'/\mathbb{Q}$ finite (i.e., number fields), then $h_{\mathbb{K}}(x) = h_{\mathbb{K}'}(x)$.

- $\forall \sigma \in \mathrm{Gal}(\mathbb{K}^{alg}/\mathbb{K})$, $h_{\mathbb{K}}(x) = h_{\mathbb{K}}(\sigma(x))$.

Proof of $(*)$: identical to the proof in dimension 1:

For each $p \in M_{\mathbb{Q}}$, we set $M_{\mathbb{K},p} = \{v \in M_{\mathbb{K}},\ |\cdot|_v|_{\mathbb{Q}} = |\cdot|_p\}$

$(\sigma_1, \dots, \sigma_n)$ embedding of $\mathbb{K} \hookrightarrow \mathbb{C}_p$. $\qquad M_{\mathbb{K},p} = \{|\sigma_i|_p\}$

For each $v \in M_{\mathbb{K},p}$, $\mathrm{Card}\{i : |\sigma_i| = |\cdot|_v\} = [\mathbb{K}_v : \mathbb{Q}_v] = n_v$

Fix $p \in \mathcal{P} \cup \{\infty\}$. $\quad \sum_{y \in G \cdot x} \log \|y\|_p = \frac{\deg x}{[\mathbb{K}:\mathbb{Q}]} \cdot \sum_{i=1}^{n} \log \|\sigma_i(x)\|_p = (\quad) \sum_{v \in M_{\mathbb{K},p}} n_v \log \|x\|_v$.

Northcott's theorem:

$\mathcal{E}^N(M, \delta) = \{x \in \mathbb{P}^N(\mathbb{Q}^{alg}),\ \deg(x) \leq \delta,\ h(x) \leq M\}$ is finite.

Proof: without loss of generality (WLOG) $\quad x = [1 : x_1 : \dots : x_N] \quad x_i \in \mathbb{Q}^{alg}$

- $\deg_{\mathbb{Q}}(x_i) \leq \deg(x) \leq \delta$.

- $\log \max\{1, |x_i|_v\} \geq \log \max\{1, |x_i|_v\}$

$\log_{\mathbb{Q}}(\cdot) \le \log \cdots$

- $\log \max_i \{1, |x_i|_v\} \geq \log \max \{1, |x_j|_v\}$

$\Rightarrow M \geq h(x) \geq h(x_j) = h([1:x_j])$

$\mathcal{E}^N(M,\delta) \leq \underbrace{\mathcal{E}^1(M,\delta) \times \cdots \times \mathcal{E}^1(M,\delta)}_{N \text{ times.}}$. We already proved the Northcott property for $N=1$, and we are done $\square$

2) Canonical height

$\mathbb{K}/\mathbb{Q}$ number field, $f : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ of degree $d \geq 2$

Thm: the sequence $\frac{1}{d^n} h \circ f^n : \mathbb{P}^N(\mathbb{K}^{alg}) \to \mathbb{R}_+$ is converging uniformly to a function : $h_f : \mathbb{P}^N(\mathbb{K}^{alg}) \to \mathbb{R}_+$ s.t.

- $h_f \circ f = d \cdot h_f$
- $\sup |h_f - h| < \infty$
- $\forall \sigma \in Gal(\mathbb{K}^{alg}/\mathbb{K})$, $h_f(\sigma(x)) = h_f(x)$.

Consequences

$\mathrm{Preper}(f, \mathbb{K}) = \{h_f = 0\} \cap \mathbb{P}^N(\mathbb{K})$

Proof: $\subseteq$ did before.

$\supseteq$ by Northcott's theorem, applied to $\delta = [\mathbb{K}:\mathbb{Q}]$ and $M = \sup |h_f - h|$.

By Northcott, $\mathcal{E}^N(\delta, M)$ is finite

$x \in \mathbb{P}^N(\mathbb{K}) \cap \{h_f = 0\} \Rightarrow \deg(x) \leq \delta$, $h(x) \leq M + h_f(x) \leq M$.

Observe now that $\forall n \geq 0$, $f^n(x) \in \{h_f(0)\} \cap \mathbb{P}^N(\mathbb{K})$.

$\Rightarrow \mathrm{Orbit}_f(x) \subseteq \mathcal{E}^N(\delta, M)$ finite $\Rightarrow x \in \mathrm{Preper}(f, \mathbb{K})$ $\square$

Rem: the same holds for $\mathbb{K}^{alg}$: $\mathrm{Preper}(f, \mathbb{K}^{alg}) = \{h_f = 0\} \cap \mathbb{P}^N(\mathbb{K}^{alg})$: apply the previous case to any finite extension of $\mathbb{K}$.

Rem: $\mathcal{E}^N_f(\delta, M) = \{x \in \mathbb{P}^N(\mathbb{K}^{alg}), \deg(x) \leq \delta, h_f(x) \leq M\}$ is finite.
(property b):

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(property b):

$$\mathcal{E}^N_f(\delta, M) \subseteq \mathcal{E}^N(\delta \cdot [\mathbb{K}:\mathbb{Q}], M + \sup|h_f - h|)$$

$$\mathcal{E}^N(\delta', M') = \{x \in \mathbb{P}^N(\mathbb{Q}^{alg}), \deg_{\mathbb{Q}}(x) \le \delta, h(x) \le M\}.$$

Rem: $h_{\mathbb{K}}(x) = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \sum_{v \in M_{\mathbb{K}}} n_v \log \|x\|_v \quad x = [x_0 : \ldots : x_N]$.

Observe that for each i s.t. $x_i \neq 0$ - $\{v \in M_{\mathbb{K}} : \log |x_i|_v \neq 1\}$ is finite.

Application (KRONECKER's theorem)

$x \in (\mathbb{Q}^{alg})^N$ $x = (x_1 \ldots x_N)$. x_i algebraic integers and $\forall \sigma \in Gal(\mathbb{Q}^{alg}/\mathbb{Q})$, $|\sigma(x_i)|_\infty \le 1$. Then $|x_i|$ is a root of unity.

Proof: hypothesis $\Rightarrow h(x) = 0$. Take $d \geq 1$.

$\Phi_d(x) = [x_0^d : \ldots : x_N^d]$ $\quad \Phi_d : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$.

$h \circ \Phi_d = d \cdot h. \quad \Rightarrow h_{\Phi_d} = h \quad \forall d$.

Take $d = 2$: $h(x) = 0 \Rightarrow x$ is preperiodic for Φ_2. $\Rightarrow x_i^{2^\ell} = x_i^{2^{\ell+j}}$ $\ell \geq 0, j > 0$

Rem: Fix a number field $\mathbb{K}/\mathbb{Q}$ containing $G \cdot (x_i)$

hypothesis $\overset{(*)}{\Rightarrow} \forall v \in M_{\mathbb{K}}$, $|x_i|_v \le 1 \Rightarrow h_{\mathbb{K}}(x) = 0$.

if $v \in M_{\mathbb{K},\infty}$, follows from $|\sigma(x_i)|_\infty \le 1$

if $v \in M_{\mathbb{K},p}$, " " $\overset{*}{\Rightarrow}$

Proof of $\overset{(*)}{\Rightarrow}$: $y \in \mathbb{K}$ algebraic integer $\Rightarrow |y|_v \le 1$ $\forall v \in M_{\mathbb{K},p}$, $p \in P$.

Proof: $y^d + a_1 y^{d-1} + \ldots + a_d = 0$ $a_i \in \mathbb{Z}$. (polynomial given by $y \in \mathbb{Q}^{alg}$).

$\forall p \in P$, $|a_i|_p \le 1$. If $|y| > 1 \Rightarrow |0| = |y^d + \ldots + a_d| = |y|^d > 1$. Contradiction $\square$

Proof of the theorem.

It follows from the next proposition:

$\exists C$ s.t. $\left|\frac{1}{d} h \circ f(x) - h(x)\right| \le C \quad \forall x \in \mathbb{P}^N(\mathbb{K}^{alg}) \qquad (*)$

Suppose this is true, set $h_n = \frac{1}{d^n} h \circ f^n$.

$|h_{n+1}(x) - h_n(x)| = \left|\frac{1}{d^{n+1}} h \circ f^{n+1}(x) - \frac{1}{d^n} h \circ f^n(x)\right| = \frac{1}{d^n} \cdot |h \circ f(f^n(x)) - h(f^n(x))| \le \frac{C}{d^n}$.

suppose this is true, set $h_n = \frac{1}{d^n} h\circ f$.

$$|h_{n+1}(x) - h_n(x)| = \left|\frac{1}{d^{n+1}} h\circ f^{n+1}(x) - \frac{1}{d^n} h\circ f^n(x)\right| = \frac{1}{d^n}\cdot |h\circ f(f^n(x)) - h(f^n(x)| \leq \frac{C}{d^n}.$$

Since $\sum \frac{C}{d^n} < \infty$, $h_n(x)$ converges to a limit $h_f(x)$

$|h_f(x) - h_n(x)| \leq \frac{C'}{d^n}$ for some C'. If $n=0$, $|h\circ f(x) - h(x)| \leq C'$.

$$h_f\circ f(x) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(f(x))) = \lim_{n\to\infty} \frac{d}{d^{n+1}} h(f^{n+1}(x)) = d\cdot h_f(x).$$

The Proposition ($*$) is a consequence of Nullstellensatz

Recall $f(x) = [P_0(x) : \ldots : P_N(x)]$. (suppose $\mathbb{K} \hookrightarrow \mathbb{C}_p$).

- P_i homogeneous polynomials of degree $d \geq 2$ (in $N+1$ variables)

$\bigcap_{i=0}^{N} P_i^{-1}(0) = (0) \subseteq \mathbb{A}_{\mathbb{K}}^{N+1}$.

Goal: estimate $|P_i(x)|$ in terms of $|x|$.

$P_i(x) = \sum_{|J|=d} a_{i,J} x^J$ $\quad J = (j_0, \ldots, j_N)$ $\quad |J| = j_0 + \ldots + j_N$, $a_{iJ} \in \mathbb{K}$.

$x^J = x_0^{j_0} \cdots x_N^{j_N}$.

$v \in M_{\mathbb{K}}$ $\quad |P_i|_v = \max_J |a_{i,J}|$.

Choose $x \in \mathbb{P}^N(\mathbb{K})$, $v \in M_{\mathbb{K}}$. $\log \|f(x)\|_v$ appears in $h\circ f(x)$.

- $\log \|f(x)\|_v \underset{\substack{\text{abuse}\\\text{of notation}}}{=} \{\log |P_0(x)|_v, \ldots, \log |P_N(x)|_v\}$.

$$|P_i(x)|_v = \left|\sum_{|J|=d} a_{i,J} x^J\right|_v \underset{\text{if } v \text{ is N.A.}}{\leq} |P_i|_v \|x\|_v^d \quad \text{since } |x^J|_v \leq \max\{|x_0|, \ldots, |x_N|\}^d.$$

$$\underset{v\in M_{\mathbb{K},\infty}}{\leq} C(d)\, |P_i|_v \|x\|_v^d \qquad C(d) = \#\{J \in \mathbb{N}^d_- \;\; |J| = d\}.$$

$$\log \|f(x)\|_v \leq d \log \|x\|_v + \log \|f\|_v + \begin{cases} 0 & v \text{ N.A.} \\ \log C(d) & v \text{ Arch.}\end{cases}$$

$$\|f\|_v = \max\{|P_0|_v, \dots, |P_N|_v\}$$

$$h(f(x)) = \frac{1}{[K:\mathbb{Q}]} \cdot \sum_{v \in M_K} n_v \log \|f(x)\|_v \leq \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v (d \log \|x\|_v + A_v(f)).$$

$A_v(f) = 0$ except for finitely many v.

$$\leq d\, h(x) + \underbrace{C(f)}_{= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v A_v(f)}$$

Rem: For the upper bound, we didn't use $\bigcap_i P_i^{-1}(0) = (0)$

we use it for the lower bound.

- The lower bound uses in an essential way the Nullstellensatz

$\mathfrak{a} := (P_0, \dots, P_N) \subseteq K[x_0, \dots, x_N]$, $V(\mathfrak{a}) = \bigcap_{i=0} (P_i = 0) = (0) = V(x_0, \dots, x_N)$.

$\sqrt{\mathfrak{a}} = \sqrt{(x_0, \dots, x_N)} \Rightarrow \exists\, i \geq 1,\ (x_0, \dots, x_N)^e \subseteq \mathfrak{a}$

$$x_0^e = \sum_{i=0}^N g_{0,i} P_i \qquad x_1^e = \sum_{i=0}^N g_{1,i} P_i \quad \dots \quad x_N^e = \sum_{i=0}^N g_{N,i} P_i \qquad g_{j,i} \in K[x_0 \dots x_N]$$

Since P_i and x_j^e are homogeneous, you may suppose that $g_{j,i}$ are also homogeneous of degree $e-d$

Explanation: write $g_{0,i} = \sum_k \underbrace{g_{0,i}^{(k)}}_{\text{homogeneous of degree } k}$.

$$\sum_{i=0}^N g_{0,i}^{(k)} P_i = 0 \quad \text{except for } k = e-d.$$

$x = (x_0, \dots, x_N) \in K^{N+1}$, $v \in M_K$

$$\|x\|_v^e = \max |x_i|_v^e \leq C_v \cdot \max |g_{ji}|_v \cdot \max |P_i|_v.$$

$$C_v = \begin{cases} 1 & \text{if } v \text{ is NA.} \\ C(d) & \text{as before if } v \text{ is Archi.} \end{cases}$$

$$\text{Max}\, |g_{ji}(x)|_v \leq C_{g,f} \max |x_i|^{e-d} \leq C \|x\|^{e-d}.$$

$$\rightsquigarrow \|x\|_v^e \leq C_v \cdot C(f) \cdot \|x\|^{e-d} \cdot \|f(x)\| \quad \text{divide by } \|x\|^e.$$

$$\Rightarrow \|x\|^d \leq C_v \cdot C(f) \cdot \|f(x)\|$$

$\Rightarrow \quad \|x\|^d \le C_v \cdot C(f) \cdot \|f(x)\|$

We need to check: $\{ v \in M_K , C_v \cdot C(f) \neq 1 \}$ is infinite.

$\Rightarrow d \log \|x\|_v \le C_v' + \log \|f(x)\|_v$, where $C_v' = 0$ except for finitely many v.

$\Rightarrow d\, h(x) \le C + h(f(x))$. □

Conjecture (LEHMER)

$\exists K > 0$ if $x \in (\mathbb{Q}^{\deg})^N$ s.t. one of the coords is not a root of unity:

$\deg(x)\, h(x) \ge K$.

OSTROWOLSKI (1979)

$$\exists K \quad h(x) \ge \frac{K}{D} \left(\frac{\log \log (3D)}{\log (3D)} \right)^3 \qquad D = \deg x.$$

Best Bounds so far

(SILVERMAN)

Conjecture (Dynamical LEHMER)

$\forall f : \mathbb{P}^N_K \circlearrowleft$ endo of degree $d \ge 2$, $K/\mathbb{Q}$ is finite.

$\exists K_f$, $\deg_K(x) \cdot h_f(x) \ge K \quad \forall x \in \mathbb{P}^N(K^\infty)$, $h_f(x) \neq 0$.

Rem: careful, in the proof we should work on an extension of K that contains all coeff $g_{j,i}$.