Dynamical system on valuation space

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Remember yesterday

•
$$P : \mathbb{C}^2 \to \mathbb{C}^2$$
 polynomial, dominant.

• $d_n = \deg(P^n), d_\infty = \lim_n d_n^{1/n}$

Theorem

- Either P = (Q(x), R(x, y)) is a skew product;
- Or $d_{\infty}^n \leq d_n \leq C \cdot d_{\infty}^n$

 d_{∞} is a quadratic integer.

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Method

V₁ = {ν : C[x, y] → R centered at ∞, ν(φ) < 0, A(ν) < 0} P_{*}ν(φ) = ν(φ ∘ P).

Study the dynamics of $P_* : \mathcal{V}_1 \rightarrow \mathcal{V}_1$

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Key results

Theorem

 \mathcal{V}_1 is a tree

Theorem (Eigenvaluation)

 ${\sf P}_*
u = \lambda
u$ for some $u \in {\mathcal V}_1$

Theorem (Structure of valuations in \mathcal{V}_1)

Suppose $\nu \in \mathcal{V}_1$

- *Either* $C_1(-\deg) \le \nu \le C_2(-\deg)$
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- 2 The valuative space is a tree
 - The elements
 - The topology
- 3 Global valuations
 - Definition of \mathcal{V}_1
 - Thinness
 - Proof of the structure theorem

Oynamics of P_{*}

- Fixed point theorem
- Attracting eigenvaluation

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The elements The topology

Valuations

Definition

Valuation $\nu : \mathbb{C}[x, y] \setminus \{0\} \to \mathbb{R}$

•
$$u|_{\mathbb{C}^*} \equiv 0;$$

•
$$\nu(\phi_1\phi_2) = \nu(\phi_1) + \nu(\phi_2);$$

•
$$\nu(\phi_1 + \phi_2) \ge \min\{\nu(\phi_1), \nu(\phi_2)\};$$

• ν centered at infinity.

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The elements The topology

Examples



Quasimonomial valuation



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- Monomial valuation. $\nu_s(\sum a_{ij}x^iy^j) = \min\{is_1 + js_2, a_{ij} \neq 0\}$
- Divisorial valuation
- Quasimonomial or Abhyankhar valuations.
- Zariski or infinitely singular valuations

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The elements The topology

Pencil valuations

• *C* with one place at infinity $C = P^{-1}(0)$

- Moh $\Rightarrow C_{\lambda} = P^{-1}(\lambda)$ has one place at infinity.
- $\nu_C(Q) = -\frac{(C \cdot Q^{-1}(0))_{\mathbb{C}^2}}{\deg(C)}$ curve valuation

•
$$\nu_{|C|}(Q) = \min_{\lambda} \nu_{C_{\lambda}}(Q)$$

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Topology

• $\mathcal{V} = \overline{\{ \text{ normalized valuations } \min\{\nu(x), \nu(y)\} = -1 \} }$

- Order relation $\nu \leq \mu \Leftrightarrow \forall \phi, \nu(\phi) \leq \mu(\phi)$
- Compact for the pointwise convergence.

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The elements The topology

Geometry

Theorem

 (\mathcal{V},\leq) is a tree:

• - deg is the unique minimal element;

• $(\{-\deg \leq \cdot \leq \nu\}, \leq) \simeq ([0, 1], \leq)$

Quasimonomial segments: { $\pi_*\nu_{(s_1,s_2)}$ s.t. $a_1s_1 + a_2s_2 = -1$ }.

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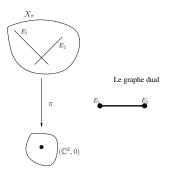
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The elements The topology

Why \mathcal{V} is a tree?

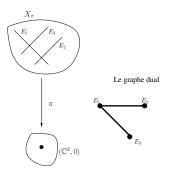


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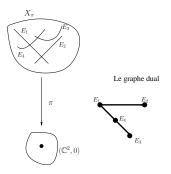


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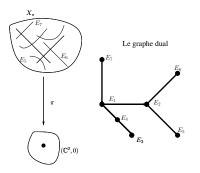


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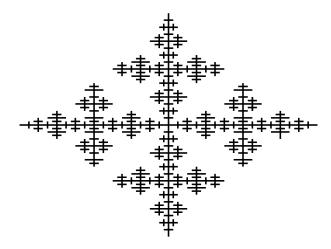


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 $\begin{array}{l} \text{Definition of } \mathcal{V}_1 \\ \text{Thinness} \\ \text{Proof of the structure theorem} \end{array}$

Idea

Ex. P(x, y) = (x, xy), $P_* \nu_{s,t} = \nu_{s,s+t}$ hence $P_* \mathcal{V} \not\subset \mathcal{V}$

- Valuations are local object
- Some carry global informations:
 - - deg
 - pencil valuation

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A new valuation space

Definition

$$\mathcal{V}_1 = \overline{\{\nu \in \mathcal{V}, \, \nu(\phi) < \mathbf{0} \, \forall \phi, \text{ and } A(\nu) < \mathbf{0}\}}$$

Theorem

 \mathcal{V}_1 is a closed subtree of \mathcal{V} .

Theorem

 $\nu \in \mathcal{V}_1$

- Either $C_1(-\deg) \le \nu \le C_2(-\deg)$
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Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Thinness

$\pi: X \to \mathbb{P}^2, E \subset \pi^{-1}L_{\infty}$ • $A(\operatorname{div}_E) = \operatorname{div}_E(\pi^* dx \wedge dy) + 1$ • $A(t\nu) = tA(\nu)$

Charles Favre Dynamics and valuations

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• $A(t\nu) = tA(\nu)$
• $A(-\operatorname{deg}) = -2$

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Theorem

For all
$$\nu \in \mathcal{V}$$
, $A : [-\deg, \nu] \rightarrow [-2, A(\nu)]$ is a bijection

Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Proof

Theorem

 $\overline{\{\nu \in \mathcal{V}, \nu(\phi) < 0 \,\forall \phi, \text{ and } A(\nu) < 0\}}$ is a closed subtree of \mathcal{V} .

Proof.

- $\nu \mapsto \nu(\phi)$ is increasing
- $\nu \mapsto A(\nu)$ is increasing

Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Proof

Theorem

 $\overline{\{\nu \in \mathcal{V}, \nu(\phi) < \mathbf{0} \, \forall \phi, \text{ and } \mathbf{A}(\nu) < \mathbf{0}\}}$ is a closed subtree of \mathcal{V} .

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Charles Favre Dynamics and valuations

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Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Proof



Theorem

 $\nu \in \mathcal{V}_1$

- Either $C_1(-\deg) \le \nu \le C_2(-\deg)$
- Or ν is a rational pencil valuation

Idea of proof. $p = \text{center of } \nu$.

- $\nu \rightsquigarrow \{P_k\}$ key polynomials
 - Monomial $\rightsquigarrow \{X, Y\}$
 - Quasim. $\rightsquigarrow \{X, Y, X^p + \theta Y^q, \cdots, P_N\}$
- $P_k \in \mathbb{C}[x, y]$ irreducible, analytically irreducible at p

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Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

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Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Key polynomials

Theorem $\nu \rightsquigarrow \{P_k\}$ $\nu \in \mathcal{V}_1 \Rightarrow P_k^{-1}(0)$ has one place at infinity . $\sup \frac{\nu(\phi)}{\deg(\phi)} = \max \frac{\nu(P_k)}{\deg(P_k)}$.

induction on the number of key pol.

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Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Key polynomials

Theorem

$$\begin{array}{l} \nu \rightsquigarrow \{P_k\}\\ \nu \in \mathcal{V}_1 \Rightarrow P_k^{-1}(0) \text{ has one place at infinity }.\\ \sup \frac{\nu(\phi)}{\deg(\phi)} = \max \frac{\nu(P_k)}{\deg(P_k)} \ . \end{array}$$

induction on the number of key pol.

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Definition of \mathcal{V}_1 Thinness Proof of the structure theorem

Key polynomials

Theorem

induction on the number of key pol.

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induction on the number of key pol.

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Fixed point theorem Attracting eigenvaluation

Basics

•
$$P_*\nu(\phi) = \nu(\phi \circ P)$$

$$\bullet P_*\nu = d(P,\nu) \times P_\bullet\nu$$

•
$$P_{\bullet}: \mathcal{V}_1 \to \mathcal{V}_1$$
 continuous

• $d(P,\nu) = -\min\{\nu(x \circ P), \nu(y \circ P)\} \ge 0$

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Fixed point theorem Attracting eigenvaluation

Basics

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$$P_*\nu(\phi) = \nu(\phi \circ P)$$

•
$$A(P_*\nu) = A(\nu) + \nu(\operatorname{Jac}(P))$$

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$$d(P, \nu) = -\min\{\nu(x \circ P), \nu(y \circ P)\} \ge 0$$

Fixed point theorem Attracting eigenvaluation

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$$P_*\nu = d(P,\nu) \times P_*\nu$$

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• $P_*\nu = d(P,\nu) \times P_*\nu$
• $P_* : \mathcal{V}_1 \to \mathcal{V}_1 \text{ continuous}$
• $d(P,\nu) = -\min\{\nu(x \circ P), \nu(y \circ P)\} \ge 0$

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Fixed point theorem Attracting eigenvaluation

Local degree

• $d(P, \nu) = -\min\{\nu(x \circ P), \nu(y \circ P)\}$



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Fixed point theorem Attracting eigenvaluation

Local degree

•
$$d(P, \cdot) : \mathcal{V}_1 \to \mathbb{R}_+$$

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-ixed point theorem Attracting eigenvaluation

Local degree

•
$$d(P, \cdot) : \mathcal{V}_1 \to \mathbb{R}_+$$

•
$$d(P, -\deg) = \deg(P)$$

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Fixed point theorem Attracting eigenvaluation

Local degree

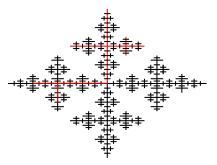
- $d(P, \cdot) : \mathcal{V}_1 \to \mathbb{R}_+$
- $d(P, -\deg) = \deg(P)$
- $d(P, \cdot)$ decreasing and locally cst outside a finite tree.

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Fixed point theorem Attracting eigenvaluation

Local degree

- $d(P, \cdot) : \mathcal{V}_1 \to \mathbb{R}_+$
- *d*(*P*, − deg) = deg(*P*)
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Fixed point theorem Attracting eigenvaluation

Fixed pt thm

Theorem (Eigenvaluation)

 $P_{\bullet}: \mathcal{V}_1 \rightarrow \mathcal{V}_1$ has a fixed point $P_* \nu = \lambda \nu$ for some $\nu \in \mathcal{V}_1$

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Fixed point theorem Attracting eigenvaluation

Fixed pt thm

Theorem (Eigenvaluation)

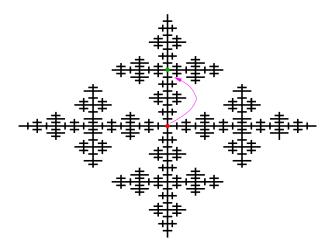
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Fixed point theorem Attracting eigenvaluation

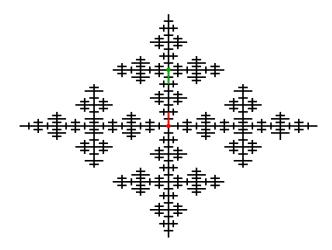
Proof



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Fixed point theorem Attracting eigenvaluation

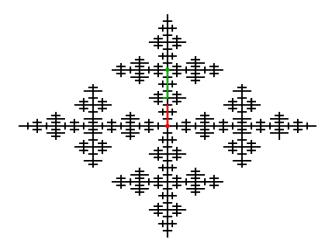
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Fixed point theorem Attracting eigenvaluation

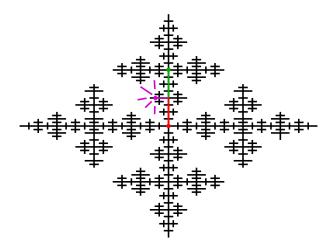
Proof



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Fixed point theorem Attracting eigenvaluation

Proof



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Fixed point theorem Attracting eigenvaluation

Summary

• $P_*\nu = \lambda \nu$

- ν rational pencil \Rightarrow *P* skew product
- $C_1(-\deg) \leq \nu \leq C_2(-\deg)$
 - $\lambda^n/\deg(P^n) \in [C_2, C_1]$

•
$$d_{\infty} = \lambda$$

Why d_{∞} is a quadratic integer?

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Fixed point theorem Attracting eigenvaluation

Summary

•
$$P_*\nu = \lambda \nu$$

• ν rational pencil \Rightarrow *P* skew product

C₁(-deg) ≤ ν ≤ C₂(-deg)
 λⁿ/deg(Pⁿ) ∈ [C₂, C₁]
 d_∞ = λ

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 is a quadratic integer?

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Fixed point theorem Attracting eigenvaluation

Attracting eigenvaluation

Theorem

 $P_{\bullet}\nu_{\star} = \nu_{\star}.$ There exists $U \ni \nu_{\star}$ such that $P_{\bullet}^{n}\nu \to \nu_{\star}$ for all $\nu \in U.$

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Fixed point theorem Attracting eigenvaluation

Attracting eigenvaluation

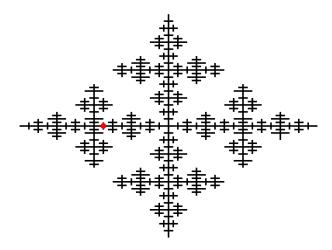
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Fixed point theorem Attracting eigenvaluation

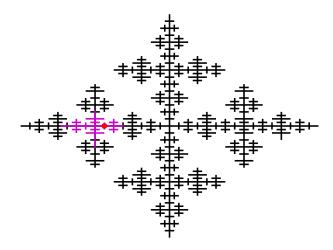
Animation



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Fixed point theorem Attracting eigenvaluation

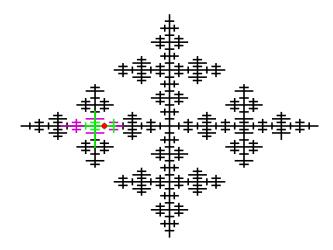
Animation



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Fixed point theorem Attracting eigenvaluation

Animation



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Fixed point theorem Attracting eigenvaluation

Idea of proof

For a suitable parameterization on \mathcal{V}_1

$$P_{\bullet}: \alpha \mapsto \frac{a\alpha + b}{c\alpha + d}$$

with $a, b, c, d \in \mathbb{N}$.

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Fixed point theorem Attracting eigenvaluation

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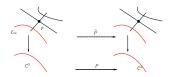
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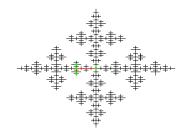
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Fixed point theorem Attracting eigenvaluation

Consequences in picture





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Fixed point theorem Attracting eigenvaluation

Towards monomialization

Theorem

There exists $\pi: X \to \mathbb{P}^2$ and $p \in \pi^{-1}(L_{\infty})$ such that

• P is holomorphic at p

• Critical set of P is included in $\pi^{-1}(L_{\infty})$ and contracted to p.

Normal form for (P, p)

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Fixed point theorem Attracting eigenvaluation

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Fixed point theorem Attracting eigenvaluation

Rigid germs

Definition

 $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is rigid if $\bigcup_n \operatorname{Crit} (f^n)$ has normal crossing singularities.

Theorem (Favre 2000)

Suppose f rigid and not invertible. Then f =

- $(\alpha z, wz^q + P(z))$
- $(\alpha z, W^p)$
- $(z^{p}, \lambda w z^{q} + P(z))$
- (α*z*, *z^cw^d*)
- $(Z^a W^b, Z^c W^d)$

Fixed point theorem Attracting eigenvaluation

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- (αz, z^cw^d)

o ...

• $(z^a w^b, z^c w^d)$

Fixed point theorem Attracting eigenvaluation

Conclusion

- When ν_{*} is divisorial or an end point then d(P, ν_{*}) is an integer
- When ν_{*} is irrational qm.
 (P, p) is locally monomial
- Local computation \Rightarrow d_{∞} is a quadratic integer.

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Fixed point theorem Attracting eigenvaluation

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