A Classification of Well-Posed Kinetic Layer Problems

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Abstract

In the first part of this paper, we study the half space boundary value problem for the Boltzmann equation with an incoming distribution, obtained when considering the boundary layer arising in the kinetic theory of gases as the mean free path tends to zero. We linearize it about a drifting Maxwellian and prove that, as conjectured by Cercignani [4], the problem is well-posed when the drift velocity u exceeds the sound speed c, but that one (respectively four, five) additional conditions must be imposed when $0 < u < c$ (respectively $-c < u < 0$ and $u < -c$).

In the second part, we show that the well-posedness and the asymptotic behavior results for kinetic layers equations with prescribed incoming flux can be extended to more general and realistic boundary conditions.

1. Kinetic Layer Problems with Incoming Flux

1.1. Introduction. We consider the boundary layer problem arising in the kinetic theory of gases when the mean free path tends to zero. The resulting half-space problem for the Boltzmann equation is

$$\xi_1 \frac{\partial F}{\partial x} = Q(F, F), \quad x > 0, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \tag{1.1.1}$$

where Q is the collision operator defined by

$$Q(F, F) = \int_{R^3 \times S^2} (F(\eta')F(\xi') - F(\eta)F(\xi)) q(\eta - \xi, \omega)\, d\eta\, d\omega \tag{1.1.2}$$

with

$$\begin{aligned} \eta' &= \eta - ((\eta - \xi) \cdot \omega)\omega, \\ \xi' &= \xi + ((\eta - \xi) \cdot \omega)\omega. \end{aligned} \tag{1.1.3}$$

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We restrict ourselves to collision kernels for hard sphere gas satisfying the angular cut-off assumption as proposed by Grad [9]:

$$q(V, \omega) = \sigma|V \cdot \omega|. \tag{1.1.4}$$

We are interested in solutions F such that

$$\lim_{x \to +\infty} F(x, \xi) = M_\infty(\xi), \tag{1.1.5}$$

where

$$M_\infty(\xi) = \frac{\rho_\infty}{(2\pi T_\infty)^{3/2}} \exp\left\{ - \frac{|\xi - u_\infty|^2}{2T_\infty} \right\} \tag{1.1.6}$$

is the Maxwellian distribution whose parameters $(\rho_\infty, u_\infty, T_\infty)$ describe the macroscopic flow to which we match the boundary layer. Linearizing around M_∞ in the form $F = M_\infty + M_\infty^{1/2} f$, equation (1.1.1) has the form

$$\xi_1 \frac{\partial f}{\partial x} + L_M f = 0 \tag{1.1.7}$$

with

$$L_M f = 2M_\infty^{-1/2} Q\left(M_\infty, M_\infty^{1/2} f\right). \tag{1.1.8}$$

It is clear that no changes arise if u_∞ has components along the axis orthogonal to x, but that the x-component of u_∞, denoted by u, can provide significant changes.

Shifting the velocities by changing ξ_1 to $\xi_1 + u$, equation (1.1.7) can be rewritten as

$$(\xi_1 + u) \frac{\partial f}{\partial x} + Lf = 0 \tag{1.1.9}$$

with

$$Lf = 2M^{-1/2} Q\left(M, M^{1/2} f\right) \tag{1.1.10}$$

and

$$M(\xi) = \frac{\rho_\infty}{(2\pi T_\infty)^{3/2}} \exp\left\{ - \frac{|\xi|^2}{2T_\infty} \right\}. \tag{1.1.11}$$

We look for bounded solutions of (1.1.9) with a given distribution ϕ of incoming particles at $x = 0$:

$$f(0, \xi) = \phi(\xi), \qquad \xi_1 + u > 0. \tag{1.1.12}$$

Equations (1.1.9), (1.1.12) with $u = 0$ have been studied by Bardos, Caflisch, Nicolaenko [2] and Cercignani [4]. They proved that the problem is well posed when the mass flux defined by $\int \xi_1 M^{1/2} f(x, \xi)\, d\xi$ (which is a constant in x) is specified. When $u \neq 0$, Cercignani [4] conjectured that the number of additional conditions to ensure the well-posedness of the problem depends on the value of u compared to the Mach number of the flow at infinity $c = \sqrt{\frac{5}{3} T_\infty}$ and that it is simply related to the signature of the quadratic form

$$P(f, f) = \int_{\mathbf{R}^3} (\xi_1 + u) f^2\, d\xi \tag{1.1.13}$$

(which can be viewed as the linearized entropy flux), in the nullspace $N(L)$ of L. More precisely, when $u > \sqrt{\frac{5}{3} T_\infty}$, the problem is well posed for any incoming distribution ϕ. When $0 < u < \sqrt{\frac{5}{3} T_\infty}$, it is well posed when looking for solutions with vanishing mass flux (which is no longer constant) at infinity. The conjecture claims also that, when $-\sqrt{\frac{5}{3} T_\infty} < u < 0$, (respectively $u < -\sqrt{\frac{5}{3} T_\infty}$), one has to give four (respectively five) additional conditions.

The proof of a similar conjecture was given by Arthur and Cercignani [1] and Greenberg and Van der Mee [10] in the case of the linearized Bhatganar-Gross-Krook (BGK) model.

Cercignani's conjecture can be supported by formally looking at the compressible Euler equations for (ρ, v, T) linearized around the constant velocity u, ($u \neq 0, \pm \sqrt{\frac{5}{3} T_\infty}$). It is a hyperbolic system with characteristic values $\lambda_1 = \lambda_2 = \lambda_3 = u$, $\lambda_4 = u + \sqrt{\frac{5}{3} T_\infty}$, $\lambda_5 = u - \sqrt{\frac{5}{3} T_\infty}$. The values of the conserved quantities along the corresponding characteristics are determined by the initial conditions for incoming characteristics and boundary conditions for outgoing ones. For example, when $u > \sqrt{\frac{5}{3} T_\infty}$, there are five outgoing characteristics and one has to give the values of (ρ, v, T) at the boundary. When $0 < u < \sqrt{\frac{5}{3} T_\infty}$, there is one incoming characteristic and one has to impose four boundary conditions.

The aim of this work is to give a rigorous proof of Cercignani's conjecture for equations (1.1.9), (1.1.12). It is based on energy type estimates and on the construction of a solution as the limit, as $B \to \infty$, of solutions f_B defined in the slab $[0, B]$. The main new tool to handle such a problem, since it is related to the determination of admissible boundary conditions for a hyperbolic system, will the linearized entropy flux (the entropy for the Boltzmann equation being the H function):

$$\int (\xi_1 + u) f(x, \xi)^2\, d\xi.$$

Essentially, we shall have to choose a boundary condition at $x = B$ for f_B such that the projection of the limit solution on $N(L)$ has a positive linearized entropy flux at infinity.

The following subsections are organized as follows: in subsection 1.2, we give some preliminaries and state the main results; in subsection 1.3, the problem in the slab $[0, B]$ is studied; the solution in $[0, \infty[$ is constructed in subsection 1.4; subsection 1.5 is devoted to the problem of uniqueness; in subsection 1.6, we deal with the special cases $u = \pm\sqrt{\frac{5}{3}T_\infty}$ and in subsection 1.7, we reformulate the invariant relations in a more general and intrinsic way.

1.2. Preliminaries and results. The linearized operator L defined in (1.1.8) is non-negative and selfadjoint on $L^2(\mathbb{R}^3)$. Due to the interaction law (1.1.4), it can be split as

$$L = \nu(\xi) - K, \tag{1.2.1}$$

where the collision frequency $\nu(\xi)$ satisfies, for hard sphere balls,

$$\nu_0(1 + |\xi|) \leqq \nu(\xi) \leqq \nu_1(1 + |\xi|), \tag{1.2.2}$$

ν_0 and ν_1 being constants which depend on T_∞, and K a compact operator on $L^2(\mathbb{R}^3)$ which can be defined by

$$Kv(\xi) = \int_{\mathbb{R}^3} k(\xi, \eta) v(\eta)\, d\eta \tag{1.2.3}$$

with

$$\int_{\mathbb{R}^3} k(\xi, \eta)^2\, d\eta \leqq C(1 + |\xi|)^{-1}. \tag{1.2.4}$$

The domain of L is

$$D(L) = \left\{ f \in L^2(\mathbb{R}^3),\ \nu(\xi)^{1/2} f \in L^2(\mathbb{R}^3) \right\} \tag{1.2.5}$$

and its nullspace $N(L)$ is a five-dimensional subspace spanned by $\tilde{X}_\alpha$, $\alpha = 0, 1, \cdots, 4$, with

$$\tilde{X}_0(\xi) = \frac{\xi_2}{\sqrt{T_\infty}} M^{1/2},$$

$$\tilde{X}_1(\xi) = \frac{\xi_3}{\sqrt{T_\infty}} M^{1/2},$$

$$\tilde{X}_2(\xi) = \left(\sqrt{\tfrac{5}{2}} - \frac{|\xi|^2}{\sqrt{10}\, T_\infty} \right) M^{1/2},$$

$$\tilde{X}_3(\xi) = \left(\frac{1}{\sqrt{2T_\infty}} \xi_1 + \frac{|\xi|^2}{\sqrt{30}\, T_\infty} \right) M^{1/2},$$

$$\tilde{X}_4(\xi) = \left(-\frac{1}{\sqrt{2T_\infty}} \xi_1 + \frac{|\xi|^2}{\sqrt{30}\, T_\infty} \right) M^{1/2}.$$

The collision invariants have been chosen in such a way that

$$P(\tilde{X}_\alpha, \tilde{X}_\beta) = 0, \qquad \alpha \neq \beta, \tag{1.2.6}$$

and that

$$\int \tilde{X}_\alpha \tilde{X}_\beta \, d\xi = 0, \qquad \alpha \neq \beta,$$

that is the $\tilde{X}_\alpha$ have been constructed in order to form an orthogonal basis of $N(L)$ with respect to both the usual scalar product and P. Note that

$$\begin{aligned} P(\tilde{X}_i, \tilde{X}_i) &= \rho_\infty u, \qquad i = 0,1,2, \\ P(\tilde{X}_3, \tilde{X}_3) &= \rho_\infty\left(u + \sqrt{\tfrac{5}{3}T_\infty}\right), \\ P(\tilde{X}_4, \tilde{X}_4) &= \rho_\infty\left(u - \sqrt{\tfrac{5}{3}T_\infty}\right). \end{aligned} \tag{1.2.7}$$

Consequently, the signature σ of the restriction of the quadratic form P to $N(L)$ is:

$(5,0)$ if $u > \sqrt{\frac{5}{3}T_\infty}$,
$(4,0)$ if $u = \sqrt{\frac{5}{3}T_\infty}$,
$(4,1)$ if $0 < u < \sqrt{\frac{5}{3}T_\infty}$,
$(1,1)$ if $u = 0$,
$(1,4)$ if $-\sqrt{\frac{5}{3}T_\infty} < u < 0$,
$(0,4)$ if $u = -\sqrt{\frac{5}{3}T_\infty}$,
$(0,5)$ if $u < -\sqrt{\frac{5}{3}T_\infty}$.

For convenience, we denote $X_\alpha = \tilde{X}_\alpha / \sqrt{|P(\tilde{X}_\alpha, \tilde{X}_\alpha)|}$ in the nondegenerate cases. We recall that any function f can be split in a unique way as

$$f = w_f + q_f, \tag{1.2.8}$$

where $q_f \in N(L)$ is its hydrodynamic part and $w_f \in R(L) = N(L)^\perp$ its kinetic part. We also have, for some $\mu > 0$,

$$\text{for all } f \in D(L), \quad \int_{\mathbf{R}^3} fLf \, d\xi \geqq \mu \int_{\mathbf{R}^3} \nu(\xi) w_f^2 \, d\xi. \tag{1.2.9}$$

Finally, we introduce the notations

$$\|f\|^2 = \int_{\mathbf{R}^3} f(\xi)^2 \, d\xi, \tag{1.2.10}$$

$$|||f|||^2 = \int_{\mathbf{R}^+ \times \mathbf{R}^3} f(x, \xi)^2 \, dx \, d\xi, \tag{1.2.11}$$

and

$$\|f\|_{2,\gamma}^2 = \int_{\mathbf{R}^3} \sup_{x \geqq 0} e^{2\gamma x} f(x, \xi)^2 \nu(\xi) \, d\xi. \tag{1.2.12}$$

The space of functions f equipped with the norm $\|f\|_{2,\gamma}$ is denoted by $L^2(\mathbb{R}^3, L^\infty_\gamma(\mathbb{R}^+))$.

Let us now state the results.

Consider the system

$$(u+\xi_1)\frac{\partial f}{\partial x} + Lf = 0, \qquad x > 0, \tag{1.2.13}$$

$$f(0,\xi) = \phi(\xi), \qquad u + \xi_1 > 0. \tag{1.2.14}$$

We are going to consider any function ϕ satisfying one of the following two conditions:

$$\int_{\xi_1+u>0} (\xi_1+u)\phi(\xi)^2\, d\xi \leqq k_\phi < \infty, \tag{1.2.15}$$

$$\int_{\xi_1+u>0} \nu(\xi)\phi(\xi)^2\, d\xi \leqq k_\phi < \infty. \tag{1.2.15'}$$

Denote

$$I = \{0,1,2,3,4\},$$

$$I^+ = \{\alpha, P(\mathrm{X}_\alpha, \mathrm{X}_\alpha) > 0\}, \tag{1.2.16}$$

$$I^- = \{\alpha, P(\mathrm{X}_\alpha, \mathrm{X}_\alpha) < 0\}, \tag{1.2.17}$$

$$I^0 = \{\alpha, P(\mathrm{X}_\alpha, \mathrm{X}_\alpha) = 0\}. \tag{1.2.18}$$

THEOREM 1.2.1. *Let λ_α be given real constants for $\alpha \in I^-$ and ϕ satisfying* (1.2.15), *then the system* (1.2.13)–(1.2.14) *has a unique solution $f \in L^\infty(dx, L^2(|\xi_1 + u|\, d\xi))$ such that*

$$P(f, \mathrm{X}_\alpha) = \lambda_\alpha, \qquad \alpha \in I^-,\ x > 0. \tag{1.2.19}$$

Moreover, we have

$$P(f, \mathrm{X}_\alpha) = 0, \qquad \alpha \in I^0,\ x > 0, \tag{1.2.20}$$

and there exists a unique $q^\infty \in N(L)$ such that

$$f - q^\infty \in L^\infty\big(e^{\gamma x}\, dx, L^2(|\xi_1 + u|\, d\xi)\big), \tag{1.2.21}$$

for small enough $\gamma > 0$.

In addition, if ϕ *satisfies* (1.2.15′),

$$f - q^{\infty} \in L^{\infty}\left(e^{\gamma x}\,dx, L^2(\nu\,d\xi)\right). \tag{1.2.21′}$$

Although the difference between the weights $|\xi_1 + u|$ and $\nu(\xi)$ does not appear relevant at the present stage, it will be of significant importance in Section 2 for accomodation boundary conditions.

In the following subsections 1.3, 1.4 and 1.5, we shall restrict our attention to cases where the quadratic form P is nondegenerate.

1.3. The approximate problem in the slab $[0, B]$. Consider the problem

$$(u + \xi_1)\frac{\partial f_B}{\partial x} + Lf_B = 0, \qquad 0 < x < B, \tag{1.3.1}$$

$$f_B(0, \xi) = \phi(\xi), \qquad u + \xi_1 > 0. \tag{1.3.2}$$

One has to impose a boundary condition at $x = B$. Notice that, when $u = 0$, a natural condition is a reflection condition (see [2], [4])

$$f_B(B, \xi) = f_B(B, R\xi), \tag{1.3.3}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $R\xi = (-\xi_1, \xi_2, \xi_3)$. It ensures existence of a solution such that $P(f_B, f_B)(B) = 0$. This property is preserved when taking the limit as $B \to \infty$. The aim of the lemma below is (in the nondegenerated cases $u \neq 0$, $\pm\sqrt{\frac{5}{3}T_\infty}$) to write boundary conditions at $x = B$ which ensure $P(f_B, f_B)(B) \geqq 0$; they will provide existence of a solution f_B such that certain quantities independent of x (namely $P(f_B, X_\alpha)$ for $\alpha \in I^-$) vanish at $x = B$. These properties will be preserved when passing to the limit as $B \to \infty$.

LEMMA 1.3.1. *Assume that P is nondegenerate; then there exists a linear subspace G of $L^2(|\xi_1 + u|\,d\xi)$ such that*

(i) *for all $g \in G$, $P(g, g) \geq 0$ and G is maximal with respect to this property,*
(ii) *for all $g \in G$ and all $\alpha \in I^-$, $P(g, X_\alpha) = 0$,*
(iii) *for all $\alpha \in I^+$, $X_\alpha \in G$.*

Proof: First decompose the space $V = L^2(|\xi_1 + u|\,d\xi)$ into

$$V = N(L) \oplus W,$$

where W is the orthogonal complement of $N(L)$ with respect to the form P. Let S be the set of subspaces of W where P is positive. The subspace G is constructed as follows:

$$G = \text{span}\{X_i,\ i \in I^+\} + X, \qquad X \subset W,$$

where X is a maximal element of S. To prove the existence of X it is sufficient, by Zorn's lemma, to prove that S is inductive. Clearly, if A is a totally ordered subset of S, the subspace of all linear finite combinations of vectors belonging to a set of A is an upper bound for A. Now, G satisfies properties (ii) and (iii) of the lemma. To prove that it is maximal, use a contradiction argument: suppose that P is positive on a subspace $G_1 \subset V$ that contains G and denote by G_1' its projection (orthogonal with respect to P) on W. For any $g \in G_1$, one has

$$g = g' + \sum_{i \in I^+} k_i X_i + \sum_{i \in I^-} k_i X_i, \qquad g' \in G_1'.$$

Clearly, $g' + \sum_{i \in I^-} k_i X_i \in G_1$; thus $P(g', g') \geqq 0$. But X is maximal and hence $G_1' = X$ and $\sum_{i \in I^-} k_i X_i \in G_1$. Computing $P(\sum_{i \in I^-} k_i X_i, \sum_{i \in I^-} k_i X_i)$, one obtains that $k_i = 0$ for $i \in I^-$. Thus $G_1 = G$ and G is maximal.

To construct a solution of

$$(1.3.4) \qquad (u + \xi_1)\frac{\partial f_B}{\partial x} + Lf_B = 0, \qquad 0 < x < B,$$

$$(1.3.5) \qquad f_B(0, \xi) = \phi(\xi), \qquad u + \xi_1 > 0,$$

$$(1.3.6) \qquad f_B(B, \xi) \in G,$$

we consider the penalized system (we drop the indices B)

$$(1.3.7) \qquad (u + \xi_1)\frac{\partial f^\varepsilon}{\partial x} + Lf^\varepsilon + \varepsilon f^\varepsilon = 0, \qquad 0 < x < B,$$

$$(1.3.8) \qquad f^\varepsilon(0, \xi) = \phi(\xi), \qquad u + \xi_1 > 0,$$

$$(1.3.9) \qquad f^\varepsilon(B, \xi) \in G.$$

It is classical to prove that (1.3.7)–(1.3.9) has a unique solution with $\nu^{1/2} f^\varepsilon \in L^2([0, B] \times \mathbb{R}^3)$. To get uniform estimates with respect to ε, one first writes

$$(1.3.10) \qquad \int_{[0, B] \times \mathbb{R}^3} \nu(\xi) w_{f^\varepsilon}^2 \, dx \, d\xi \leqq \frac{1}{\mu} k_\phi .$$

To show that q_{f^ε} remains bounded in $L^2([0, B] \times \mathbb{R}^3)$, one uses a contradiction argument: suppose that

$$(1.3.11) \qquad A^\varepsilon = \int_{[0, B] \times \mathbb{R}^3} q_{f^\varepsilon}^2 \, dx \, d\xi \to \infty, \quad \varepsilon \to 0,$$

and denote $g^\varepsilon = f^\varepsilon / A^\varepsilon$. It is bounded in $L^2([0, B] \times \mathbb{R}^3)$ and satisfies

$$(1.3.12) \qquad (u + \xi_1)\frac{\partial g^\varepsilon}{\partial x} + \varepsilon g^\varepsilon + \frac{1}{A^\varepsilon} L w_{f^\varepsilon} = 0.$$

Thus,

$$(1.3.13) \qquad (u + \xi_1)\frac{\partial g^\varepsilon}{\partial x} \to 0, \quad \varepsilon \to 0.$$

Using a compactness theorem of Golse, Perthame, Sentis [8], there exists a subsequence of g^ε such that q_{g^ε} (thus g^ε) strongly converges in $L^2([0, B] \times \mathbb{R}^3)$ and for a.e. x to a function $Z = \sum_{\beta=0}^4 b_\beta \mathrm{X}_\beta \in N(L)$ independent of x. To prove that Z is identically 0, one multiplies equation (1.3.12) by X_α and integrates over $u + \xi_1 > 0$ and $(0, x)$; setting

$$P^+(f, g) = \int_{u+\xi_1>0} (u + \xi_1) f(\xi) g(\xi)\, d\xi, \tag{1.3.14}$$

one has

$$\begin{aligned} &P^+(q_{g^\varepsilon}, \mathrm{X}_\alpha) + \frac{1}{A_\varepsilon} P^+(w_{f^\varepsilon}, \mathrm{X}_\alpha) + \frac{1}{A_\varepsilon} \int_0^x \int_{u+\xi_1>0} (L w_{f^\varepsilon}) \mathrm{X}_\alpha \, dx\, d\xi \\ &\qquad + \varepsilon \int_0^x \int_{u+\xi_1>0} g^\varepsilon \mathrm{X}_\alpha \, dx\, d\xi = \frac{1}{A_\varepsilon} P^+(\phi, \mathrm{X}_\alpha). \end{aligned} \tag{1.3.15}$$

Taking the limit as $\varepsilon \to 0$, we see that $P^+(Z, \mathrm{X}_\alpha) = 0$. Noticing that the form P^+ is positive definite on $N(L)$, we have $b_\beta = 0$; thus $Z = 0$, which contradicts $|||q_{g^\varepsilon}||| = 1$. Consequently, q_{g^ε} remains bounded in $L^2([0, B] \times \mathbb{R}^3)$. We can pass to the limit as $\varepsilon \to 0$ and obtain a solution f_B of (1.3.4)–(1.3.5) such that $\nu^{1/2} f_B \in L^2([0, B] \times \mathbb{R}^3)$.

Let us check that f_B satisfies (1.3.6). From the positivity of the quadratic form P on G and the property of maximality of G, one obtains that G is a closed (convex) subspace of $L^2(|\xi_1 + u|\, d\xi)$. It is thus weakly closed. To prove that $f_B \in G$, it is sufficient to prove that the sequence f^ε remains bounded in $L^2(|\xi_1 + u|\, d\xi)$ uniformly in ε. In view of the previous estimates and equation (1.3.7), $\nu^{1/2} f^\varepsilon$ and $(\xi_1 + u)\, \partial_x f^\varepsilon$ remain bounded in $L^2((0, B) \times \mathbb{R}^3)$. Thus, the function

$$g(x) = \int |\xi_1 + u| f^\varepsilon(\xi)^2 \, d\xi \tag{1.3.16}$$

is bounded in $W^{1,1}(0, B)$. The injection from $W^{1,1}(0, B)$ into $\overline{C}(I)$ being continuous, we have proved that

$$\|f^\varepsilon\|_{L^2(|\xi_1 + u|\, d\xi)} < C, \tag{1.3.17}$$

independently of ε.

Uniform estimates with respect to B for the solution f_B of (1.3.4)–(1.3.6).

Proposition 1.3.1. *The solution f_B of* (1.3.4)–(1.3.6) *which can be written in the form*

$$f_B = w_B + q_B = w_B + \sum_{\alpha=0}^{4} a_\alpha^B(x) \mathrm{X}_\alpha \tag{1.3.18}$$

satisfies the uniform estimates

$$\int_0^B \int_{\mathbf{R}^3} \nu(\xi) w_B^2 \, d\xi \, dx \leqq \frac{1}{\mu^2} k_\phi, \tag{1.3.19}$$

$$|a_\alpha^B(x)| \leqq C_\phi\left(1 + \sqrt{x} + \|\nu^{1/2} w_B\|\right), \tag{1.3.20}$$

$$\int_{\mathbf{R}^3} (\xi_1 + u) f_B \mathrm{X}_\alpha \, d\xi = 0, \qquad \alpha \in I^-, \tag{1.3.21}$$

and, for any $\gamma < \mu/C'_\mu$ *(where* C'_μ *is a positive constant depending on* μ*),*

$$\int_0^B e^{2\gamma x} \int_{\mathbf{R}^3} \nu(\xi) w_B^2 \, d\xi \, dx \leqq C_\phi. \tag{1.3.22}$$

Proof: Estimate (1.3.19) is obtained as usual by multiplying equation (1.3.4) by f_B and integrating over $[0, B] \times \mathbf{R}^3$. To obtain (1.3.20), one multiples equation (1.3.4) by X_α and integrates over $u + \xi_1 > 0$ and $[0, x]$:

$$\begin{aligned} P^+(q_B, \mathrm{X}_\alpha) + P^+(w_B, \mathrm{X}_\alpha) &= P^+(\phi, \mathrm{X}_\alpha) \\ &\quad + \int_0^x \int_{u+\xi_1>0} L(w_B) \mathrm{X}_\alpha \, d\xi \, dx. \end{aligned} \tag{1.3.23}$$

One has

$$\begin{aligned} &\int_0^x \int_{u+\xi_1>0} L w_B \mathrm{X}_\alpha \, d\xi \, dx \\ &\leqq \left(\int_0^x \int_{u+\xi_1>0} \nu w_B^2 \, d\xi \, dx \right)^{1/2} \left(\int_0^x \int_{u+\xi_1>0} \nu \mathrm{X}_\alpha^2 \, d\xi \, dx \right)^{1/2} \\ &\leqq C_\alpha \sqrt{x} \left(\int_0^x \int_{u+\xi_1>0} \nu w_B^2 \, d\xi \, dx \right)^{1/2} \leqq \frac{1}{\mu} C_\alpha \sqrt{x} \sqrt{k_\phi}, \end{aligned} \tag{1.3.24}$$

and similarly,

$$|P^+(w_B, \mathrm{X}_\alpha)| \leqq C'_\alpha \|\nu^{1/2} w_B\|. \tag{1.3.25}$$

Thus,

$$|P^+(w_B, \mathrm{X}_\alpha)| \leqq C\left(1 + \sqrt{x} + \|\nu^{1/2} w_B\|\right), \tag{1.3.26}$$

where C depends only on ϕ and μ. P^+ being positive definite on $N(L)$, one immediately obtains (1.3.20).

Multiplying equation (1.3.4) by X_α and integrating over ξ, one sees that the quantities $P(f_B, X_\alpha)$ are independent of x. From the boundary conditions (1.3.6), these constants, denoted by k_α^B, are equal to 0 for $\alpha \in I^-$.

The form P being nondegenerate, there exists a unique $q_B^\infty \in N(L)$ such that

$$P(q_B^\infty, X_\alpha) = P(f_B, X_\alpha) = k_\alpha^B. \tag{1.3.27}$$

The function $\bar{f}_B = f_B - q_B^\infty$ satisfies

$$(u + \xi_1)\frac{\partial \bar{f}_B}{\partial x} + Lw_B = 0. \tag{1.3.28}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\left(\int (u + \xi_1)\bar{f}_B^2 e^{2\gamma x}\, d\xi\right)_0^B - \gamma \int_0^B \int_{\mathbf{R}^3} (u + \xi_1)\bar{f}_B^2 e^{2\gamma x}\, d\xi\, dx \\ &\quad + \int_0^B \int_{\mathbf{R}^3} \nu w_B^2 e^{2\gamma x}\, d\xi\, dx \leqq 0. \end{aligned} \tag{1.3.29}$$

From the definition of q_B^∞, it is clear that $q_B^\infty \in G$. Thus $\bar{f}_B(B, \xi) \in G$ and $P(\bar{f}_B, \bar{f}_B)(B) \geqq 0$. Also,

$$\int_{\mathbf{R}^3} (u + \xi_1)\bar{f}_B^2(0, \xi)\, d\xi \leqq \int_{u+\xi_1>0} (u + \xi_1)(\phi - q_B^\infty)^2(\xi)\, d\xi. \tag{1.3.30}$$

The coefficients of q_B^∞ of the basis $\{X_\alpha\}$ can be written in the form

$$P(q_B^\infty, X_\alpha) = \int_0^1 (P(w_B, X_\alpha) + P(q_B, X_\alpha))\, dx \tag{1.3.31}$$

and are thus bounded independently of B by estimates (1.3.20)–(1.3.21). Consequently,

$$\int_{\mathbf{R}^3} (u + \xi_1)\bar{f}_B(0, \xi)^2\, d\xi \leqq C_\phi. \tag{1.3.32}$$

Now,

$$\begin{aligned} P(\bar{f}_B, \bar{f}_B) &= P(w_B, w_B) + 2P(w_B, q_B - q_B^\infty) \\ &\quad + P(q_B - q_B^\infty, q_B - q_B^\infty). \end{aligned} \tag{1.3.33}$$

Using (1.3.27), a simple computation gives

$$P(q_B - q_B^\infty, q_B - q_B^\infty) = -P(w_B, q_B - q_B^\infty) \tag{1.3.34}$$

and

$$\begin{aligned} P(w_B, q_B - q_B^\infty) &= \sum_{\alpha=0}^{4} P(w_B, \mathrm{X}_\alpha) P(q_B - q_B^\infty, \mathrm{X}_\alpha) P(\mathrm{X}_\alpha, \mathrm{X}_\alpha) \\ &= -\sum_{\alpha=0}^{4} P(w_B, \mathrm{X}_\alpha)^2 P(\mathrm{X}_\alpha, \mathrm{X}_\alpha). \end{aligned} \tag{1.3.35}$$

Thus,

$$\int_0^B P(\bar{f}_B, \bar{f}_B) e^{2\gamma x}\, dx \leqq C_1 \nu_0 \int_0^B e^{2\gamma x} \int_{\mathbf{R}^3} (1 + |\xi|) w_B^2 \, d\xi \, dx. \tag{1.3.36}$$

Therefore estimate (1.3.22) holds for $0 < \gamma < C_1\nu_0$. This completes the proof of Proposition 1.3.1.

1.4. Construction of a solution in $[0, \infty]$. We now prove the existence part of Theorem 1.2.1 in the nondegenerate cases.

Proof: First notice that we can restrict our attention to the case where the constants λ_α are equal to zero. Indeed, consider f defined by

$$f = g + \sum_{\alpha \in I^-} \lambda_\alpha \mathrm{X}_\alpha, \tag{1.4.1}$$

where g is the solution of

$$(\xi_1 + u)\frac{\partial g}{\partial x} + Lg = 0, \tag{1.4.2}$$

$$g(0, \xi) = \phi(\xi) - \sum_{\alpha \in I^-} \lambda_\alpha \mathrm{X}_\alpha, \qquad \xi_1 > 0, \tag{1.4.3}$$

$$\int (\xi_1 + u) \mathrm{X}_\alpha g \, d\xi = 0, \qquad \alpha \in I^-; \tag{1.4.4}$$

f is a solution of (1.2.13), (1.2.14), (1.2.19).

The solution f is constructed as the limit, as $B \to \infty$, of the solutions f_B of (1.3.4)–(1.3.6): from estimates (1.3.19)–(1.3.22), there exists a sequence $B_n \to \infty$ such that

$$w_{B_n} \to w \text{ weakly in } L^2(e^{\gamma x} \otimes \nu\, d\xi), \qquad a_\alpha^B \to a_\alpha \text{ weakly in } L^2_{\mathrm{loc}}(dx)$$

and $f = w + \sum_{\alpha=0}^{4} a_\alpha \mathrm{X}_\alpha$ is a weak solution of (1.2.13) in $L^2_{\mathrm{loc}}(dx \otimes \nu\, d\xi)$. It satisfies

$$P(f, \mathrm{X}_\alpha) = k_\alpha, \qquad \alpha \in I, \tag{1.4.5}$$

where the constants k_α are the limits of k_α^B as $B \to \infty$. They are finite because the k_α^B are bounded uniformly with respect to B. By construction, $k_\alpha = 0$ for $\alpha \in I^-$.

Let $q^\infty \in N(L)$ be defined by

$$P(q^\infty, X_\alpha) = k_\alpha. \tag{1.4.6}$$

One has

$$P(q^\infty - q, X_\alpha) = P(w, X_\alpha); \tag{1.4.7}$$

thus,

$$q^\infty - q \in L^2(e^{\gamma x}\, dx \otimes \nu\, d\xi), \tag{1.4.8}$$

and estimate (1.2.22) is proved.

L^∞ ESTIMATE WHEN ϕ SATISFIES (1.2.15). The function $\bar f = f - q^\infty$ satisfies equation (1.2.13). Following [4], we use the integral form of (1.2.13) and denote $\lambda = \nu(\xi)/(\xi_1 + u)$; we have

$$\bar f(y, \xi)e^{\lambda y} - \bar f(x, \xi)e^{\lambda x} = \int_x^y \frac{1}{\xi_1 + u} K\bar f(s, \xi) e^{\lambda s}\, ds. \tag{1.4.9}$$

First consider the case $\xi_1 + u < 0$ and let x go to infinity. Equation (1.4.9) has the form

$$e^{\gamma y}\bar f(y, \xi) = -\int_y^\infty \frac{1}{\xi_1 + u} K\bar f(s, \xi) e^{\gamma s} e^{(\lambda - \gamma)(s - y)}\, ds. \tag{1.4.10}$$

Thus we have

$$e^{\gamma y}|\bar f(y, \xi)| \leqq \left(\int_0^\infty K\bar f(s, \xi)^2 e^{2\gamma s}\, ds \right)^{1/2} \frac{1}{|\xi_1 + u|} \left(\frac{1}{2|\lambda - \gamma|} \right)^{1/2}. \tag{1.4.11}$$

Consequently, using (1.2.22),

$$\begin{aligned} &\int_{\xi_1 + u < 0} e^{2\gamma y}|\xi_1 + u|\, |\bar f(y, \xi)|^2\, d\xi \\ &\quad \leqq C\left(\int_\xi \int_0^\infty K\bar f(s, \xi)^2 e^{2\gamma s}\, ds\, d\xi \right) \\ &\quad \leqq C^{st}. \end{aligned} \tag{1.4.12}$$

For the case $\xi_1 + u > 0$, one takes $x = 0$ in (1.4.9):

$$\begin{aligned} e^{\gamma y}\bar f(y, \xi) &= (\phi - q^\infty)(\xi) e^{(\gamma - \lambda)y} \\ &\quad + \int_0^y \frac{1}{\xi_1 + u} K\bar f(s, \xi) e^{\gamma s} e^{(\lambda - \gamma)(s - y)}\, ds. \end{aligned} \tag{1.4.13}$$

For γ small enough, $(\lambda - \gamma)$ is always positive:

$$
\begin{aligned}
e^{\gamma y}|\bar{f}(y,\xi)| &\leqq |(\phi - q^{\infty})(\xi)|e^{(\gamma-\lambda)y} \\
&\quad + \left(\int_0^{\infty} \frac{1}{\xi_1 + u} K\bar{f}(s,\xi)^2 e^{2\gamma s}\, ds\right)^{1/2} \frac{C}{|\xi_1 + u|^{1/2}}.
\end{aligned} \tag{1.4.14}
$$

And so, using relation (1.2.22),

$$
\int_{\xi_1 + u > 0} e^{2\gamma y}|\xi_1 + u|\,|\bar{f}(u,\xi)|^2\, d\xi \leqq C^{st}. \tag{1.4.15}
$$

Combining (1.4.12) and (1.4.15) we obtain the estimate (1.2.21).

L^{∞} ESTIMATE WHEN ϕ SATISFIES (1.2.15′). First consider the case $\xi_1 + u < 0$. To estimate the right-hand side of (1.4.10), one separates the cases $-1 < \xi_1 + u < 0$ and $\xi_1 + u < -1$. For $\xi_1 + u < -1$, one writes

$$
\begin{aligned}
&\left|\int_y^{\infty} \frac{1}{\xi_1 + u} K\bar{f}(s,\xi) e^{\gamma s} e^{(\lambda-\gamma)(s-y)}\, ds\right| \\
&\quad \leqq C_{\gamma}\left(\int_0^{\infty} K\bar{f}(s,\xi)^2 e^{2\gamma s}\, ds\right)^{1/2}.
\end{aligned} \tag{1.4.16}
$$

For $-1 < \xi_1 + u < 0$, one splits up the neighborhood of y:

$$
\int_y^{\infty} = \int_y^{y+\varepsilon} + \int_{y+\varepsilon}^{\infty} \tag{1.4.17}
$$

with

$$
\begin{aligned}
&\left|\int_{y+\varepsilon}^{\infty} \frac{\nu^{1/2}}{\xi_1 + u} K\bar{f}(s,\xi) e^{\gamma s} e^{(\lambda-\gamma)(s-y)}\, ds\right| \\
&\quad \leqq C\frac{1}{\varepsilon^{1/2}}\left(\int_0^{\infty} K\bar{f}(s,\xi)^2 e^{2\gamma s}\, ds\right)^{1/2}
\end{aligned} \tag{1.4.18}
$$

and

$$
\begin{aligned}
&\left|\int_y^{y+\varepsilon} \frac{\nu^{1/2}}{\xi_1 + u} K\bar{f}(s,\xi) e^{\gamma s} e^{(\lambda-\gamma)(s-y)}\, ds\right| \\
&\quad \leqq C\varepsilon^{\alpha}\frac{\nu^{1/2}}{|\xi_1 + u|^{\alpha}\nu(\xi)^{1-\alpha}} \sup_s |K\bar{f}(s,\xi) e^{\gamma s}|.
\end{aligned} \tag{1.4.19}
$$

Thus, for $\alpha < \frac{1}{2}$,

$$\begin{aligned}
&\int_{-1<\xi_1+u<0} \nu(\xi) \sup_y |e^{\gamma y}\bar{f}(y,\xi)|^2 \, d\xi \\
(1.4.20) \qquad &\leqq C\frac{1}{\varepsilon}\int_0^\infty \int_{\mathbf{R}^3} K\bar{f}(s,\xi)^2 e^{2\gamma s}\, ds \\
&\quad + C\varepsilon^{2\alpha}\int_{-1<\xi_1+u<0} \frac{1}{|\xi_1+u|^{2\alpha}} \sup_s |K\bar{f}(s,\xi)e^{\gamma s}|^2 \, d\xi.
\end{aligned}$$

The second term of the right-hand side of (1.4.20) is given by

$$\begin{aligned}
&\int_{-1<\xi_1+u<0} \frac{1}{|\xi_1+u|^{2\alpha}} \sup_s \left(\int k(\xi,\eta)\bar{f}(s,\eta)e^{\gamma s}\, d\eta\right)^2 d\xi \\
(1.4.21) \qquad &\leqq \left(\int \sup_s |\bar{f}(s,\eta)e^{\gamma s}|^2 \, d\eta\right) \int_{-1<\xi_1+u<0} \frac{1}{|\xi_1+u|^{2\alpha}} \frac{1}{1+|\xi|}\, d\xi. \\
&\leqq C\|\bar{f}\|_{2,\gamma}, \qquad \alpha < \tfrac{1}{2}.
\end{aligned}$$

Combining (1.4.20) and (1.4.21), one gets

$$(1.4.22) \qquad \left\|1_{\xi_i+u\leqq 0}\bar{f}\right\|_{2,\gamma} \leqq \frac{C}{\varepsilon^{1/2}}|||e^{\gamma x}\bar{f}||| + C\varepsilon^\alpha \|\bar{f}\|_{2,\gamma}.$$

For the case $\xi_1 + u > 0$, one uses (1.4.13).

Again, for $\xi_1 + u > 1$,

$$\begin{aligned}
&\left|\int_0^y \frac{\nu^{1/2}}{\xi_1+u} K\bar{f}(s,\xi)e^{\gamma s}e^{(\lambda-\gamma)(s-y)}\, ds\right| \\
(1.4.23) \qquad & \\
&\leqq C\left(\int_0^\infty K\bar{f}(s,\xi)^2 e^{2\gamma s}\, ds\right)^{1/2}.
\end{aligned}$$

For $0 < \xi_1 + u < 1$,

$$\begin{aligned}
&\left|\int_0^{y-\varepsilon} \frac{\nu^{1/2}}{\xi_1+u} K\bar{f}(s,\xi)e^{\gamma s}e^{(\lambda-\gamma)(s-y)}\, ds\right| \\
(1.4.24) \qquad & \\
&\leqq \frac{C_\gamma}{\varepsilon^{1/2}}\left(\int_0^\infty K\bar{f}(s,\xi)^2 e^{2\gamma s}\, ds\right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{y-\varepsilon}^y \frac{\nu^{1/2}}{\xi_1+u} K\bar{f}(s,\xi)e^{\gamma s}e^{(\lambda-\gamma)(s-y)}\, ds\right| \\
(1.4.25) \qquad & \\
&\leqq C\varepsilon^\alpha \frac{1}{|\xi_1+u|^\alpha \nu(\xi)^{1-\alpha}} \sup_s |K\bar{f}(s,\xi)e^{\gamma s}|.
\end{aligned}$$

Thus

$$\left\|1_{\xi_1+u\geq 0}\bar{f}\right\|_{2,\gamma} \leqq \int_{\xi_1+u\geqq 0}\left(\nu|\phi - q^\infty|^2(\xi)\,d\xi\right)^{1/2} + \frac{C}{\varepsilon^{1/2}}|||e^{\gamma x}\bar{f}||| + C\varepsilon^\alpha\|\bar{f}\|_{2,\gamma}. \tag{1.4.26}$$

Combining (1.4.22) and (1.4.26), we get

$$\|\bar{f}\|_{2,\gamma} \leqq \|\phi - q^\infty\|_{L^2(\nu\,d\xi;\,\xi_1+u>0)} + \frac{C}{\varepsilon^{1/2}}|||e^{\gamma x}\bar{f}||| + C\varepsilon^\alpha\|\bar{f}\|_{2,\gamma}. \tag{1.4.27}$$

Choosing ε such that $C\varepsilon^\alpha < 1$ and using estimate (1.2.22), and the fact that the coefficients of q^∞ of the basis $\{X_\alpha\}$ are finite, one has estimate (1.2.21′).

1.5. Uniqueness.

THEOREM 1.5.1. *Let $f \in L^\infty(dx; L^2(|\xi_1 + u|\,d\xi))$ be such that*

$$(\xi_1 + u)\frac{\partial f}{\partial x} + Lf = 0, \qquad x > 0, \tag{1.5.1}$$

$$f(0,\xi) = 0, \qquad \xi_1 > 0, \tag{1.5.2}$$

$$\int(\xi_1 + u)X_\alpha f\,d\xi = 0, \qquad \alpha \in I^-. \tag{1.5.3}$$

Then $f = 0$.

Proof: First we write

$$f = w_f + q_f = w_f + \sum_{\alpha=0}^{4} a_\alpha(x)X_\alpha(\xi). \tag{1.5.4}$$

One has

$$P(f,f)(x) - P(f,f)(0) + \int_0^x\int_{\mathbf{R}^3}(1 + |\xi|)w_f^2 \leqq 0. \tag{1.5.5}$$

Using (1.5.3), we have, for $\alpha \in I^-$,

$$a_\alpha(x) = P(X_\alpha, w_f); \tag{1.5.6}$$

thus,

$$P(f,f) = P(w_f, w_f) + \sum_{\alpha\in I^-} P(w_f, X_\alpha)^2 + \sum_{\alpha\in I^+}\left(a_\alpha + P(w_f, X_\alpha)\right)^2 - \sum_{\alpha\in I^+} P(w_f, X_\alpha)^2, \tag{1.5.7}$$

and

$$P(f,f) \geqq P(w_f, w_f) - \sum_{\alpha\in I^+} P(w_f, X_\alpha)^2. \tag{1.5.8}$$

By the hypothesis, $P(f, f) \in L^\infty(\mathbb{R}^+)$; thus $\nu^{1/2} w_f \in L^2(\mathbb{R}^+ \times \mathbb{R}^3)$. It follows that there exists a sequence $x_n \to \infty$ such that $P(X_\alpha, w_f)(x_n) \to 0$ and $P(w_f, w_f)(x_n) \to 0$. But $P(f, f)(x)$ is a decreasing function of x and $P(f, f)(0) \leq 0$ because of (1.5.2). Thus $P(f, f)(x)$ is identically equal to 0. Consequently, w_f is also identically equal to 0 and q is a constant (with respect to x). This constant is equal to 0 from the condition at $x = 0$. Thus the function f is identically 0.

Remark 1.5.1. In the case $0 < u < \sqrt{\frac{5}{3} T_\infty}$, the condition $P(f, X_1) = 0$ is equivalent to $m_f = \int \xi_1 f M^{1/2} \, d\xi = 0$ as $x \to \infty$ because $f \to 0$ as $x \to \infty$. It is in this form that Čercignani [4] conjectured that the problem was well posed.

1.6. The degenerated cases $u = 0, \pm \sqrt{\frac{5}{3} T_\infty}$. We are now interested in the special cases $u = 0, \pm \sqrt{\frac{5}{3} T_\infty}$, where the quadratic form P is degenerate.

LEMMA 1.6.1. *If f is a bounded solution of* (1.2.13), *it satisfies*

$$P(f, X_\alpha) = 0 \quad \textit{for all} \quad \alpha \in I^0. \tag{1.6.1}$$

Proof: Notice that, for $u = 0, \pm \sqrt{\frac{5}{3} T_\infty}$,

$$\text{for all } \alpha \in I^0 \text{ and all } \beta \in I, \quad P(X_\alpha, X_\beta) = 0, \tag{1.6.2}$$

but $N(L)^\perp = R(L)$ and thus

$$(\xi_1 + u) X_\alpha \in R(L), \qquad \alpha \in I^0. \tag{1.6.3}$$

Following [4], if f is a bounded solution of (1.2.13), it satisfies, for any $\alpha \in I^0$,

$$\begin{aligned} 0 &= \frac{d}{dx} P\big(f, L^{-1}((\xi_1 + u) X_\alpha)\big) + \int L f L^{-1}\big((\xi_1 + u) X_\alpha\big) \, d\xi \\ &= \frac{d}{dx} P\big(f, L^{-1}((\xi_1 + u) X_\alpha)\big) + P(f, X_\alpha), \end{aligned} \tag{1.6.4}$$

because L is selfadjoint. Moreover, the quantities $P(f, X_\alpha)$ are constant. If such a constant were not zero, it would imply that the corresponding scalar product $P(f, L^{-1}((\xi_1 + u) X_\alpha))$ which appears in (1.6.4) has a linear growth which is in contradiction with the assumption that $f \in L^\infty(dx, L^2(|\xi_1 + u| \, d\xi))$. Thus, $P(f, X_\alpha) = 0$, for $\alpha \in I^0$.

LEMMA 1.6.2. *The matrix A defined by*

$$A_{\alpha\beta} = \begin{cases} P\big(X_\alpha, X_\beta\big), & \alpha \in I^+ \cup I^0, \quad \beta \in I^+, \\ P\big(X_\alpha, L^{-1}((\xi_1 + u) X_\beta)\big), & \alpha \in I^+ \cup I^0, \quad \beta \in I^0, \end{cases} \tag{1.6.5}$$

is invertible.

Proof: We write A in the form of four blocks defined above. Since the X_α form a P-orthogonal set, the upper right block is zero. Moreover, the submatrix $A_{\alpha\beta}$ for $(\alpha, \beta) \in I^0 \times I^0$ defines a positive symmetric quadratic form because L is selfadjoint and satisfies the coercivity property (1.2.9). Then the matrix A is invertible.

We are now going to prove Theorem 1.2.1 in the case where P is degenerate.

Proof: To prove the existence of a solution, let us come back to subsections 1.3–1.4, and make slight modifications to fit the degenerate cases. First we adapt Lemma 1.3.1 by replacing I^- by $I^- \cup I^0$ in (ii) and I^+ by $I^+ \cup I^0$ in (iii). The construction of a solution f_B of (1.3.4)–(1.3.6) by considering the penalized system is extended without modifications. We thus have a solution f_B satisfying estimates (1.3.19)–(1.3.20). Using Lemma 1.6.2, it is possible to construct $q_B^\infty \in G \cap N(L)$ such that

$$P(X_\alpha, q_B^\infty) = P(X_\alpha, f_B) \quad \text{for all} \quad \alpha \in I^+, \tag{1.6.6}$$

$$\begin{aligned} &P\big(L^{-1}((\xi_1 + u)X_\alpha), q_B^\infty\big) \\ &\qquad = P\big(L^{-1}((\xi_1 + u)X_\alpha), f_B\big) \quad \text{for all} \quad \alpha \in I^0. \end{aligned} \tag{1.6.7}$$

Notice that the last quantity is constant (see equation (1.6.4) and use the fact that $P(f, X_\alpha)$, which is constant, is equal to zero for $\alpha \in I^0$ at $x = B$ because $f(B, \cdot) \in G$).

Equations (1.6.6)–(1.6.7) together with estimates (1.3.19)–(1.3.20) and a trick similar to (1.3.31) prove that the coefficients of q_B^∞ of the basis of $N(L)$ remain bounded independently of B. In order to estimate the second term of the left-hand side of (1.3.31), we notice that, from the definition of q_B^∞, $q_B - q_B^\infty$ is controlled in terms of w_B. Thus, for γ small enough, we have estimate (1.3.22).

The construction of a solution in $[0, \infty[$ and the L^∞ estimate are extended to the degenerate cases without modifications.

We prove uniqueness as in subsection 1.5, following the proof of Theorem 1.5.1. Equalities (1.5.4)–(1.5.6) still hold and we use Lemma 1.6.1 to prove that (1.5.7) is also true in the degenerate case. We complete the proof as in subsection 1.5.

1.7. An abstract formulation of the invariant relations. In Theorem 1.2.1, we have proved that, for any λ_α, $\alpha \in I^-$, there exists a unique bounded solution of (1.2.13)–(1.2.14) satisfying

$$P(f, X_\alpha) = \lambda_\alpha, \qquad \alpha \in I^-. \tag{1.7.1}$$

This condition clearly depends on the choice of the basis X_α in $N(L)$.

If we look carefully at the above proofs, it appears that the essential point to ensure that problem (1.2.13), (1.2.14), (1.2.19) is well posed is the positivity of the entropy flux at infinity. This, however, is independent of the spectral decomposition of the quadratic form P restricted to $N(L)$, and can be formulated without reference to any special basis like the X_α in the previous subsections.

THEOREM 1.7.1. *Let H be a subspace of $N(L)$ on which the form P is positive and which is maximal with respect to this property. For any $l \in N(L)$, there exists a unique solution f of* (1.2.13)–(1.2.14) *in $L^\infty(dx; L^2(\nu\, d\xi))$ such that*

$$\lim_{x\to\infty} (f - l) \in H. \tag{1.7.2}$$

Before turning to the proof of this theorem (which is an extension of the proof given in the previous subsections), let us give some remarks and lemmas.

Remark 1.7.1. Such subspaces H exist. Take for example $H = \operatorname{span}(X_\alpha, \alpha \in I^0 \cup I^+)$.

This theorem generalizes Theorem 1.2.1: choose H given above and define $l = \sum_{\alpha\in I^-}\lambda_\alpha X_\alpha$. Moreover, it is by no means a straightforward consequence of our previous results even in the nondegenerate case. To justify this last statement, it is sufficient to construct a subspace H that contains an isotropic vector h of P (for example, in the case $0 < u < \sqrt{\frac{5}{3}T_\infty}$, $h = X_\alpha + X_\beta$, with $\alpha \in I^+$ and $\beta \in I^-$). Such a space cannot be obtained by a direct application of our previous arguments.

In the same way, for $u = 0$, the results of Bardos et al. [2] and Cercignani [4] are not, strictly speaking, corollaries of Theorem 1.2.1. However, they can be included, together with Theorem 1.2.1 in the same frame and stated as Theorem 1.7.1. Moreover, it turns out that the abstract condition defined below in Lemma 1.7.1 and the constructions of [2] and [4] are the same in the special case $u = 0$. Indeed, the subspace

$$G = \left\{ f \in L^2(|\xi_1|\, d\xi),\ f_B(B, \xi) = f_B(B, R\xi) \right\} \tag{1.7.3}$$

satisfies condition (i) of Lemma 1.3.1. Let us check the maximality property: first, if $f \in G$, then $P(f, f) = 0$. Let

$$F = \left\{ f \in L^2(|\xi_1|\, d\xi),\ f_B(B, \xi) = f_B(B, R\xi),\ P(f, f) \geqq 0 \right\}. \tag{1.7.4}$$

For $f \in F$, decompose $f = f^+ + f^-$, with

$$f^+(\xi) = \tfrac{1}{2}(f(\xi) + f(R\xi)) \quad \text{and} \quad f^-(\xi) = \tfrac{1}{2}(f(\xi) + f(R\xi));$$

$f \in F$ and $f^+ \in F$ lead to $f^- \in F$. For any even function θ and any real number λ, $\theta + \lambda f^- \in F$, therefore,

$$0 \leqq P(f, f) = 4\lambda \int_{\xi_1 > 0} \xi_1 \theta f^-\, d\xi.$$

Thus $f^- = 0$, and G is maximal.

We state now some remarks concerning the quadratic form P.

Let H be a subspace of $N(L)$ that contains the elements of the nullspace $N_L(P) = N(L) \cap N(L)^{\perp P}$ restricted to $N(L)$ of P, and

$$H^{\perp P} = \{ f \in N(L) / \text{ for all } h \in H,\ P(f,h) = 0 \} \tag{1.7.5}$$

its orthogonal complement with respect to P in $N(L)$. One has

$$H = (H^{\perp P})^{\perp P}. \tag{1.7.6}$$

Let H be as in Theorem 1.7.1. Then the dimension of the subspace H is determined by u and is equal to Card$(I^0 \cup I^+)$. Indeed H necessarily contains the nullspace of P, because it is maximal, and thus the X_α for $\alpha \in I^0$. Now consider the restriction of P to the subspace H' of H where it is definite. From the positivity of P on H and the maximality of H, one gets

$$\begin{gathered} H \cap \operatorname{span}(\mathrm{X}_\alpha, \alpha \in I^-) = \{0\}, \\ H^\perp \cap \operatorname{span}(\mathrm{X}_\alpha, \alpha \in I^+) = \{0\}. \end{gathered} \tag{1.7.7}$$

Moreover, for H satisfying the assumptions of Theorem 1.7.1, the subspace G of Lemma 1.3.1 is constructed as follows:

$$G = H + X, \qquad X \subset W, \tag{1.7.8}$$

where X is a maximal element of S (W and S being defined in the proof of Lemma 1.3.1).

The goal of the next lemma is to formulate the statement $f(B, \cdot) \in G$ in terms of invariant quantities.

LEMMA 1.7.1. *If $f(B, \cdot) \in G$, then*

$$f(B, \cdot) \in G \Rightarrow \left(\text{for all } h \in H^{\perp P},\ P(h, f)(B) = 0\right), \tag{1.7.9}$$

the inverse assertion being true if $f(B, \cdot) \in N(L)$.

Proof of Theorem 1.7.1. Let us return to the proof of Theorem 1.2.1 and modify it as necessary. For the existence proof, we take $l = 0$. Estimates (1.3.4) to (1.3.20) are still valid and one has to replace equation (1.3.21) by

$$\int_{\mathbf{R}^3} (\xi_1 + u) f_B h \, d\xi = 0, \quad \text{for all} \quad h \in H^\perp. \tag{1.7.10}$$

To prove estimate (1.3.22), one has to construct q_B^∞. Let us solve, in $N(L)$, the system

$$P(q_B^\infty, h) = P(f, h) \quad \text{for all} \quad h \in N(L), \tag{1.7.11}$$

$$P\big(q_B^\infty, L^{-1}((\xi_1 + u)h)\big) = P(f, h) \quad \text{for all} \quad h \in N_L(P) \tag{1.7.12}$$

Equation (1.7.11) represents five minus $\dim N_L(P)$ nontrivial equations. For any $h \in N_L(P)$ and $n \in N(L)$, one has $P(h, n) = 0$. Thus, $(\xi_1 + u)h$ belongs to $N(L)^\perp$ and $L^{-1}((\xi_1 + u)h)$ is defined. The function q_B^∞ which belongs to the five-dimensional space $N(L)$ is then defined by five equations. To prove that the system (1.7.11)–(1.7.12) is well posed, let us consider the associated homogeneous system referred to as (1.7.11′)–(1.7.12′) and prove that it has only the trivial solution. Equation (1.7.11′) ensures that the solution denoted by $\tilde{q}_B^\infty \in N_L(P)$ and equation (1.7.12′) with $h = \tilde{q}_B^\infty$ leads to

$$\int (\xi_1 + u)\tilde{q}_B^\infty L^{-1}((\xi_1 + u)\tilde{q}_B^\infty)\, d\xi = 0 \tag{1.7.13}$$

and thus, from the coercivity property (1.2.9), to $\tilde{q}_B^\infty = 0$.

Equation (1.7.11), together with Lemma 1.7.1 and the above remarks, ensures that $q_B^\infty \in H$ and thus the first term of the left-hand side of (1.3.29) computed at $x = B$ is positive.

Computations similar to (1.3.31)–(1.3.35) ensure that q_B^∞ is bounded uniformly with respect to B and that $(q_B - q_B^\infty)(x, \cdot)$ can be controlled by $w_B(x, \cdot)$ in $L^2(\nu\, d\xi)$-norm.

The proof of uniqueness is obtained as before.

2. Accomodation Boundary Conditions

2.1. Introduction and main results. Space vehicle aerodynamics has raised a new interest in understanding the presumably complex interaction between a rarefied gas flow and a body surface. In this context, the special case of specular reflection

$$F(x, \xi) = F(x, \xi - 2(\xi \cdot n_x)n_x)$$

(where n_x is the exterior unit normal vector at point x of the wall) appears rather academic. In particular, one has to take into account: matter ablation at the wall, thermalization of molecules impinging the wall, etc. To this end, one is led to introduce the following type of boundary conditions (see Cercignani [3] and Ferziger-Kaper [7]):

$$F(x, \xi) = \int_{\xi' \cdot n_x < 0} \frac{|\xi' \cdot n_x|}{|\xi \cdot n_x|} R(\xi' \to \xi, x) F(x, \xi')\, d\xi' + S(x, \xi), \qquad \xi \cdot n_x > 0. \tag{2.1.1}$$

The term $S(x, \xi)$ in the right-hand side of (2.1.1) is to model the surface emission in the flow due to the wall deterioration and the scattering kernel $R(\xi' \to \xi, x)$ is to model the complex reflection mechanism at the wall (including for example thermalization processes). A special case of (2.1.1) is the well-known Maxwell boundary condition given by

$$\begin{aligned} R(\xi' \to \xi, x) &= (1-\alpha)\delta(\xi' - \xi + 2n_x \cdot \xi) + \alpha|\xi \cdot n_x| M_w, \\ &\qquad \xi \cdot n_x > 0,\ \xi' \cdot n_x < 0, \end{aligned} \tag{2.1.2}$$

where M_w is the thermalization Maxwellian at the wall, and α is the "accomodation coefficient".

In this work, we shall mainly consider the boundary condition (2.1.1), with additional assumptions on the scattering kernel that can be derived from physical arguments (see [3]):

$$R(\xi' \to \xi, x) \geqq 0, \tag{2.1.3a}$$

$$\int_{\xi \cdot n_x > 0} R(\xi' \to \xi, x)\, d\xi = 1, \qquad \xi' \cdot n_x < 0; \tag{2.1.3b}$$

there exists a Maxwellian state M_w such that

$$\begin{aligned} &|\xi \cdot n_x| M_w(\xi') R(\xi' \to \xi, x) \\ &\quad = |\xi \cdot n_x| M_w(\xi) R(-\xi \to -\xi', x), \quad \xi \cdot n_x > 0,\ \xi' \cdot n_x < 0. \end{aligned} \tag{2.1.3c}$$

Condition (2.1.3c) is referred to as "the law of reciprocity" and ensures that M_w satisfies the boundary condition (2.1.1) with $S(x, \xi) \equiv 0$. The following additional property can be derived from the above properties (see [3]).

PROPOSITION 2.1.1. *For any function F satisfying the boundary condition* (2.1.1) *with a null source term* ($S(x, \xi) \equiv 0$), *one has*

$$\int \xi \cdot n_x F^2 M_w^{-1}\, d\xi \leqq 0 \tag{2.1.4}$$

with equality if and only if

either F is a.e. proportional to M_w,

or $R(\xi' \to \xi)$ is proportional to a Dirac mass.

The proof of this proposition is given in [3]. Throughout the present work, we shall exclude the latter situation, corresponding to the case of a specular reflection (i.e., (2.1.2) with $\alpha = 0$).

Let us now state our main result. We consider the half-space problem:

$$\xi_1 \frac{\partial f}{\partial x} + Lf = 0, \qquad x > 0, \tag{2.1.5}$$

(where L is the linearized collision operator around the Maxwellian state M_w, and $F = M_w + M_w^{1/2} f$; see Section 1) supplemented with the transformed boundary condition derived from (2.1.1) and the prescribed mass flux condition:

$$\int_{\xi_1 < 0} |\xi_1| f(0, \xi) M_w^{1/2}(\xi)\, d\xi = \lambda, \tag{2.1.6}$$

where λ is a given constant. Notice that, since M_w satisfies the boundary condition (2.1.1), it is equivalent to require that $M_w^{1/2} f$ satisfies (2.1.1), i.e.,

$$f(x, \xi) = \int_{\xi' \cdot n_x < 0} \frac{|\xi' \cdot n_x|}{|\xi \cdot n_x|} R(\xi' \to \xi, x) \frac{M_w^{1/2}(\xi')}{M_w^{1/2}(\xi)} f(x, \xi')\, d\xi + s(x, \xi), \qquad \xi \cdot n_x > 0, \tag{2.1.1'}$$

where $s = M_w^{-1/2} S$. We have

THEOREM 2.1.1. *For any constant λ, and any source term $s(\xi)$ in $L^2(\xi_1 > 0; |\xi_1|\, d\xi)$, there exists a unique solution f of the problem* (2.1.5), (2.1.1′), (2.1.6) *in $L^\infty(dx; L^2(|\xi_1|\, d\xi))$. This solution f has the following asymptotic behavior as $x \to +\infty$: there exists a unique q_f^∞ in the nullspace of L such that*

$$f - q_f^\infty \in L^\infty\big(e^{\gamma x}\, dx; L^2(|\xi_1|\, d\xi)\big)$$

for any small enough $\gamma > 0$.

The following subsections are organized as follows:
In subsection 2.2, we prove a compactness property for the Albedo operator.
In subsection 2.3, we give the proof of Theorem 2.1.1.

2.2. Compactness of the Albedo operator. The compactness of the Albedo operator is a somewhat general property for half-space problems in kinetic theory (in the simple case of the transport equation with isotropic scattering, a computational proof using for example Chandrasekhar's calculus can be given; see Chandrasekhar [5]).

LEMMA 2.2.1. *Let us consider the equation* (2.1.5), *together with the mass flux condition*

$$\int \xi_1 f(x, \xi) M_w^{1/2}(\xi)\, d\xi = m_f. \tag{2.2.1}$$

The Albedo operator A is defined by

$$A: f(0, \xi) \to f(0, -\xi), \qquad \xi_1 > 0. \tag{2.2.2}$$

The operator A is compact on $L^2(\xi_1 > 0, |\xi_1|\, d\xi)$.

Proof: Denote by V the space for half-densities on which A is defined: $V = L^2(\xi_1 > 0, |\xi_1|\, d\xi_1)$. Let $f_n(0, \cdot)$ be a sequence of V that converges weakly to $f(0, \cdot)$, associated to solutions $f_n(x, \xi)$ of the problem (2.1.5)–(2.2.1) and $f(x, \xi)$, respectively. Notice that the continuous mapping

$$f_n(0, \cdot) \to q_{f_n}^{\infty}$$

is of finite rank, and therefore compact from V into $N(L) \subset L^2(\nu\, d\xi)$ (see Section 1). Thus, there exists a sequence of $q_{f_n}^{\infty}$ that converges strongly to q_f^{∞}.

We recall that, following our assumptions, the sequence $f_n(x, \xi) - q_{f_n}^{\infty}$ is bounded in $L^2(dx \otimes \nu\, d\xi)$, which in turn implies that

$$\xi_1 \frac{\partial}{\partial x}\left(f_n - q_{f_n}^{\infty}\right) + \nu\left(f_n - q_{f_n}^{\infty}\right) \tag{2.2.3}$$

is bounded in $L^2(dx \otimes d\xi)$ since K is in particular bounded in $L^2(d\xi)$. From this, we deduce that $K(f_n - q_{f_n}^{\infty})$ converges to $K(f - q_f^{\infty})$ strongly in $L^2_{\mathrm{loc}}(dx; L^2(d\xi))$ (using the fact that K is compact on $L^2(d\xi)$ and the averaging results of Golse, Lions, Perthame and Sentis; see Dautray-Lions [6]).

We then use the classical integral representation for the solution of (2.1.5)–(2.2.1): introducing $g_n = (f_n - q_{f_n}^{\infty}) - (f - q_f^{\infty})$, we have

$$g_n(0, \xi) = -\int_0^{\infty} \frac{1}{|\xi_1|} e^{-\nu s/|\xi_1|} (K g_n)(s, \xi)\, ds, \qquad \xi_1 < 0. \tag{2.2.4}$$

We point out that

$$\begin{aligned} \int_{\xi_1 < 0} |\xi_1| g_n^2(0, \xi)\, d\xi &\leqq \int_{\xi_1 < 0} |\xi_1| \left(\int_0^{\infty} \frac{e^{-2\nu s/|\xi_1|}}{|\xi_1|^2}\, ds \right) \\ &\quad \cdot \left(\int_0^{\infty} (K g_n)^2(s, \xi)\, ds \right) d\xi \\ &\leqq C \int_0^{\infty} \int (K g_n)^2(s, \xi)\, d\xi\, ds. \end{aligned} \tag{2.2.5}$$

We know that $K g_n \to 0$ strongly in $L^2_{\mathrm{loc}}(dx; L^2(d\xi))$ and $K g_n$ is uniformly bounded in $L^2(e^{\gamma x}\, dx \otimes d\xi)$. Thus, we have

$$g_n(0, \cdot) \to 0$$

in $L^2(\xi_1 < 0; |\xi_1|\, d\xi)$, which proves the announced compactness.

LEMMA 2.2.2. *Consider equation* (2.1.5) *together with zero mass flux condition*

$$\int \xi_1 f(x, \xi) M_w^{1/2}(\xi)\, d\xi = 0.$$

The corresponding albedo operator

$$A_0 : \quad f(0, \xi) \to f(0, -\xi), \qquad \xi_1 > 0,$$

is a contraction on $L^2(\xi_1 > 0; |\xi_1|\, d\xi)$.

Proof: The linearized entropy flux $P(f, f)$ goes to zero at infinity (see [2] and Section 1) and is nonincreasing with respect to x. Therefore, $P(f, f)(x = 0) \geqq 0$, and thus

$$\|A_0 f\|_{L^2(|\xi_1|\, d\xi)} \leqq \|f\|_{L^2(|\xi_1|\, d\xi)};$$

whence the announced conclusion follows.

2.3. The fixed point result. We first introduce a suitable framework to reduce the existence part in Theorem 2.1.1 to a fixed point result. Let Π be the orthogonal projection in V on $(\mathbb{R}\, M_w^{1/2})^\perp$ and denote by $\mathcal{R}$ the operator defined by

$$f(\xi) \to \int_{\xi_1' > 0} \frac{|\xi_1'|}{|\xi_1|} R(-\xi' \to \xi) \frac{M_w^{1/2}(\xi')}{M_w^{1/2}(\xi)} f(\xi')\, d\xi'; \tag{2.3.1}$$

$\mathcal{R}$ acts on $L^2(\xi_1\, d\xi; \xi_1 > 0)$ and, from (2.1.4), $\mathcal{R}$ is a contracting mapping since M_w satisfies the boundary condition (2.1.1) with $S \equiv 0$, $M_w^{1/2}$ is an eigenvector of $\mathcal{R}$ for the eigenvalue 1. Moreover, according to the law of reciprocity (2.1.3c), $\mathcal{R}$ is a selfadjoint operator on $L^2(\xi_1 > 0; \xi_1\, d\xi)$ equipped with the natural scalar product, and $\mathcal{R}$ induces a strictly contracting mapping on $(\mathbb{R}\, M_w^{1/2})^\perp$, that is $\|\mathcal{R}x\| < \|x\|$, for any $x \in (\mathbb{R}\, M_w^{1/2})^\perp$. We consider the following problem:

$$\text{(P)} \quad \begin{aligned} &\xi_1 \frac{\partial v}{\partial x} + Lv = 0, \qquad x > 0, \\ &v(0, \xi) = \beta M_w^{1/2} + \mathcal{R} \circ \Pi u + s, \qquad \xi_1 > 0, \\ &\int \xi_1 M_w^{1/2} v\, d\xi = \int_{\xi_1 > 0} \xi_1 M_w^{1/2} s\, d\xi, \end{aligned}$$

where β is to be fixed later, in order to satisfy (2.1.6). Consider the mapping T defined by

$$Tu(\xi) = v(0, -\xi), \qquad \xi_1 > 0. \tag{2.3.2}$$

is continuous on V (see [2]). Then we have the following result:

PROPOSITION 2.3.1. *For any $\beta \in \mathbb{R}$, the mapping T has a unique fixed point in V.*

Before going into the proof, let us show that the existence part of Theorem 2.1.1 follows from Proposition 2.3.1.

Let $v \in V$ be a fixed point of T with a parameter β that will be chosen later. There exists a function $f \in L^\infty(dx; L^2(|\xi_1|\,d\xi))$ which is a solution of (P); in particular, f satisfies

$$f(0, -\xi) = u(\xi), \qquad \xi_1 > 0, \tag{2.3.3}$$

$$f(0, \xi_1 > 0) = \beta M_w^{1/2} + \mathcal{R} \circ \Pi u + s. \tag{2.3.4}$$

Introducing the decomposition $u = \Pi u + CM_w^{1/2}$ in the latter equation, using the fact that $M_w^{1/2}$ is invariant under the action of $\mathcal{R}$, and knowing the prescribed mass flux of f in terms of s as imposed in system (P), we conclude that $\beta = C$, the coordinate of u on $M_w^{1/2}$. To obtain exactly equation (2.1.6) for the mass flux, we notice that

$$\begin{aligned} \int_{\xi_1 < 0} \xi_1 M_w^{1/2} f(0, \xi)\, d\xi &= -\int_{\xi_1 > 0} \xi_1 M_w^{1/2} u\, d\xi \\ &= -C \int_{\xi_1 > 0} \xi_1 M_w^{1/2}\, d\xi; \end{aligned} \tag{2.3.5}$$

thus, it is enough to adjust the parameter β in order to fit λ.

Now we prove Proposition 2.3.1.

Proof of Proposition 2.3.1. Consider $v_0 = v - \lambda \xi_1 M_w^{1/2}$ with

$$\lambda = \frac{\int_{\xi_1 > 0} \xi_1 M_w^{1/2} s\, d\xi}{\int_{\xi_1 > 0} \xi_1^2 M_w\, d\xi};$$

it is clear that v_0 solves problem (P) with s replaced by $s - \lambda \xi_1 M_w^{1/2}$ and zero mass flux condition. Therefore,

$$Tu = A_0\{\beta M_w^{1/2} + \mathcal{R} \circ \Pi u + s - \lambda \xi_1 M_w^{1/2}\} + \lambda \xi_1 M_w^{1/2}.$$

Observe now that the Fréchet derivative of the (affine) operator T is equal to $A_0 \circ \mathscr{R} \circ \Pi$ which, by Lemma 2.2.2 is a strictly contracting mapping on V, and, by Lemma 2.2.1, is compact. Therefore, $N(I - (A_0 \circ \mathscr{R} \circ \Pi)^*) = 0$, which proves the announced conclusion by the Fredholm alternative.

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