Arch. Rational Mech. Anal. 153 (2000) 177–204. © Springer-Verlag 2000

The Acoustic Limit for the Boltzmann Equation

CLAUDE BARDOS, FRANÇOIS GOLSE & C. DAVID LEVERMORE

Communicated by P.-L. LIONS

Abstract

The acoustic equations are the linearization of the compressible Euler equations about a spatially homogeneous fluid state. We first derive them directly from the Boltzmann equation as the formal limit of moment equations for an appropriately scaled family of Boltzmann solutions. We then establish this limit for the Boltzmann equation considered over a periodic spatial domain for bounded collision kernels. Appropriately scaled families of DiPerna-Lions renormalized solutions are shown to have fluctuations that converge entropically (and hence strongly in L^1) to a unique limit governed by a solution of the acoustic equations for all time, provided that its initial fluctuations converge entropically to an appropriate limit associated to any given L^2 initial data of the acoustic equations. The associated local conservation laws are recovered in the limit.

1. Introduction

The endeavor to understand how fluid dynamical equations arise from kinetic theory originates in the founding works of MAXWELL [21] and BOLTZMANN [10]. While there has been considerable success at the formal level, full mathematical justifications have proved elusive. Here we establish the so-called acoustic fluid dynamical limit for the classical Boltzmann equation considered over a periodic spatial domain for bounded collision kernels. We do so in the physical setting of DiPerna-Lions renormalized solutions.

1.1. The Boltzmann Equation

The state of a fluid composed of identical point particles confined to a spatial domain $\Omega \subset \mathbf{R}^D$ is described at the kinetic level by a mass density F over the single-particle phase space $\Omega \times \mathbf{R}^D$. At any instant of time $t \geqq 0$ and point

$(x, v) \in \Omega \times \mathbf{R}^D$, $F(t, x, v)\, dv\, dx$ is understood to give the mass of the particles that occupy the infinitesimal volume $dv\, dx$ about the point (x, v). To remove complications due to boundaries, we take Ω to be the periodic box $\mathbf{T}^D = \mathbf{R}^D/\mathbf{Z}^D$. If the particles interact only through a conservative interparticle force with a finite range, then at low densities this range will be much smaller than the interparticle spacing. In that regime all but binary collisions can be neglected and the evolution of $F = F(t, x, v)$ is governed by the classical Boltzmann equation [13]:

$$\partial_t F + v\cdot\nabla_x F = \mathcal{B}(F, F), \qquad F(0, x, v) = F^{\text{in}}(x, v) \geqq 0. \tag{1.1}$$

The Boltzmann collision operator $\mathcal{B}(F, F)$ acts only on the v argument of F. It is formally given by

$$\mathcal{B}(F, F) = \iint_{\mathbf{S}^{D-1}\times\mathbf{R}^D} (F_1' F' - F_1 F)\, b(v_1 - v, \omega)\, d\omega\, dv_1, \tag{1.2}$$

where v_1 ranges over $\mathbf{R}^D$ endowed with its Lebesgue measure dv_1 while ω ranges over the unit sphere $\mathbf{S}^{D-1} = \{\omega \in \mathbf{R}^D \,:\, |\omega| = 1\}$ endowed with its rotationally invariant unit measure $d\omega$. The F, F_1, F' and F_1' appearing in the integrand are understood to mean $F(t, x, \cdot)$ evaluated at the velocities v, v_1, v' and v_1' respectively, where the primed velocities are defined by

$$v' = v + \omega\,\omega\cdot(v_1 - v), \qquad v_1' = v_1 - \omega\,\omega\cdot(v_1 - v), \tag{1.3}$$

for any given $(\omega, v_1, v) \in \mathbf{S}^{D-1}\times\mathbf{R}^D\times\mathbf{R}^D$. Quadratic operators like $\mathcal{B}$ are extended by polarization to be bilinear and symmetric.

The unprimed and primed velocities are possible velocities for a pair of particles either before and after, or after and before, they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during collisions is expressed as

$$v + v_1 = v' + v_1', \qquad |v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2. \tag{1.4}$$

Equation (1.3) represents the general solution of these $D + 1$ equations for the $4D$ unknowns v, v_1, v', and v_1' in terms of the $3D - 1$ parameters (ω, v_1, v).

The Boltzmann kernel b is a nonnegative, locally integrable function. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(v_1 - v, \omega) = |v_1 - v|\, \Sigma(|v_1 - v|, |\mu_c|), \qquad \mu_c = \frac{\omega\cdot(v_1 - v)}{|v_1 - v|}, \tag{1.5}$$

where Σ is the specific differential cross-section. It will be assumed that there exists a constant $C < \infty$ that is independent of ω such that b satisfies the bounds

$$0 < b(v_1 - v, \omega) \leqq C\left(1 + |v_1 - v|^2\right) \quad \text{almost everywhere.} \tag{1.6}$$

This condition is met by classical Boltzmann kernels with a small deflection cut-off (see [12, Chapter II.4, 5]). Additional technical requirements on b will be imposed later.

1.2. *Fluid Dynamical Approximations*

Fluid dynamical regimes are those where the mean free path is small compared to the macroscopic length scales. Formal derivations of the compressible Euler equations are rather direct. Formal derivations of other fluid dynamical equations, such as the compressible Navier-Stokes equations, are more subtle. Early derivations of the Navier-Stokes equations rested on arguments as to how the various terms in a kinetic equation balance each other. These balance arguments seemed arbitrary to some, so HILBERT [18] proposed that such derivations should be based on a systematic asymptotic expansion. This expansion takes the form of a power series in a nondimensional parameter $\varepsilon \ll 1$, now called the Knudsen number, that is a ratio of the mean free path to the macroscopic length scales. With the Knudsen number introduced [3], the initial-value problem for the Boltzmann equation (1.1) takes the nondimensional form

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\varepsilon} \mathcal{B}(F, F), \qquad F(0, x, v) = F^{\text{in}}(x, v) \geqq 0. \tag{1.7}$$

A bit later ENSKOG [16] proposed a somewhat different asymptotic expansion, now called the Chapman-Enskog expansion, in the same small parameter ε. Either the Hilbert or the Chapman-Enskog expansion yields at successive orders: the compressible Euler equations, the compressible Navier-Stokes equations, the Burnett equations, and the so-called super-Burnett equations (see [17]).

Justification of these formal approximations has proved difficult in part because many basic well-posedness and regularity questions remain open for both these fluid equations and the Boltzmann equation. The problem is exacerbated by the fact that to bound the error of the asymptotic expansions requires the control of successively higher order spatial derivatives of the fluid variables, thereby requiring unphysical restrictions to a meager subset of all physically natural initial data and possibly to finite periods of time. For example, the compressible Euler equations have been derived from the Boltzmann equation by CAFLISCH [11] using a method based on the Hilbert expansion; this derivation holds for smooth initial data and for as long as the limiting solution of the compressible Euler system is smooth. Because solutions of the compressible Euler equations are known to become singular in finite time for a very general class of initial data (see [22]), Caflisch's result [11] is about the best one can hope for by appealing to the Hilbert expansion.

Two approaches to circumventing these difficulties have emerged recently. First, some authors have studied direct derivations of incompressible Stokes, Navier-Stokes, and Euler equations [1, 2, 6–8, 14, 9, 23, 24] about which more is known. Second, some authors have abandoned the traditional expansion-based derivations in favor of moment-based formal derivations [2, 6, 7, 9], which put fewer demands on the well-posedness and regularity theory. Here we embrace both of these approaches.

To begin with, we will also consider a fluid dynamical limit obtained through a scaling in which the density F is close to a spatially homogeneous Maxwellian $M = M(v)$. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{D/2}} \exp(-\tfrac{1}{2}|v|^2). \tag{1.8}$$

This corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and the bulk velocity equal to 0. If the compressible Euler equations are linearized about this state, then one obtains the acoustic equations

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u &= 0, & \rho(0,x) &= \rho^{\mathrm{in}}(x), \\ \partial_t u + \nabla_x (\rho + \theta) &= 0, & u(0,x) &= u^{\mathrm{in}}(x), \\ \tfrac{D}{2} \partial_t \theta + \nabla_x \cdot u &= 0, & \theta(0,x) &= \theta^{\mathrm{in}}(x). \end{aligned} \tag{1.9}$$

This is about the simplest system of fluid dynamical equations imaginable, being essentially the wave equation. We will show how it can be formally derived from the Boltzmann equation. We will also employ a moment-based formal derivation that will enable us to establish the acoustic limit within the class of DiPerna-Lions global weak solutions to the classical Boltzmann equation.

The existence of global weak solutions to the classical Boltzmann equation for all initial data within the entropy class was first established by DiPerna and Lions [15]. Their theory has the virtue of considering physically natural classes of initial data. However, it suffers deficiencies in that its solutions are not known either to be unique, or to satisfy all the local conservation laws one would formally expect. These solutions were subsequently studied in the incompressible Navier-Stokes limit [3] and the incompressible Euler limit [9] with partial success, and in the linearized Boltzmann limit [19] with complete success. In those studies a notion of *entropic convergence* was used as a natural tool for obtaining strong convergence results for fluctuations about an absolute Maxwellian. This paper establishes the convergence of such fluctuations of DiPerna-Lions solutions to so-called infinitesimal Maxwellians that have the form

$$\rho + u \cdot v + \theta(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}), \tag{1.10}$$

and that are each governed by an L^2 solution (ρ, u, θ) of the acoustic equations (1.9). Here, the notion of entropic convergence again plays a major role.

Loosely stated, our main result (Proposition 4.2, announced in [4]) is the following: for any L^2 initial data for the acoustic equations and any sequence of DiPerna-Lions solutions whose initial fluctuations about M converge entropically to the infinitesimal Maxwellian associated with that L^2 initial data, the fluctuations of the DiPerna-Lions solutions converge entropically to the infinitesimal Maxwellian associated with the L^2 solution of the acoustic initial-value problem (1.9) for all positive values of time. The key points being made are that the limit of the DiPerna-Lions Boltzmann dynamics maps *onto* the L^2 acoustic dynamics, and that the limit is strong. The main obstacle we overcome is that the DiPerna-Lions solutions are not known to satisfy local conservation laws of either momentum or energy.

The next section contains preliminary material regarding the Boltzmann equation and the formal derivation of the acoustic equations. Section 3 lays the analytical groundwork. It includes a statement of the DiPerna-Lions result and of the basic results on fluctuations from [3]. These propositions are fully stated for completeness. Their proofs can essentially be found in [15] and [3], and so are not reproduced here. It also reintroduces the notion of entropic convergence. Section 4 establishes the acoustic limit. The associated local conservation laws of momentum and energy are recovered only in the limit. In a companion paper [5] we establish a similar result for an incompressible Stokes limit, also announced in [4].

2. Formal Preliminaries

In this section are recalled the basic formal properties of the Boltzmann equation, together with a formal derivation of the acoustic equations by a moment-based method, in the style of [2]. The notation introduced here is a subset of that in the first section of [3].

2.1. Formal Structure of the Boltzmann Equation

It is natural to introduce the relative density, $G = G(t, x, v)$, defined by $F = MG$. Recasting the initial-value problem (1.7) for G yields

$$\partial_t G + v \cdot \nabla_x G = \frac{1}{\varepsilon} \mathcal{Q}(G, G), \tag{2.1a}$$

$$G(0, x, v) = G^{\text{in}}(x, v) \geqq 0, \tag{2.1b}$$

where the collision operator is now given by

$$\mathcal{Q}(G, G) = \iint_{\mathbf{S}^{D-1}\times\mathbf{R}^D} (G_1' G' - G_1 G)\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1. \tag{2.2}$$

We take the nondimensionalization with the normalizations

$$\int_{\mathbf{S}^{D-1}} d\omega = 1, \qquad \int_{\mathbf{R}^D} M dv = 1, \qquad \int_{\mathbf{T}^D} dx = 1, \tag{2.3}$$

associated with the domains $\mathbf{S}^{D-1}$, $\mathbf{R}^D$, and $\mathbf{T}^D$ respectively, and

$$\iiint_{\mathbf{S}^{D-1}\times\mathbf{R}^D\times\mathbf{R}^D} b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\, M dv = 1, \tag{2.4}$$

associated with the Boltzmann kernel.

Because $M dv$ is a positive unit measure on $\mathbf{R}^D$, we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$\langle \xi \rangle = \int_{\mathbf{R}^D} \xi(v)\, M dv. \tag{2.5}$$

Because $b(v_1-v,\omega)\,d\omega\,M_1dv_1\,Mdv$ is a positive unit measure on $\mathbf{S}^{D-1}\times\mathbf{R}^D\times\mathbf{R}^D$, we denote by $\langle\!\langle\Xi\rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(v,v_1,\omega)$,

$$\langle\!\langle\Xi\rangle\!\rangle = \iiint_{\mathbf{S}^{D-1}\times\mathbf{R}^D\times\mathbf{R}^D} \Xi(v,v_1,\omega)\,b(v_1-v,\omega)\,d\omega\,M_1dv_1\,Mdv. \tag{2.6}$$

We now present the basic formal structure of the Boltzmann equation in the setting of this notation for later reference. All of these results are standard and their proofs can be essentially found in [12, Chapter II.6, 7].

The structure of the Boltzmann equation derives from properties of the Boltzmann collision operator (2.2) relating to conservation, dissipation, and Galilean symmetry. The key to these properties is the following identity discovered by Boltzmann:

$$\begin{aligned}\langle\xi\,\mathcal{Q}(G,G)\rangle &= \langle\!\langle\xi\,(G_1'G'-G_1G)\rangle\!\rangle\\ &= \tfrac14\langle\!\langle(\xi+\xi_1-\xi'-\xi_1')\,(G_1'G'-G_1G)\rangle\!\rangle,\end{aligned} \tag{2.7}$$

for every $\xi=\xi(v)$ and $G=G(v)$ for which the integrals make sense. Here we will recall just those properties related to conservation and dissipation.

First, upon successively setting $\xi = 1, v_1, \cdots, v_D, |v|^2$ into the Boltzmann identity (2.7), the microscopic conservation laws (1.4) yield the conservation laws

$$\langle\mathcal{Q}(G,G)\rangle=0,\qquad \langle v\,\mathcal{Q}(G,G)\rangle=0,\qquad \langle\tfrac12|v|^2\mathcal{Q}(G,G)\rangle=0, \tag{2.8}$$

for every $G=G(v)$ for which the integrals make sense. It can be shown that these are essentially all the quantities conserved by $\mathcal{Q}(G,G)$. More precisely, the following statements are equivalent:

$$\begin{aligned}&\text{(i)}\quad \langle\xi\,\mathcal{Q}(G,G)\rangle=0\\ &\qquad\ \text{for every } G=G(v) \text{ for which the integral makes sense;}\\ &\text{(ii)}\quad \xi\in\operatorname{span}\{1,v_1,\cdots,v_D,|v|^2\}.\end{aligned} \tag{2.9}$$

If G solves the Boltzmann equation (2.1), then (2.8) implies that G satisfies local conservation laws of mass, momentum, and energy:

$$\begin{aligned}\partial_t\langle G\rangle + \nabla_x\cdot\langle v\,G\rangle &= 0,\\ \partial_t\langle v\,G\rangle + \nabla_x\cdot\langle v\otimes v\,G\rangle &= 0,\\ \partial_t\langle\tfrac12|v|^2G\rangle + \nabla_x\cdot\langle v\tfrac12|v|^2G\rangle &= 0.\end{aligned} \tag{2.10}$$

Integrating these over space and time yields the global conservation laws of mass, momentum, and energy:

$$\begin{aligned}\int_{\mathbf{T}^D}\langle G(t)\rangle\,dx &= \int_{\mathbf{T}^D}\langle G^{\text{in}}\rangle\,dx,\\ \int_{\mathbf{T}^D}\langle v\,G(t)\rangle\,dx &= \int_{\mathbf{T}^D}\langle v\,G^{\text{in}}\rangle\,dx,\\ \int_{\mathbf{T}^D}\langle\tfrac12|v|^2G(t)\rangle\,dx &= \int_{\mathbf{T}^D}\langle\tfrac12|v|^2G^{\text{in}}\rangle\,dx.\end{aligned} \tag{2.11}$$

Next, upon setting $\xi = -\log(G)$ into identity (2.7), Boltzmann observed that the resulting integrand is nonnegative, and hence obtained the dissipation law

$$-\langle \log(G)\, \mathcal{Q}(G,G)\rangle = \tfrac{1}{4}\left\langle\!\!\left\langle \log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle \geqq 0 \tag{2.12}$$

for every $G = G(v)$ for which the integrals make sense. He then characterized the equilibria of the collision operator. He found that for any $G = G(v)$ for which the integrals make sense, the following statements are equivalent:

$$\begin{aligned} &\text{(i)} && \langle \log(G)\, \mathcal{Q}(G,G)\rangle = 0\,;\\ &\text{(ii)} && \mathcal{Q}(G,G) = 0\,;\\ &\text{(iii)} && \log(G) \in \operatorname{span}\{1, v_1, \cdots, v_D, |v|^2\}. \end{aligned} \tag{2.13}$$

Equilibria characterized by (iii) that have finite mass, momentum, and energy density can be written as $G = M_{(\rho,u,\theta)}/M$, where $M_{(\rho,u,\theta)}$ are the classical Maxwellians defined by

$$M_{(\rho,u,\theta)}(v) \equiv \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right), \tag{2.14}$$

and where the density $\rho \geqq 0$, the velocity $u \in \mathbf{R}^D$, and the temperature $\theta > 0$ are determined by the relations

$$\rho = \langle G\rangle, \qquad \rho u = \langle v\, G\rangle, \qquad \tfrac{1}{2}\rho|u|^2 + \tfrac{D}{2}\rho\,\theta = \langle \tfrac{1}{2}|v|^2 G\rangle. \tag{2.15}$$

In particular, $G = 1$ is the unique equilibrium associated with initial data G^{in} that satisfies

$$\int_{\mathbf{T}^D} \langle G^{\mathrm{in}}\rangle\, dx = 1, \qquad \int_{\mathbf{T}^D} \langle v\, G^{\mathrm{in}}\rangle\, dx = 0, \qquad \int_{\mathbf{T}^D} \langle \tfrac{1}{2}|v|^2 G^{\mathrm{in}}\rangle\, dx = \tfrac{D}{2}. \tag{2.16}$$

This is consistent with the choice of absolute Maxwellian M made in (1.8).

Now, if G solves the Boltzmann equation (2.1), then the dissipation law (2.12) implies that G satisfies the local entropy dissipation law

$$\begin{aligned} \partial_t \langle (G\log(G) - G + 1)\rangle &+ \nabla_x\cdot \langle v\,(G\log(G) - G + 1)\rangle \\ &= -\frac{1}{\varepsilon}\frac{1}{4}\left\langle\!\!\left\langle \log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle \leqq 0. \end{aligned}$$

Integrating this over space and time gives the global entropy equality

$$H(G(t)) + \frac{1}{\varepsilon}\int_0^t R(G(s))\, ds = H(G^{\mathrm{in}}), \tag{2.17}$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_{\mathbf{T}^D} \langle (G\log(G) - G + 1)\rangle\, dx, \tag{2.18}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\mathbf{T}^D} \frac{1}{4} \left\langle\!\!\left\langle \log\left(\frac{G_1' G'}{G_1 G}\right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle dx. \tag{2.19}$$

This choice of H as the relative entropy functional (2.18) is based on the fact that its integrand is a nonnegative strictly convex function of G with a minimum value of 0 at $G = 1$. Thus for any G,

$$H(G) \geqq 0, \qquad \text{and} \qquad H(G) = 0 \quad \text{if and only if} \quad G = 1. \tag{2.20}$$

This relative entropy provides a natural measure of the proximity of G to that equilibrium.

2.2. Formal Derivation of the Acoustic Equations

Before describing the mathematical apparatus necessary to establish the acoustic limit, we give below a formal theorem in the style of [2] whose proof will serve as a guideline.

We suppose there exists a family G_ε of nonnegative weak solutions of the Boltzmann equation (2.1) whose fluctuations about the equilibrium value $G = 1$ are of order ε^m as $\varepsilon \to 0$ for some $m > 0$. We then introduce the scaled fluctuations g_ε by

$$G_\varepsilon = 1 + \varepsilon^m g_\varepsilon. \tag{2.21}$$

By (2.1) the fluctuations g_ε will then be weak solutions of

$$\partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon = \varepsilon^{m-1} \mathcal{Q}(g_\varepsilon, g_\varepsilon), \tag{2.22}$$

where $\mathcal{L}$ is the linearized collision operator defined by

$$\mathcal{L}\tilde{g} = -2\mathcal{Q}(1, \tilde{g}).$$

This operator has a nonnegative, self-adjoint extension over $L^2(M dv)$. Possible limits of the fluctuations g_ε as $\varepsilon \to 0$ are then governed by the following theorem.

Proposition 2.1 (Formal Acoustic Limit Theorem**).** *Let $m > 0$. Let G_ε be a family of nonnegative weak solutions of the Boltzmann equation (2.1) whose fluctuations g_ε given by (2.21) converge in the sense of distributions to a function $g \in L^\infty(dt; L^2(M\, dv\, dx))$ as $\varepsilon \to 0$. Furthermore, assume that the moments*

$$\langle g_\varepsilon \rangle, \qquad \langle v\, g_\varepsilon \rangle, \qquad \langle v \otimes v\, g_\varepsilon \rangle, \qquad \langle v|v|^2 g_\varepsilon \rangle, \tag{2.23a}$$

satisfy the local conservation laws and converge in the sense of distributions as $\varepsilon \to 0$ to the corresponding moments

$$\langle g \rangle, \qquad \langle v\, g \rangle, \qquad \langle v \otimes v\, g \rangle, \qquad \langle v|v|^2 g \rangle, \tag{2.23b}$$

and that

$$\mathcal{L} g_\varepsilon \to \mathcal{L} g, \qquad \varepsilon^m \mathcal{Q}(g_\varepsilon, g_\varepsilon) \to 0, \tag{2.24}$$

in the sense of distributions as $\varepsilon \to 0$. *Then* g *has the form of the infinitesimal Maxwellian*

$$g = \rho + u\cdot v + \theta\left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right), \tag{2.25}$$

where (ρ, u, θ) *solve the acoustic equations (1.9).*

Proof. First, multiply (2.22) by ε and let $\varepsilon \to 0$. By the convergences assumed in (2.23) and (2.24), we obtain

$$\mathcal{L}g = 0. \tag{2.26}$$

It is known (see for example [12, Chapter IV.1]) that the L^2 extension of the linearized collision operator has its nullspace given by $\mathrm{Null}(\mathcal{L}) = \mathrm{span}\{1, v_1, \cdots, v_D, |v|^2\}$. Because the limiting fluctuation g is assumed to belong to $L^\infty(dt; L^2(M\,dv\,dx))$, (2.26) implies that it must have the form of an infinitesimal Maxwellian (2.25).

Second, by the local conservation laws (2.10) the fluctuations g_ε will satisfy

$$\begin{aligned}
\partial_t\langle g_\varepsilon\rangle + \nabla_x\cdot\langle v\, g_\varepsilon\rangle &= 0,\\
\partial_t\langle v\, g_\varepsilon\rangle + \nabla_x\cdot\langle v\otimes v\, g_\varepsilon\rangle &= 0,\\
\partial_t\langle \tfrac{1}{2}|v|^2 g_\varepsilon\rangle + \nabla_x\cdot\langle v\tfrac{1}{2}|v|^2 g_\varepsilon\rangle &= 0.
\end{aligned} \tag{2.27}$$

The theorem then follows by letting $\varepsilon \to 0$ in these equations using the convergences assumed in (2.23) and then using the limiting form of g given by (2.25). □

3. Analytical Preliminaries

Going beyond the formal derivation of the last section requires clarification of (1) the notion of a solution for the Boltzmann equation, and (2) the sense in which the phase-space density is to be close to the background absolute Maxwellian. The first is provided by the theory of global solutions of DiPerna-Lions, while the second is provided by the theory of fluctuations developed in [3].

3.1. Global Solutions

DiPerna and Lions [15] proved the existence of a temporally global weak solution to the Boltzmann equation over the whole space $\mathbf{R}^D$ for any initial data satisfying natural physical bounds. As they pointed out, with only slight modifications their theory can be extended to the periodic box $\mathbf{T}^D$. It gives the existence of a global weak solution to a class of formally equivalent initial-value problems obtained by dividing the Boltzmann equation (2.1a) by normalizing functions $N = N(G) > 0$:

$$\big(\partial_t + v\cdot\nabla_x\big)\Gamma(G) = \frac{1}{\varepsilon}\frac{\mathcal{Q}(G,G)}{N(G)}, \tag{3.1a}$$

$$G(0,x,v) = G^{\mathrm{in}}(x,v), \tag{3.1b}$$

where each N is continuous over $[0,\infty)$ and satisfies a bound $(1+z)/N(z) \leqq C_N$ over $z \geqq 0$ for some constant $C_N < \infty$, and where $\Gamma'(z) = 1/N(z)$. Their solutions lie in $C([0,\infty); w\text{-}L^1(M dv\, dx))$, where the prefix "$w$-" on a space indicates that the space is endowed with its weak topology. They say that $G \geqq 0$ is a weak solution of (3.1) provided that it is initially equal to G^{in}, and that it satisfies the normalized Boltzmann equation (3.1a) in the sense that for every $\chi \in L^\infty(M dv; C^1(\mathbf{T}^D))$ and every $[t_1, t_2] \subset [0,\infty)$ it satisfies

$$\int_{\mathbf{T}^D} \langle \Gamma(G(t_2))\, \chi \rangle \, dx - \int_{\mathbf{T}^D} \langle \Gamma(G(t_1))\, \chi \rangle \, dx - \int_{t_1}^{t_2} \int_{\mathbf{T}^D} \langle \Gamma(G)\, v \cdot \nabla_x \chi \rangle \, dx\, dt \\ = \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\mathbf{T}^D} \left\langle \frac{\mathcal{Q}(G,G)}{N(G)}\, \chi \right\rangle dx\, dt. \tag{3.2}$$

They show that if G is a weak solution of (3.1) for one such N and satisfies certain bounds then it is a weak solution for every such N. They call such solutions renormalized solutions of the Boltzmann equation (2.1).

Proposition 3.1 (DiPerna-Lions Renormalized Solutions**).** *Given any initial data* G^{in} *in the entropy class*

$$E(M dv\, dx) = \left\{ G^{\text{in}} \geqq 0 \,:\, H(G^{\text{in}}) < \infty \right\},$$

there exists at least one $G \geqq 0$ *in* $C([0,\infty); w\text{-}L^1(M dv\, dx))$ *that is a weak solution of* (3.1) *such that:*

$$\begin{aligned} &\tfrac{\mathcal{Q}^-(G,G)}{1+N} \in L^\infty(dt; L^1(M dv\, dx)), \\ &\tfrac{\mathcal{Q}^+(G,G)}{1+N} \in L^1_{\text{loc}}(dt; L^1(M dv\, dx)), \end{aligned} \tag{3.3}$$

where $\mathcal{Q}^-$ *and* $\mathcal{Q}^+$ *are the source and sink components of the collision operator* (2.2)

$$\begin{aligned} \mathcal{Q}^+(G,G) &= \iint_{\mathbf{S}^{D-1}\times\mathbf{R}^D} G_1' G'\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1, \\ \mathcal{Q}^-(G,G) &= \iint_{\mathbf{S}^{D-1}\times\mathbf{R}^D} G_1 G\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1. \end{aligned} \tag{3.4}$$

Moreover, G also satisfies the global entropy inequality

$$H(G(t)) + \frac{1}{\varepsilon} \int_0^t R(G)\, ds \leqq H(G^{\text{in}}), \tag{3.5}$$

a weak form of the local conservation law of mass

$$\partial_t \langle G \rangle + \nabla_x \cdot \langle v\, G \rangle = 0, \tag{3.6}$$

the global conservation law of momentum

$$\int_{\mathbf{T}^D} \langle v\, G(t) \rangle \, dx = \int_{\mathbf{T}^D} \langle v\, G^{\text{in}} \rangle \, dx, \tag{3.7}$$

and finally, the global energy inequality

$$\int_{\mathbf{T}^D} \langle \tfrac{1}{2}|v|^2 G(t) \rangle \, dx \leqq \int_{\mathbf{T}^D} \langle \tfrac{1}{2}|v|^2 G^{\text{in}} \rangle \, dx. \tag{3.8}$$

The finiteness of the entropy is enough to insure the integrability of the conserved densities. However, the DiPerna-Lions theory does not assert the local conservation of momentum, the global conservation of energy, the global entropy equality (2.17), or a local entropy inequality; nor does it assert the uniqueness of the solution.

3.2. Controlling Fluctuations

In order to derive fluid dynamical equations from the Boltzmann equation for regimes near a background absolute Maxwellian, be they the acoustic, Stokes, or incompressible Navier-Stokes equations, one needs a proper definition of the sense in which these limits hold. While L^2 based spaces are natural for these fluid equations, the natural setting for global solutions of the Boltzmann equation are rather weighted L^1 or $L\log(L)$ spaces. These two different types of spaces were reconciled in the limit of small fluctuations about a background equilibrium in [3]. Here we do not need the entire theory developed there. We have extracted the relevant parts below.

Let $G_\varepsilon \geqq 0$ be a family of DiPerna-Lions renormalized solutions to the scaled Boltzmann initial-value problem (2.1) such that the initial data $G_\varepsilon^{\mathrm{in}}$ satisfies the entropy bound

$$H(G_\varepsilon^{\mathrm{in}}) \leqq C^{\mathrm{in}}\varepsilon^{2m}, \tag{3.9}$$

for some fixed $C^{\mathrm{in}} > 0$. Consider the families of fluctuations g_ε and $g_\varepsilon^{\mathrm{in}}$ defined by the relations

$$G_\varepsilon = 1 + \varepsilon^m g_\varepsilon, \qquad G_\varepsilon^{\mathrm{in}} = 1 + \varepsilon^m g_\varepsilon^{\mathrm{in}}. \tag{3.10}$$

The DiPerna-Lions entropy inequality (3.5) and the entropy bound (3.9) are consistent with this order of fluctuation about the equilibrium $G = 1$. More specifically, below it will be shown that these families of fluctuations are of order one.

With this in mind, we choose to work with a DiPerna-Lions normalization in the form

$$N_\varepsilon = N(G_\varepsilon) = \tfrac{2}{3} + \tfrac{1}{3}G_\varepsilon = 1 + \tfrac{1}{3}\varepsilon^m g_\varepsilon, \tag{3.11}$$

One reason for this choice is that formally $N_\varepsilon \to 1$ as ε tends to zero; thus, the normalizing factor will conveniently disappear from all algebraic expressions considered in this limit. Another reason lies in simplification of the specifics encountered during some subsequent estimates. Of course, our main results are independent of this particular choice of normalization. Given this choice, the normalized Boltzmann equation (3.1) becomes

$$\partial_t\gamma_\varepsilon + v\cdot\nabla_x\gamma_\varepsilon = \frac{1}{\varepsilon}\frac{\mathcal{Q}(G_\varepsilon, G_\varepsilon)}{N_\varepsilon}, \tag{3.12}$$

where we have introduced γ_ε by

$$\gamma_\varepsilon = \frac{1}{\varepsilon^m}\Gamma(G_\varepsilon) = \frac{3}{\varepsilon^m}\log\bigl(1 + \tfrac{1}{3}\varepsilon^m g_\varepsilon\bigr). \tag{3.13}$$

Because γ_ε formally behaves like g_ε for small ε, it should be thought of as the normalized form of the fluctuations g_ε.

The first objective is to characterize properties of the limit of the fluctuations g_ε. The *a priori* estimates needed are found in the combination of the entropy inequality (3.5) and the entropy bound (3.9) assumed for the initial data:

$$H(G_\varepsilon(t)) + \frac{1}{\varepsilon}\int_0^t R(G_\varepsilon)\,ds \leqq H(G_\varepsilon^{\mathrm{in}}) \leqq C^{\mathrm{in}}\varepsilon^{2m}. \tag{3.14}$$

As can be seen from (2.20) and the comment thereafter, the terms involving the entropy H measure the proximity of G_ε and $G_\varepsilon^{\mathrm{in}}$ to the absolute equilibrium value of 1.

As in [3], the relative entropy can be recast as

$$H(G_\varepsilon) = \int_{\mathbf{T}^D} \langle h(\varepsilon^m g_\varepsilon)\rangle\,dx, \tag{3.15}$$

where the integrand is written in terms of the convex function

$$h(z) = (1+z)\log(1+z) - z. \tag{3.16}$$

Because $h(z) \sim \frac{1}{2}z^2$ as $z \to 0$, one easily sees that $H(G_\varepsilon)$ asymptotically behaves almost like an L^2 norm of g_ε as ε tends to zero. This observation lies behind the following proposition, which follows from Propositions 3.1, 3.4, and 3.8 of [3] and is set in the notation therein.

Proposition 3.2 (Controlling Fluctuations Lemma**).** *Let* $G_\varepsilon \geqq 0$ *be a family in* $C([0,\infty); w\text{-}L^1(M dv\,dx))$ *that satisfies the entropy inequality and the bound* (3.14) *with* $G_\varepsilon^{\mathrm{in}} = G_\varepsilon(0)$. *Let* g_ε *and* $g_\varepsilon^{\mathrm{in}}$ *be the corresponding fluctuations* (3.10). *Then*

(a) *The family* $\{(1+|v|^2)\,g_\varepsilon\}_{\varepsilon>0}$ *is bounded in* $L^\infty(dt; L^1(M dv\,dx))$ *and relatively compact in* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\,dx))$.

(b) *For each* $t \geqq 0$ *the family* $\{(1+|v|^2)\,g_\varepsilon(t)\}_{\varepsilon>0}$ *is relatively compact in* $w\text{-}L^1(M dv\,dx)$.

(c) *If* g *is a* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(M dv\,dx))$ *limit point of the family* $\{g_\varepsilon\}_{\varepsilon>0}$ *as* $\varepsilon \to 0$ *then* $g \in L^\infty(dt; L^2(M dv\,dx))$ *and for almost every* $t \geqq 0$ *satisfies the inequality*

$$\begin{aligned}\int_{\mathbf{T}^D} \tfrac{1}{2}\langle g^2(t)\rangle\,dx &\leqq \liminf_{\varepsilon\to 0} \int \Big\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon(t))\Big\rangle\,dx \\ &\leqq \liminf_{\varepsilon\to 0} \int \Big\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{\mathrm{in}})\Big\rangle\,dx \leqq C^{\mathrm{in}}.\end{aligned} \tag{3.17}$$

(d) *Moreover,* g *has the form of an infinitesimal Maxwellian*

$$g = \rho + u\cdot v + \theta\left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right), \tag{3.18}$$

where $(\rho, u, \theta) \in L^\infty(dt; L^2(dx; \mathbf{R}\times\mathbf{R}^D\times\mathbf{R}))$ *and for almost every* $t \geqq 0$ *satisfies*

$$\int_{\mathbf{T}^D} \langle g^2(t)\rangle\,dx = \int_{\mathbf{T}^D} \left(\rho(t)^2 + |u(t)|^2 + \tfrac{D}{2}\theta(t)^2\right)dx. \tag{3.19}$$

Statement (a) is Proposition 3.1 (1) of [3]. Statement (b) is proved essentially the same way. Statement (c) follows from Proposition 3.4 (2) of [3] – more specifically, from the inequality (3.31) there. Finally, statement (d) is Proposition 3.8 of [3], which makes full use of the bound on the dissipation term in (3.14).

The above proposition does not take into account the fact that the g_ε will eventually represent fluctuations of the number density in the Boltzmann equation; only the entropy and entropy dissipation bounds (3.14) provide the needed weak compactness.

Proposition 3.2 shows how the L^2 setting for the macroscopic variables arises from the limiting form of the entropy inequality (3.14) applied to fluctuations of the number density. The notion of "entropic convergence" introduced in [3] and recalled below will strengthen this view by using the entropy inequality not only to produce bounds on, but also to measure the distance from the asymptotic state.

Definition 1. Let $G_\varepsilon \geqq 0$ be a family in $L^1(M dv\, dx)$ and let g_ε be the corresponding fluctuations as in (3.10). The family g_ε is said to *converge entropically at order* ε^m to $g \in L^2(M dv\, dx)$ if and only if

$$g_\varepsilon \to g \text{ in } w\text{-}L^1(M dv\, dx), \text{ and } \lim_{\varepsilon\to 0} \int_{\mathbf{T}^D} \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon) \right\rangle dx = \int_{\mathbf{T}^D} \tfrac{1}{2}\langle g^2 \rangle\, dx. \tag{3.20}$$

It was shown in Proposition 4.11 of [3] that entropic convergence is stronger than norm convergence in $L^1((1+|v|^2)M dv\, dx))$. This notion immediately leads to the following sharpening of inequality (3.17) in Proposition 3.2.

Proposition 3.3 (Dissipation Inequality Corollary**).** *Let* $G_\varepsilon \geqq 0$ *be a family in* $C([0,\infty); w\text{-}L^1(M dv\, dx))$ *that satisfies the entropy inequality and bound* (3.14), *where* $G_\varepsilon^{\text{in}} = G_\varepsilon(0)$ *has fluctuations* $g_\varepsilon^{\text{in}}$ *that converge entropically at order* ε^m *as* $\varepsilon \to 0$ *to some* $g^{\text{in}} \in L^2(M dv\, dx)$. *Let* g_ε *be the corresponding fluctuations* (3.10) *and* g *be a weak limit. Then, for almost every* $t \geqq 0$,

$$\int_{\mathbf{T}^D} \tfrac{1}{2}\langle g^2(t)\rangle\, dx \leqq \liminf_{\varepsilon\to 0} \int \left\langle \frac{1}{\varepsilon^{2m}} h(\varepsilon^m g_\varepsilon^{\text{in}}) \right\rangle dx = \int_{\mathbf{T}^D} \tfrac{1}{2}\langle g^{\text{in}\,2}\rangle\, dx. \tag{3.21}$$

In particular, if g^{in} *is an infinitesimal Maxwellian of the form*

$$g^{\text{in}} = \rho^{\text{in}} + u^{\text{in}}\cdot v + \theta^{\text{in}} \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2} \right), \tag{3.22}$$

where $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbf{R}\times\mathbf{R}^D\times\mathbf{R})$, *then*

$$\int_{\mathbf{T}^D} \langle g^{\text{in}\,2}\rangle\, dx = \int_{\mathbf{T}^D} \left(\rho^{\text{in}\,2} + |u^{\text{in}}|^2 + \tfrac{D}{2}\theta^{\text{in}\,2}\right) dx,$$

and for almost every $t \geqq 0$,

$$\int_{\mathbf{T}^D} \left(\rho(t)^2 + |u(t)|^2 + \tfrac{D}{2}\theta(t)^2\right) dx \leqq \int_{\mathbf{T}^D} \left(\rho^{\text{in}\,2} + |u^{\text{in}}|^2 + \tfrac{D}{2}\theta^{\text{in}\,2}\right) dx. \tag{3.23}$$

Remark. It is clear from (3.20) that the assumption that the initial fluctuations $g_\varepsilon^{\mathrm{in}}$ converge entropically at order ε^m as $\varepsilon \to 0$ to some g^{in} that is in $L^2(M dv\, dx)$ implies that those fluctuations satisfy the entropy bound (3.9).

The significance of Proposition 3.3 becomes more apparent upon noticing that for every $m > 0$ and every $g^{\mathrm{in}} \in L^2(M dv\, dx)$ there are families $G_\varepsilon^{\mathrm{in}}$ in the entropy class $E(M dv\, dx)$ with fluctuations $g_\varepsilon^{\mathrm{in}}$ that converge entropically at order ε^m as $\varepsilon \to 0$ to g^{in}. For example, it was pointed out in [19] that one can take $g_\varepsilon^{\mathrm{in}} = \max\{g^{\mathrm{in}},\ -\varepsilon^{-m}\}$.

One can say more when g^{in} is an infinitesimal Maxwellian (3.22).

Proposition 3.4 (Realizability of the Initial Data Lemma**).** *Let $m > 0$ and let $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(dx; \mathbf{R} \times \mathbf{R}^D \times \mathbf{R})$ satisfy the normalizations*

$$\int_{\mathbf{T}^D} \rho^{\mathrm{in}}\, dx = 0, \qquad \int_{\mathbf{T}^D} u^{\mathrm{in}}\, dx = 0, \qquad \int_{\mathbf{T}^D} \theta^{\mathrm{in}}\, dx = 0. \tag{3.24}$$

Then there exists a family of local Maxwellians $G_\varepsilon^{\mathrm{in}}$ that satisfy the normalizations

$$\int_{\mathbf{T}^D} \langle G_\varepsilon^{\mathrm{in}} \rangle\, dx = 1, \qquad \int_{\mathbf{T}^D} \langle v\, G_\varepsilon^{\mathrm{in}} \rangle\, dx = 0, \qquad \int_{\mathbf{T}^D} \langle \tfrac{1}{2}|v|^2 G_\varepsilon^{\mathrm{in}} \rangle\, dx = \tfrac{D}{2}, \tag{3.25}$$

and whose fluctuations, $g_\varepsilon^{\mathrm{in}} = \varepsilon^{-m}(G_\varepsilon^{\mathrm{in}} - 1)$, converge entropically at order ε^m as $\varepsilon \to 0$ to the infinitesimal Maxwellian g^{in} given by (3.22).

Proof. Let $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(dx; \mathbf{R} \times \mathbf{R}^D \times \mathbf{R})$ satisfy (3.24). Let $j \in C_c^\infty(\mathbf{R}^D)$ be a mollifying function:

$$j \geqq 0, \qquad \operatorname{supp}(j) \subset B_{\frac{1}{2}}(0), \qquad \int_{\mathbf{R}^D} j(x)\, dx = 1.$$

For every $\varepsilon \in (0, 1]$ define $j_\varepsilon \in C^\infty(\mathbf{T}^D)$ by

$$j_\varepsilon(x) = \frac{1}{\varepsilon^m} \sum_{z \in \mathbf{Z}^D} j\left(\frac{x + z}{\varepsilon^{m/D}} \right).$$

The assumption on the support of j guarantees that the supports of the various terms in the above sum never overlap for $0 < \varepsilon \leqq 1$. Then j_ε is a mollifying family over $\mathbf{T}^D$. Define

$$\rho_\varepsilon^{\mathrm{in}} = j_\varepsilon * \rho^{\mathrm{in}},$$

where the symbol "$*$" designates the convolution over $\mathbf{T}^D$. The Cauchy-Schwarz inequality gives

$$\|\rho_\varepsilon^{\mathrm{in}}\|_{L^\infty} \leqq \|j_\varepsilon\|_{L^2} \|\rho^{\mathrm{in}}\|_{L^2} = \frac{1}{\varepsilon^{m/2}} \|j_1\|_{L^2} \|\rho^{\mathrm{in}}\|_{L^2},$$

whereby it is clear that for all $\varepsilon \in (0, 1]$ sufficiently small one has $1 + \varepsilon^m \rho_\varepsilon^{\mathrm{in}} > \frac{1}{2}$. For all such ε define

$$u_\varepsilon^{\mathrm{in}} = \frac{j_\varepsilon * u^{\mathrm{in}}}{1 + \varepsilon^m \rho_\varepsilon^{\mathrm{in}}}, \qquad \theta_\varepsilon^{\mathrm{in}} = \frac{j_\varepsilon * \theta^{\mathrm{in}}}{1 + \varepsilon^m \rho_\varepsilon^{\mathrm{in}}} - \varepsilon^m \tfrac{1}{D} |u_\varepsilon^{\mathrm{in}}|^2.$$

Several more applications of the Cauchy-Schwarz inequality give

$$\begin{aligned}\|u_\varepsilon^{\mathrm{in}}\|_{L^\infty} &\leqq 2\|j_\varepsilon\|_{L^2}\|u^{\mathrm{in}}\|_{L^2} = \frac{2}{\varepsilon^{m/2}}\|j_1\|_{L^2}\|u^{\mathrm{in}}\|_{L^2},\\ \|\theta_\varepsilon^{\mathrm{in}}\|_{L^\infty} &\leqq 2\|j_\varepsilon\|_{L^2}\|\theta^{\mathrm{in}}\|_{L^2} + \varepsilon^m \frac{1}{D}\|u_\varepsilon^{\mathrm{in}}\|_{L^\infty}^2\\ &\leqq \frac{2}{\varepsilon^{m/2}}\|j_1\|_{L^2}\|\theta^{\mathrm{in}}\|_{L^2} + \frac{4}{D}\|j_1\|_{L^2}^2\|u^{\mathrm{in}}\|_{L^2}^2 0.\end{aligned}$$

It is therefore clear that for all $\varepsilon \in (0, 1]$ sufficiently small one has $1 + \varepsilon^m \theta_\varepsilon^{\mathrm{in}} > \frac{1}{2}$. Now for all such ε define

$$G_\varepsilon^{\mathrm{in}} = M_{(1+\varepsilon^m \rho_\varepsilon^{\mathrm{in}}, \varepsilon^m u_\varepsilon^{\mathrm{in}}, 1+\varepsilon^m \theta_\varepsilon^{\mathrm{in}})}/M.$$

Direct calculations show that this satisfies the normalizations (3.25). One can also easily check that the associated fluctuations converge entropically at order ε^m as $\varepsilon \to 0$ to the infinitesimal Maxwellian (3.22). □

4. Establishing the Acoustic Limit

4.1. Mathematical Statement of the Acoustic Limit

In the previous section we introduced all the notions contained in the mathematical statement of the otherwise formal Proposition 2.1. The proof of Proposition 2.1 itself suggests that all that remains to be done is to pass to the limit in the local conservation laws (2.27). Unfortunately, these local conservation laws are not guaranteed by the DiPerna-Lions theory of renormalized solutions. In order to circumvent that difficulty, we will rely on the two following technical assumptions:

(A1) $$m > 1;$$

(A2) $$b \in L^\infty(d\omega\, M_1 dv_1).$$

Assumption (A2) is satisfied by the Boltzmann kernels corresponding to either Maxwell molecules or soft cutoff potentials (see [12], Chapter II.4-5 and II.9 for a thorough discussion of these matters). While the relations (2.10) may not be satisfied by the renormalized solutions of the Boltzmann equation (2.1), the defects are proved to vanish in the limit as $\varepsilon \to 0$ thanks to the assumptions above.

Thus, consider a family $\{G_\varepsilon^{\mathrm{in}}\}$ of nonnegative measurable functions in the entropy class $E(M dv\, dx)$ that satisfies the bounds

$$G_\varepsilon^{\mathrm{in}} \geqq 0, \qquad H(G_\varepsilon^{\mathrm{in}}) = O(\varepsilon^{2m}), \tag{4.1}$$

as well as the normalizations

$$\int_{\mathbf{T}^D} \langle G_\varepsilon^{\mathrm{in}}\rangle\, dx = 1, \qquad \int_{\mathbf{T}^D} \langle v\, G_\varepsilon^{\mathrm{in}}\rangle\, dx = 0, \qquad \int_{\mathbf{T}^D} \langle |v|^2 G_\varepsilon^{\mathrm{in}}\rangle\, dx = D. \tag{4.2}$$

For each $\varepsilon > 0$, let G_ε be a renormalized solution of the scaled Boltzmann equation (2.1) with initial data $G_\varepsilon^{\text{in}}$. Consider the fluctuations of G_ε around 1 at the scale ε^m, i.e.

$$g_\varepsilon = \frac{1}{\varepsilon^m}(G_\varepsilon - 1), \qquad g_\varepsilon^{\text{in}} = \frac{1}{\varepsilon^m}(G_\varepsilon^{\text{in}} - 1). \tag{4.3}$$

The first main result in this paper shows that the acoustic equations (1.9) describe all possible limit points of the family G_ε as $\varepsilon \to 0$ under the sole assumption (4.1), (4.2).

Proposition 4.1 (Weak Acoustic Limit Theorem**).** *Assume* (A1) *and* (A2). *Let* $G_\varepsilon^{\text{in}}$ *be any family in the entropy class* $E(M dv\, dx)$ *satisfying the bounds* (4.1) *as well as the normalizations* (4.2). *Let* G_ε *be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation* (2.1) *that have* $G_\varepsilon^{\text{in}}$ *as initial values. Then the family of fluctuations* g_ε *is relatively compact in* $w\text{-}L^1_{\text{loc}}(dt; L^1((1+|v|^2)M\, dv\, dx))$ *while the associated family of moments*

$$(\langle g_\varepsilon \rangle,\ \langle v\, g_\varepsilon \rangle,\ \langle (\tfrac{1}{D}|v|^2 - 1)\, g_\varepsilon \rangle)$$

is relatively compact in $C([0,\infty); w\text{-}L^1(dx; \mathbf{R} \times \mathbf{R}^D \times \mathbf{R}))$. *Any limit point* g *of the family* g_ε *as* $\varepsilon \to 0$ *is an infinitesimal Maxwellian of the form*

$$g = \rho + u\cdot v + \theta(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}), \tag{4.4}$$

where (ρ, u, θ) *is the solution of the acoustic equations* (1.9) *with initial data*

$$(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) = \lim_{\varepsilon_n \to 0} (\langle g_{\varepsilon_n}^{\text{in}} \rangle,\ \langle v\, g_{\varepsilon_n}^{\text{in}} \rangle,\ \langle (\tfrac{1}{D}|v|^2 - 1) g_{\varepsilon_n}^{\text{in}} \rangle) \tag{4.5}$$

for every sequence ε_n *such that* $g_{\varepsilon_n} \to g$ *in* $w\text{-}L^1_{\text{loc}}(dt; L^1((1+|v|^2)M dv dx))$ *while* $\varepsilon_n \to 0$.

The second main result is an amplification of Proposition 4.1 when the initial fluctuations are known to converge entropically to some infinitesimal Maxwellian; it shows that any physically natural solution of the acoustic equations (1.9) is indeed a strong hydrodynamic limit of renormalized solutions of the Boltzmann equation (2.1).

Proposition 4.2 (Strong Acoustic Limit Theorem**).** *Assume* (A1) *and* (A2). *Let* $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbf{R} \times \mathbf{R}^D \times \mathbf{R})$ *satisfy the normalizations*

$$\int_{\mathbf{T}^D} \rho^{\text{in}}\, dx = 0, \qquad \int_{\mathbf{T}^D} u^{\text{in}}\, dx = 0, \qquad \int_{\mathbf{T}^D} \theta^{\text{in}}\, dx = 0. \tag{4.6}$$

Let $G_\varepsilon^{\text{in}}$ *be any family in the entropy class* $E(M dv\, dx)$ *whose fluctuations* $g_\varepsilon^{\text{in}}$ *satisfy the normalizations* (4.2) *and converge entropically at order* ε^m *as* $\varepsilon \to 0$ *to the infinitesimal Maxwellian*

$$g^{\text{in}} = \rho^{\text{in}} + u^{\text{in}}\cdot v + \theta^{\text{in}}(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}). \tag{4.7}$$

Let G_ε *be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation* (2.1) *that have* $G_\varepsilon^{\text{in}}$ *as initial values. Then, as* $\varepsilon \to 0$, *the family of* *fluctuations g_ε converges entropically at order ε^m for every $t \geqq 0$ and in w-$L^1_{\mathrm{loc}}(dt;$ $L^1((1+|v|^2)M dv dx))$ to an infinitesimal Maxwellian g of the form* (4.4) *where* $(\rho, u, \theta) \in C([0,\infty); L^2(dx; \mathbf{R}\times\mathbf{R}^D\times\mathbf{R}))$ *is the solution of the acoustic equations* (1.9) *with initial data* $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$. *Moreover,*

$$(\langle g_\varepsilon\rangle,\ \langle v\, g_\varepsilon\rangle,\ \langle(\tfrac{1}{D}|v|^2-1)g_\varepsilon\rangle) \to (\rho, u, \theta)$$

in $C([0,\infty); w\text{-}L^1(dx; \mathbf{R}\times\mathbf{R}^D\times\mathbf{R}))$.

The normalizations (4.2) on the initial data for the Boltzmann equation and the assumed entropic convergence of their initial fluctuations around 1 entail the normalizations (4.6) of the initial data for the acoustic equations. Conversely, Proposition 3.4 shows that all L^2 initial data of the acoustic equations satisfying (4.6) are indeed entropic limits of initial fluctuations for the Boltzmann equation in the manner described in Proposition 4.2.

This discussion shows that, at variance with the classical methods based on either the Hilbert or Chapman-Enskog expansion, the strategy proposed in this paper adresses *all* physically natural initial data for the Boltzmann equation as well as for its hydrodynamic limit, here the system of acoustics.

Assuming Proposition 4.1, the proof of Proposition 4.2 is given below; it is a direct consequence of the weak compactness of fluctuations stated in Proposition 4.1, with additional arguments provided by Propositions 3.2, 3.3, and 3.4.

Proof of Proposition 4.2. Let $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$ be an initial data for the acoustic equations in $L^2(dx; \mathbf{R}\times\mathbf{R}^D\times\mathbf{R})$, satisfying the normalizations (4.6). By Proposition 3.4, there indeed exists families $G_\varepsilon^{\mathrm{in}}$ in the entropy class $E(M\,dv\,dx)$ whose fluctuations satisfy the normalizations (4.2) and converge entropically at order ε^m to the infinitesimal Maxwellian (4.7) as $\varepsilon \to 0$. Consider therefore such families $G_\varepsilon^{\mathrm{in}}$ of initial data for the Boltzmann equation (2.1). Because the family of initial fluctuations $g_\varepsilon^{\mathrm{in}} \to g^{\mathrm{in}}$ in w-$L^1(M\,dv\,dx)$ as $\varepsilon \to 0$, it follows from Proposition 4.1 that the family g_ε is relatively compact in w-$L^1_{\mathrm{loc}}(dt; w\text{-}L^1((1+|v|^2)M\,dv\,dx))$ and that any of its limit points as $\varepsilon \to 0$ is of the form (4.4) with (ρ, u, θ) the unique solution of the system of acoustics (1.9) with initial data $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$. Thus $g_\varepsilon \to g$ given by (4.4) in w-$L^1_{\mathrm{loc}}(dt; w\text{-}L^1((1+|v|^2)M\,dv\,dx))$ as $\varepsilon \to 0$, by compactness and uniqueness of the limit point. By Proposition 3.2 (c), Proposition 3.3, and (3.19), one has

$$\begin{aligned}
\tfrac{1}{2}\int_{\mathbf{T}^D} \big(\rho(t)^2 + |u(t)|^2 + \tfrac{D}{2}\theta(t)^2\big)\,dx &= \tfrac{1}{2}\int_{\mathbf{T}^D} \langle g(t)^2\rangle\,dx \\
&\leqq \liminf_{\varepsilon\to 0} \int_{\mathbf{T}^D} \langle \frac{1}{\varepsilon^{2m}}\, h(\varepsilon^m g_\varepsilon(t))\rangle\,dx \\
&\leqq \limsup_{\varepsilon\to 0} \int_{\mathbf{T}^D} \langle \frac{1}{\varepsilon^{2m}}\, h(\varepsilon^m g_\varepsilon(t))\rangle\,dx \\
&\leqq \tfrac{1}{2}\int_{\mathbf{T}^D} \langle g^{\mathrm{in}\,2}\rangle\,dx \\
&= \tfrac{1}{2}\int_{\mathbf{T}^D} \big(\rho^{\mathrm{in}\,2} + |u^{\mathrm{in}}|^2 + \tfrac{D}{2}\theta^{\mathrm{in}\,2}\big)\,dx.
\end{aligned}$$

The quantity on the left-hand side of the chain of inequalities above is invariant under the evolution of the system of acoustics. Therefore all inequalities above are in fact equalities, which in turn implies that the convergence $g_\varepsilon \to g$ is entropic of order ε^m for all $t \geqq 0$. The proof is complete. □

The proof of Proposition 4.1 occupies the remaining part of Section 4. Its main step consists in establishing the limiting conservation laws (2.27); this is done in Section 4.4 (see Proposition 4.7 below). The proof of Proposition 4.7 itself depends upon controlling the collision integrals so as to dispose of the defects in the relations (2.10). These controls are provided in Section 4.3 (see Propositions 4.5 and 4.6 below) and depend themselves upon a decomposition of fluctuations, the flat-sharp decomposition, introduced in [3] and recalled in Section 4.2 below.

4.2. The Flat-Sharp Decomposition of Fluctuations

As stated in Proposition 3.2, the limiting fluctuations are estimated in $L^\infty(dt;\, L^2(M\,dv\,dx))$ by the entropy inequality. However, the fluctuations are not known to be bounded in $L^\infty(dt;\, L^2(M\,dv\,dx))$ before the limit as $\varepsilon \to 0$ is taken. The absence of such a bound was addressed in [3] (section 3) by the introduction of the following decomposition for the normalization $N_\varepsilon = 1 + \frac{1}{3}\varepsilon^m g_\varepsilon$:

$$g_\varepsilon = {}^\flat g_\varepsilon + \varepsilon^m\, {}^\sharp g_\varepsilon, \qquad {}^\flat g_\varepsilon = \frac{g_\varepsilon}{N_\varepsilon}, \quad {}^\sharp g_\varepsilon = \frac{g_\varepsilon^2}{3N_\varepsilon}. \tag{4.8}$$

The second term in the decomposition (4.8) is precisely the obstruction to proving such a uniform bound. Notice that this term is nonnegative: this observation will be crucial in what follows. Thus ${}^\flat g_\varepsilon$ is a $L^\infty(dt;\, L^2(M\,dv\,dx))$ substitute for g_ε; another natural one is the quantity naturally involved in the renormalized form of the Boltzmann equation

$$\gamma_\varepsilon = \frac{3}{\varepsilon^m}\log(1 + \tfrac{1}{3}\varepsilon^m g_\varepsilon). \tag{4.9}$$

The various properties of this decomposition are recalled in

Proposition 4.3 (The Flat-Sharp Decomposition Lemma**).** *Assume that the family of initial data $G_\varepsilon^{\text{in}}$ satisfies the bounds* (4.1) *and the normalizations* (4.2). *Then the flat-sharp decomposition* (4.8) *of the fluctuations g_ε has the following properties:*

(a) *the family ${}^\flat g_\varepsilon$ is bounded in $L^\infty(dt;\, L^2(M\,dv\,dx))$;*
(b) *the family ${}^\sharp g_\varepsilon$ is bounded in $L^\infty(dt;\, L^1(M\,dv\,dx))$ and, for any $\alpha > 0$, the family $\varepsilon^\alpha |v|^2\, {}^\sharp g_\varepsilon$ converges to* 0 *in $L^\infty(dt;\, L^1(M\,dv\,dx))$;*
(c) *the family γ_ε is bounded in $L^\infty(dt;\, L^2(M\,dv\,dx))$;*
(d) $(1+|v|^2)|g_\varepsilon - \gamma_\varepsilon| \to 0$ *in $L^\infty(dt;\, L^1(M\,dv\,dx))$ as $\varepsilon \to 0$;*
(e) *for any sequence $\varepsilon_n \to 0$ such that*

$$g_{\varepsilon_n} \to g \quad \textit{in } w\text{-}L^1_{\text{loc}}(dt;\, w\text{-}L^1((1+|v|^2)M\,dv\,dx))$$

the corresponding subsequences

$${}^\flat g_{\varepsilon_n} \textit{ and } \gamma_{\varepsilon_n} \textit{ both converge to } g \textit{ in } w\text{-}L^1_{\text{loc}}(dt;\, w\text{-}L^2(M\,dv\,dx)).$$

Statements (a), (c), (d) and the first half of statement (b) follow from Corollary 3.2 of [3]. Statement (e) is a straightforward consequence of statements (b) and (d). As for the second half of statement (b), it is a direct application of Proposition 3.2 in [3].

4.3. Controls of the Collision Integral

In this section are gathered some crucial preparations for the proofs of Propositions 4.1 and 4.2. As recalled in Section 3.1, the local conservation laws of momentum and energy do not hold for renormalized solutions of the Boltzmann equation. This destroys the argument given in the “formal proof” of Section 2.

One can however circumvent this difficulty by considering moments of the Boltzmann equation in renormalized form; the resulting equations are no longer conservation laws because renormalization and velocity averaging are not commuting operations. Specifically, we find

$$\partial_t \langle \chi \gamma_\varepsilon \rangle + \nabla_x \cdot \langle v \chi \gamma_\varepsilon \rangle = \frac{1}{\varepsilon}\Big\langle \Big(1 - \frac{1}{N_\varepsilon}\Big) \chi \mathcal{L} g_\varepsilon \Big\rangle + \varepsilon^{m-1} \Big\langle \chi \frac{\mathcal{Q}(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \Big\rangle \tag{4.10}$$

when $\chi \equiv \chi(v)$ is one of the collision invariants $1, v_1, \ldots, v_D$ and $|v|^2$ or a linear combination thereof with constant coefficients.

The purpose of the present subsection is to study the limit as $\varepsilon \to 0$ of the right-hand side of (4.10).

First, both terms on the right-hand side of (4.10) can be recast in a way that clearly indicates that they are of the same nature. Indeed, the obvious formula

$$1 - \frac{1}{N_\varepsilon} = \frac{1}{3}\varepsilon^m \frac{g_\varepsilon}{N_\varepsilon}$$

shows that

$$\frac{1}{\varepsilon}\Big\langle \Big(1 - \frac{1}{N_\varepsilon}\Big) \chi \mathcal{L} g_\varepsilon \Big\rangle = \frac{1}{3}\varepsilon^{m-1} \Big\langle\!\!\Big\langle \chi \frac{g_\varepsilon (g_\varepsilon + g_{\varepsilon 1} - g'_\varepsilon - g'_{\varepsilon 1})}{N_\varepsilon} \Big\rangle\!\!\Big\rangle, \tag{4.11}$$

while

$$\varepsilon^{m-1} \Big\langle \chi \frac{\mathcal{Q}(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \Big\rangle = \varepsilon^{m-1} \Big\langle\!\!\Big\langle \chi \frac{g'_\varepsilon g'_{\varepsilon 1} - g_\varepsilon g_{\varepsilon 1}}{N_\varepsilon} \Big\rangle\!\!\Big\rangle. \tag{4.12}$$

The next Proposition shows that the first term in the right-hand side of (4.10) converges to 0 as $\varepsilon \to 0$.

Proposition 4.4. *Assume* (A1) *and* (A2). *Set* $\chi(v) = \frac{1+|v|^2}{1+D}$. *Assume that the family of initial data* $G_\varepsilon^{\mathrm{in}}$ *satisfies the bound* (4.1) *and the normalizations* (4.2). *Then*

(a) *as* $\varepsilon \to 0$, *the family*

$$\varepsilon^{m-1} \Big\langle\!\!\Big\langle \chi \frac{g_\varepsilon^2}{N_\varepsilon} \Big\rangle\!\!\Big\rangle \to 0 \quad \textit{in } L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^D));$$

(b) *as* $\varepsilon \to 0$, *the family*

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle \to 0 \quad \textit{in } L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^D));$$

(c) *as* $\varepsilon \to 0$, *the families*

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{|g_\varepsilon||g'_\varepsilon|}{N_\varepsilon} \right\rangle\!\!\right\rangle \quad \textit{and} \quad \varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{|g_\varepsilon||g'_{\varepsilon 1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle$$

both converge to 0 *in* $L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^D))$.

Proof. By assumption (A2)

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{g_\varepsilon^2}{N_\varepsilon} \right\rangle\!\!\right\rangle \leqq \|b\|_{L^\infty}\varepsilon^{m-1}\left\langle \chi \frac{g_\varepsilon^2}{N_\varepsilon} \right\rangle = \|b\|_{L^\infty}\varepsilon^{m-1}\langle \chi\, {}^\sharp g_\varepsilon \rangle$$

so that statement (a) follows from Proposition 4.3(b).

Assumption (A2) again implies that

$$\begin{aligned}\varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle &\leqq \varepsilon^{m-1}(1+D)\left\langle\!\!\left\langle \chi\chi_1 \frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle \\ &\leqq (1+D)\|b\|_{L^\infty}\varepsilon^{m-1}\left\langle \chi \frac{|g_\varepsilon|}{N_\varepsilon} \right\rangle \langle \chi |g_\varepsilon| \rangle\end{aligned} \tag{4.13}$$

Now consider the new normalization $\mathcal{N}_\varepsilon = 1 + \frac{1}{3}\varepsilon^m |g_\varepsilon|$. Because $G_\varepsilon \geqq 0$ a.e., the first normalization is bounded below: $N_\varepsilon = \frac{2}{3} + \frac{1}{3}G_\varepsilon \geqq \frac{2}{3}$. Besides, N_ε and $\mathcal{N}_\varepsilon$ coincide wherever $g_\varepsilon \geqq 0$ while, at points where $g_\varepsilon < 0$, one has $\varepsilon^m|g_\varepsilon| \leqq 1$ because $G_\varepsilon \geqq 0$. Thus, at points where $g_\varepsilon < 0$, $\mathcal{N}_\varepsilon \leqq \frac{4}{3}$, so that $N_\varepsilon \geqq \frac{2}{3} \geqq \frac{1}{2}\mathcal{N}_\varepsilon$. Putting all this together leads to the following inequalities which hold independently of the sign of g_ε:

$$\tfrac{1}{2}\mathcal{N}_\varepsilon \leqq N_\varepsilon \leqq \mathcal{N}_\varepsilon. \tag{4.14}$$

Thus, by (4.13),

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle \leqq 2(1+D)\|b\|_{L^\infty}\varepsilon^{m-1}\left\langle \chi \frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon} \right\rangle \langle \chi |g_\varepsilon| \rangle. \tag{4.15}$$

Observe that $\chi M dv$ is a probability measure on $\mathbf{R}^D$. Further, the map $z \mapsto z/(1+\frac{1}{3}z)$ is concave. The Jensen inequality then implies that

$$\left\langle \chi \frac{\varepsilon^m |g_\varepsilon|}{\mathcal{N}_\varepsilon} \right\rangle \leqq \frac{\varepsilon^m \langle \chi |g_\varepsilon| \rangle}{1 + \frac{1}{3}\varepsilon^m \langle \chi |g_\varepsilon| \rangle}$$

which, when used in (4.15) leads to

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi \frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle \leqq 2(1+D)\|b\|_{L^\infty}\varepsilon^{m-1}\frac{\langle \chi |g_\varepsilon| \rangle^2}{1 + \frac{1}{3}\varepsilon^m \langle \chi |g_\varepsilon| \rangle}. \tag{4.16}$$

Next observe that the map $z \mapsto z^2/(1+\frac{1}{3}z)$ is convex. A further application of the Jensen inequality shows that

$$\frac{\varepsilon^{2m}\langle\chi|g_\varepsilon|\rangle^2}{1+\frac{1}{3}\varepsilon^m\langle\chi|g_\varepsilon|\rangle} \leqq \left\langle\chi\,\frac{\varepsilon^{2m}|g_\varepsilon|^2}{1+\frac{1}{3}\varepsilon^m|g_\varepsilon|}\right\rangle,$$

which, when substituted in (4.16), gives

$$\varepsilon^{m-1}\left\langle\!\!\left\langle\chi\,\frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon}\right\rangle\!\!\right\rangle \leqq 2(1+D)\|b\|_{L^\infty}\varepsilon^{m-1}\left\langle\chi\,\frac{|g_\varepsilon|^2}{1+\frac{1}{3}\varepsilon^m|g_\varepsilon|}\right\rangle. \tag{4.17}$$

Finally, the second inequality in (4.14) yields

$$\varepsilon^{m-1}\left\langle\!\!\left\langle\chi\,\frac{|g_\varepsilon||g_{\varepsilon 1}|}{N_\varepsilon}\right\rangle\!\!\right\rangle \leqq 2(1+D)\|b\|_{L^\infty}\varepsilon^{m-1}\langle\chi\,{}^\sharp g_\varepsilon\rangle. \tag{4.18}$$

Statement (b) then follows from Proposition 4.3 (b).

It remains to prove statement (c). By inequality (4.14),

$$\varepsilon^{m-1}\left\langle\!\!\left\langle\chi\,\frac{|g_\varepsilon||g'_\varepsilon|}{N_\varepsilon}\right\rangle\!\!\right\rangle \leqq 2\varepsilon^{m-1}\left\langle\!\!\left\langle\chi\,\frac{|g_\varepsilon||g'_\varepsilon|}{\mathcal{N}_\varepsilon}\right\rangle\!\!\right\rangle. \tag{4.19}$$

At this point, we need to recall the geometry of collisions. According to (1.3)

$$v' = v - (v - v_1)\cdot\omega\omega.$$

This suggests the decomposition of both v and v' in the direct orthogonal sum $\mathbf{R}^D = \mathbf{R}\omega \oplus (\mathbf{R}\omega)^\perp$:

$$v = v^\parallel + v^\perp, \qquad v' = v_1^\parallel + v^\perp. \tag{4.20}$$

Because the decomposition (4.20) is orthogonal, it is easy to check the relations

$$M(v)dv = M(v^\parallel)dv^\parallel \otimes M(v^\perp)dv^\perp, \quad M(v_1)dv_1 = M(v_1^\parallel)dv_1^\parallel \otimes M(v_1^\perp)dv_1^\perp.$$

For simplicity, we denote below the centered, reduced Gaussian volume element in the Euclidian space by the single symbol $d_{\mathbf{G}}$, without mention of the space dimension.

Of course, these decompositions depend on ω; however, for notational convenience, we shall refrain from indicating the ω dependence in $v^\parallel$, $v_1^\parallel$ and $v^\perp$. The integral on the right-hand side of (4.19) can thus be estimated by

$$\begin{aligned}\varepsilon^{m-1}\left\langle\!\!\left\langle\chi\,\frac{|g_\varepsilon||g'_\varepsilon|}{N_\varepsilon}\right\rangle\!\!\right\rangle &\leqq 2\varepsilon^{m-1}\|b\|_{L^\infty} \times \iiint d_{\mathbf{G}}v^\perp d_{\mathbf{G}}v_1^\perp d\omega \\ &\quad \cdot \iint \chi(v^\parallel+v^\perp)\frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^\parallel+v^\perp)|g_\varepsilon|(v_1^\parallel+v^\perp)d_{\mathbf{G}}v^\parallel d_{\mathbf{G}}v_1^\parallel. \end{aligned} \tag{4.21}$$

Let us first estimate the inner integral on the right-hand side of (4.21):

$$\begin{aligned}
&\iint \chi(v^{\|}+v^{\perp})\frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})|g_\varepsilon|(v_1^{\|}+v^{\perp})d_{\mathbf{G}}v^{\|}d_{\mathbf{G}}v_1^{\|}\\
&\leqq \frac{1}{1+D}(1+|v^{\perp}|^2)\iint \frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})|g_\varepsilon|(v_1^{\|}+v^{\perp})d_{\mathbf{G}}v^{\|}d_{\mathbf{G}}v_1^{\|}\\
&+\frac{2}{1+D}\iint \frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})|g_\varepsilon|(v_1^{\|}+v^{\perp})\tfrac{1}{2}(1+|v^{\|}|^2)d_{\mathbf{G}}v^{\|}d_{\mathbf{G}}v_1^{\|}.
\end{aligned}\tag{4.22}$$

Using the Jensen inequality as in the proof of statement (b) (see (4.17) in particular), we obtain

$$\iint \frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})|g_\varepsilon|(v_1^{\|}+v^{\perp})d_{\mathbf{G}}v^{\|}d_{\mathbf{G}}v_1^{\|}\leqq \int \frac{|g_\varepsilon|^2}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})d_{\mathbf{G}}v^{\|},$$

while

$$\begin{aligned}
&\iint \frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})|g_\varepsilon|(v_1^{\|}+v^{\perp})\tfrac{1}{2}(1+|v^{\|}|^2)d_{\mathbf{G}}v^{\|}d_{\mathbf{G}}v_1^{\|}\\
&\qquad\leqq 2\int \frac{|g_\varepsilon|^2}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})\tfrac{1}{2}(1+|v^{\|}|^2)d_{\mathbf{G}}v^{\|}.
\end{aligned}$$

Using these last two inequalities in (4.22) shows that

$$\begin{aligned}
&\iint \chi(v^{\|}+v^{\perp})\frac{|g_\varepsilon|}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})|g_\varepsilon|(v_1^{\|}+v^{\perp})d_{\mathbf{G}}v^{\|}d_{\mathbf{G}}v_1^{\|}\\
&\leqq \frac{1+|v^{\perp}|^2}{1+D}\int \frac{|g_\varepsilon|^2}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})d_{\mathbf{G}}v^{\|}+\frac{4}{1+D}\int \frac{|g_\varepsilon|^2}{\mathcal{N}_\varepsilon}(v^{\|}+v^{\perp})\tfrac{1}{2}(1+|v^{\|}|^2)d_{\mathbf{G}}v^{\|}.
\end{aligned}\tag{4.23}$$

It suffices then to estimate the inner integral in (4.21) as done in (4.23) to arrive at

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi\frac{|g_\varepsilon||g_\varepsilon'|}{N_\varepsilon}\right\rangle\!\!\right\rangle \leqq 6\|b\|_{L^\infty}\varepsilon^{m-1}\left\langle \chi\frac{|g_\varepsilon|^2}{\mathcal{N}_\varepsilon}\right\rangle.\tag{4.24}$$

By a similar method, one can prove that

$$\varepsilon^{m-1}\left\langle\!\!\left\langle \chi\frac{|g_\varepsilon||g_{\varepsilon 1}'|}{N_\varepsilon}\right\rangle\!\!\right\rangle \leqq 6\|b\|_{L^\infty}\varepsilon^{m-1}\left\langle \chi\frac{|g_\varepsilon|^2}{\mathcal{N}_\varepsilon}\right\rangle.\tag{4.25}$$

Statement (c) follows then from (4.24), (4.25) and Proposition 4.3 (b). This concludes the proof of Proposition 4.4. □

The next Proposition summarizes the conclusions of Proposition 4.4 in the form that they are actually used below.

Proposition 4.5. *Assume that the family of initial data $G_\varepsilon^{\text{in}}$ satisfies the bound* (4.1) *and the normalizations* (4.2). *Then when $\chi \equiv \chi(v)$ is one of the collision invariants* $1, v_1, \ldots, v_D, |v|^2$ *or a linear combination thereof with constant coefficients,*

$$\frac{1}{\varepsilon}\left\langle\left(1-\frac{1}{N_\varepsilon}\right)\chi\mathcal{L}g_\varepsilon\right\rangle \quad \textit{and} \quad \varepsilon^{m-1}\left\langle \chi\frac{\mathcal{Q}^-(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\right\rangle$$

both converge to 0 *as* $\varepsilon \to 0$ *in* $L^\infty(dt;L^1(dx))$.

It remains to control the gain part of the collision integral (4.12). Instead of proving directly that

$$\varepsilon^{m-1}\Big\langle \chi\,\frac{\mathcal{Q}^+(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\Big\rangle \to 0$$

as $\varepsilon \to 0$ in $L^\infty(dt; L^1(dx))$, we use the flat-sharp decomposition of g_ε in the gain term above and show in Proposition 4.6 below that all the resulting quadratic expressions vanish in the limit as $\varepsilon \to 0$, except for the one involving only sharp terms, namely

$$\varepsilon^{3m-1}\Big\langle \chi\,\frac{\mathcal{Q}^+({}^\sharp g_\varepsilon,\ {}^\sharp g_\varepsilon)}{N_\varepsilon}\Big\rangle.$$

However, this term happens to be nonnegative, a property which we use in Proposition 4.8 below to prove eventually that it also vanishes in the limit as $\varepsilon \to 0$.

Proposition 4.6. *Assume that the family of initial data $G_\varepsilon^{\mathrm{in}}$ satisfies the bound* (4.1) *and the normalizations* (4.2). *Then when $\chi \equiv \chi(v)$ is one of the collision invariants $1, v_1, \ldots, v_D, |v|^2$ or a linear combination thereof with constant coefficients,*

$$\varepsilon^{m-1}\Big\langle \chi\,\frac{\mathcal{Q}^+(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\Big\rangle - \varepsilon^{3m-1}\Big\langle \chi\,\frac{\mathcal{Q}^+({}^\sharp g_\varepsilon,\ {}^\sharp g_\varepsilon)}{N_\varepsilon}\Big\rangle \to 0 \quad \textit{in } L^\infty(dt; L^1(dx)). \tag{4.26}$$

Proof. First, one has

$$\begin{aligned}
&\varepsilon^{m-1}\Big\langle \chi\,\frac{\mathcal{Q}^+(g_\varepsilon, g_\varepsilon)}{N_\varepsilon}\Big\rangle - \varepsilon^{m-1}\Big\langle \chi\,\frac{\mathcal{Q}^+(\varepsilon^m\,{}^\sharp g_\varepsilon,\ \varepsilon^m\,{}^\sharp g_\varepsilon)}{N_\varepsilon}\Big\rangle \\
&\quad = \varepsilon^{m-1}\Big\langle\!\Big\langle \chi\,\frac{{}^\flat g_\varepsilon'\,{}^\flat g_{\varepsilon\,1}'}{N_\varepsilon}\Big\rangle\!\Big\rangle + \varepsilon^{m-1}\Big\langle\!\Big\langle \chi\,\frac{\varepsilon^m\,{}^\sharp g_\varepsilon'\,{}^\flat g_{\varepsilon\,1}'}{N_\varepsilon}\Big\rangle\!\Big\rangle + \varepsilon^{m-1}\Big\langle\!\Big\langle \chi\,\frac{\varepsilon^m\,{}^\flat g_\varepsilon'\,{}^\sharp g_{\varepsilon\,1}'}{N_\varepsilon}\Big\rangle\!\Big\rangle
\end{aligned} \tag{4.27}$$

Then, in the case where $\chi(v) = 1 + |v|^2$, using the fact that $N_\varepsilon = \frac{2}{3} + \frac{1}{3}G_\varepsilon \geqq \frac{2}{3}$, one has

$$\begin{aligned}
\Big\langle\!\Big\langle \chi\,\frac{|{}^\flat g_\varepsilon'||{}^\flat g_{\varepsilon\,1}'|}{N_\varepsilon}\Big\rangle\!\Big\rangle &\leqq \tfrac{3}{2}\big\langle\!\big\langle (1+|v|^2+|v_1|^2)|{}^\flat g_\varepsilon'||{}^\flat g_{\varepsilon\,1}'|\big\rangle\!\big\rangle \\
&= \tfrac{3}{2}\big\langle\!\big\langle (1+|v|^2+|v_1|^2)|{}^\flat g_\varepsilon||{}^\flat g_{\varepsilon\,1}|\big\rangle\!\big\rangle \\
&\leqq \tfrac{3}{2}\|b\|_{L^\infty}\langle \chi|{}^\flat g_\varepsilon|\rangle^2 \\
&\leqq \tfrac{3}{2}\|b\|_{L^\infty}(D^2+4D+1)\langle |{}^\flat g_\varepsilon|^2\rangle.
\end{aligned}$$

Thus, as $\varepsilon \to 0$

$$\left|\varepsilon^{m-1}\Big\langle\!\Big\langle \chi\,\frac{{}^\flat g_\varepsilon'\,{}^\flat g_{\varepsilon\,1}'}{N_\varepsilon}\Big\rangle\!\Big\rangle\right| \leqq \tfrac{3}{2}\|b\|_{L^\infty}(D^2+4D+1)\varepsilon^{m-1}\langle |{}^\flat g_\varepsilon|^2\rangle \to 0 \tag{4.28}$$

in $L^\infty(dt; L^1(dx))$ by Proposition 4.3 (a).

By the same token

$$\begin{aligned}\left\langle\!\!\left\langle \chi \frac{\varepsilon^m |\,^\sharp g'_\varepsilon||\,^\flat g'_{\varepsilon\,1}|}{N_\varepsilon} \right\rangle\!\!\right\rangle &\leqq \tfrac{3}{2}\langle\!\langle (1+|v|^2+|v_1|^2)\varepsilon^m |\,^\sharp g'_\varepsilon||\,^\flat g'_{\varepsilon\,1}|\rangle\!\rangle \\ &= \tfrac{3}{2}\langle\!\langle (1+|v|^2+|v_1|^2)\varepsilon^m |\,^\sharp g_\varepsilon||\,^\flat g_{\varepsilon\,1}|\rangle\!\rangle \\ &\leqq \tfrac{9}{2}\langle\!\langle (1+|v|^2+|v_1|^2)|\,^\sharp g_\varepsilon|\rangle\!\rangle \leqq \tfrac{9}{2}\|b\|_{L^\infty}(1+D)\langle \chi\,|\,^\sharp g_\varepsilon|\rangle,\end{aligned}$$

where the penultimate inequality above rests on the estimate $|\varepsilon^m\,^\flat g_\varepsilon| \leqq 3$ inherited from the bounds

$$-\tfrac{3}{2} \leqq \frac{z}{1+\frac{1}{3}z} \leqq 3, \qquad z > -1.$$

Therefore, as $\varepsilon \to 0$

$$\left| \varepsilon^{m-1} \left\langle\!\!\left\langle \chi \frac{\varepsilon^m\,^\sharp g'_\varepsilon\,^\flat g'_{\varepsilon\,1}}{N_\varepsilon} \right\rangle\!\!\right\rangle \right| \leqq \tfrac{9}{2}\|b\|_{L^\infty}(1+D)\varepsilon^{m-1}\langle \chi\,|\,^\sharp g_\varepsilon|\rangle \to 0 \tag{4.29}$$

in $L^\infty(dt; L^1(dx))$ by Proposition 4.3 (b).

Estimates (4.28), (4.29) show that the first two terms on the right-hand side of (4.27) converge to 0 in $L^\infty(dt; L^1(dx))$ as $\varepsilon \to 0$. The third term on the right-hand side of (4.27) is accommodated as in (4.29), which completes the proof. □

4.4. The Limiting Local Conservation Laws

With the preparations contained in Propositions 4.4–4.7, we now state the proposition upon which rest the proofs of Proposition 4.1 and 4.3. This proposition removes the gap in the "formal proof" of the acoustic limit, namely the fact that renormalized solutions of the Boltzmann equation do not in general satisfy either the local conservation law of momentum or that of energy.

Proposition 4.7. *Assume that the family of initial data $G_\varepsilon^{\text{in}}$ satisfies the bound* (4.1) *and the normalizations* (4.2). *Assume further that the family of fluctuations g_ε converges to g in w^*-$L^\infty(dt; w$-$L^1((1+|v|^2)M\,dv\,dx))$ as $\varepsilon \to 0$. Then*

$$\partial_t \langle \chi g\rangle + \nabla_x \cdot \langle v\chi g\rangle = 0 \tag{4.30}$$

when $\chi = \chi(v)$ is any one of the collision invariants $1, v_1, \ldots, |v|^2$ or a linear combination thereof with constant coefficients.

The key argument in the proof of Proposition 4.7 has been isolated in Proposition 4.8 below.

Proposition 4.8. *Assume that the family of initial data $G_\varepsilon^{\text{in}}$ satisfies the bound* (4.1) *and the normalizations* (4.2). *Then, as $\varepsilon \to 0$,*

$$\varepsilon^{3m-1} \frac{\mathcal{Q}^+(\,^\sharp g_\varepsilon,\,^\sharp g_\varepsilon)}{N_\varepsilon} \to 0 \text{ in } L^1_{\text{loc}}(\mathbf{R}_+; L^1((1+|v|^2)Mdvdx)). \tag{4.31}$$

Although we have not been able to control the term appearing in (4.31) in the same fashion as in Section 4.4, it so happens that this term is nonnegative and would contribute an unphysical growth of energy were it not vanishingly small in the limit as $\varepsilon \to 0$.

Proof of Proposition 4.8. Choose any $T > 0$ and set $\chi(v) = 1 + |v|^2$ in (4.10). Integrating (4.10) over $[0, T] \times \mathbf{T}^D$ and reshuffling the terms leads to

$$\begin{aligned}
&\int_0^T \int \left\langle \chi \frac{\varepsilon^{3m-1} \mathcal{Q}^+({}^\sharp g_\varepsilon, {}^\sharp g_\varepsilon)}{N_\varepsilon} \right\rangle dx\, dt \\
&\quad = \varepsilon^{m-1} \int_0^T \int \left\langle \chi \frac{\mathcal{Q}^-(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \right\rangle dx\, dt - \frac{1}{\varepsilon} \int_0^T \int \left\langle \left(1 - \frac{1}{N_\varepsilon}\right) \chi \mathcal{L} g_\varepsilon \right\rangle dx\, dt \\
&\qquad - \int_0^T \int \left[\varepsilon^{m-1} \left\langle \chi \frac{\mathcal{Q}^+(g_\varepsilon, g_\varepsilon)}{N_\varepsilon} \right\rangle - \varepsilon^{m-1} \left\langle \chi \frac{\mathcal{Q}^+(\varepsilon^m\, {}^\sharp g_\varepsilon, \varepsilon^m\, {}^\sharp g_\varepsilon)}{N_\varepsilon} \right\rangle \right] dx\, dt \\
&\qquad + \left[\int \langle \chi \gamma_\varepsilon(T) \rangle\, dx - \int \langle \chi \gamma_\varepsilon(0) \rangle\, dx \right]. \qquad (4.32)
\end{aligned}$$

On the right-hand side of (4.32), the first two integrals converge to 0 as $\varepsilon \to 0$ by Proposition 4.5 while the third integral converges to 0 by Proposition 4.6. The last term between brackets on the right-hand side of (4.32) can be recast as

$$\begin{aligned}
&\int \langle \chi \gamma_\varepsilon(T) \rangle\, dx - \int \langle \chi \gamma_\varepsilon(0) \rangle\, dx \\
&\quad = \int \langle \chi [\gamma_\varepsilon(T) - g_\varepsilon(T)] \rangle\, dx - \int \langle \chi [\gamma_\varepsilon(0) - g_\varepsilon^{\mathrm{in}}] \rangle\, dx \\
&\qquad + \frac{1}{\varepsilon^m} \left[\int \langle \chi G_\varepsilon(T) \rangle\, dx - \int \langle \chi G_\varepsilon(0) \rangle\, dx \right]. \qquad (4.33)
\end{aligned}$$

On the right-hand side of (4.33), the two first integrals converge to 0 as $\varepsilon \to 0$ by Proposition 4.3 (d), while the term between brackets is nonpositive by the energy inequality (3.8). Therefore

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \int_0^T \int \left\langle \chi \frac{\varepsilon^{3m-1} \mathcal{Q}^+({}^\sharp g_\varepsilon, {}^\sharp g_\varepsilon)}{N_\varepsilon} \right\rangle dx\, dt \\
&\quad = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^m} \left[\int \langle \chi G_\varepsilon(T) \rangle\, dx - \int \langle \chi G_\varepsilon(0) \rangle\, dx \right] \leqq 0.
\end{aligned} \qquad (4.34)$$

But the integrand on the left-hand side of (4.34) is nonnegative, because the collision cross-section entering the definition of $\mathcal{Q}^+$ is nonnegative (1.6) while the functions ${}^\sharp g_\varepsilon$ in the decomposition (4.8) also are nonnegative. Therefore the inequality (4.34) implies the convergence (4.31). $\square$

With this last preparation, the proof of Proposition 4.7 is a mere formality.

Proof of Proposition 4.7. Write (4.10) in the form

$$\partial_t\langle\chi\gamma_\varepsilon\rangle + \nabla_x\cdot\langle v\chi\gamma_\varepsilon\rangle = \frac{1}{\varepsilon}\Big\langle\Big(1-\frac{1}{N_\varepsilon}\Big)\chi\mathcal{L}g_\varepsilon\Big\rangle - \varepsilon^{m-1}\Big\langle\chi\,\frac{\mathcal{Q}^-(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\Big\rangle$$
$$+\left[\varepsilon^{m-1}\Big\langle\chi\,\frac{\mathcal{Q}^+(g_\varepsilon,g_\varepsilon)}{N_\varepsilon}\Big\rangle - \varepsilon^{3m-1}\Big\langle\chi\,\frac{\mathcal{Q}^+({}^\sharp g_\varepsilon,{}^\sharp g_\varepsilon)}{N_\varepsilon}\Big\rangle\right] + \Big\langle\chi\,\frac{\varepsilon^{3m-1}\mathcal{Q}^+({}^\sharp g_\varepsilon,{}^\sharp g_\varepsilon)}{N_\varepsilon}\Big\rangle, \tag{4.35}$$

where $\chi\equiv\chi(v)$ is any one of the collision invariants 1, $v_1,\ldots,v_D$ or $|v|^2$. The left-hand side of (4.35) converges to

$$\partial_t\langle\chi g\rangle + \nabla_x\cdot\langle v\chi g\rangle \tag{4.36}$$

in $\mathcal{D}(\mathbf{R}_+^*\times\mathbf{R}^D)$ as $\varepsilon\to 0$ by Proposition 4.3(e). The first two terms on the right-hand side of (4.35) converge to 0 in $L^\infty(dt;L^1(dx))$ as $\varepsilon\to 0$ by Proposition 4.5. The third term on the right-hand side of (4.35) converges to 0 in $L^\infty(dt;L^1(dx))$ as $\varepsilon\to 0$ by Proposition 4.6. Finally, the last term on the right-hand side of (4.35) converges to 0 in $L^1_{\mathrm{loc}}(dt;L^1(dx))$ as $\varepsilon\to 0$ by Proposition 4.8. Thus, the expression (4.36) is 0, which establishes Proposition 4.7. □

4.5. Proof of Proposition 4.1

It follows from Proposition 3.2 (a) that the family g_ε is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt;w\text{-}L^1((1+|v|^2)M\,dv\,dx))$. Again by Proposition 3.2 (a) and Proposition 4.3 (d), the family of moments

$$(\langle\gamma_\varepsilon\rangle,\langle v\gamma_\varepsilon\rangle,\langle\tfrac{1}{D}(|v|^2-D)\gamma_\varepsilon\rangle) \tag{4.37}$$

is bounded in $L^\infty(dt;L^1(dx))$. The system (4.30) and Proposition 4.3 (c) imply that the family (4.37) is also bounded in $W^{1,\infty}([0,\infty);W^{-1,1}(\mathbf{T}^D))$. By Proposition 3.2 (b) and Ascoli's theorem in the form stated in Appendix C of [20] as Lemma C.1, the family (4.37) is relatively compact in $C([0,\infty);w\text{-}L^1(dx;\mathbf{R}\times\mathbf{R}^D\times\mathbf{R}))$. This and Proposition 4.3 (d) show that the family of moments

$$(\langle g_\varepsilon\rangle,\langle vg_\varepsilon\rangle,\langle\tfrac{1}{D}(|v|^2-D)g_\varepsilon\rangle) \tag{4.38}$$

is relatively compact in $C([0,\infty);w\text{-}L^1(dx))$. Let g be a limit point of g_ε as $\varepsilon\to 0$ and let $\varepsilon_n\to 0$ define a subsequence $g_{\varepsilon_n}\to g$ as $n\to+\infty$. By Proposition 4.7, g must satisfy the system of moment equations (4.30). By Proposition 3.2 (d), g is a local infinitesimal Maxwellian of the form (4.4). Substituting the form (4.4) into the local conservation laws (4.30) shows that the parameters (ρ,u,θ) of g must satisfy the system of acoustics (1.9) – this being the essence of the formal proof of the acoustic limit in Section 2. Finally, the subsequence

$$(\langle g_{\varepsilon_n}\rangle,\langle vg_{\varepsilon_n}\rangle,\langle\tfrac{1}{D}(|v|^2-D)g_{\varepsilon_n}\rangle)\to(\rho,u,\theta)$$

in $C([0,\infty);w\text{-}L^1(dx))$. That this convergence is uniform locally in $t\in[0,\infty)$ implies in particular that

$$(\langle g^{\mathrm{in}}_{\varepsilon_n}\rangle,\langle vg^{\mathrm{in}}_{\varepsilon_n}\rangle,\langle\tfrac{1}{D}(|v|^2-D)g^{\mathrm{in}}_{\varepsilon_n}\rangle)\to(\rho,u,\theta)|_{t=0}$$

in $w\text{-}L^1(dx)$, which establishes the initial data (4.5). □

5. Conclusions

Propositions 4.1 and 4.2 are very likely the optimal formulations of the acoustic limit of the Boltzmann equation, save for assumptions (A1) and (A2). Both assumptions (A1) and (A2) reflect the fact that our results use the entropy inequality (3.14) to propagate the entropy bound (3.9) to any positive time, thereby neglecting the bound on the entropy dissipation rate $R(G_\varepsilon)$ provided by (3.14). We therefore feel that these assumptions are of a purely technical nature. Indeed, (A1) requires that $m > 1$ while the formal derivation (Proposition 2.1) allows for every $m > 0$. Likewise, (A2) excludes the natural case of a hard-sphere gas (see [13]) or more generally that of cutoff potentials harder than that for Maxwell molecules (see [12, Chapter II.4, 5]). Possibly both (A1) and (A2) could be dispensed with by an appropriate use of the bound on the entropy dissipation estimate (3.14).

Acknowledgements. C.B. and F.G. were supported by the TMR network ERB FMRX CT97 0157. F.G. thanks the University of Arizona for its support during his visits there. C.D.L. was supported by the NSF under grant DMS-9803753 at the University of Arizona. He thanks the Ecole Normale Supérieure and the Université Paris VII for their support during his visits there.

References

1. C. Bardos, F. Golse, & D. Levermore, Sur les limites asymptotiques de la théorie cinétique conduisant à la dynamique des fluides incompressibles, *C.R. Acad. Sci. Paris* **309**, (1989) 727–732.
2. C. Bardos, F. Golse, & D. Levermore, Fluid Dynamic Limits of Kinetic Equations I: Formal Derivations, *J. Stat. Phys.* **63**, (1991) 323–344.
3. C. Bardos, F. Golse, & C. D. Levermore, Fluid Dynamic Limits of Kinetic Equations II: Convergence Proofs for the Boltzmann Equation, *Commun. Pure & Appl. Math.* **46**, (1993) 667–753.
4. C. Bardos, F. Golse, & C. D. Levermore, Acoustic and Stokes Limits for the Boltzmann Equation, *C.R. Acad. Sci. Paris* **327**, (1999) 323–328.
5. C. Bardos, F. Golse, & C. D. Levermore, The Stokes Limit for the Boltzmann Equation, preprint.
6. C. Bardos & C. D. Levermore, Kinetic Equations and an Incompressible Fluid Dynamical Limit that Recovers Viscous Heating, (in preparation 1999).
7. C. Bardos & S. Ukai, The Classical Incompressible Navier-Stokes Limit of the Boltzmann Equation, *Math. Models & Meth. Appl. Sci.* **1**, (1991) 235–257.
8. A. V. Bobylev, Quasistationary Hydrodynamics for the Boltzmann Equation, *J. Stat. Phys.* **80**, (1995) 1063–1083.
9. F. Bouchut, F. Golse, and M. Pulvirenti, *Kinetic Equations and Asymptotic Theories*, L. Desvillettes & B. Perthame eds., (Gauthier-Villars Elsevier, Paris 1999).
10. L. Boltzmann, Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen *Sitzungs. Akad. Wiss. Wien* **66**, (1872) 275–370; English translation in *Kinetic Theory, vol. 2* S.G. Brush ed. (Pergamon Press, London 1966) 88–174.
11. R. Caflisch, The Fluid Dynamical Limit of the Nonlinear Boltzmann Equation, *Commun. Pure & Appl. Math.* **33**, (1980) 651–666.
12. C. Cercignani, *The Boltzmann Equation and its Applications*, (Springer-Verlag, New York 1988).
13. C. Cercignani, R. Illner, & M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, (Springer-Verlag, New York 1994).

14. A. DEMASI, R. ESPOSITO, AND J. LEBOWITZ, Incompressible Navier-Stokes and Euler Limits of the Boltzmann Equation, *Commun. Pure & Appl. Math.* **42**, (1990) 1189–1214.
15. R. J. DIPERNA & P.-L. LIONS, On the Cauchy Problem for the Boltzmann Equation: Global Existence and Weak Stability Results, *Annals of Math.* **130**, (1990) 321–366.
16. D. ENSKOG, Kinetische Theorie der Vorgänge in mässig verdünnten Gasen, I. Allgemeiner Teil, Uppsala, Almqvist & Wiksell; English translation in *Kinetic Theory, vol. 3* S. G. BRUSH ed. (Pergamon Press, Oxford 1972) 125–225.
17. H. GRAD, Principles of the Kinetic Theory of Gases, *Handbuch der Physik vol. 12* S. FLÜGGE ed. (Springer-Verlag, Berlin 1958) 205–294.
18. D. HILBERT, Begründung der kinetischen Gastheorie, *Math. Annalen* **72,** (1912) 562–577; English translation in *Kinetic Theory, vol. 3* S. G. BRUSH ed. (Pergamon Press, Oxford 1972) 89–101.
19. C. D. LEVERMORE, Entropic Convergence and the Linearized Limit for the Boltzmann Equation, *Commun. P.D.E.* **18**, (1993) 1231–1248.
20. P.-L. LIONS, *Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models,* (Oxford University Press, New York 1996)
21. J. C. MAXWELL, On the Dynamical Theory of Gases, *Phil. Trans. Roy. Soc. London* **157**, (1866) 49–88.
22. T. SIDERIS, Formation of Singularities in Three Dimensional Compressible Fluids, *Commun. Math. Phys.* **101**, (1985) 475–485.
23. Y. SONE, Asymptotic Theory of a Steady Flow of Rarefied Gas Past Bodies for Small Knudsen Numbers, in *Advances in Kinetic Theory and Continuum Mechanics,* R. GATIGNOL & SOUBBARAMAYER eds. (Springer-Verlag, Berlin 1991) 19–31.
24. S. UKAI, The Incompressible Limit and the Initial Layer of the Compressible Euler Equation, *J. Math. Kyoto Univ.* **26**, (1986) 323–331.

Université Paris VII
& CMLA, Ecole Normale Supérieure Cachan
61 avenue du Président Wilson,
94235 Cachan cedex, France

Université Paris VII
& Institut Universitaire de France
DMA, Ecole Normale Supérieure
45 rue d'Ulm
75230 Paris cedex 05, France

and

Department of Mathematics
University of Arizona
Tucson, Arizona 85721, USA

(*Accepted October 22, 1999*)