Stokes-Fourier and Acoustic Limits for the Boltzmann Equation: Convergence Proofs

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To Claude Bardos, en témoignage d'admiration et d'amitié

Abstract

We establish a Stokes-Fourier limit for the Boltzmann equation considered over any periodic spatial domain of dimension 2 or more. Appropriately scaled families of DiPerna-Lions renormalized solutions are shown to have fluctuations that globally in time converge weakly to a unique limit governed by a solution of Stokes-Fourier motion and heat equations provided that the fluid moments of their initial fluctuations converge to appropriate L^2 initial data of the Stokes-Fourier equations. Both the motion and heat equations are recovered in the limit by controlling the fluxes and the local conservation defects of the DiPerna-Lions solutions with dissipation rate estimates. The scaling of the fluctuations with respect to Knudsen number is essentially optimal. The assumptions on the collision kernel are little more than those required for the DiPerna-Lions theory and that the viscosity and heat conduction are finite. For the acoustic limit, these techniques also remove restrictions to bounded collision kernels and improve the scaling of the fluctuations. Both weak limits become strong when the initial fluctuations converge entropically to appropriate L^2 initial data.

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1 Introduction

The endeavor to understand how fluid dynamical equations can be derived from kinetic theory goes back to the founding works of Maxwell [23] and Boltzmann [9]. Most of these derivations are well understood at several formal levels by now, and yet their full mathematical justifications are still missing. Here we establish a so-called Stokes-Fourier fluid dynamical limit for the classical Boltzmann equation considered over any periodic spatial domain of dimension 2 or more. In the same setting, we also significantly extend our previous result that established the so-called acoustic limit [5].

The Stokes-Fourier system is the linearization about the zero state of an incompressible Navier-Stokes-Fourier system. It governs (ρ, u, θ), the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrium values. After a suitable choice of units, these fluctuations satisfy the incompressibility and Boussinesq relations

$$\nabla_x \cdot u = 0\,, \qquad \rho + \theta = 0\,, \tag{1.1}$$

while their evolution is given by the motion and heat equations

$$\begin{aligned} &\partial_t u + \nabla_x p = \nu \Delta_x u\,, \qquad && u(x,0) = u^{\mathrm{in}}(x)\,, \\ &\frac{D+2}{2}\partial_t \theta = \kappa \Delta_x \theta\,, \qquad && \theta(x,0) = \theta^{\mathrm{in}}(x)\,, \end{aligned} \tag{1.2}$$

where $\nu > 0$ is the kinematic viscosity and $\kappa > 0$ is the thermal conductivity. This is one of the simplest systems of fluid dynamical equations imaginable, being essentially a system of heat equations. It may be derived directly from the Boltzmann equation as the formal limit of moment equations for an appropriately scaled family of Boltzmann solutions as the Knudsen number tends to zero.

Here we establish the Stokes-Fourier limit, henceforth referred to as simply the Stokes limit, in the physical setting of DiPerna-Lions renormalized solutions of the Boltzmann equation [15]. Whether such solutions always satisfy the local conservation laws of momentum and energy that one would formally expect to be satisfied has been an outstanding open problem since that work appeared. In our earlier work with Bardos on the Stokes limit [3], the local momentum conservation law was therefore assumed. The present work both removes this rather large assumption and enlarges the class of collision kernels from that considered in [3].

The present work also improves upon the result given in our later work with Bardos [4]. Without assuming local momentum conservation, that work recovered the Stokes motion equation by using energy and relative entropy estimates to remove the local momentum conservation law defect in the limit, but at the cost of being restricted to bounded collision kernels (such as is the case for Maxwell molecules). It required, moreover, that the fluctuations be scaled to be an order smaller than the square of the Knudsen number. This is far smaller than what one expects to be optimal from formal derivations of the Stokes equations, namely, that the fluctuations should only be required to be of an order smaller than the Knudsen number [2]. (One formally derives the incompressible Navier-Stokes system when the fluctuations are of the same order as the Knudsen number.) Here we recover both the motion and the heat equation of the Stokes limit by controlling the local conservation defects of the DiPerna-Lions solutions with dissipation rate estimates. Our scaling of the fluctuations with respect to Knudsen number is now essentially optimal.

Recently Lions and Masmoudi [22] elegantly recovered the Stokes motion equation. They showed that DiPerna-Lions renormalized solutions satisfy the formally expected local momentum conservation up to the divergence of a nonnegative definite, matrix-valued defect measure. They control this measure by an entropy bound which shows that the measure vanishes in the Stokes limit. Their scaling of the fluctuations with respect to Knudsen number is also essentially optimal. (They also use this defect measure and entropy bound to get an improved partial result for the incompressible Euler limit.) However, as in all the other results mentioned above, they do not recover the heat equation. There are two reasons for this. First, it is unknown whether DiPerna-Lions solutions satisfy local energy conservation up to the divergence of a defect measure or how to control such a measure in the Stokes scaling should it exist. Second, even if local energy conservation were assumed, the techniques they used to control the momentum flux would fail to control the heat flux. We therefore do not use their approach here. Rather, the dissipation rate estimates that we develop here to control the heat flux and remove the local energy conservation defects of DiPerna-Lions solutions do the same for the motion equation.

The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units, in this model the fluid fluctuations (ρ, u, θ) satisfy

$$\begin{aligned}
&\partial_t \rho + \nabla_x \cdot u = 0\,, && \rho(x,0) = \rho^{\mathrm{in}}(x)\,,\\
&\partial_t u + \nabla_x (\rho + \theta) = 0\,, && u(x,0) = u^{\mathrm{in}}(x)\,,\\
&\frac{D}{2}\partial_t \theta + \nabla_x \cdot u = 0\,, && \theta(x,0) = \theta^{\mathrm{in}}(x)\,.
\end{aligned} \tag{1.3}$$

This is also one of the simplest systems of fluid dynamical equations imaginable, being essentially the wave equation. Like the Stokes system, it may be derived directly from the Boltzmann equation as the formal limit of moment equations for an appropriately scaled family of Boltzmann solutions as the Knudsen number tends to zero.

In earlier work with Bardos [4, 5] we established the acoustic limit in the setting of DiPerna-Lions renormalized solutions. That work removed the local momentum and energy conservation law defects with energy and relative entropy estimates, but at the cost of being restricted to bounded collision kernels, and with a scaling of the fluctuations with respect to Knudsen number that was far from optimal. The dissipation rate estimates developed here to remove the local conservation defects of DiPerna-Lions solutions both allow the restriction to bounded collision kernels to be dropped and improve the scaling of the fluctuations from being of an order smaller than the Knudsen number to being of an order smaller than the square root of the Knudsen number. While this scaling is a considerable improvement, it is still far from what one formally expects to be optimal, namely, that the fluctuations merely vanish with the Knudsen number. This gap must be bridged before one can hope to fully establish the compressible Euler limit.

For both the Stokes and acoustic limits we show that appropriately scaled families of DiPerna-Lions solutions have fluctuations whose weak limit points are governed for all time by solutions of the corresponding fluid equations with L^2 initial data. Conversely, we show that every L^2 initial data for the fluid equations have scaled families of DiPerna-Lions initial data whose fluctuations converge entropically (and hence strongly in L^1) to an appropriate limit associated to the L^2 fluid initial data. Moreover, every corresponding scaled family of DiPerna-Lions solutions has fluctuations that converge entropically to a unique limit governed for all time by the solution of the fluid equations. In this sense we obtain a uniqueness result for DiPerna-Lions solutions in both the Stokes and acoustic limits.

The next section contains preliminary material regarding the Boltzmann equation. Section 3 gives the formal scalings that lead from the Boltzmann equation to the acoustic and Stokes limits. Section 4 reviews the DiPerna-Lions theory of global solutions [15] and the theory of fluctuations [3]. Propositions that are essentially found in these works are fully stated for completeness, but their proofs are omitted. Section 5 presents precise statements of our main results. Section 6 reintroduces the notion of entropic convergence and uses it to strengthen the limits in our main results. Section 7 gives the proof of the acoustic limit modulo an estimate that removes the local conservation defects. Section 8 gives the proof of the Stokes limit modulo two estimates: one that shows convergence of the fluxes, and one that removes the local conservation defects. Section 9 establishes the estimates that control the local momentum and energy conservation defects for both the Stokes and acoustic limits. There are three new estimates given here—all derived using Young's inequality techniques. Section 10 establishes the estimates that control the momentum and heat fluxes for the Stokes limit. The key estimate is new. It controls the fluxes with the dissipation rate rather as well as the relative entropy. In fact, it controls moments with respect to every power of the velocity for the Stokes limit. Section 11 makes some concluding remarks.

2 Boltzmann Equation Preliminaries

Our starting point is the Boltzmann equation. In this section we collect the basic facts we need. These will include its nondimensionalization and its formal conservation and dissipation laws.

2.1 The Boltzmann Equation

Here we will introduce the Boltzmann equation only so far as to set our notation and to make precise some of our assumptions regarding the collision kernel. While our notation is essentially that of [3], our assumptions on the collision kernel are weaker and more natural than those of [3]. More complete introductions to the Boltzmann equation can be found in [10, 12, 13, 17].

The state of a fluid composed of identical point particles confined to a spatial domain $\Omega \subset \mathbb{R}^D$ is described at the kinetic level by a mass density F over the single-particle phase space $\mathbb{R}^D \times \Omega$. At any instant of time $t \geq 0$ and point $(v, x) \in \mathbb{R}^D \times \Omega$, $F(v, x, t)\, dv\, dx$ is understood to give the mass of the particles that occupy any infinitesimal volume $dv\, dx$ centered at the point (v, x). To remove complications due to boundaries, we take Ω to be the periodic domain $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}^D$, where $\mathbb{L}^D \subset \mathbb{R}^D$ is any D-dimensional lattice.

If the particles interact only through a conservative interparticle force with a finite range, then at low densities this range will be much smaller than the interparticle spacing. In that regime all but binary collisions can be neglected when $D \geq 2$, and the evolution of $F = F(v, x, t)$ is governed by the classical Boltzmann equation [13]:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)\,, \quad F(v, x, 0) = F^{\text{in}}(v, x) \geq 0\,. \tag{2.1}$$

The Boltzmann collision operator $\mathcal{B}$ acts only on the v argument of F. It is formally given by

$$\mathcal{B}(F, F) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (F_1' F' - F_1 F) b(\omega, v_1 - v) d\omega\, dv_1\,, \tag{2.2}$$

where v_1 ranges over $\mathbb{R}^D$ endowed with its Lebesgue measure dv_1, while ω ranges over the unit sphere $\mathbb{S}^{D-1} = \{\omega \in \mathbb{R}^D : |\omega| = 1\}$ endowed with its rotationally invariant unit measure $d\omega$. The F_1', F', F_1, and F appearing in the integrand designate $F(\cdot, x, t)$ evaluated at the velocities v_1', v', v_1, and v, respectively, where the primed velocities are defined by

$$v_1' = v_1 - \omega\omega \cdot (v_1 - v)\,, \quad v' = v + \omega\omega \cdot (v_1 - v)\,, \tag{2.3}$$

for any given $(\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$. Notice that when $D = 1$ these reduce to $v_1' = v$ and $v' = v_1$, whereby $\mathcal{B}$ vanishes identically. This reflects the restriction to $D \geq 2$. Quadratic operators like $\mathcal{B}$ are extended by polarization to be bilinear and symmetric.

The unprimed and primed velocities are possible velocities for a pair of particles either before and after, or after and before, they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during collisions is expressed as

$$v + v_1 = v' + v_1' \,, \quad |v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2 \,. \tag{2.4}$$

These equations have the trivial solution $v_1' = v_1$ and $v' = v$. Equation (2.3) represents the general nontrivial solution of these $D + 1$ equations for the $4D$ unknowns v_1', v', v_1, and v in terms of the $3D - 1$ parameters (ω, v_1, v).

The collision kernel b is a positive, locally integrable function. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v) = |v| \Sigma(|\omega \cdot \hat{v}|, |v|) \,,$$

where $\hat{v} = v/|v|$ and Σ is the specific differential cross section. This symmetry implies that the quantity $\int b(\omega, v)\, d\omega$ will be a function of $|v|$ only. The DiPerna-Lions theory requires that b satisfies

$$\lim_{|v|\to\infty} \frac{1}{1 + |v|^2} \iint_{\mathbb{S}^{D-1}\times K} b(\omega, v_1 - v) d\omega\, dv_1 = 0 \tag{2.5}$$

for every compact set $K \subset \mathbb{R}^D$. In addition, we assume that there exist constants $C_b \in (0, \infty)$ and $\beta \in [0, 1]$ such that b satisfies

$$\int_{\mathbb{S}^{D-1}} b(\omega, v) d\omega \leq C_b \left(1 + \frac{1}{2}|v|^2\right)^\beta \quad \text{almost everywhere} \,. \tag{2.6}$$

This condition implies (2.5) whenever $\beta < 1$. It holds for some $\beta \leq \frac{1}{2}$ for those b that are classically derived from a so-called hard interparticle potential with a small deflection cutoff; see [13, chap. II.4,5]. In particular, condition (2.6) holds with $\beta = 0$ for Maxwell molecules and $\beta = \frac{1}{2}$ for hard spheres. Some of our results impose additional conditions on b. These conditions also hold for those b that are classically derived from a hard interparticle potential with a small deflection cutoff.

2.2 Nondimensionalized Form

We will work with the nondimensionalized form of the Boltzmann equation that was used in [3]. The form is motivated by the fact that the Stokes (1.1)–(1.2) and acoustic (1.3) systems can be formally derived from the Boltzmann equation through a scaling in which the density F is close to a spatially homogeneous Maxwellian $M = M(v)$ that has the same total mass, momentum, and energy as the initial data F^{in}. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{D/2}} \exp\left(-\frac{1}{2}|v|^2\right). \tag{2.7}$$

This corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equal to 0, and is consistent with the form of both the Stokes system given by (1.1–1.2) and the acoustic system given by (1.3).

It is natural to introduce the relative density, $G = G(v, x, t)$, defined by $F = MG$. Recasting the initial-value problem (2.1) for G yields

$$\partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon}\mathcal{Q}(G, G)\,, \quad G(v, x, 0) = G^{\text{in}}(v, x)\,, \tag{2.8}$$

where the collision operator is now given by

$$\mathcal{Q}(G, G) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (G_1' G' - G_1 G) b(\omega, v_1 - v) d\omega\, M_1\, dv_1\,, \tag{2.9}$$

with the nondimensional collision kernel b being normalized so that

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} b(\omega, v_1 - v) d\omega\, M_1\, dv_1\, M\, dv = 1\,. \tag{2.10}$$

The positive, nondimensional parameter ϵ is the Knudsen number, which is the ratio of the mean-free-path to the macroscopic length scale determined by setting the volume of $\mathbb{T}^D$ to unity [3].

This nondimensionalization has the normalizations

$$\int_{\mathbb{S}^{D-1}} d\omega = 1\,, \quad \int_{\mathbb{R}^D} M\, dv = 1\,, \quad \int_{\mathbb{T}^D} dx = 1\,, \tag{2.11}$$

associated with the domains $\mathbb{S}^{D-1}$, $\mathbb{R}^D$, and $\mathbb{T}^D$, respectively, (2.10) associated with the collision kernel b, and

$$\begin{gathered} \iint_{\mathbb{R}^D\times\mathbb{T}^D} G^{\text{in}} M\, dv\, dx = 1\,, \qquad \iint_{\mathbb{R}^D\times\mathbb{T}^D} v G^{\text{in}} M\, dv\, dx = 0\,, \\ \iint_{\mathbb{R}^D\times\mathbb{T}^D} \frac{1}{2}|v|^2 G^{\text{in}} M\, dv\, dx = \frac{D}{2}\,, \end{gathered} \tag{2.12}$$

associated with the initial data G^{in}.

Because $M\, dv$ is a positive unit measure on $\mathbb{R}^D$, we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v) M\, dv\,. \tag{2.13}$$

Because

$$d\mu = b(\omega, v_1 - v)d\omega\, M_1\, dv_1\, M\, dv$$

is a positive unit measure on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle\langle \Xi \rangle\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$,

$$\langle\langle \Xi \rangle\rangle = \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_1, v)d\mu\, . \tag{2.14}$$

The measure $d\mu$ is invariant under the coordinate transformations

$$(\omega, v_1, v) \mapsto (\omega, v, v_1)\, , \qquad (\omega, v_1, v) \mapsto (\omega, v_1', v')\, . \tag{2.15}$$

These, and compositions of these, are called $d\mu$-symmetries.

2.3 Formal Conservation and Dissipation Laws

We now list for later reference the basic conservation and entropy dissipation laws that are formally satisfied by solutions to the Boltzmann equation. Derivations of these laws in this nondimensional setting are outlined in [3] and can, up to notational differences, be found in [12, sec. II.6-7)], [17, sec. 1.4], or [10].

First, if G solves the Boltzmann equation (2.8), then G satisfies local conservation laws of mass, momentum, and energy:

$$\begin{aligned}
\partial_t \langle G \rangle + \nabla_x \cdot \langle vG \rangle &= 0\, ,\\
\partial_t \langle vG \rangle + \nabla_x \cdot \langle v \otimes vG \rangle &= 0\, ,\\
\partial_t \left\langle \frac{1}{2}|v|^2 G \right\rangle + \nabla_x \cdot \left\langle v \frac{1}{2}|v|^2 G \right\rangle &= 0\, .
\end{aligned} \tag{2.16}$$

Integrating these over space and time while recalling the normalizations (2.12) of G^{in} yields the global conservation laws of mass, momentum, and energy:

$$\begin{aligned}
\int_{\mathbb{T}^D} \langle G(t) \rangle dx &= \int_{\mathbb{T}^D} \langle G^{\text{in}} \rangle dx = 1\, ,\\
\int_{\mathbb{T}^D} \langle vG(t) \rangle dx &= \int_{\mathbb{T}^D} \langle vG^{\text{in}} \rangle dx = 0\, ,\\
\int_{\mathbb{T}^D} \left\langle \frac{1}{2}|v|^2 G(t) \right\rangle dx &= \int_{\mathbb{T}^D} \left\langle \frac{1}{2}|v|^2 G^{\text{in}} \right\rangle dx = \frac{D}{2}\, .
\end{aligned} \tag{2.17}$$

Second, if G solves the Boltzmann equation (2.8), then G satisfies the local entropy dissipation law

$$\begin{aligned}
&\partial_t \langle (G \log(G) - G + 1) \rangle + \nabla_x \cdot \langle v(G \log(G) - G + 1) \rangle =\\
&\qquad\qquad -\frac{1}{\epsilon} \left\langle\!\!\left\langle \frac{1}{4} \log\left(\frac{G_1' G'}{G_1 G}\right)(G_1' G' - G_1 G) \right\rangle\!\!\right\rangle \leq 0\, .
\end{aligned} \tag{2.18}$$

Integrating this over space and time gives the global entropy equality

$$H(G(t)) + \frac{1}{\epsilon}\int_0^t R(G(s))ds = H(G^{\mathrm{in}})\,, \tag{2.19}$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_{\mathbb{T}^D} \langle (G\log(G) - G + 1)\rangle dx\,, \tag{2.20}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\mathbb{T}^D} \left\langle\!\!\left\langle \frac{1}{4}\log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle dx\,. \tag{2.21}$$

3 Formal Scalings and Derivations

Fluid dynamical regimes are those where the mean free path is small compared to the macroscopic length scales, i.e., where the Knudsen number ϵ is small. Formal derivations of the compressible Euler system are rather direct. Formal derivations of other fluid dynamical systems, such as the compressible Navier-Stokes system, are more subtle. (Indeed, some situations cannot be described by directly using the compressible Navier-Stokes system: These are referred to as "ghost effects" in [26] and are somehow related to the discussion in [6]). Hilbert [19] proposed that at the formal level all derivations of fluid dynamics should be based on a systematic asymptotic expansion in ϵ. A somewhat different asymptotic expansion in ϵ, now called the *Chapman-Enskog expansion*, was proposed a bit later by Enskog [16]. The Chapman-Enskog expansion yields at successive orders the compressible Euler system and the compressible Navier-Stokes system; see [18].

Justification of these formal approximations has proven difficult in part because many basic well-posedness and regularity questions remain open for both these fluid systems and the Boltzmann equation. The problem is exacerbated by the fact that to bound the error of the asymptotic expansions requires the control of successively higher-order spatial derivatives of the fluid variables, thereby requiring unphysical restrictions to a meager subset of all physically natural initial data and possibly to finite periods of time. For example, Caflisch used a method based on the Hilbert expansion to justify the compressible Euler system from the Boltzmann equation [11]. His result requires smooth initial data and holds for as long as the limiting solution of the compressible Euler system is smooth. Because solutions of the compressible Euler system are known to become singular in finite time for a very general class of initial data (see [24]), such a result is about the best one can hope for by appealing to such an expansion.

Two approaches to circumvent these difficulties have emerged recently. First, some authors have studied direct derivations of linear or weakly nonlinear fluid dynamical systems, such as incompressible Stokes, Navier-Stokes, and Euler systems [1, 2, 3, 6, 7, 8, 10, 14, 21, 22, 25, 27], about which more is known. Second, some authors have abandoned the traditional expansion-based derivations in favor of moment-based formal derivations [2, 3, 6, 7, 10, 21, 22], which put fewer demands on the well-posedness and regularity theory. In [5] we embraced both of these approaches when first establishing the acoustic limit in a far more restrictive setting. We do so again here when establishing the Stokes limit and extending the acoustic limit.

Both the Stokes (1.1)–(1.2) and the acoustic (1.3) systems can be formally derived from the Boltzmann equation through a scaling in which the density F is close to the absolute Maxwellian M. More precisely, we consider families of solutions parametrized by the Knudsen number ϵ that have the form

$$G_\epsilon^{\text{in}} = 1 + \delta_\epsilon g_\epsilon^{\text{in}}\,, \quad G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,, \tag{3.1}$$

where the fluctuations g_ϵ^{in} and g_ϵ are bounded while $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \quad \text{as } \epsilon \to 0\,. \tag{3.2}$$

The common practice of past works was to set $\delta_\epsilon = \epsilon^m$ for some $m > 0$, but we will not do so here in order to clarify how close the scalings in our analytical results are to those that are formally optimal. Toward this end, we outline moment-based formal derivations of both the Stokes and acoustic limits in this more general setting. They go further than the derivations given in [2] and [5], respectively.

In these derivations we assume that g_ϵ converges formally to g, where the limiting function is in $L^\infty(dt; L^2(M\,dv\,dx))$, and that all formally small terms vanish. For example, we express the global conservation laws (2.17), which are the same for both derivations, in terms of g_ϵ and then formally let $\epsilon \to 0$ to obtain

$$\int_{\mathbb{T}^D} \langle g(t)\rangle dx = 0\,, \quad \int_{\mathbb{T}^D} \langle v g(t)\rangle dx = 0\,, \quad \int_{\mathbb{T}^D} \left\langle \frac{1}{2}|v|^2 g(t)\right\rangle dx = 0\,. \tag{3.3}$$

Henceforth, the two derivations differ.

3.1 Acoustic Formal Derivation

It is most natural to derive the acoustic limit first because its derivation is simpler and requires no additional assumptions regarding either the scaling or the collision kernel. One considers a family of formal solutions G_ϵ to the scaled Boltzmann initial-value problem

$$\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon}\mathcal{Q}(G_\epsilon, G_\epsilon)\,, \quad G_\epsilon(v,x,0) = G_\epsilon^{\text{in}}(v,x)\,, \tag{3.4}$$

whose fluctuations g_ϵ are given by (3.1) for some $\delta_\epsilon > 0$ that vanishes with ϵ as in (3.2). The derivation has two steps.

The first step determines the form of the limiting function g. Observe that by (3.4) the fluctuations g_ϵ satisfy

$$\epsilon(\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon) + \mathcal{L} g_\epsilon = \delta_\epsilon \mathcal{Q}(g_\epsilon, g_\epsilon)\,, \tag{3.5}$$

where the linearized collision operator $\mathcal{L}$ is formally defined by

$$\mathcal{L}\tilde{g} = -2\mathcal{Q}(1, \tilde{g}) . \tag{3.6}$$

We define $\mathcal{L}$ to be the unique nonnegative, self-adjoint extension over $L^2(M\,dv)$ of this formal operator. By letting $\epsilon \to 0$ in (3.5), one finds that $\mathcal{L}g = 0$. It is known (see, for example, [12, chap. IV.1]) that the null space of $\mathcal{L}$ is given by $\mathrm{Null}(\mathcal{L}) = \mathrm{Span}\{1, v_1, \dots, v_D, |v|^2\}$. Because the limit g is assumed to belong to $L^\infty(dt; L^2(M\,dv\,dx))$, we conclude that g has the form of a so-called infinitesimal Maxwellian, namely,

$$g = \rho + u \cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right) \tag{3.7}$$

for some (ρ, u, θ) in $L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$.

The second step shows that the evolution of (ρ, u, θ) is governed by the acoustic system (1.3). Observe that the fluctuations g_ϵ formally satisfy the local conservation laws

$$\begin{aligned} &\partial_t \langle g_\epsilon \rangle + \nabla_x \cdot \langle v g_\epsilon \rangle = 0 , \\ &\partial_t \langle v g_\epsilon \rangle + \nabla_x \cdot \langle v \otimes v g_\epsilon \rangle = 0 , \\ &\partial_t \left\langle \frac{1}{2}|v|^2 g_\epsilon \right\rangle + \nabla_x \cdot \left\langle v \frac{1}{2}|v|^2 g_\epsilon \right\rangle = 0 . \end{aligned} \tag{3.8}$$

By letting $\epsilon \to 0$ in these equations and using the infinitesimal Maxwellian form of g given by (3.7), one then finds that (ρ, u, θ) solves the local conservation laws of the acoustic system (1.3). By the formal continuity in time of the densities in (3.8), one finds that

$$(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) = \lim_{\epsilon \to 0} \left(\langle g_\epsilon^{\mathrm{in}} \rangle, \langle v g_\epsilon^{\mathrm{in}} \rangle, \left\langle \left(\frac{1}{D}|v|^2 - 1 \right) g_\epsilon^{\mathrm{in}} \right\rangle \right) , \tag{3.9}$$

provided we assume that the limits on the right-hand side exist in the sense of distributions for some $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$.

The above formal derivation can be stated more precisely as follows:

THEOREM 3.1 (Formal Acoustic Limit Theorem) *Let G_ϵ be a family of distribution solutions of the scaled Boltzmann initial-value problem* (3.4) *with initial data G_ϵ^{in} that satisfy the normalizations* (2.12). *Let G_ϵ^{in} and G_ϵ have fluctuations g_ϵ^{in} and g_ϵ given by* (3.1) *that are bounded families for some $\delta_\epsilon > 0$ that vanishes with ϵ as in* (3.2). *Also*:

(i) *Assume that the local conservation laws* (3.8) *are also satisfied in the sense of distributions for every g_ϵ.*

(ii) *Assume that the family g_ϵ converges in the sense of distributions as $\epsilon \to 0$ to $g \in L^\infty(dt; L^2(M\,dv\,dx))$. Assume furthermore that $\mathcal{L}g_\epsilon \to \mathcal{L}g$, that the moments*

$$\langle g_\epsilon \rangle\,, \quad \langle v g_\epsilon \rangle\,, \quad \langle v \otimes v g_\epsilon \rangle\,, \quad \langle v|v|^2 g_\epsilon \rangle\,,$$

converge to the corresponding moments

$$\langle g \rangle\,, \quad \langle v g \rangle\,, \quad \langle v \otimes v g \rangle\,, \quad \langle v|v|^2 g \rangle\,,$$

and that every formally small term vanishes, all in the sense of distributions as $\epsilon \to 0$.

(iii) *Assume that for some $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ the family g_ϵ^{in} satisfies* (3.9) *in the sense of distributions.*

Then g is the unique local infinitesimal Maxwellian (3.7) *determined by the solution (ρ, u, θ) of the acoustic system* (1.3) *with the initial data $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})$ obtained from* (3.9).

3.2 Stokes Formal Derivation

The acoustic system differs from the Boltzmann equation in one very important respect. Stationary solutions of the acoustic system (1.3) are exactly those that formally satisfy the incompressibility and Boussinesq relations (1.1). The acoustic system over $\mathbb{T}^D$ therefore has stationary solutions that vary in space, while the Boltzmann equation over $\mathbb{T}^D$ does not. It is clear that the time scale at which the acoustic system was derived was not long enough to see the evolution of these solutions.

It was shown in [2] that by considering the Boltzmann equation with a longer time scale, one can give formal moment derivations of three fluid dynamical systems, depending on the limiting behavior of the ratio δ_ϵ/ϵ as $\epsilon \to 0$.

- When $\delta_\epsilon/\epsilon \to 0$, one considers time scales of order $1/\epsilon$, and an incompressible Stokes system is derived.
- When $\delta_\epsilon/\epsilon \to 1$ (or any other nonzero number), one considers time scales of order $1/\epsilon$, and an incompressible Navier-Stokes system is derived.
- When $\delta_\epsilon/\epsilon \to \infty$, one considers time scales of order $1/\delta_\epsilon$, and an incompressible Euler system is derived.

The common practice of past works was to set $\delta_\epsilon = \epsilon^m$, in which case $m > 1$ leads to Stokes, $m = 1$ to Navier-Stokes, and $0 < m < 1$ to Euler. Each derivation yields motion and temperature equations that, when supplemented by the incompressibility and Boussinesq relations, govern the evolution of (ρ, u, θ).

In particular, to derive the Stokes system one considers a family of formal solutions G_ϵ to the scaled Boltzmann initial-value problem

$$\epsilon \partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon)\,, \quad G_\epsilon(v, x, 0) = G_\epsilon^{\text{in}}(v, x)\,, \tag{3.10}$$

whose fluctuations g_ϵ are given by (3.1) for some $\delta_\epsilon > 0$ that satisfies

$$\frac{\delta_\epsilon}{\epsilon} \to 0 \quad \text{as } \epsilon \to 0\,. \tag{3.11}$$

The derivation has six steps.

The first step shows that the limiting g is an infinitesimal Maxwellian. Observe that by (3.10) the fluctuations g_ϵ satisfy

$$\epsilon\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L} g_\epsilon = \frac{\delta_\epsilon}{\epsilon}\mathcal{Q}(g_\epsilon, g_\epsilon)\,. \tag{3.12}$$

After multiplying this equation by ϵ and letting $\epsilon \to 0$, we argue as in the first step of the acoustic limit derivation to conclude g has the form (3.7) for some (ρ, u, θ) in $L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$.

The second step shows that (ρ, u, θ) satisfy the incompressibility and Boussinesq relations. Observe that the fluctuations g_ϵ formally satisfy the local conservation laws

$$\begin{aligned} \epsilon\partial_t \langle g_\epsilon \rangle + \nabla_x \cdot \langle v g_\epsilon \rangle &= 0\,, \\ \epsilon\partial_t \langle v g_\epsilon \rangle + \nabla_x \cdot \langle v \otimes v g_\epsilon \rangle &= 0\,, \\ \epsilon\partial_t \left\langle \frac{1}{2}|v|^2 g_\epsilon \right\rangle + \nabla_x \cdot \left\langle v \frac{1}{2}|v|^2 g_\epsilon \right\rangle &= 0\,. \end{aligned} \tag{3.13}$$

By letting $\epsilon \to 0$ in these equations and using the infinitesimal Maxwellian form of g given by (3.7), one then finds that

$$\nabla_x \cdot u = 0\,, \qquad \nabla_x(\rho + \theta) = 0\,.$$

The first equation is the incompressibility relation, while the second says $\rho + \theta$ is a function of time only. By global energy conservation laws of (3.3) one thereby concludes that

$$\rho + \theta = \int_{\mathbb{T}^D} (\rho + \theta)\,dx = \frac{2}{D}\int_{\mathbb{T}^D} \left\langle \frac{1}{2}|v|^2 g \right\rangle dx = 0\,.$$

Hence, (ρ, u, θ) satisfy the incompressibility and Boussinesq relations (1.1). Notice that the Boussinesq relation implies g is an infinitesimal Maxwellian of the form

$$g = u \cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right) \tag{3.14}$$

for some (u, θ) in $L^\infty(dt; L^2(dx; \mathbb{R}^D \times \mathbb{R}))$.

The next three steps show that the evolution of (u, θ) is governed by the motion and heat equations. The difficulty here is that when the local conservation laws are written so that the time derivatives are order 1, the fluxes become order $1/\epsilon$. This difficulty is overcome by the following strategy [2]. Observe that the momentum and a linear combination of the mass and energy local conservation laws from (3.13) can be expressed as

$$\begin{aligned} &\partial_t \langle v g_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle A g_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \left\langle \frac{1}{D} |v|^2 g_\epsilon \right\rangle = 0 \,, \\ &\partial_t \left\langle \left(\frac{1}{2} |v|^2 - \frac{D+2}{2} \right) g_\epsilon \right\rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle B g_\epsilon \rangle = 0 \,, \end{aligned} \tag{3.15}$$

where the matrix-valued function A and the vector-valued function B are defined by

$$A(v) = v \otimes v - \frac{1}{D} |v|^2 I \,, \qquad B(v) = \frac{1}{2} |v|^2 v - \frac{D+2}{2} v \,. \tag{3.16}$$

It is clear that $A \in L^2(M\,dv; \mathbb{R}^{D\times D})$ and $B \in L^2(M\,dv; \mathbb{R}^D)$. As is common when studying incompressible fluid dynamical limits, the momentum equation will be integrated against divergence-free test functions. The last term in its flux will thereby be eliminated, and one only has to pass to the limit in the flux terms of (3.15) that involve A and B, namely, in the terms

$$\frac{1}{\epsilon} \langle A g_\epsilon \rangle \,, \qquad \frac{1}{\epsilon} \langle B g_\epsilon \rangle \,. \tag{3.17}$$

There is a chance that these terms have a limit because each entry of A and B is in $\mathrm{Null}(\mathcal{L})^\perp$ while g_ϵ converges to g, which is in $\mathrm{Null}(\mathcal{L})$. The next two steps show that these terms indeed have a limit.

The third step evaluates the limit for moments of the form $\langle \mathcal{L}\xi g_\epsilon \rangle / \epsilon$ for every $\xi \in \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Null}(\mathcal{L})^\perp$, where $\mathrm{Dom}(\mathcal{L}) \subset L^2(M\,dv)$ is the domain of $\mathcal{L}$. Because $\mathcal{L}$ is formally symmetric, one has

$$\langle \mathcal{L}\xi g_\epsilon \rangle = \langle \xi \mathcal{L} g_\epsilon \rangle \,. \tag{3.18}$$

Upon multiplying (3.12) by ξ and integrating, one obtains

$$\epsilon \partial_t \langle \xi g_\epsilon \rangle + \nabla_x \cdot \langle v \xi g_\epsilon \rangle + \frac{1}{\epsilon} \langle \xi \mathcal{L} g_\epsilon \rangle = \frac{\delta_\epsilon}{\epsilon} \langle \xi \mathcal{Q}(g_\epsilon, g_\epsilon) \rangle \,. \tag{3.19}$$

By letting $\epsilon \to 0$ in this equation and using the infinitesimal Maxwellian form of g given by (3.14), one finds that, in the sense of distributions,

$$\frac{1}{\epsilon} \langle \mathcal{L}\xi g_\epsilon \rangle \to -\langle \xi v \cdot \nabla_x g \rangle = -\langle \xi A \rangle : \nabla_x u - \langle \xi B \rangle \cdot \nabla_x \theta \,, \tag{3.20}$$

where the matrix-valued function A and the vector-valued function B are defined by (3.16).

The fourth step determines the limit of the flux terms (3.17). At this point we assume that for some $\ell > 0$ the operator $\mathcal{L}$ satisfies the coercivity estimate

$$\ell \langle \xi^2 \rangle \le \langle \xi \mathcal{L} \xi \rangle \quad \text{for every } \xi \in \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Null}(\mathcal{L})^\perp \,. \tag{3.21}$$

This estimate holds for every linearized collision operator that arises from a classical hard potential with a small deflection cutoff. This assumption is equivalent to assuming that the Fredholm alternative holds for $\mathcal{L}$, namely, that $\mathrm{Range}(\mathcal{L}) =$

$\mathrm{Null}(\mathcal{L})^\perp$. In particular, it implies that unique $\phi \in L^2(M\,dv;\mathbb{R}^{D\times D})$ and $\psi \in L^2(M\,dv;\mathbb{R}^D)$ exist which solve

$$\begin{aligned} \mathcal{L}\phi &= A\,, \quad \phi \in \mathrm{Null}(\mathcal{L})^\perp \text{ entrywise}\,, \\ \mathcal{L}\psi &= B\,, \quad \psi \in \mathrm{Null}(\mathcal{L})^\perp \text{ entrywise}\,. \end{aligned} \tag{3.22}$$

Then by letting ξ in (3.20) be the entries of ϕ and ψ, one finds that

$$\begin{aligned} &\frac{1}{\epsilon}\langle A g_\epsilon\rangle \to -\langle \phi\otimes A\rangle : \nabla_x u = -\nu\big(\nabla_x u + (\nabla_x u)^{\mathsf{T}}\big)\,, \\ &\frac{1}{\epsilon}\langle B g_\epsilon\rangle \to -\langle \psi\otimes B\rangle \cdot \nabla_x \theta = -\kappa\nabla_x\theta\,, \end{aligned} \tag{3.23}$$

where kinematic viscosity ν and thermal conductivity κ are given by

$$\nu = \frac{1}{(D-1)(D+2)}\,\langle \phi : \mathcal{L}\phi\rangle\,, \quad \kappa = \frac{1}{D}\langle \psi\cdot\mathcal{L}\psi\rangle\,. \tag{3.24}$$

In this step the coercivity assumed in (3.21) has been used only to assert the existence of ϕ and ψ, something that could have been asserted by assuming much less. The full power of coercivity will be used in the sixth step.

The fifth step shows that the evolution of (u,θ) is governed by the Stokes motion and heat equations (1.2). The fluctuations g_ϵ formally satisfy the local conservation laws (3.15). Hence, when letting $\epsilon \to 0$ in these equations, we use the infinitesimal Maxwellian form of g given by (3.14) to evaluate the limiting densities while we use (3.23) to evaluate the limiting fluxes. We find that (u,θ) satisfies (1.2). If we let Π denote the orthogonal projection from $L^2(dx;\mathbb{R}^D)$ onto divergence-free vector fields, then by the formal continuity in time of the densities in (3.15), one finds that

$$(u^{\mathrm{in}},\theta^{\mathrm{in}}) = \lim_{\epsilon\to 0}\left(\Pi\langle v g_\epsilon^{\mathrm{in}}\rangle, \left\langle\left(\frac{1}{D+2}|v|^2 - 1\right) g_\epsilon^{\mathrm{in}}\right\rangle\right)\,, \tag{3.25}$$

provided we assume that the limit on the right-hand side exists in the sense of distributions for some $(u^{\mathrm{in}},\theta^{\mathrm{in}}) \in L^2(dx;\mathbb{R}^D\times\mathbb{R})$.

The sixth step determines the limit of the difference of g_ϵ from its infinitesimal Maxwellian, $\mathcal{P}g_\epsilon$, where $\mathcal{P}$ is the orthogonal projection from $L^2(M\,dv)$ onto $\mathrm{Null}(\mathcal{L})$, which for every $\tilde g \in L^2(M\,dv)$ is given by

$$\mathcal{P}\tilde g = \langle \tilde g\rangle + \langle v\tilde g\rangle\cdot v + \left\langle\left(\frac{1}{D}|v|^2 - 1\right)\tilde g\right\rangle\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right). \tag{3.26}$$

The Fredholm alternative implies that for every $\tilde\xi \in L^2(M\,dv)$ there is a unique $\xi \in \mathrm{Dom}(\mathcal{L})$ that solves $\mathcal{L}\xi = \mathcal{P}^\perp\tilde\xi$ with $\mathcal{P}\xi = 0$, where $\mathcal{P}^\perp = \mathcal{I} - \mathcal{P}$. Hence, for every $\tilde\xi \in L^2(M\,dv)$ one has

$$\langle \tilde\xi\mathcal{P}^\perp g_\epsilon\rangle = \langle g_\epsilon\mathcal{P}^\perp\tilde\xi\rangle = \langle g_\epsilon\mathcal{L}\xi\rangle\,.$$

One thereby sees that as $\epsilon \to 0$, (3.20) yields

$$\begin{aligned} \frac{1}{\epsilon}\langle \tilde{\xi}\mathcal{P}^{\perp} g_\epsilon \rangle &\to -\,\langle \xi A \rangle : \nabla_x u - \langle \xi B \rangle \cdot \nabla_x \theta \\ &= -\,\langle \tilde{\xi}\phi \rangle : \nabla_x u - \langle \tilde{\xi}\psi \rangle \cdot \nabla_x \theta \,. \end{aligned} \tag{3.27}$$

Hence, as $\epsilon \to 0$ we have the distribution limit

$$\frac{1}{\epsilon}\mathcal{P}^{\perp} g_\epsilon \to -\phi : \nabla_x u - \psi \cdot \nabla_x \theta \,. \tag{3.28}$$

The right-hand side is exactly the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with the Stokes scaling.

The above formal derivation can be stated more precisely as follows:

THEOREM 3.2 (Formal Stokes Limit Theorem) *Let b be a collision kernel for which $\mathcal{L}$ satisfies the coercivity estimate* (3.21)*. Let G_ϵ be a family of distribution solutions of the scaled Boltzmann initial-value problem* (3.10) *with initial data G_ϵ^{in} that satisfy the normalizations* (2.12)*. Let G_ϵ^{in} and G_ϵ have fluctuations g_ϵ^{in} and g_ϵ given by* (3.1) *that are bounded families for some $\delta_\epsilon > 0$ that scales with ϵ as* (3.11)*. Also*:

(i) *Assume that the local conservation laws* (3.13) *and the moment equation* (3.19) *for every $\xi \in L^2(M\,dv)$ are also satisfied in the sense of distributions for every g_ϵ.*

(ii) *Assume that g_ϵ converges in the sense of distributions as $\epsilon \to 0$ to $g \in L^\infty(dt; L^2(M\,dv\,dx))$. Assume furthermore that $\mathcal{L}g_\epsilon \to \mathcal{L}g$, that for every $\xi \in L^2(M\,dv)$ the moments $\langle \xi g_\epsilon \rangle$ converge to $\langle \xi g \rangle$, and that every formally small term vanishes, all in the sense of distributions as $\epsilon \to 0$.*

(iii) *Assume that for some $(u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbb{R}^D \times \mathbb{R})$ the family g_ϵ^{in} satisfies* (3.25) *in the sense of distributions.*

Then g is the unique local infinitesimal Maxwellian (3.14) *determined by the solution (u, θ) of the Stokes system* (1.1)–(1.2) *with ν and κ given by* (3.24) *and initial data $(u^{\text{in}}, \theta^{\text{in}})$ obtained from* (3.25)*. Moreover, the family $\mathcal{P}^{\perp} g_\epsilon$ of the deviations of g_ϵ from the infinitesimal Maxwellians satisfies the limit* (3.28) *in the sense of distributions.*

4 Analytic Setting

In order to mathematically justify the fluid dynamical limits that were derived formally in the last section, two things must be made precise: (1) the notion of solution for the Boltzmann equation and (2) the sense in which the solutions fluctuate about the absolute Maxwellian. Ideally, the solutions should be global while the bounds and scalings should be physically natural. We therefore work in the setting of the DiPerna-Lions theory of renormalized solutions. The theory has the virtues of considering the physically natural class of initial data, and consequently, of yielding global solutions. These solutions have been used to study the incompressible Navier-Stokes limit [3, 21] and the incompressible Euler limit [10, 22] with only partial success, the acoustic limit [4, 5] and the Stokes limit [4, 22] with considerable success, and the linearized Boltzmann limit [20] with complete success. These works have developed the theory introduced in [3], which uses the relative entropy and the entropy dissipation rate to control the fluctuations about the absolute Maxwellian.

We present the basic facts we need about these theories in the following general setting that allows a unified analysis of many aspects of both the acoustic and Stokes limits. The scaled Boltzmann initial-value problems for both the acoustic (3.4) and Stokes (3.10) limits can be put into the general form

$$\tau_\epsilon \partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon) \,, \quad G_\epsilon(v, x, 0) = G_\epsilon^{\mathrm{in}}(v, x) \,, \tag{4.1}$$

where $1/\tau_\epsilon$ is the time scale being considered. One sets $\tau_\epsilon = 1$ for the acoustic limit and $\tau_\epsilon = \epsilon$ for the Stokes limit.

4.1 Global Solutions

DiPerna and Lions [15] proved the global existence of a type of weak solution to the Boltzmann equation over the whole space $\mathbb{R}^D$ for any initial data satisfying natural physical bounds. As they pointed out, with only slight modifications their theory can be extended to the periodic box $\mathbb{T}^D$.

The DiPerna-Lions theory does not yield solutions that are known to solve the Boltzmann in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by dividing the Boltzmann equation in (4.1) by normalizing functions $N = N(G) > 0$:

$$(\tau_\epsilon \partial_t + v \cdot \nabla_x) \Gamma(G) = \frac{1}{\epsilon} \frac{\mathcal{Q}(G, G)}{N(G)} \,, \quad G(v, x, 0) = G^{\mathrm{in}}(v, x) \geq 0 \,, \tag{4.2}$$

where $\Gamma'(Z) = \frac{1}{N}(Z)$. Here each normalizing function N is a positive-valued, continuous function over $[0, \infty)$ that for some constant $C_N < \infty$ satisfies the bound

$$\frac{1}{N(Z)} \leq \frac{C_N}{1 + Z} \quad \text{for every } Z \geq 0 \,. \tag{4.3}$$

Their solutions lie in $C([0, \infty); w\text{-}L^1(M\, dv\, dx))$, where the prefix "$w$-" on a space indicates that the space is endowed with its weak topology. They say that $G \geq 0$ is a weak solution of (4.2) provided that it is initially equal to G^{in}, and that it satisfies (4.2) in the sense that for every $Y \in L^\infty(dv; C^1(\mathbb{T}^D))$ and every $[t_1, t_2] \subset [0, \infty)$

it satisfies

$$\tau_\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_2))Y\rangle dx - \tau_\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_1))Y\rangle dx - \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \langle \Gamma(G) v \cdot \nabla_x Y\rangle dx\, dt = \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \left\langle \frac{\mathcal{Q}(G,G)}{N(G)} Y \right\rangle dx\, dt\,. \tag{4.4}$$

They show that if G is a weak solution of (4.2) for one such N, and if G satisfies certain bounds, then it is a weak solution of (4.2) for every such N. They call such solutions *renormalized solutions* of the Boltzmann equation (2.8).

Specifically, cast in our setting, their theory yields the following:

PROPOSITION 4.1 (DiPerna-Lions Renormalized Solutions) *Let b satisfy condition* (2.5). *Given any initial data* G^{in} *in the entropy class*

$$E(M\, dv\, dx) = \{G^{\text{in}} \ge 0 : H(G^{\text{in}}) < \infty\}\,, \tag{4.5}$$

there exists at least one $G \ge 0$ *in* $C([0,\infty); w\text{-}L^1(M\, dv\, dx))$ *that is a weak solution of* (4.2) *for every normalizing function* N. *Moreover,* G *satisfies the global entropy inequality*

$$H(G(t)) + \frac{1}{\epsilon\tau_\epsilon} \int_0^t R(G(s))ds \le H(G^{\text{in}})\,, \tag{4.6}$$

a weak form of the local conservation law of mass

$$\tau_\epsilon \partial_t \langle G\rangle + \nabla_x \cdot \langle vG\rangle = 0\,, \tag{4.7}$$

the global conservation law of momentum

$$\int_{\mathbb{T}^D} \langle vG(t)\rangle dx = \int_{\mathbb{T}^D} \langle vG^{\text{in}}\rangle dx\,, \tag{4.8}$$

and, finally, the global energy inequality

$$\int_{\mathbb{T}^D} \left\langle \frac{1}{2}|v|^2 G(t) \right\rangle dx \le \int_{\mathbb{T}^D} \left\langle \frac{1}{2}|v|^2 G^{\text{in}} \right\rangle dx\,. \tag{4.9}$$

DiPerna-Lions renormalized solutions are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Boltzmann equation. In particular, the theory does not assert either the local conservation of momentum in (2.16), the global conservation of energy in (2.17), the global entropy equality (2.19), or even a local entropy inequality; nor does it assert the uniqueness of the solution. Nevertheless, as stated here, it provides enough control to establish the Stokes limit. Specifically, we do not need the strengthening of the theory recently given by Lions and Masmoudi in [22].

4.2 Fluctuations

In order to derive fluid dynamical equations from the Boltzmann equation for regimes near an absolute Maxwellian, be they the acoustic, Stokes, or incompressible Navier-Stokes equations, one needs a proper setting in which these limits hold. While L^2-based spaces are natural for these fluid equations, natural settings for the Boltzmann equation are $L\log(L)$ spaces. These different types of spaces were reconciled in the limit of small fluctuations about an absolute Maxwellian in [3].

We will consider families G_ϵ of DiPerna-Lions renormalized solutions to (4.1) such that $G_\epsilon^{\text{in}} \geq 0$ satisfies the entropy bound

$$H(G_\epsilon^{\text{in}}) \leq C^{\text{in}}\delta_\epsilon^2 \tag{4.10}$$

for some $C^{\text{in}} < \infty$ and $\delta_\epsilon > 0$ that satisfies the scaling (3.2). For this scaling the DiPerna-Lions entropy inequality (4.6) becomes

$$H(G_\epsilon(t)) + \frac{1}{\eta_\epsilon^2}\int_0^t R(G_\epsilon(s))ds \leq H(G_\epsilon^{\text{in}}) \leq C^{\text{in}}\delta_\epsilon^2\,, \tag{4.11}$$

where $\eta_\epsilon^2 = \epsilon\tau_\epsilon$. One has $\eta_\epsilon = \epsilon^{1/2}$ for the acoustic limit and $\eta_\epsilon = \epsilon$ for the Stokes limit. We will therefore assume that η_ϵ satisfies $\epsilon \leq \eta_\epsilon \leq \epsilon^{1/2}$.

The relative entropy functional H given by (2.20) has an integrand that is a nonnegative strictly convex function of G with a minimum value of 0 at $G = 1$. Thus for any G,

$$H(G) \geq 0 \quad \text{and} \quad H(G) = 0 \quad \text{if and only if } G = 1\,. \tag{4.12}$$

It thereby provides a natural measure of the proximity of G to that equilibrium. We therefore consider the families g_ϵ^{in} and g_ϵ of fluctuations about $G = 1$ defined by the relations

$$G_\epsilon^{\text{in}} = 1 + \delta_\epsilon g_\epsilon^{\text{in}}\,, \qquad G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,. \tag{4.13}$$

One easily sees that H asymptotically behaves like half the square of the L^2-norm of these fluctuations as $\epsilon \to 0$. Hence, (4.11) is consistent with these fluctuations being of order 1. Just as the relative entropy H controls the fluctuations g_ϵ, the dissipation rate R given by (2.21) controls the scaled collision integrands defined by

$$q_\epsilon = \frac{1}{\eta_\epsilon\delta_\epsilon}\big(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon\big)\,. \tag{4.14}$$

Once again, (4.11) is consistent with these scaled integrands being of order 1. The following shows more.

LEMMA 4.2 (Fluctuations Lemma) *Let* $\delta_\epsilon > 0$ *vanish with* ϵ *as* (3.2). *Let* $G_\epsilon \geq 0$ *be a family of functions in* $C([0,\infty); w\text{-}L^1(M\,dv\,dx))$ *that satisfies the entropy bound* (4.11) *with* $G_\epsilon^{\text{in}} = G_\epsilon(0)$. *Let* g_ϵ^{in}, g_ϵ, *and* q_ϵ *be given by* (4.13)–(4.14). *Define*

$$N_\epsilon = \frac{2}{3} + \frac{1}{3}G_\epsilon = 1 + \frac{1}{3}\delta_\epsilon g_\epsilon\,. \tag{4.15}$$

Then, adopting the notation $\sigma = 1 + |v|^2$, *we have the following*:

(i) *The family* g_ϵ *is bounded in* $L^\infty(dt; L^1(\sigma M\, dv\, dx))$, *relatively compact in* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$, *and relatively compact in* $w\text{-}L^1(\sigma M\, dv\, dx)$ *pointwise in* t *for each* $t \geq 0$.

(ii) *The family* q_ϵ / N_ϵ *is relatively compact in* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma\, d\mu\, dx))$.

(iii) *If* g^{in} *is a* $w\text{-}L^1(\sigma M\, dv\, dx)$ *limit point of the family* g^{in}_ϵ *as* $\epsilon \to 0$, *then* $g^{\mathrm{in}} \in L^2(M\, dv\, dx)$, *and one has*

$$\frac{1}{2} \int_{\mathbb{T}^D} \langle g^{\mathrm{in}\,2} \rangle dx \leq \liminf_{\epsilon \to 0} \frac{1}{\delta_\epsilon^2} H(G^{\mathrm{in}}_\epsilon) \leq C^{\mathrm{in}} . \tag{4.16}$$

(iv) *If* g *is a* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ *limit point of the family* g_ϵ *and* q *is jointly a* $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma\, d\mu\, dx))$ *limit point of the family* q_ϵ / N_ϵ *as* $\epsilon \to 0$, *then* $g \in L^\infty(dt; L^2(M\, dv\, dx))$, $q \in L^2(d\mu\, dx\, dt)$, *and* q *inherits the symmetries of* q_ϵ. *Moreover, for almost every* $t \geq 0$ *one has*

$$\frac{1}{2} \int_{\mathbb{T}^D} \langle g(t)^2 \rangle dx \leq \liminf_{\epsilon \to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)) , \tag{4.17}$$

while for every $t \geq 0$ *one has*

$$\frac{1}{4} \int_0^t \int_{\mathbb{T}^D} \langle\!\langle q(s)^2 \rangle\!\rangle dx\, ds \leq \liminf_{\epsilon \to 0} \frac{1}{\eta_\epsilon^2 \delta_\epsilon^2} \int_0^t R(G_\epsilon(s))\, ds . \tag{4.18}$$

(v) *The family* g_ϵ *satisfies the nonlinear estimates*

$$\int_{\mathbb{T}^D} \left\langle \frac{g_\epsilon^2}{N_\epsilon} \right\rangle (t) dx \leq 2C^{\mathrm{in}} \quad \textit{for every } t \geq 0 , \tag{4.19}$$

$$\sigma \frac{g_\epsilon^2}{N_\epsilon} = O(|\log(\delta_\epsilon)|) \quad \textit{in } L^\infty(dt; L^1(M\, dv\, dx)) \textit{ as } \epsilon \to 0 . \tag{4.20}$$

(vi) *Let* g *be as in* (iv); *then* g *has the form of an infinitesimal Maxwellian*,

$$g = \rho + u \cdot v + \theta \left(\frac{1}{2}|v|^2 - \frac{D}{2} \right) \tag{4.21}$$

for some $(\rho, u, \theta) \in L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$.

Assertion (i) is essentially proposition 3.1(1) of [3]. Assertion (ii) is proposition 3.4(1) of [3]. Assertion (iii) is essentially contained in proposition 3.1(2) of [3]. Assertion (iv) consolidates proposition 3.1(2) and proposition 3.4(2) of [3]. Assertion (v) consolidates proposition 3.2(3) and proposition 3.3 of [3]. Estimate (4.20) is the key nonlinear estimate from [3]. Assertion (vi) is proposition 3.8 of [3]. It is a consequence of assertions (i), (ii), and (v).

It should be noted that in this proposition G_ϵ is only required to satisfy the entropy bound (4.11), while δ_ϵ is only required to satisfy (3.2), $\delta_\epsilon \to 0$ as $\epsilon \to 0$, which is minimal. In particular, G_ϵ is not required to solve the Boltzmann equation in any sense.

This result shows that the initial entropy bound (4.10) on G_ϵ^{in} provides a notion of smallness that insures the family of fluctuations g_ϵ will have limit points g, and that these limit points will be in $L^\infty(dt; L^2(M\,dv\,dx))$, both of which were assumed in the formal theorems. It shows moreover that the entropy inequality (4.11) is enough to conclude that every limit point must be an infinitesimal Maxwellian, which was the conclusion of the first step in the proof of each formal theorem. We therefore employ this notion of smallness in the results below.

5 The Weak Limit Theorems

In striving to mathematically justify any fluid dynamical limit, the goal should be to obtain results that reflect the best physical understanding of the problem. In the context of justifying the Stokes and acoustic limits, we take the formal theorems of Section 3 as our best physical understanding of the problem. Their proofs simply make more precise the traditional balance arguments of kinetic theory, which date to Maxwell [23]. Our goal is therefore to remove as many assumptions as possible from these formal theorems while leaving their conclusions unchanged for as large a class of solutions as physics allows. More specifically, our goals are to

- work within the class of DiPerna-Lions renormalized solutions,
- use only global bounds on fluctuations in terms of the relative entropy H and δ_ϵ,
- make the requirements on how δ_ϵ scales with ϵ as close as possible to those required by the formal theorems,
- eliminate all the assumptions labeled (i) and (ii) in the formal theorems, and
- minimize any assumptions on the collision kernel b beyond those required for the DiPerna-Lions theory and those required by the formal theorems.

With these goals in mind, we now state our main results precisely.

5.1 Weak Stokes Limit Theorem

We state our main result for the Stokes limit first, because it comes closest to what is expected from the corresponding formal result, Proposition 3.2. Its proof will be given in Section 8.2.

THEOREM 5.1 (Weak Stokes Limit Theorem) *Let b be a collision kernel that satisfies conditions* (2.5)–(2.6) *and for which $\mathcal{L}$ satisfies the coercivity estimate* (3.21) *and the domain condition*

$$\mathrm{Dom}(\mathcal{L}) \subset \left\{\xi \in L^2(M\,dv) : \langle\langle \xi^2 \rangle\rangle < \infty\right\}. \tag{5.1}$$

Let G_ϵ^{in} be a family in the entropy class $E(M\,dv\,dx)$ that satisfies the normalizations (2.12) *and the entropy bound* (4.10) *for some $C^{\mathrm{in}} < \infty$ and $\delta_\epsilon > 0$ that* *scales with ϵ as*

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon}|\log(\delta_\epsilon)|^\beta \to 0 \quad \text{as } \epsilon \to 0 \tag{5.2}$$

for the β that arises in condition (2.6)*.*

Assume, moreover, that for some $(u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbb{R}^D \times \mathbb{R})$ the family of initial fluctuations g_ϵ^{in} given by (4.13) *satisfies*

$$(u^{\text{in}}, \theta^{\text{in}}) = \lim_{\epsilon \to 0} \left(\Pi \langle v g_\epsilon^{\text{in}} \rangle, \left\langle \left(\frac{1}{D+2}|v|^2 - 1 \right) g_\epsilon^{\text{in}} \right\rangle \right) \tag{5.3}$$

in the sense of distributions, where Π is the orthogonal projection from $L^2(dx; \mathbb{R}^D)$ onto divergence-free vector fields.

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (3.10) *that have G_ϵ^{in} as initial values. Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ given by* (4.13) *satisfies*

$$g_\epsilon \to u \cdot v + \theta \left(\frac{1}{2}|v|^2 - \frac{D+2}{2} \right) \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(\sigma M \, dv \, dx)) \,, \tag{5.4}$$

$$\frac{1}{\epsilon}\mathcal{P}^\perp g_\epsilon \to -\nabla_x u : \phi - \nabla_x \theta \cdot \psi \qquad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(\sigma M \, dv \, dx)) \,, \tag{5.5}$$

where $(u, \theta) \in C([0, \infty); L^2(dx; \mathbb{R}^D \times \mathbb{R}))$ is the unique solution of the Stokes system (1.1)–(1.2) *with ν and κ given by* (3.24) *and with initial data $(u^{\text{in}}, \theta^{\text{in}})$ obtained from* (5.3)*. In addition, one has that*

$$\begin{aligned} \Pi \langle v g_\epsilon \rangle &\to u \quad \text{in } C([0, \infty); \mathcal{D}'(\mathbb{T}^D; \mathbb{R}^D)) \,, \\ \left\langle \left(\frac{1}{D+2}|v|^2 - 1 \right) g_\epsilon \right\rangle &\to \theta \quad \text{in } C([0, \infty); w\text{-}L^1(dx; \mathbb{R})) \,, \end{aligned} \tag{5.6}$$

and that (u, θ) satisfies

$$\int_{\mathbb{T}^D} u \, dx = 0 \,, \quad \int_{\mathbb{T}^D} \theta \, dx = 0 \,. \tag{5.7}$$

This result improves upon earlier Stokes limit results in three ways. First, it establishes the heat equation. No earlier work had done this because of difficulties that arise in controlling the heat flux and in proving that local energy conservation holds in the limit. Only the results in [5] and [21] established the motion equation without assuming the local momentum conservation law is satisfied by DiPerna-Lions solutions.

Second, its scaling assumption (5.2) is better. The scaling assumption in [5] is essentially

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon^2}|\log(\delta_\epsilon)| \to 0 \quad \text{as} \quad \epsilon \to 0 \,,$$

which differs from (5.2) by at least a factor of ϵ. The assumption in [21] is essentially (5.2) but with $\beta = 1$.

Third, this result places conditions on b that are both weaker and more natural than those in [5] and [21]. The result in [5] assumed b was bounded, which excludes all the classical collision kernels except the one for Maxwell molecules. The result in [21] assumed the conditions introduced in [3]. These are extremely complicated and are not easy to verify for any classical collision kernel save the one for Maxwell molecules.

The assumptions made by this result differ from those made by the formal result (Proposition 3.2) in a number of ways. Here we make three additional assumptions regarding the collision kernel b, namely, that it satisfies the DiPerna-Lions condition (2.5), that it satisfies the bound (2.6), and that $\mathrm{Dom}(\mathcal{L})$ satisfies (5.1). These are all natural assumptions in that they are satisfied for classical hard potentials with a small deflection cutoff. The assumed bound (2.6) and the assumed coercivity estimate (3.21) are technical in nature and can be weakened at the expense of giving up some of the theorem's conclusions.

A more significant difference is that here the scaling assumption (5.2) on δ_ϵ is generally more restrictive than the one in the formal result (3.11). However, it is not much more restrictive. Indeed, when $\delta_\epsilon = \epsilon^m$ they both require the same thing, namely, that $m > 1$. In this sense, assumption (5.2) is essentially optimal. Of course, assumptions (3.11) and (5.2) become identical when $\beta = 0$, which is the case for Maxwell molecules. The difference between the two assumptions when $\beta > 0$ is important, however, because it reflects technical difficulties that likely must be overcome in any complete result for the incompressible Navier-Stokes scaling.

Of course, the biggest difference between this result and the formal result is the absence of any assumptions here regarding either the convergence or the compactness of the family of fluctuations g_ϵ. The only convergence assumption made here is that the family of initial fluctuations g_ϵ^{in} satisfies (5.3) in the sense of distributions. By assertions (i) and (iii) of the fluctuations lemma (Proposition 4.2), we may always pass to a subfamily of g_ϵ^{in} that satisfies this assumption. The question of exactly what $(u^{\mathrm{in}}, \theta^{\mathrm{in}})$ can be realized as a limit (5.3) will be addressed in Section 6.

Finally, we remark that the limits asserted in (5.4)–(5.5) capture the first and second nontrivial terms in the Chapman-Enskog expansion. What makes it remarkable is the fact that the first limit need not be strong before obtaining the second. Conditions under which both these become strong limits will be given in Section 6.1.

5.2 Weak Acoustic Limit Theorem

We now state our main result for the acoustic limit. It does not come as close to what one expects from its corresponding formal result, Theorem 3.1, as our main result for the Stokes limit did. But it is far better in this regard than our previous result [5]. Its proof will be given in Section 7.1.

THEOREM 5.2 (Weak Acoustic Limit Theorem) *Let b be a collision kernel that satisfies the conditions* (2.5)–(2.6).

Let G_ϵ^{in} be a family in the entropy class $E(M\,dv\,dx)$ that satisfies the normalizations (2.12) *and the entropy bound* (4.10) *for some $C^{\text{in}} < \infty$ and $\delta_\epsilon > 0$ that satisfies*

$$\delta_\epsilon \to 0 \quad \textit{and} \quad \frac{\delta_\epsilon}{\epsilon^{1/2}} |\log(\delta_\epsilon)|^{\beta/2} \to 0 \quad \textit{as } \epsilon \to 0 \tag{5.8}$$

for the β that arises in condition (2.6).

Assume, moreover, that for some $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ the family of fluctuations g_ϵ^{in} given by (4.13) *satisfies*

$$(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) = \lim_{\epsilon \to 0} \left(\langle g_\epsilon^{\text{in}} \rangle, \langle v g_\epsilon^{\text{in}} \rangle, \left\langle \left(\frac{1}{D}|v|^2 - 1 \right) g_\epsilon^{\text{in}} \right\rangle \right) \tag{5.9}$$

in the sense of distributions.

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (3.4) *that have G_ϵ^{in} as initial values.*

Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ given by (4.13) *satisfies*

$$g_\epsilon \to \rho + u \cdot v + \theta \left(\frac{1}{2}|v|^2 - \frac{D}{2} \right) \quad \textit{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(\sigma M\,dv\,dx))\,, \tag{5.10}$$

where $(\rho, u, \theta) \in C([0,\infty); L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ is the unique solution of the acoustic system (1.3) *with initial data $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})$ obtained from* (5.9)*. In addition, one has that*

$$\left(\langle g_\epsilon \rangle, \langle v g_\epsilon \rangle, \left\langle \left(\frac{1}{D}|v|^2 - 1 \right) g_\epsilon \right\rangle \right) \to (\rho, u, \theta) \tag{5.11}$$

in $C([0,\infty); w\text{-}L^1(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ and that (ρ, u, θ) satisfies

$$\int_{\mathbb{T}^D} \rho\,dx = 0\,, \quad \int_{\mathbb{T}^D} u\,dx = 0\,, \quad \int_{\mathbb{T}^D} \theta\,dx = 0\,. \tag{5.12}$$

This result improves upon the acoustic limit result in [5] in two ways. First, its scaling assumption (5.8) is much better. The scaling assumption in [5] is essentially

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon} |\log(\delta_\epsilon)| \to 0 \quad \text{as } \epsilon \to 0\,,$$

which differs from (5.8) by at least a factor of $\epsilon^{1/2}$. Second, it places conditions on b that are far weaker. The result in [5] assumed b was bounded, which excludes all the classical collision kernels except the one for Maxwell molecules.

The assumptions made by this result differ from those made by the formal result (Proposition 3.1) in several ways. Here we assume that the collision kernel b satisfies conditions (2.5)–(2.6). These assumptions are satisfied for classical hard potentials with a small deflection cutoff. The assumed bound (2.6) is technical in nature and can be weakened to embrace the classical soft potentials without giving up any of the theorem's conclusions.

One very big difference is that here the scaling assumption (5.8) on δ_ϵ is far from the one in the formal result (3.2). When $\delta_\epsilon = \epsilon^m$, it requires that $m > \frac{1}{2}$, whereas the formal one requires $m > 0$. This more restrictive requirement arises from the way in which we remove the local conservation law defects of the DiPerna-Lions solutions. This rather significant gap must be bridged before one can hope to fully establish the compressible Euler limit and may have to be bridged before the incompressible Euler limit is fully established.

Another big difference between this result and the formal result is the absence of any assumptions here regarding either the convergence or the compactness of the family of fluctuations g_ϵ. The only convergence assumption made here is that the family of initial fluctuations g_ϵ^{in} satisfies (5.9) in the sense of distributions. By assertions (i) and (iii) of the fluctuations lemma (Proposition 4.2), we may always pass to a subfamily of g_ϵ^{in} that satisfies this assumption. The question of exactly what $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})$ can be realized as a limit (5.9) will be addressed in Section 6.

6 The Strong Limit Theorems

In earlier studies the notion of *entropic convergence*, first introduced in [3], was used as a natural tool for obtaining strong convergence results for fluctuations about an absolute Maxwellian [4, 5, 10, 20, 21, 22]. With it, the entropy inequality can be used not only to produce bounds on the fluctuations but also to measure the distance of the fluctuations from their asymptotic state. It plays a similar role here for the Stokes and acoustic limits.

DEFINITION 6.1 Let $G_\epsilon \geq 0$ be a family in $L^1(M\,dv\,dx)$ and let g_ϵ be the corresponding fluctuations as in (3.1). The family g_ϵ is said to *converge entropically of order* δ_ϵ to $g \in L^2(M\,dv\,dx)$ if and only if

$$(6.1) \qquad g_\epsilon \to g \quad \text{in } w\text{-}L^1(M\,dv\,dx) \quad \text{and} \quad \lim_{\epsilon \to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon) = \int_{\mathbb{T}^D} \frac{1}{2} \langle g^2 \rangle \, dx\,.$$

It is clear that if g_ϵ converges entropically of order δ_ϵ, then G_ϵ satisfies the entropy bound $H(G_\epsilon) = O(\delta_\epsilon^2)$. This definition requires that the bound asserted by Proposition 4.2 on the L^2-norm of g by a lim inf be sharpened to an equality with a limit. It was shown in proposition 4.11 of [3] that entropic convergence is stronger than norm convergence in $L^1(\sigma M\,dv\,dx)$. It was shown in proposition 3.4 of [5] that given any $\delta_\epsilon > 0$ that satisfies (3.2) and any $(\rho, u, \theta) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ that satisfies the normalizations (5.12), the associated infinitesimal Maxwellian (3.7) has families of fluctuations g_ϵ that converge to it entropically of order δ_ϵ. In particular, every L^2 initial data for either the Stokes or acoustic system can be realized as a limit (even a strong limit) as in (5.3) or (5.9), respectively.

Loosely stated, the results of this section are the following: Given any L^2 initial data for either the Stokes or acoustic system and any sequence of DiPerna-Lions solutions whose initial fluctuations converge entropically to the infinitesimal Maxwellian associated with the L^2 fluid initial data, we prove that at every positive time the fluctuations of the DiPerna-Lions solutions converge entropically to the infinitesimal Maxwellian associated with the unique L^2 solution of the fluid system. The key points are that the limit of the DiPerna-Lions Boltzmann dynamics maps *onto* the L^2 fluid dynamics and that the limit is strong.

6.1 Strong Stokes Limit Theorem

In parallel with the last section, we state our result for the Stokes limit first. It turns the weak limits asserted by Theorem 5.1 into strong limits by simply assuming that the initial fluctuations converge entropically to an appropriate infinitesimal Maxwellian. Its proof will be given in Section 8.3.

THEOREM 6.2 (Strong Stokes Limit Theorem) *Let b and δ_ϵ be as in Theorem* 5.1. *Given any $(u^{\mathrm{in}}, \theta^{\mathrm{in}})$ in $L^2(dx; \mathbb{R}^D \times \mathbb{R})$ that satisfies*

$$\nabla_x \cdot u^{\mathrm{in}} = 0\,, \qquad \int_{\mathbb{T}^D} u^{\mathrm{in}}\, dx = 0\,, \qquad \int_{\mathbb{T}^D} \theta^{\mathrm{in}}\, dx = 0\,.$$

Let G_ϵ^{in} be any family in the entropy class $E(M\, dv\, dx)$ that satisfies the normalizations (2.12) *and whose family of fluctuations g_ϵ^{in} converge entropically of order δ_ϵ as $\epsilon \to 0$ to the infinitesimal Maxwellian*

$$g^{\mathrm{in}} = u^{\mathrm{in}} \cdot v + \theta^{\mathrm{in}} \left(\frac{1}{2}|v|^2 - \frac{D+2}{2} \right).$$

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (3.10) *that have G_ϵ^{in} as initial values. Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ given by* (4.13) *satisfies*

$$g_\epsilon(t) \to u(t) \cdot v + \theta(t) \left(\frac{1}{2}|v|^2 - \frac{D+2}{2} \right) \quad \textit{entropically of order } \delta_\epsilon \textit{ for every } t \geq 0\,, \tag{6.2}$$

$$g_\epsilon \to u \cdot v + \theta \left(\frac{1}{2}|v|^2 - \frac{D+2}{2} \right) \quad \textit{in } L^1_{\mathrm{loc}}(dt; L^1(\sigma M\, dv\, dx))\,, \tag{6.3}$$

$$\frac{q_\epsilon}{N_\epsilon} \to \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^{\mathsf{T}}\big) : \Phi + \nabla_x \theta \cdot \Psi \quad \textit{in } L^1_{\mathrm{loc}}(dt; L^1(\sigma\, d\mu\, dx))\,, \tag{6.4}$$

where $(u, \theta) \in C([0, \infty); L^2(dx; \mathbb{R}^D \times \mathbb{R}))$ is the unique solution of the Stokes system (1.1)–(1.2) *with ν and κ given by* (3.24) *and initial data $(u^{\mathrm{in}}, \theta^{\mathrm{in}})$.*

This result shows that every physically natural solution of the Stokes system is a strong limit of renormalized solutions of the Boltzmann equation. In contrast with earlier results [3, 4, 22], it asserts entropic convergence everywhere in time rather than almost everywhere.

6.2 Strong Acoustic Limit Theorem

We now state the corresponding result for the acoustic limit. It turns the weak limits asserted by the weak acoustic limit theorem, Proposition 5.2, into strong limits by simply assuming that the initial fluctuations converge entropically to an appropriate infinitesimal Maxwellian. Its proof will be given in Section 7.2.

THEOREM 6.3 (Strong Acoustic Limit Theorem) *Let b and δ_ϵ be as in Theorem* 5.2. *Given any* $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ *that satisfies*

$$\int_{\mathbb{T}^D} \rho^{\mathrm{in}}\, dx = 0\,, \quad \int_{\mathbb{T}^D} u^{\mathrm{in}}\, dx = 0\,, \quad \int_{\mathbb{T}^D} \theta^{\mathrm{in}}\, dx = 0\,.$$

Let G_ϵ^{in} be any family in the entropy class $E(M\,dv\,dx)$ that satisfies the normalizations (2.12) *and whose family of fluctuations g_ϵ^{in} converge entropically of order δ_ϵ as $\epsilon \to 0$ to the infinitesimal Maxwellian*

$$g^{\mathrm{in}} = \rho^{\mathrm{in}} + u^{\mathrm{in}} \cdot v + \theta^{\mathrm{in}}\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right).$$

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (3.4) *that have G_ϵ^{in} as initial values. Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ given by* (4.13) *satisfies*

$$g_\epsilon(t) \to \rho(t) + u(t) \cdot v + \theta(t)\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right) \quad \text{entropically of order } \delta_\epsilon \text{ for every } t \geq 0\,, \tag{6.5}$$

$$g_\epsilon \to \rho + u \cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right) \quad \text{in } L^1_{\mathrm{loc}}(dt; L^1(\sigma M\,dv\,dx))\,, \tag{6.6}$$

where $(\rho, u, \theta) \in C([0,\infty); L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ *is the solution of the acoustic system with initial data* $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$.

Analogous to the corresponding Stokes result, this result shows that every physically natural solution of the acoustic system is a strong limit of renormalized solutions of the Boltzmann equation. This fact is perhaps more striking in this setting because physically natural solutions of the acoustic system are generally quite weak, being an orbit of a strongly continuous unitary group over $L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$. Also like the Stokes result, one way it improves upon the earlier result in [4, 5] is by asserting entropic convergence everywhere in time rather than just almost everywhere.

7 Establishing the Acoustic Limit

Here we give proofs of both the weak and strong theorems for the acoustic limit that were stated in the previous two sections. We give these proofs first because they are more straightforward than those of any theorem leading to an incompressible model such as the Stokes system.

7.1 Proof of the Weak Acoustic Limit Theorem

Our proof of the weak acoustic limit Theorem 5.2 closely follows that of the formal acoustic limit Theorem 3.1. That proof has two steps: (1) showing that limiting fluctuations are infinitesimal Maxwellians and (2) passing to the limit in the local conservation laws. The analogue of the first step has essentially already been realized by assertion (vi) of the fluctuations lemma, Lemma 4.2.

The analogue of the second step is not as easy to realize because DiPerna-Lions solutions are not known to satisfy the momentum and energy local conservation laws. We therefore have to recover these local conservation laws in the limit. This is done by taking the velocity moments of the renormalized Boltzmann equation with respect to v and $|v|^2$ and showing that the resulting conservation defects vanish as $\epsilon \to 0$. This is the same basic strategy that we used in our previous work [5], except that the means by which we assert that the conservation defects vanish are based on the entropy dissipation rate control in the present paper, while only the energy and entropy control were used in [5].

Framework for the Proof

For the acoustic limit one considers the Boltzmann equation (4.1) with $\tau_\epsilon = 1$. One then has that $\eta_\epsilon = \epsilon^{1/2}$, whereby the fluctuations g_ϵ and q_ϵ given by (4.13)–(4.14) are

$$g_\epsilon = \frac{G_\epsilon - 1}{\delta_\epsilon} \quad \text{and} \quad q_\epsilon = \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon^{1/2} \delta_\epsilon} . \tag{7.1}$$

By assertion (i) of the fluctuations lemma, the family g_ϵ is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$. We will show that the family g_ϵ is convergent by showing that all of its subsequences converge to the same limit point.

Consider any subsequence of the family g_ϵ, still abusively denoted g_ϵ. It will also be relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$. We will show that this sequence is convergent by showing that it has a unique limit point. Indeed, let g be any $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ limit point of the sequence g_ϵ. Assertion (vi) of the fluctuations lemma states that g is an infinitesimal Maxwellian given by (4.21) for some (ρ, u, θ) that belongs to $L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$. By passing to the limit in the renormalized Boltzmann equation, we will show that (ρ, u, θ) must be a weak solution of the acoustic system (1.3) with initial data $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$ that is uniquely obtained from (5.9). Such weak solutions of the acoustic system are in $C([0, \infty); L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$. Moreover, they are uniquely determined by their initial data. The limiting infinitesimal Maxwellian g is thereby uniquely determined. However, g was an arbitrary $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ limit point of an arbitrary subsequence of the original family g_ϵ. We can then conclude that the original family g_ϵ converges to g in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ as $\epsilon \to 0$, which would establish (5.10).

Approximate Local Conservation Laws

All that remains to be done to establish (5.10) is to show that (ρ, u, θ) is the aforementioned weak solution of the acoustic system by passing to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (4.2). We choose to use the normalization of that equation given by

$$\Gamma(Z) = 3\log\left(\frac{2}{3} + \frac{1}{3}Z\right), \quad N(Z) = \frac{2}{3} + \frac{1}{3}Z\,. \tag{7.2}$$

After setting $\tau_\epsilon = 1$ and dividing by δ_ϵ, equation (4.2) becomes

$$\partial_t \gamma_\epsilon + v \cdot \nabla_x \gamma_\epsilon = \frac{1}{\epsilon^{1/2}} \iint \frac{q_\epsilon}{N_\epsilon} b(\omega, v_1 - v)\, d\omega\, M_1\, dv_1 \tag{7.3}$$

where

$$\gamma_\epsilon = \frac{3}{\delta_\epsilon}\log\left(1 + \frac{1}{3}\delta_\epsilon g_\epsilon\right), \quad N_\epsilon = 1 + \frac{1}{3}\delta_\epsilon g_\epsilon\,. \tag{7.4}$$

When the moment of the renormalized Boltzmann equation (7.3) is formally taken with respect to any $\zeta \in \mathrm{Span}\{1, v_1, \ldots, v_D, |v|^2\}$, one obtains

$$\partial_t \langle \zeta \gamma_\epsilon \rangle + \nabla_x \cdot \langle v \zeta \gamma_\epsilon \rangle = \frac{1}{\epsilon^{1/2}} \left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle. \tag{7.5}$$

This fails to be a local conservation law because the so-called conservation defect on the right-hand side is generally nonzero. The idea of the proof is to show that as $\epsilon \to 0$ this conservation defect vanishes, while the left-hand side converges to the left-hand side of the local conservation law corresponding to ζ.

It can be shown from (4.4) that every DiPerna-Lions solution of (7.3) satisfies (7.5) in the sense that for every $\chi \in C^1(\mathbb{T}^D)$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned} &\int \chi \langle \zeta \gamma_\epsilon(t_2) \rangle dx - \int \chi \langle \zeta \gamma_\epsilon(t_1) \rangle dx = \\ &\qquad \int_{t_1}^{t_2} \int \nabla_x \chi \cdot \langle v \zeta \gamma_\epsilon \rangle dx\, dt + \int_{t_1}^{t_2} \int \chi \frac{1}{\epsilon^{1/2}} \left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle dx\, dt\,. \end{aligned} \tag{7.6}$$

We analyze this equation term-by-term before passing to the limit.

Removal of the Conservation Defect

The fact that the conservation defect term on the right-hand side of (7.6) vanishes as $\epsilon \to 0$ follows from the scaling assumption (5.8), the fact χ is bounded, the fact ζ is a collision invariant, and the key new estimate

$$\frac{1}{\epsilon^{1/2}} \left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle = O\left(\frac{\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}}{\epsilon^{1/2}} \right) + O(\delta_\epsilon |\log(\delta_\epsilon)|) \tag{7.7}$$

in $L^1_{\mathrm{loc}}(dt; L^1(dx))$ as $\epsilon \to 0$. Given this estimate, the argument is as follows: The scaling assumption (5.8) directly implies that the first term on the right-hand side of (7.7) vanishes as $\epsilon \to 0$. The second term also manifestly vanishes as $\epsilon \to 0$. Therefore, because χ is bounded in L^∞, one sees that

$$\left| \int_{t_1}^{t_2} \int \chi \frac{1}{\epsilon^{1/2}} \left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle dx\, dt \right| \leq \|\chi\|_{L^\infty} \int_{t_1}^{t_2} \int \left| \frac{1}{\epsilon^{1/2}} \left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle \right| dx\, dt \to 0 \tag{7.8}$$

as $\epsilon \to 0$. All that remains is to establish estimate (7.7); but this follows from the conservation defect Theorem 9.1.

Control of the Flux Term

The flux term on the right-hand side of (7.6) contains the sequence $\langle v\zeta\gamma_\epsilon\rangle$. To control this term, first observe that when one sets $z = \frac{1}{3}\delta_\epsilon g_\epsilon$ in the elementary inequality

$$(\log(1+z))^2 \leq \frac{z^2}{1+z} \quad \text{for every } z > -1\,,$$

one obtains $\gamma_\epsilon^2 \leq g_\epsilon^2/N_\epsilon$. The nonlinear bound (4.19) of Proposition 4.2 then shows that the sequence γ_ϵ is bounded in $L^\infty(dt; L^2(M\,dv\,dx))$ with

$$\int \langle \gamma_\epsilon^2(t)\rangle dx \leq 2C^{\mathrm{in}} \quad \text{for every } t \geq 0\,.$$

Because the sequence γ_ϵ is bounded in $L^\infty(dt; L^2(M\,dv\,dx))$ and $v\zeta \in L^2(M\,dv)$, the sequence

$$\langle v\zeta\gamma_\epsilon\rangle \quad \text{is relatively compact in } w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^2(dx))\,. \tag{7.9}$$

Control of the Density Terms

The density terms on the left-hand side of (7.6) contain the sequence $\langle\zeta\gamma_\epsilon\rangle$. We use the Arzela-Ascoli theorem to establish that this sequence is relatively compact in $C([0,\infty); w\text{-}L^2(dx))$.

First, because the weak form of (7.3) implies that the time-dependent function

$$t \mapsto \int \langle Y\gamma_\epsilon(t)\rangle dx \quad \text{is continuous for every } Y \in L^\infty(M\,dv; C^1(\mathbb{T}^D))\,,$$

the above bound and a standard density argument then imply that the sequence γ_ϵ is bounded in $C([0,\infty); w\text{-}L^2(M\,dv\,dx))$. The sequence $\langle\zeta\gamma_\epsilon\rangle$ is therefore bounded in $C([0,\infty); w\text{-}L^2(dx))$ because the sequence γ_ϵ is bounded in

$$C([0,\infty); w\text{-}L^2(M\,dv\,dx)) \quad \text{and} \quad \zeta \in L^2(M\,dv)\,.$$

In particular, this implies that the sequence $\langle\zeta\gamma_\epsilon\rangle$ is equibounded.

Next, observe that the sequence $\langle\zeta\gamma_\epsilon\rangle$ is also equicontinuous because the first term on the right-hand side of (7.6) can be bounded as

$$\left| \int_{t_1}^{t_2} \int \nabla_x\chi \cdot \langle v\zeta\gamma_\epsilon\rangle dx\, dt \right| \leq \|\nabla_x\chi\|_{L^\infty} \big(\langle |v|^2\zeta^2\rangle 2C^{\mathrm{in}}\big)^{1/2} |t_2 - t_1|\,,$$

while the second term vanishes by (7.8). The Arzela-Ascoli theorem then implies the sequence

$$\langle \zeta \gamma_\epsilon \rangle \quad \text{is relatively compact in } C([0,\infty); w\text{-}L^2(dx)). \tag{7.10}$$

Passing to the Limit

Lemma 4.2(i) allows us to pass to a subsequence of the sequence g_ϵ, still abusively denoted g_ϵ, such that as $\epsilon \to 0$

$$g_\epsilon \to g \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx)) \,. \tag{7.11}$$

Now consider the associated subsequence γ_ϵ. Observe that when one sets $z = \frac{1}{3}\delta_\epsilon g_\epsilon$ in the elementary inequalities

$$0 \le z - \log(1+z) \le \frac{z^2}{1+z} \quad \text{for every } z > -1 \,,$$

one obtains

$$0 \le g_\epsilon - \gamma_\epsilon \le \frac{1}{3}\delta_\epsilon \frac{g_\epsilon^2}{N_\epsilon} \,.$$

The nonlinear estimate (4.20) of Lemma 4.2 then shows that

$$g_\epsilon - \gamma_\epsilon \to 0 \quad \text{in } L^\infty(dt; L^1(\sigma M\, dv\, dx)) \text{ as } \epsilon \to 0 \,. \tag{7.12}$$

This limit and assertion (iv) of Lemma 4.2 imply that as $\epsilon \to 0$

$$\gamma_\epsilon \to g \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^2(M\, dv\, dx)) \,. \tag{7.13}$$

Then (7.9) and (7.10) imply that as $\epsilon \to 0$

$$\begin{aligned} \langle v\zeta\gamma_\epsilon \rangle &\to \langle v\zeta g \rangle \qquad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^2(dx)) \,, \\ \langle \zeta\gamma_\epsilon \rangle &\to \langle \zeta g \rangle \qquad \text{in } C([0,\infty); w\text{-}L^2(dx)) \,. \end{aligned} \tag{7.14}$$

Moreover, because the initial fluctuations g_ϵ^{in} satisfy (5.9), one sees from (7.12) that

$$\left(\langle \gamma_\epsilon^{\text{in}} \rangle, \langle v\gamma_\epsilon^{\text{in}} \rangle, \left\langle \left(\frac{1}{D}|v|^2 - 1 \right) \gamma_\epsilon^{\text{in}} \right\rangle \right) \to (\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \tag{7.15}$$

in $w\text{-}L^2(dx)$ as $\epsilon \to 0$, where we define $\gamma_\epsilon^{\text{in}} = \gamma_\epsilon(0)$.

Now taking limits in (7.6) as $\epsilon \to 0$ leads to

$$\int \chi \langle \zeta g(t_2) \rangle dx - \int \chi \langle \zeta g(t_1) \rangle dx = \int_{t_1}^{t_2} \int \nabla_x \chi \cdot \langle v\zeta g \rangle dx\, dt \,, \tag{7.16}$$

which is the weak form of the local conservation law

$$\partial_t \langle \zeta g \rangle + \nabla_x \cdot \langle v\zeta g \rangle = 0 \,.$$

When one sets $\zeta = 1, v_1, \ldots, v_D$ and $(\frac{1}{2}|v|^2 - \frac{D}{2})$ into this equation and uses the infinitesimal Maxwellian form of g given by (4.21), the resulting system for (ρ, u, θ) coincides with the acoustic system (1.3) with initial data $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})$ given by (5.9). This establishes (5.10).

When one then sets $\zeta = 1, v_1, \ldots, v_D$ and $(\frac{1}{D}|v|^2 - 1)$ into the second line of (7.13) and combines it with (7.12), the limits (5.11) follow.

Finally, by setting $\chi \equiv 1$ and $t_1 = 0$ in (7.16) with $\zeta = 1, v_1, \ldots, v_D$, and $(\frac{1}{D}|v|^2 - 1)$, one sees that for every $t \geq 0$

$$\int \langle \zeta g(t) \rangle dx = \int \langle \zeta g(0) \rangle dx = 0\,,$$

because of the normalizations (2.12). This establishes (5.12) and concludes the proof.

7.2 Proof of the Strong Acoustic Limit Theorem

The fact that $g_\epsilon^{\text{in}} \to g^{\text{in}}$ entropically of order δ_ϵ as $\epsilon \to 0$ implies that g_ϵ^{in} satisfies (5.9). The weak acoustic limit theorem therefore implies that the family g_ϵ, which is contained in $C([0,\infty); w\text{-}L^1(M\,dv\,dx))$, converges in the topology of $w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(\sigma M\,dv\,dx))$ to the infinitesimal Maxwellian g given by (5.10), which belongs to $C([0,\infty); L^2(M\,dv\,dx))$. The definition of entropic convergence (6.1) requires us to show that for every $t > 0$ one has

$$g_\epsilon(t) \to g(t) \quad \text{in } w\text{-}L^1(M\,dv\,dx) \text{ as } \epsilon \to 0 \tag{7.17}$$

and

$$\lim_{\epsilon \to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)) = \int_{\mathbb{T}^D} \frac{1}{2} \langle g(t)^2 \rangle \, dx\,. \tag{7.18}$$

This is done by a squeezing argument.

First, because (ρ, u, θ) is the weak solution of the Cauchy problem for the acoustic system (1.3) with initial data $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})$, it satisfies the energy equality

$$\int_{\mathbb{T}^D} \left(\rho(t)^2 + |u(t)|^2 + \frac{D}{2}\theta(t)^2 \right) dx = \int_{\mathbb{T}^D} \left(\rho^{\text{in}\,2} + |u^{\text{in}}|^2 + \frac{D}{2}\theta^{\text{in}\,2} \right) dx\,.$$

Upon taking limits in the entropy inequality (4.11), using the assumed entropic convergence of the initial data, and employing the above energy equality, for every

$t > 0$ one is led to

$$\begin{aligned}
\limsup_{\epsilon\to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)) &\le \lim_{\epsilon\to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon^{\mathrm{in}}) \\
&= \frac{1}{2}\int_{\mathbb{T}^D} \langle g^{\mathrm{in}\,2}\rangle dx \\
&= \frac{1}{2}\int_{\mathbb{T}^D} \left(\rho^{\mathrm{in}\,2} + |u^{\mathrm{in}}|^2 + \frac{D}{2}\theta^{\mathrm{in}\,2}\right) dx \qquad (7.19)\\
&= \frac{1}{2}\int_{\mathbb{T}^D} \left(\rho(t)^2 + |u(t)|^2 + \frac{D}{2}\theta(t)^2\right) dx \\
&= \frac{1}{2}\int_{\mathbb{T}^D} \langle g(t)^2\rangle dx \,.
\end{aligned}$$

This inequality gives one direction of the equality (7.18).

Next, observe that (5.11) of the weak acoustic Theorem 5.2 states that for every $\zeta \in \mathrm{Span}\{1, v_1, \ldots, v_D, |v|^2\}$ one has

$$\langle \zeta \gamma_\epsilon\rangle \to \langle \zeta g\rangle \quad \text{in } C([0,\infty); w\text{-}L^2(dx)) \text{ as } \epsilon \to 0\,.$$

Let $\mathcal{P}$ be the orthogonal projection from $L^2(M\,dv)$ onto $\mathrm{Null}(\mathcal{L})$; the above convergence statement actually says that

$$\mathcal{P}\gamma_\epsilon \to g \quad \text{in } C([0,\infty); w\text{-}L^2(M\,dv\,dx)) \text{ as } \epsilon \to 0\,. \tag{7.20}$$

Now let $t > 0$ be arbitrary but fixed. The elementary inequality

$$\frac{1}{2}\left(3\log\left(1+\frac{1}{3}z\right)\right)^2 \le (1+z)\log(1+z) - z \quad \text{for every } z > -1$$

implies that for every $\epsilon > 0$ one has

$$\frac{1}{2}\int \langle \gamma_\epsilon(t)^2\rangle dx \le \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)) \le C^{\mathrm{in}}\,. \tag{7.21}$$

The family $\gamma_\epsilon(t)$ is thereby relatively compact in $w\text{-}L^2(M\,dv\,dx)$. Let $\tilde{g}$ be any $w\text{-}L^2(M\,dv\,dx)$ limit point of the family $\gamma_\epsilon(t)$. By Fatou's lemma and (7.21), one sees that

$$\frac{1}{2}\int \langle \tilde{g}^2\rangle dx \le \liminf_{\epsilon\to 0} \int \langle \gamma_\epsilon(t)^2\rangle dx \le \liminf_{\epsilon\to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t))\,. \tag{7.22}$$

On the other hand, because $\mathcal{P}\gamma_\epsilon(t) \to g(t)$ by (7.20), we have

$$\mathcal{P}\tilde{g} = \lim_{\epsilon\to 0} \mathcal{P}\gamma_\epsilon(t) = g(t) \quad \text{in } w\text{-}L^2(M\,dv\,dx)\,. \tag{7.23}$$

Hence, by employing the orthogonal decomposition $\tilde{g} = \mathcal{P}\tilde{g} + \mathcal{P}^{\perp}\tilde{g} = g(t) + \mathcal{P}^{\perp}\tilde{g}$ in combination with the bounds (7.19) and (7.22), we arrive at

$$\begin{aligned}\frac{1}{2}\int \langle g(t)^2 \rangle dx + \frac{1}{2}\int \langle (\mathcal{P}^{\perp}\tilde{g})^2 \rangle dx &\leq \liminf_{\epsilon\to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)) \\ &\leq \limsup_{\epsilon\to 0} \frac{1}{\delta_\epsilon^2} H(G_\epsilon(t)) \\ &\leq \frac{1}{2}\int \langle g(t)^2 \rangle dx\,.\end{aligned}$$

This chain of inequalities immediately implies that $\mathcal{P}^{\perp}\tilde{g} = 0$ and that the equality (7.18) is satisfied.

The fact that $\mathcal{P}^{\perp}\tilde{g} = 0$ combines with (7.23) to show that $\tilde{g} = g(t)$. The uniqueness of the limit point $\tilde{g}$ thereby implies that

$$\gamma_\epsilon(t) \to g(t) \quad \text{in } w\text{-}L^1(\sigma M\, dv\, dx) \text{ as } \epsilon \to 0\,.$$

Because $g_\epsilon - \gamma_\epsilon \to 0$ in $L^\infty(dt; L^1(\sigma M\, dv\, dx))$ as $\epsilon \to 0$, the above limit shows that (7.17) is also satisfied. Therefore we conclude that $g_\epsilon(t) \to g(t)$ entropically of order δ_ϵ for every $t \geq 0$.

Finally, by proposition 4.11 of [3] and dominated convergence, this also implies that $g_\epsilon \to g$ strongly in $L^1_{\mathrm{loc}}(dt; L^1(\sigma M\, dv\, dx))$ as announced.

8 Establishing the Stokes Limit

Here we give proofs of both the weak and strong theorems for the Stokes limit. There are three main ingredients:

- control of the fluctuations of both the phase-space densities and the collision integrands,
- removal of the local conservation law defects, and
- convergence of the momentum and heat fluxes.

The fluctuation controls needed here were developed in [3]. Some of these controls are of the same nature as those used for the acoustic limit and have already been recalled in Lemma 4.2. The additional fluctuation controls from [3] needed for the Stokes limit are gathered below in Lemma 8.1. Conservation law defects are handled in the same way as for the acoustic limit, by appealing to the conservation defect theorem, Theorem 9.1, which is stated and proved in the next section. The last item on the list above is particular to the Stokes limit and, along with the method to handle conservation defects, rests upon another new estimate on the collision integrand, whose proof is deferred until Section 10.

8.1 Control of the Stokes Fluctuations

For the Stokes limit one considers the Boltzmann equation (4.1) with $\tau_\epsilon = \epsilon$. One then has that $\eta_\epsilon = \epsilon$ whereby the fluctuations g_ϵ and q_ϵ given by (4.13)–(4.14) are

$$g_\epsilon = \frac{G_\epsilon - 1}{\delta_\epsilon}\,, \quad q_\epsilon = \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon \delta_\epsilon}\,. \tag{8.1}$$

The a priori estimates below are natural amplifications of Proposition 4.2 that result because G_ϵ are renormalized solutions of the Boltzmann equation with $\tau_\epsilon = \epsilon$.

LEMMA 8.1 (Stokes Fluctuations Lemma) *Let b be a collision kernel that satisfies conditions* (2.5)–(2.6) *and for which $\mathcal{L}$ satisfies the domain condition* (5.1).

Let $G_\epsilon \geq 0$ be a family of renormalized solutions of the scaled initial-value problem (3.10) *with initial data G_ϵ^{in} that satisfy the entropy bound* (4.10) *for some $C^{\text{in}} < \infty$ and $\delta_\epsilon > 0$ that satisfies* (3.2)*. Let g_ϵ and q_ϵ be the corresponding fluctuations and scaled collision integrands* (8.1)*.*

Let g be a w-$L^1_{\text{loc}}(dt; w\text{-}L^1(M\,dv\,dx))$ limit point of the family g_ϵ and q be jointly a w-$L^1_{\text{loc}}(dt; w\text{-}L^1(d\mu\,dx))$ limit point of the family q_ϵ/N_ϵ as $\epsilon \to 0$. Then

(i) *$g \in L^\infty(dt; L^2(M\,dv\,dx))$ and $q \in L^2(d\mu\,dx\,dt)$ satisfy*

$$v \cdot \nabla_x g = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} q b(\omega, v_1 - v) d\omega\, M_1\, dv_1\,. \tag{8.2}$$

(ii) *g has the form of an infinitesimal Maxwellian*

$$g = \rho + u \cdot v + \theta\left(\frac{1}{2}|v|^2 - \frac{D}{2}\right), \tag{8.3}$$

where $(\rho, u, \theta) \in L^\infty(dt; L^2(dx)) \cap L^2(dt; H^1(dx))$ satisfy

$$\nabla_x \cdot u = 0\,, \quad \nabla_x(\rho + \theta) = 0\,, \tag{8.4}$$

while q satisfies the relations

$$\langle\!\langle \phi q \rangle\!\rangle = \nu(\nabla_x u + (\nabla_x u)^{\mathsf{T}})\,, \quad \langle\!\langle \psi q \rangle\!\rangle = \kappa \nabla_x \theta\,, \tag{8.5}$$

and the inequality

$$\int_0^t \int_{\mathbb{T}^D} \frac{1}{2}\nu|\nabla_x u + (\nabla_x u)^{\mathsf{T}}|^2 + \kappa|\nabla_x\theta|^2 dx\,ds \leq \frac{1}{4}\int_0^t \int_{\mathbb{T}^D} \langle\!\langle q^2 \rangle\!\rangle dx\,ds \tag{8.6}$$

for every $t \geq 0$, where ϕ and ψ are given by (3.22) *while ν and κ are given by* (3.24)*.*

Assertion (i) is essentially proposition 4.1 of [3], while assertion (ii) strengthens assertion (vi) of Lemma 4.2. It consolidates propositions 4.2, 4.3, and 4.6 of [3]. The proof of each of these assertions rests on the key nonlinear estimate (4.20).

8.2 Proof of the Weak Stokes Limit Theorem 5.1

Our proof of the weak Stokes limit theorem, Theorem 5.1, closely follows that of the formal Stokes limit theorem, Theorem 3.2. That proof has six steps:

(1) showing that limiting fluctuations are infinitesimal Maxwellians,
(2) establishing the incompressibility and Boussinesq relations,
(3) evaluating the limit for moments of the form $\langle \mathcal{L}\xi g_\epsilon \rangle/\epsilon$ for every $\xi \in \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Null}(\mathcal{L})^\perp$,
(4) finding the limit of the flux terms in (3.15) that involve A and B,
(5) showing that the limiting dynamics is governed by the Stokes motion and heat equations (1.2), and
(6) finding the limit of the difference of g_ϵ from its infinitesimal Maxwellian.

The analogue of the first step has already been realized by Lemma 4.2(vi). The analogues of the remaining steps are not as easy to realize because DiPerna-Lions solutions are not known to satisfy most of the convervation laws that were used extensively in the proof of Theorem 3.2. We therefore have to recover these conservation laws in the limit. As we did for the acoustic limit, this is done by taking the velocity moments of the renormalized Boltzmann equation with respect to v and $|v|^2$ and showing that the resulting conservation defects vanish as $\epsilon \to 0$.

Framework for the Proof

The family g_ϵ is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ by assertion (i) of the fluctuations lemma. We will show that the family g_ϵ is convergent by showing that all of its subsequences converge to the same limit point.

Consider any subsequence of the family g_ϵ, still abusively denoted g_ϵ. It will also be relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$. We will show that this sequence is convergent by showing that it has a unique limit point. Indeed, let g be any $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ limit point of the sequence g_ϵ. Assertion (vi) of the fluctuations lemma states that g is an infinitesimal Maxwellian given by (4.21) for some (ρ, u, θ) that belongs to $L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$. By the analogues of steps 2 through 5 above, we will show that (ρ, u, θ) is a weak solution of the Stokes system (1.1)–(1.2) with initial data $(u^{\mathrm{in}}, \theta^{\mathrm{in}})$ that is uniquely obtained from (5.3). Because such weak solutions of the Stokes system are uniquely determined by their initial data, the limiting infinitesimal Maxwellian g is thereby uniquely determined. However, because g was an arbitrary $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ limit point of an arbitrary subsequence of the original family g_ϵ, we can conclude that the original family g_ϵ converges to g in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma M\, dv\, dx))$ as $\epsilon \to 0$, which would establish (5.4).

Approximate Local Conservation Laws

All that remains to be done is to show that (ρ, u, θ) is a weak solution of the Stokes system by passing to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (4.2). We choose to use the normalization of that equation given by

$$\Gamma(Z) = \frac{Z-1}{\frac{2}{3}+\frac{1}{3}Z}\,, \quad N(Z) = \left(\frac{2}{3}+\frac{1}{3}Z\right)^2. \tag{8.7}$$

After setting $\tau_\epsilon = \epsilon$ and dividing by δ_ϵ, equation (4.2) becomes

$$\epsilon\partial_t \frac{g_\epsilon}{N_\epsilon} + v\cdot\nabla_x \frac{g_\epsilon}{N_\epsilon} = \iint \frac{q_\epsilon}{N_\epsilon^2} b(\omega, v_1 - v)d\omega\, M_1\, dv_1 \tag{8.8}$$

where $N_\epsilon = 1 + \frac{1}{3}\delta_\epsilon g_\epsilon$.

When the moment of the renormalized Boltzmann equation (8.8) is formally taken with respect to any $\zeta \in \mathrm{Span}\{1, v_1, \ldots, v_D, |v|^2\}$, one obtains

$$\partial_t\left\langle \zeta\frac{g_\epsilon}{N_\epsilon}\right\rangle + \frac{1}{\epsilon}\nabla_x\cdot\left\langle v\zeta\frac{g_\epsilon}{N_\epsilon}\right\rangle = \frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\frac{q_\epsilon}{N_\epsilon^2}\right\rangle\!\!\right\rangle. \tag{8.9}$$

This fails to be a local conservation law because the so-called conservation defect on the right-hand side is generally nonzero. In this section we show that this conservation defect vanishes as $\epsilon \to 0$.

It can be shown from (4.4) that every DiPerna-Lions solution of (8.8) satisfies (8.9) in the sense that for every $\chi \in C^1(\mathbb{T}^D)$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned}\int \chi\left\langle \zeta\frac{g_\epsilon}{N_\epsilon}(t_2)\right\rangle dx - \int \chi\left\langle \zeta\frac{g_\epsilon}{N_\epsilon}(t_1)\right\rangle dx = \\ \int_{t_1}^{t_2}\int \frac{1}{\epsilon}\nabla_x\chi\cdot\left\langle v\zeta\frac{g_\epsilon}{N_\epsilon}\right\rangle dx\,dt + \int_{t_1}^{t_2}\int \chi\frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\frac{q_\epsilon}{N_\epsilon^2}\right\rangle\!\!\right\rangle dx\,dt\,.\end{aligned} \tag{8.10}$$

We analyze this equation term by term before passing to the limit.

Removal of the Conservation Defect

The fact that the conservation defect term on the right-hand side of (8.10) vanishes as $\epsilon \to 0$ follows from the scaling assumption (5.2), the fact χ is bounded, the fact ζ is a collision invariant, and the key new estimate

$$\frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\frac{q_\epsilon}{N_\epsilon^2}\right\rangle\!\!\right\rangle = O\left(\frac{\delta_\epsilon|\log(\delta_\epsilon)|^{\beta/2}}{\epsilon}\right) + O(\delta_\epsilon|\log(\delta_\epsilon)|) \tag{8.11}$$

in $L^1_{\mathrm{loc}}(dt; L^1(dx))$ as $\epsilon \to 0$. Given this estimate, the argument is as follows: The scaling assumption (5.2) directly implies that the first term on the right-hand side of (8.11) vanishes as $\epsilon \to 0$. The second term manifestly also vanishes as $\epsilon \to 0$. Therefore because χ is bounded in L^∞, one sees that

$$\left|\int_{t_1}^{t_2}\int \chi\frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\frac{q_\epsilon}{N_\epsilon^2}\right\rangle\!\!\right\rangle dx\,dt\right| \le \|\chi\|_{L^\infty}\int_{t_1}^{t_2}\int \left|\frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\frac{q_\epsilon}{N_\epsilon^2}\right\rangle\!\!\right\rangle\right| dx\,dt \to 0 \tag{8.12}$$

as $\epsilon \to 0$. All that remains is to establish the estimate (8.11), but this follows from Theorem 9.1.

Control of the Flux Term

In what follows, we seek to take limits in (8.10) when $\zeta = v_i$ for $i = 1, 2, \dots, D$ or $\zeta = (\frac{1}{2}|v|^2 - \frac{D+2}{2})$. With the latter choice, the flux term on right-hand side of (8.10) involves

$$\frac{1}{\epsilon}\left\langle B\frac{g_\epsilon}{N_\epsilon}\right\rangle.$$

Because there exists $\psi \in \mathrm{Dom}(\mathcal{L})$ such that $B = \mathcal{L}\psi$, this sequence is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))$. Furthermore, for any subsequence of g_ϵ such that the same subsequence of q_ϵ/N_ϵ converges to a limit point q in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma d\mu\, dx))$, one has

$$\frac{1}{\epsilon}\left\langle B\frac{g_\epsilon}{N_\epsilon}\right\rangle \to -\langle\!\langle \psi q\rangle\!\rangle \tag{8.13}$$

in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))$. Both this statement and the compactness stated before will be established later in Theorem 10.1.

With the former choice, i.e., if one chooses $\zeta = v_i$ for $i = 1, 2, \dots, D$, one consolidates the resulting equations (8.10) by taking successively as test function χ the components of a divergence-free vector field $U \in C^1(\mathbb{T}^D; \mathbb{R}^D)$. Adding the resulting equalities gives

$$\begin{aligned}\int U\cdot\left\langle v\frac{g_\epsilon}{N_\epsilon}(t_2)\right\rangle dx - \int U\cdot\left\langle v\frac{g_\epsilon}{N_\epsilon}(t_1)\right\rangle dx = \\ \int_{t_1}^{t_2}\int \frac{1}{\epsilon}\nabla_x U\cdot\left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle dx\,dt + \int_{t_1}^{t_2}\int U\cdot\frac{1}{\epsilon}\left\langle\!\!\left\langle v\frac{q_\epsilon}{N_\epsilon^2}\right\rangle\!\!\right\rangle dx\,dt\,,\end{aligned} \tag{8.14}$$

because

$$\nabla_x U\cdot\left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle = \nabla_x U\cdot\left\langle v\otimes v\frac{g_\epsilon}{N_\epsilon}\right\rangle$$

since $\nabla_x\cdot U = 0$. Again, by Theorem 10.1, the sequence

$$\frac{1}{\epsilon}\left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle$$

is relatively compact in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))$, and for any subsequence of g_ϵ such that the same subsequence of q_ϵ/N_ϵ converges to a limit point q in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma d\mu\, dx))$, one has

$$\frac{1}{\epsilon}\left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle \to -\langle\!\langle \phi q\rangle\!\rangle\,. \tag{8.15}$$

Control of the Density Terms

In what follows, we seek to take limits in (8.10) when $\zeta = v_i$ for $i = 1, 2, \dots, D$ or $\zeta = (\frac{1}{2}|v|^2 - \frac{D+2}{2})$. The corresponding density terms on the left-hand side of (8.10) are

$$\Pi\left\langle v\frac{g_\epsilon}{N_\epsilon}\right\rangle \quad \text{and} \quad \left\langle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right)\frac{g_\epsilon}{N_\epsilon}\right\rangle. \tag{8.16}$$

We use the Arzela-Ascoli theorem to establish that these sequences are relatively compact in $C([0,\infty); w\text{-}L^2(dx))$.

First, because $N_\epsilon = \frac{2}{3} + \frac{1}{3}G_\epsilon \geq \frac{2}{3}$,

$$\frac{g_\epsilon^2}{N_\epsilon^2} \leq \frac{3}{2}\frac{g_\epsilon^2}{N_\epsilon},$$

so that, by the nonlinear estimate (4.19), the sequence g_ϵ/N_ϵ is bounded in $L^\infty(dt; L^2(M\,dv\,dx))$ with

$$\int\left\langle\frac{g_\epsilon^2}{N_\epsilon^2}(t)\right\rangle dx \leq 3C^{\text{in}} \quad \text{for every } t \geq 0.$$

On the other hand, the weak form of (8.9) implies that the time-dependent function

$$t \mapsto \int\left\langle Y\frac{g_\epsilon}{N_\epsilon}(t)\right\rangle dx$$

is continuous for each $Y \in L^\infty(M\,dv; C^1(\mathbb{T}^D))$. Therefore, the bound above and a standard density argument imply that the sequence g_ϵ/N_ϵ is also bounded in $C([0,\infty); w\text{-}L^2(M\,dv\,dx))$. In particular, this implies the sequences (8.16) are equibounded.

That the sequences (8.16) are equicontinuous is less obvious than in the case of the acoustic limit. For the first sequence, this is seen from (8.14), because the second term in the right-hand side vanishes by (8.12) while the first term

$$\int_{t_1}^{t_2}\int\frac{1}{\epsilon}\nabla_x U : \left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle dx\,dt \to 0$$

as $|t_2 - t_1| \to 0$ uniformly in ϵ, by the relative compactness of the sequence

$$\frac{1}{\epsilon}\Pi\left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(dx)).$$

For the second sequence, i.e.,

$$\left\langle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right)\frac{g_\epsilon}{N_\epsilon}\right\rangle,$$

this is seen by an analogous argument bearing on (8.10) and based instead on the relative compactness of the sequence

$$\frac{1}{\epsilon}\left\langle B\frac{g_\epsilon}{N_\epsilon}\right\rangle \quad \text{in } w\text{-}L^1_{\text{loc}}(dt; w\text{-}L^1(dx)).$$

Passing to a Converging Subsequence

By assertions (i) and (ii) of Lemma 4.2, one can find a sequence $\epsilon_n \to 0$ such that

$$
\begin{aligned}
g_{\epsilon_n} &\to g \qquad \text{in } w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(\sigma M\, dv\, dx))\,,\\
\frac{q_{\epsilon_n}}{N_{\epsilon_n}} &\to q \qquad \text{in } w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(\sigma\, d\mu\, dx))\,,
\end{aligned}
\tag{8.17}
$$

Observe that because

$$
0 \le g_\epsilon - \frac{g_\epsilon}{N_\epsilon} = \frac{1}{3}\delta_\epsilon \frac{g_\epsilon^2}{N_\epsilon}\,,
$$

the nonlinear estimate (4.20) implies that

$$
g_\epsilon - \frac{g_\epsilon}{N_\epsilon} \to 0 \quad \text{in } L^\infty(dt;\, L^1(\sigma M\, dv\, dx)) \text{ as } \epsilon \to 0\,. \tag{8.18}
$$

This limit together with assertion (iv) of Lemma 4.2 imply that

$$
\frac{g_{\epsilon_n}}{N_{\epsilon_n}} \to g \quad \text{in } w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^2(M\, dv\, dx))\,.
$$

The analogous statement for $g^{\mathrm{in}}_{\epsilon_n}$ also holds.

The arguments in the last two subsections then imply that

$$
\begin{aligned}
\Pi\left\langle v\frac{g_{\epsilon_n}}{N_{\epsilon_n}}\right\rangle &\to \langle vg\rangle && \text{in } C([0,\infty);\, w\text{-}L^2(dx)),\\
\left\langle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right)\frac{g_{\epsilon_n}}{N_{\epsilon_n}}\right\rangle &\to \left\langle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right)g\right\rangle && \text{in } C([0,\infty);\, w\text{-}L^2(dx)),
\end{aligned}
\tag{8.19}
$$

as well as

$$
\begin{aligned}
\frac{1}{\epsilon_n}\left\langle A\frac{g_{\epsilon_n}}{N_{\epsilon_n}}\right\rangle &\to -\langle\!\langle \phi q\rangle\!\rangle && \text{in } w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(dx)),\\
\frac{1}{\epsilon_n}\left\langle B\frac{g_{\epsilon_n}}{N_{\epsilon_n}}\right\rangle &\to -\langle\!\langle \psi q\rangle\!\rangle && \text{in } w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(dx)).
\end{aligned}
\tag{8.20}
$$

Moreover, because g^{in}_ϵ satisfies (5.3), one sees from (8.18) that

$$
\Pi\left\langle v\frac{g^{\mathrm{in}}_{\epsilon_n}}{N^{\mathrm{in}}_{\epsilon_n}}\right\rangle \to u^{\mathrm{in}}\,, \quad \left\langle\left(\frac{1}{D+2}|v|^2 - 1\right)\frac{g^{\mathrm{in}}_{\epsilon_n}}{N^{\mathrm{in}}_{\epsilon_n}}\right\rangle \to \theta^{\mathrm{in}}\,, \tag{8.21}
$$

in $w\text{-}L^2(dx)$. In the rest of the proof, we abuse the notation g_ϵ and q_ϵ to designate the subsequences g_{ϵ_n} and q_{ϵ_n}.

Recovering the Strong Boussinesq Relation

By assertion (ii) of the Stokes fluctuations lemma, Lemma 8.1, g is of the form of a local infinitesimal Maxwellian (8.3) parametrized by its associated (fluctuation of) velocity field u, macroscopic density ρ, and temperature θ. Choosing $t_1 = 0$, $\zeta = |v|^2$, and $\chi = 1$ in (8.10) shows that

$$\int \left\langle |v|^2 \frac{g_\epsilon}{N_\epsilon}(t_2) \right\rangle dx - \int \left\langle |v|^2 \frac{g_\epsilon^{\text{in}}}{N_\epsilon^{\text{in}}} \right\rangle dx = \int_0^{t_2} \int \frac{1}{\epsilon} \left\langle\!\!\left\langle |v|^2 \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle dx\, dt\,.$$

By the conservation defect theorem, Theorem 9.1, the right-hand side of this equality vanishes with ϵ uniformly as t_2 runs through any bounded interval of time. Further, the arguments in the last three paragraphs show that the second term in the left-hand side of this equality converges to

$$\int \langle |v|^2 g^{\text{in}} \rangle dx = 0$$

because of (2.17); hence, the sequence

$$\int \left\langle |v|^2 \frac{g_\epsilon}{N_\epsilon} \right\rangle dx \to 0 \quad \text{in } L^\infty_{\text{loc}}(dt)\,.$$

Because $g_\epsilon / N_\epsilon \to g$ in w-$L^1_{\text{loc}}(dt; w\text{-}L^1(dx))$,

$$\int \left\langle |v|^2 \frac{g_\epsilon}{N_\epsilon} \right\rangle dx \to \int \langle |v|^2 g \rangle dx \quad \text{in } L^\infty_{\text{loc}}(dt)\,.$$

Hence,

$$\int \left\langle \frac{1}{D} |v|^2 g(t) \right\rangle dx = \int (\rho + \theta)(t) dx = 0$$

for almost every $t \geq 0$. Thus, for almost every $t \geq 0$, the function $x \mapsto (\rho + \theta)(x, t)$ defines an element of $L^2(dx)$ that is orthogonal to the constants; on the other hand, by (8.4), it satisfies

$$\nabla_x (\rho + \theta) = 0\,.$$

Then, a classical argument based on Fourier series shows that

$$\rho + \theta = 0 \quad \text{for almost every } (x, t) \in \mathbb{T}^D \times [0, \infty)\,. \tag{8.22}$$

By assertion (vi) of Lemma 4.2, this implies that g is in fact of the form (5.4) as predicted by the weak Stokes limit Theorem 5.1.

Recovering the Stokes Dynamics

Let $\xi \in \text{Dom}(\mathcal{L})$; because of the convergences in (8.17),

$$\frac{1}{\epsilon} \left\langle \mathcal{L}\xi \frac{g_\epsilon}{N_\epsilon} \right\rangle \to -\langle\!\langle \xi q \rangle\!\rangle = -\langle \xi v \cdot \nabla_x g \rangle \tag{8.23}$$

in w-$L^1_{\text{loc}}(dt; w\text{-}L^1(dx))$ by the moment Theorem 10.1 and assertion (i) of the Stokes fluctuations lemma. Since g is a local infinitesimal Maxwellian of the form

(5.4), the right-hand side of the convergence above can be evaluated explicitly. Doing so leads to reformulating the convergence above as

$$\frac{1}{\epsilon}\left\langle \mathcal{L}\xi \frac{g_\epsilon}{N_\epsilon}\right\rangle \to -\langle \xi A\rangle : \nabla_x u - \langle \xi B\rangle \cdot \nabla_x \theta \tag{8.24}$$

in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))$.

In particular, let ξ in (8.24) designate one of the entries of either ϕ or ψ defined in (3.22). This gives the limit of the fluxes of momentum and energy as follows:

$$\begin{aligned} &\frac{1}{\epsilon}\left\langle A\frac{g_\epsilon}{N_\epsilon}\right\rangle \to -\langle \phi\otimes A\rangle : \nabla_x u = -\nu(\nabla_x u + (\nabla_x u)^{\mathsf{T}})\,,\\ &\frac{1}{\epsilon}\left\langle B\frac{g_\epsilon}{N_\epsilon}\right\rangle \to -\langle \psi\otimes B\rangle \cdot \nabla_x \theta = -\kappa\nabla_x\theta\,, \end{aligned} \tag{8.25}$$

in $w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))$, where the viscosity ν and the thermal conductivity κ are given by (3.24).

Let $t > 0$ be arbitrary, and consider the equality (8.14) for $t_1 = 0$ and $t_2 = t$, where $U \in C^1(\mathbb{T}^D)$ designates an arbitrary divergence-free vector field. Taking limits in this equation gives, on account of (8.19), (8.21), and (8.25), that

$$\int U\cdot\langle vg(t)\rangle dx - \int U\cdot u^{\mathrm{in}}\,dx = -\nu\int_0^t\int \nabla_x U : \nabla_x u\,dx\,ds\,.$$

Likewise, consider equality (8.10) for $t_1 = 0$ and $t_2 = t$, with

$$\zeta = \left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right).$$

Taking limits in this equation gives, by using again (8.19), (8.21), and (8.25), that

$$\begin{aligned} \int \chi\left\langle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right)g(t)\right\rangle dx - \frac{D+2}{2}\int \chi\theta^{\mathrm{in}}\,dx = \\ -\kappa\int_0^t\int \nabla_x\chi\cdot\nabla_x\theta\,dx\,ds\,. \end{aligned}$$

This and the explicit form of g provided by (5.4) show that $(u,\theta) \in C([0,\infty); w\text{-}L^2(dx;\mathbb{R}^D\times\mathbb{R}))$ is the unique solution of the Cauchy problem (1.2) with initial data $(u^{\mathrm{in}},\theta^{\mathrm{in}}) \in L^2(dx;\mathbb{R}^D\times\mathbb{R})$.

Recovering the Deviation from Maxwellians

Finally, since $\mathcal{L}$ is self-adjoint on $L^2(M\,dv)$, using the tensor ϕ and the vector ψ defined in (3.22) puts the convergence (8.24) in the form

$$\left\langle(\mathcal{L}\xi)\left(\frac{1}{\epsilon}\frac{g_\epsilon}{N_\epsilon} + \nabla_x u : \phi + \nabla_x\theta\cdot\psi\right)\right\rangle \to 0$$

in $w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(dx))$. By the definition (3.22), both ϕ and ψ are entrywise orthogonal to $\mathrm{Null}(\mathcal{L})$. Hence, the convergence above means that

$$\frac{1}{\epsilon}\mathcal{P}^{\perp}\frac{g_\epsilon}{N_\epsilon} \to -\nabla_x u : \phi - \nabla_x\theta\cdot\psi$$

in $w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(dx;\, w\text{-}L^2(M\,dv)))$. Since, as recalled at the beginning of this proof, $g_\epsilon - g_\epsilon/N_\epsilon \to 0$ in $L^\infty(dt;\, L^1(\sigma M\,dv\,dx))$, the convergence above implies in turn that (5.5) holds, which concludes the proof.

8.3 Proof of the Strong Stokes Limit Theorem 6.2

The fact that $g^{\mathrm{in}}_\epsilon \to g^{\mathrm{in}}$ entropically of order δ_ϵ as $\epsilon \to 0$ implies that g^{in}_ϵ satisfies (5.3). The weak Stokes limit theorem, Theorem 5.1, therefore implies that the family g_ϵ, which is contained in $C([0,\infty);\, w\text{-}L^1(M\,dv\,dx))$, converges in the topology of $w\text{-}L^1_{\mathrm{loc}}(dt;\, w\text{-}L^1(\sigma M\,dv\,dx))$ to the infinitesimal Maxwellian g given by (5.4), which belongs to $C([0,\infty);\, L^2(M\,dv\,dx))$. The definition of entropic convergence (6.1) requires us to show that (7.17) and (7.18) are satisfied for every $t > 0$. In addition, we will show that for every $t > 0$ one has

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2\delta_\epsilon^2}\int_0^t R(G_\epsilon)ds = \frac{1}{4}\int_0^t\int_{\mathbb{T}^D}\langle\!\langle q^2\rangle\!\rangle dx\,ds\,. \tag{8.26}$$

First, because (u,θ) is the weak solution of the Cauchy problem for the Stokes system (1.1)–(1.2) with initial data $(u^{\mathrm{in}},\theta^{\mathrm{in}})$, it satisfies the energy equality

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^D}\left(|u(t)|^2 + \frac{D+2}{2}\theta(t)^2\right)dx \\ + \int_0^t\int_{\mathbb{T}^D}\left[\frac{1}{2}\nu|\nabla_x u + (\nabla_x u)^{\mathsf{T}}|^2 + \kappa|\nabla_x\theta|^2\right]dx\,ds \\ = \frac{1}{2}\int_{\mathbb{T}^D}\left(|u^{\mathrm{in}}|^2 + \frac{D+2}{2}\theta^{\mathrm{in}\,2}\right)dx\,.\end{aligned} \tag{8.27}$$

Upon taking limits in the entropy inequality (4.11), using the assumed entropic convergence of the initial data, and employing the above energy equality, for every

$t > 0$ one is led to

$$
\begin{aligned}
&\limsup_{\epsilon\to 0}\left(\frac{1}{\delta_\epsilon^2}H(G_\epsilon(t)) + \frac{1}{\epsilon^2\delta_\epsilon^2}\int_0^t R(G_\epsilon)ds\right)\\
&\quad\le \lim_{\epsilon\to 0}\frac{1}{\delta_\epsilon^2}H(G_\epsilon^{\mathrm{in}})\\
&\quad= \frac{1}{2}\int_{\mathbb{T}^D}\langle g^{\mathrm{in}\,2}\rangle dx\\
&\quad= \frac{1}{2}\int_{\mathbb{T}^D}\left(|u^{\mathrm{in}}|^2 + \frac{D+2}{2}\theta^{\mathrm{in}\,2}\right)dx\\
&\quad= \frac{1}{2}\int_{\mathbb{T}^D}\left(|u(t)|^2 + \frac{D+2}{2}\theta(t)^2\right)dx\\
&\qquad+\int_0^t\int_{\mathbb{T}^D}\left[\frac{1}{2}\nu|\nabla_x u + (\nabla_x u)^{\mathsf{T}}|^2 + \kappa|\nabla_x\theta|^2\right]dx\,ds\\
&\quad\le \frac{1}{2}\int_{\mathbb{T}^D}\langle g(t)^2\rangle dx + \frac{1}{4}\int_0^t\int_{\mathbb{T}^D}\langle\!\langle q^2\rangle\!\rangle dx\,ds\,.
\end{aligned}
\tag{8.28}
$$

Next, observe that (5.6) of Theorem 5.1 states that as $\epsilon \to 0$ one has

$$
\begin{aligned}
\Pi\langle v g_\epsilon\rangle &\to \langle v g\rangle &&\text{in } C([0,\infty);\mathcal{D}'(\mathbb{T}^D;\mathbb{R}^D))\,,\\
\left\langle\left(\frac{1}{D+2}|v|^2-1\right)g_\epsilon\right\rangle &\to \left\langle\left(\frac{1}{D+2}|v|^2-1\right)g\right\rangle &&\text{in } C([0,\infty);w\text{-}L^1(dx;\mathbb{R}))\,.
\end{aligned}
$$

On the other hand, by (7.12) and (8.18), as $\epsilon \to 0$ one then has

$$
\begin{aligned}
\Pi\langle v \gamma_\epsilon\rangle &\to \langle v g\rangle &&\text{in } C([0,\infty);w\text{-}L^2(dx))\,,\\
\left\langle\left(\frac{1}{D+2}|v|^2-1\right)\gamma_\epsilon\right\rangle &\to \left\langle\left(\frac{1}{D+2}|v|^2-1\right)g\right\rangle &&\text{in } C([0,\infty);w\text{-}L^2(dx))\,.
\end{aligned}
$$

Now let $\mathcal{P}_I$ denote the projection of $L^2(M\,dv\,dx)$ onto the incompressible fluid modes that for each $\tilde g \in L^2(M\,dv\,dx)$ is defined by

$$
\mathcal{P}_I\tilde g = \Pi\langle v\tilde g\rangle\cdot v + \left\langle\left(\frac{1}{D+2}|v|^2-1\right)\tilde g\right\rangle\left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right),
$$

where Π is the orthogonal projection of $L^2(dx;\mathbb{R}^D)$ onto divergence-free vector fields. By construction, $\mathcal{P}_I$ is an orthogonal projection over the space $L^2(M\,dv\,dx)$. The last convergence actually says that

$$
\mathcal{P}_I\gamma_\epsilon \to g \quad \text{in } C([0,\infty);w\text{-}L^2(M\,dv\,dx)) \text{ as } \epsilon\to 0\,. \tag{8.29}
$$

Now let $t > 0$ be arbitrary but fixed; the bound (7.21) then shows that the family $\gamma_\epsilon(t)$ is relatively compact in w-$L^2(M\,dv\,dx)$. Let $\tilde{g}$ be any w-$L^2(M\,dv\,dx)$ limit point of the family $\gamma_\epsilon(t)$. By Fatou's lemma and (7.21), one sees that

$$\frac{1}{2}\int \langle \tilde{g}^2\rangle dx \le \liminf_{\epsilon\to 0}\int \langle \gamma_\epsilon(t)^2\rangle dx \le \liminf_{\epsilon\to 0}\frac{1}{\delta_\epsilon^2}H(G_\epsilon(t))\,. \tag{8.30}$$

On the other hand, because $\mathcal{P}_I\gamma_\epsilon(t) \to g(t)$ by (8.29), we have

$$\mathcal{P}_I\tilde{g} = \lim_{\epsilon\to 0}\mathcal{P}_I\gamma_\epsilon(t) = g(t) \quad \text{in } w\text{-}L^2(M\,dv\,dx)\,. \tag{8.31}$$

Hence, by employing the orthogonal decomposition $\tilde{g} = \mathcal{P}_I\tilde{g} + \mathcal{P}_I^\perp\tilde{g} = g(t) + \mathcal{P}_I^\perp\tilde{g}$ in combination with the bound (8.30), we arrive at

$$\frac{1}{2}\int \langle g(t)^2\rangle dx + \frac{1}{2}\int \langle (\mathcal{P}_I^\perp\tilde{g})^2\rangle dx \le \liminf_{\epsilon\to 0}\frac{1}{\delta_\epsilon^2}H(G_\epsilon(t))\,. \tag{8.32}$$

By combining inequalities (8.28) and (8.32) with inequality (4.18) of Lemma 4.2, one can conclude that $\mathcal{P}_I^\perp\tilde{g} = 0$ and that equalities (7.18) and (8.26) are satisfied.

The fact that $\mathcal{P}_I^\perp\tilde{g} = 0$ combines with (8.31) to show that $\tilde{g} = g(t)$. The uniqueness of the limit point $\tilde{g}$ thereby implies that

$$\gamma_\epsilon(t) \to g(t) \quad \text{in } w\text{-}L^1(\sigma M\,dv\,dx) \text{ as } \epsilon \to 0\,.$$

Because $g_\epsilon - \gamma_\epsilon \to 0$ in $L^\infty(dt; L^1(\sigma M\,dv\,dx))$ as $\epsilon \to 0$, the above limit shows that (7.17) is also satisfied. Therefore we conclude that $g_\epsilon(t) \to g(t)$ entropically of order δ_ϵ for every $t \ge 0$.

Finally, by Proposition 4.11 of [3] and dominated convergence, this also implies that $g_\epsilon \to g$ strongly in $L^1_{\mathrm{loc}}(dt; L^1(\sigma M\,dv\,dx))$ as announced. Moreover, equality (8.26) and the relations (8.5) imply that

$$q = \frac{1}{2}(\nabla_x u + (\nabla_x u)^{\mathsf{T}}) : \Phi + \nabla_x\theta\cdot\Psi\,. \tag{8.33}$$

A further consequence of equation (8.26) is that the convergence (6.4) is strong in $L^1_{\mathrm{loc}}(dt; L^1(\sigma d\mu\,dx))$. The proof of this statement is given [3, pp. 738–739] and is similar to that of proposition 4.11 of [3]; namely, the fact that entropic convergence implies strong convergence in $L^1(\sigma M\,dv\,dx)$.

9 Control of the Conservation Defects

In this section we derive the conservation defect bounds (7.7) and (8.11) that were used in the acoustic and Stokes scalings, respectively, to establish momentum and energy conservation laws from the scaled Boltzmann equation in the limit as $\epsilon \to 0$. These bounds are obtained as special cases of a more general result that can be viewed as an extension of the theory of fluctuations [3]. Just as for the fluctuations lemma (Lemma 4.2), here we work in a setting in which δ_ϵ is only required to satisfy (3.2), $\delta_\epsilon \to 0$ as $\epsilon \to 0$, while G_ϵ is only required to satisfy the entropy inequality (4.11). In particular, G_ϵ is not required to solve the Boltzmann equation in any sense. We prove the following:

THEOREM 9.1 (Conservation Defect Theorem) *Let the collision kernel b satisfy the bound* (2.6) *for some $\beta \in [0,1]$. Let $\delta_\epsilon > 0$ vanish as $\epsilon \to 0$. Let $\tau_\epsilon > 0$ be bounded as $\epsilon \to 0$ and set $\eta_\epsilon = (\epsilon\tau_\epsilon)^{1/2}$. Let $G_\epsilon \geq 0$ be a family of functions in $C([0,\infty); w\text{-}L^1(M\,dv\,dx))$ that satisfies the entropy bound* (4.11). *Let g_ϵ, q_ϵ, and N_ϵ be given by* (4.13), (4.14), *and* (4.15), *respectively. Let $\zeta \in \operatorname{Span}\{1, v_1, \ldots, v_D, |v|^2\}$. Then for $n = 1$ and $n = 2$ one has the estimate*

$$\left\langle\!\!\left\langle \zeta\,\frac{q_\epsilon}{N_\epsilon^n} \right\rangle\!\!\right\rangle = O(\delta_\epsilon|\log(\delta_\epsilon)|^{\beta/2}) + O(\eta_\epsilon\delta_\epsilon|\log(\delta_\epsilon)|) \tag{9.1}$$

in $L^1_{\mathrm{loc}}(dt; L^1(dx))$ as $\epsilon \to 0$.

Remark. Given this result, the acoustic defect bound (7.7) is obtained from (9.1) by setting $\eta_\epsilon = \epsilon^{1/2}$, $n = 1$, and dividing the result by $\epsilon^{1/2}$, while the Stokes defect bound (8.11) is obtained by setting $\eta_\epsilon = \epsilon$, $n = 2$, and dividing by ϵ.

9.1 Proof of the Conservation Defect Theorem 9.1

The case $n = 1$ is treated first. The proof simply exploits the $d\mu$-symmetries (2.15) and the fact that ζ is a collision invariant to decompose the defect into three parts, each of which is then shown to vanish as $\epsilon \to 0$. The case $n = 2$ proceeds similarly, with each part being dominated by the same function that dominates the corresponding part from the $n = 1$ case. The estimates on these dominating functions are obtained from the entropy inequality (4.11) through the bound on the dissipation rate and the use of Young's inequality in the style of [3]. They are proven in the next subsection.

For the case $n = 1$, begin with the elementary decomposition

$$\left\langle\!\!\left\langle \zeta\,\frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \zeta\left(1 - \frac{1}{N_{\epsilon 1}}\right)\frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\,\frac{q_\epsilon}{N_{\epsilon 1}N_\epsilon} \right\rangle\!\!\right\rangle. \tag{9.2}$$

By using the $d\mu$-symmetries and the fact that ζ is a collision invariant, the second term on the right-hand side of (9.2) can be recast as

$$\begin{aligned}
\left\langle\!\!\left\langle \zeta\,\frac{q_\epsilon}{N_{\epsilon 1}N_\epsilon} \right\rangle\!\!\right\rangle &= \frac{1}{2}\left\langle\!\!\left\langle (\zeta_1 + \zeta)\,\frac{q_\epsilon}{N_{\epsilon 1}N_\epsilon} \right\rangle\!\!\right\rangle \\
&= \frac{1}{4}\left\langle\!\!\left\langle (\zeta_1 + \zeta)\left(\frac{1}{N_{\epsilon 1}N_\epsilon} - \frac{1}{N'_{\epsilon 1}N'_\epsilon}\right) q_\epsilon \right\rangle\!\!\right\rangle \\
&= \frac{1}{4}\left\langle\!\!\left\langle (\zeta_1 + \zeta)\,\frac{N'_{\epsilon 1}N'_\epsilon - N_{\epsilon 1}N_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle.
\end{aligned} \tag{9.3}$$

Now observe that

$$\begin{aligned}
N'_{\epsilon 1}N'_\epsilon - N_{\epsilon 1}N_\epsilon &= \frac{2}{9}\delta_\epsilon\big(g'_{\epsilon 1} + g'_\epsilon - g_{\epsilon 1} - g_\epsilon\big) + \frac{1}{9}\big(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon\big) \\
&= -\frac{2}{9}\delta_\epsilon^2\big(g'_{\epsilon 1}g'_\epsilon - g_{\epsilon 1}g_\epsilon\big) + \frac{1}{3}\eta_\epsilon\delta_\epsilon q_\epsilon\,,
\end{aligned}$$

whereby (9.3) decomposes as

$$\left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle = -\frac{1}{18}\delta_\epsilon^2 \left\langle\!\!\left\langle (\zeta_1 + \zeta) \frac{g'_{\epsilon 1} g'_\epsilon - g_{\epsilon 1} g_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle + \frac{1}{12}\eta_\epsilon \delta_\epsilon \left\langle\!\!\left\langle (\zeta_1 + \zeta) \frac{q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle . \tag{9.4}$$

By again using the $d\mu$-symmetries and the fact that ζ is a collision invariant, the integrals in the terms on the right-hand side of (9.4) can be brought into the forms

$$\begin{aligned} \left\langle\!\!\left\langle (\zeta_1 + \zeta) \frac{g'_{\epsilon 1} g'_\epsilon - g_{\epsilon 1} g_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle &= 2 \left\langle\!\!\left\langle (\zeta'_1 + \zeta') \frac{g'_{\epsilon 1} g'_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle \\ &= 4 \left\langle\!\!\left\langle \frac{\zeta' g'_{\epsilon 1} g'_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle , \\ \left\langle\!\!\left\langle (\zeta_1 + \zeta) \frac{q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle &= 2 \left\langle\!\!\left\langle \frac{\zeta q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle . \end{aligned} \tag{9.5}$$

Upon combining (9.2), (9.4), and (9.5), we arrive at the decomposition

$$\left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle = \frac{1}{3} \left\langle\!\!\left\langle \zeta \frac{\delta_\epsilon g_{\epsilon 1}}{N_{\epsilon 1} N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle - \frac{2}{9} \left\langle\!\!\left\langle \zeta' \frac{\delta_\epsilon^2 g'_{\epsilon 1} g'_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} q_\epsilon \right\rangle\!\!\right\rangle + \frac{1}{6} \left\langle\!\!\left\langle \zeta \frac{\eta_\epsilon \delta_\epsilon q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle . \tag{9.6}$$

Because for every $\zeta \in \operatorname{Span}\{1, v_1, \ldots, v_D, |v|^2\}$ there exists a constant $C < \infty$ such that $|\zeta| \le C\sigma$ where $\sigma(v) \equiv 1 + |v|^2$, the result for the case $n = 1$ will follow upon first establishing the bounds

$$\sigma \frac{\delta_\epsilon g_{\epsilon 1}}{N_{\epsilon 1} N_\epsilon} q_\epsilon = O\left(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}\right) , \tag{9.7}$$

$$\sigma' \frac{\delta_\epsilon^2 g'_{\epsilon 1} g'_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} q_\epsilon = O\left(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}\right) , \tag{9.8}$$

$$\sigma \frac{\eta_\epsilon \delta_\epsilon q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = O\left(\eta_\epsilon \delta_\epsilon |\log(\eta_\epsilon \delta_\epsilon)|\right) , \tag{9.9}$$

in $L^1_{\mathrm{loc}}(dt; L^1(d\mu\, dx))$ as $\epsilon \to 0$ and then observing that

$$\begin{aligned} \eta_\epsilon \delta_\epsilon |\log(\eta_\epsilon \delta_\epsilon)| &\le \eta_\epsilon \delta_\epsilon |\log(\eta_\epsilon)| + \eta_\epsilon \delta_\epsilon |\log(\delta_\epsilon)| \\ &= O\left(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}\right) + O\left(\eta_\epsilon \delta_\epsilon |\log(\delta_\epsilon)|\right) . \end{aligned}$$

But the bounds (9.7)–(9.9) follow directly from Lemmas 9.2, 9.3, and 9.4, respectively, which are stated and proved in the next subsection.

The case $n = 2$ follows similarly. Begin with the elementary decomposition

$$\left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon^2} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \zeta \left(1 - \frac{1}{N_{\epsilon 1}^2}\right) \frac{q_\epsilon}{N_\epsilon^2} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_{\epsilon 1}^2 N_\epsilon^2} \right\rangle\!\!\right\rangle . \tag{9.10}$$

By using the $d\mu$-symmetries and the fact that ζ is a collision invariant as we did in (9.3), the second term on the right-hand side of (9.10) can be recast as

$$\left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_{\epsilon 1}^2 N_\epsilon^2} \right\rangle\!\!\right\rangle = \frac{1}{4}\left\langle\!\!\left\langle (\zeta_1+\zeta)\frac{N'_{\epsilon 1}N'_\epsilon - N_{\epsilon 1}N_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\left(\frac{1}{N_{\epsilon 1}N_\epsilon}+\frac{1}{N'_{\epsilon 1}N'_\epsilon}\right) q_\epsilon \right\rangle\!\!\right\rangle . \tag{9.11}$$

Because the second factor in parentheses is invariant under the $d\mu$-symmetries, this factor is just carried along when we proceed as we did in going from (9.3) to (9.5). Upon combining the result with (9.10), we arrive at the analogue of (9.6), the decomposition

$$\begin{aligned}\left\langle\!\!\left\langle \zeta \frac{q_\epsilon}{N_\epsilon^2} \right\rangle\!\!\right\rangle = {}& \frac{1}{3}\left\langle\!\!\left\langle \zeta \frac{\delta_\epsilon g_{\epsilon 1}}{N_{\epsilon 1}N_\epsilon} q_\epsilon \left(\frac{1}{N_\epsilon}+\frac{1}{N_{\epsilon 1}N_\epsilon}\right)\right\rangle\!\!\right\rangle \\ & - \frac{2}{9}\left\langle\!\!\left\langle \zeta' \frac{\delta_\epsilon^2 g'_{\epsilon 1} g'_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} q_\epsilon \left(\frac{1}{N_{\epsilon 1}N_\epsilon}+\frac{1}{N'_{\epsilon 1}N'_\epsilon}\right)\right\rangle\!\!\right\rangle \\ & + \frac{1}{6}\left\langle\!\!\left\langle \zeta \frac{\eta_\epsilon \delta_\epsilon q_\epsilon^2}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} \left(\frac{1}{N_{\epsilon 1}N_\epsilon}+\frac{1}{N'_{\epsilon 1}N'_\epsilon}\right)\right\rangle\!\!\right\rangle .\end{aligned} \tag{9.12}$$

Because the factors in parentheses are bounded functions, by arguing as was done for the case $n = 1$, the result for the case $n = 2$ will also follow upon establishing (9.7)–(9.9). The proof of Proposition 9.1 will therefore be complete upon proving Lemmas 9.2, 9.3, and 9.4.

9.2 Dissipation Rate Control Lemmas

The proofs of Lemmas 9.2, 9.3, and 9.4 all crucially use the fact that the entropy inequality (4.11) implies that the dissipation rate R satisfies the bound

$$\frac{1}{\eta_\epsilon^2 \delta_\epsilon^2}\int_0^\infty R(G_\epsilon)dt \le C^{\text{in}} .$$

More specifically, following [3], these proofs use the definition of R (2.21) and of q_ϵ (4.14) to re-express this bound as

$$\frac{1}{\eta_\epsilon^2 \delta_\epsilon^2}\int_0^\infty \int \left\langle\!\!\left\langle \frac{1}{4} r\left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right) G_{\epsilon 1}G_\epsilon \right\rangle\!\!\right\rangle dx\, dt \le C^{\text{in}} , \tag{9.13}$$

where the function r is defined over $z > -1$ by

$$r(z) = z\log(1+z) . \tag{9.14}$$

The function r is strictly convex over $z > -1$.

The proofs of Lemmas 9.2 and 9.3 are each based on a delicate use of the classical Young inequality satisfied by r and its Legendre dual, r^*, namely, the inequality

$$pz \le r^*(p) + r(z) \quad \text{for every } p \in \mathbb{R} \text{ and } z > -1 .$$

Upon choosing

$$p = \frac{\eta_\epsilon \delta_\epsilon y}{\alpha} \quad \text{and} \quad z = \frac{\eta_\epsilon \delta_\epsilon |q_\epsilon|}{G_{\epsilon 1}G_\epsilon}$$

and noticing that $r(|z|) \leq r(z)$ for every $z > -1$, for every positive α and y one obtains

$$y|q_\epsilon| \leq \frac{\alpha}{\eta_\epsilon^2\delta_\epsilon^2} r^*\left(\frac{\eta_\epsilon\delta_\epsilon y}{\alpha}\right) G_{\epsilon 1} G_\epsilon + \frac{\alpha}{\eta_\epsilon^2\delta_\epsilon^2} r\left(\frac{\eta_\epsilon\delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon \,. \tag{9.15}$$

This inequality will be the starting point for the proofs of Lemmas 9.2 and 9.3.

These proofs also use the fact, recalled from [3], that r^* is superquadratic in the sense

$$r^*(\lambda p) \leq \lambda^2 r^*(p) \quad \text{for every } p > 0 \text{ and } \lambda \in [0, 1]\,. \tag{9.16}$$

LEMMA 9.2 *Let β, δ_ϵ, η_ϵ, g_ϵ, q_ϵ, and N_ϵ be as in Proposition* 9.1. *Then*

$$\sigma \frac{\delta_\epsilon g_{\epsilon 1}}{N_{\epsilon 1} N_\epsilon} q_\epsilon = O\big(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}\big) \quad \text{in } L^1_{\mathrm{loc}}(dt; L^1(d\mu\, dx)) \text{ as } \epsilon \to 0\,.$$

PROOF: For the proof of this lemma we first set

$$y = \frac{1}{\delta_\epsilon} \frac{\sigma}{N_\epsilon} \left| 1 - \frac{1}{N_{\epsilon 1}} \right| = \frac{1}{3} \frac{\sigma |g_{\epsilon 1}|}{N_{\epsilon 1} N_\epsilon}\,. \tag{9.17}$$

in (9.15) and then apply the superquadratic property (9.16) with

$$\lambda = \frac{\eta_\epsilon \delta_\epsilon |g_{\epsilon 1}|}{\alpha N_{\epsilon 1} N_\epsilon} \quad \text{and} \quad p = \frac{\sigma}{3}\,,$$

where we note that $\lambda \leq 1$ whenever $\eta_\epsilon \leq \frac{2}{9}\alpha$. This leads to

$$\begin{aligned} &\frac{1}{\delta_\epsilon} \frac{\sigma}{N_\epsilon} \left| 1 - \frac{1}{N_{\epsilon 1}} \right| |q_\epsilon| \\ &\quad \leq \frac{1}{\alpha} \frac{g_{\epsilon 1}^2}{N_{\epsilon 1}^2 N_\epsilon^2} r^*\left(\frac{\sigma}{3}\right) G_{\epsilon 1} G_\epsilon + \frac{\alpha}{\eta_\epsilon^2 \delta_\epsilon^2} r\left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon \\ &\quad \leq \frac{3^3}{2^3 \alpha} \frac{g_{\epsilon 1}^2}{N_{\epsilon 1}} r^*\left(\frac{\sigma}{3}\right) + \frac{4\alpha}{\eta_\epsilon^2 \delta_\epsilon^2} \frac{1}{4} r\left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon\,. \end{aligned} \tag{9.18}$$

By the elementary inequality

$$\left(1 + \frac{1}{2}|v_1 - v|^2\right) \leq (1 + |v_1|^2)(1 + |v|^2) = \sigma_1 \sigma\,,$$

the bound (2.6) on the collision kernel b implies

$$\int b(\omega, v_1 - v)\, d\omega \leq C_b \sigma_1^\beta \sigma^\beta\,. \tag{9.19}$$

Let $T \in [0, \infty)$ and integrate both sides of (9.18) with respect to $d\mu\, dx\, dt$ over the set $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$. By using (9.19) one then obtains

$$
\begin{aligned}
\frac{1}{\delta_\epsilon} \int_0^T \int \left\langle\!\!\left\langle \frac{\sigma}{N_\epsilon} \left| 1 - \frac{1}{N_{\epsilon 1}} \right| |q_\epsilon| \right\rangle\!\!\right\rangle dx\, dt \le \\
\frac{3^3}{2^3 \alpha} C_b \int_0^T \int \left\langle \frac{\sigma_1^\beta g_{\epsilon 1}^2}{N_{\epsilon 1}} \right\rangle dx\, dt \left\langle \sigma^\beta r^* \left(\frac{\sigma}{3} \right) \right\rangle + 4\alpha C^{\mathrm{in}} .
\end{aligned}
\tag{9.20}
$$

Interpolation between the nonlinear entropy estimates (4.19) and (4.20) shows that

$$
\int \left\langle \frac{\sigma_1^\beta g_{\epsilon 1}^2}{N_{\epsilon 1}} \right\rangle dx = \int \left\langle \frac{\sigma^\beta g_\epsilon^2}{N_\epsilon} \right\rangle dx = O(|\log(\delta_\epsilon)|^\beta)
\tag{9.21}
$$

in $L^\infty(dt)$, while $\langle \sigma^\beta r^*(\frac{1}{3}\sigma)\rangle < \infty$ because $r^*(p) = O(e^p)$ as $p \to \infty$; therefore Lemma 9.2 follows from (9.20) by optimizing over α and multiplying the result by δ_ϵ. □

LEMMA 9.3 *Let β, δ_ϵ, η_ϵ, g_ϵ, q_ϵ, and N_ϵ be as in Theorem* 9.1. *Then*

$$
\sigma' \frac{\delta_\epsilon^2 g_{\epsilon 1}' g_\epsilon'}{N_{\epsilon 1}' N_\epsilon' N_{\epsilon 1} N_\epsilon} q_\epsilon = O\big(\delta_\epsilon |\log(\delta_\epsilon)|^{\beta/2}\big) \quad \textit{in } L^1_{\mathrm{loc}}(dt; L^1(d\mu\, dx)) \textit{ as } \epsilon \to 0 .
$$

PROOF: For the proof of this lemma, we first set

$$
y = \frac{1}{9} \frac{\delta_\epsilon \sigma' |g_{\epsilon 1}'| |g_\epsilon'|}{N_{\epsilon 1}' N_\epsilon' N_{\epsilon 1} N_\epsilon}
\tag{9.22}
$$

in (9.15) and then apply the superquadratic property (9.16) with

$$
\lambda = \frac{1}{3} \frac{\eta_\epsilon \delta_\epsilon^2 |g_{\epsilon 1}'| |g_\epsilon'|}{\alpha N_{\epsilon 1}' N_\epsilon' N_{\epsilon 1} N_\epsilon} \quad \text{and} \quad p = \frac{\sigma'}{3} ,
$$

where we note that $\lambda \le 1$ whenever $\eta_\epsilon \le \frac{4}{27}\alpha$. This leads to

$$
\begin{aligned}
&\frac{1}{9} \frac{\delta_\epsilon \sigma' |g_{\epsilon 1}'| |g_\epsilon'|}{N_{\epsilon 1}' N_\epsilon' N_{\epsilon 1} N_\epsilon} |q_\epsilon| \\
&\quad \le \frac{1}{3^2 \alpha} \frac{\delta_\epsilon^2 g_{\epsilon 1}'^2 g_\epsilon'^2}{N_{\epsilon 1}'^2 N_\epsilon'^2 N_{\epsilon 1}^2 N_\epsilon^2} r^* \left(\frac{\sigma'}{3} \right) G_{\epsilon 1} G_\epsilon + \frac{\alpha}{\eta_\epsilon^2 \delta_\epsilon^2} r \left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon} \right) G_{\epsilon 1} G_\epsilon \\
&\quad \le \frac{9^2}{8^2 \alpha} \frac{g_{\epsilon 1}'^2}{N_{\epsilon 1}'^2} r^* \left(\frac{\sigma'}{3} \right) + \frac{4\alpha}{\eta_\epsilon^2 \delta_\epsilon^2} \frac{1}{4} r \left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon} \right) G_{\epsilon 1} G_\epsilon .
\end{aligned}
$$

Let $T \in [0, \infty)$ and integrate both sides of the above inequality over the set $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$ with respect to $d\mu\, dx\, dt$. By using (9.19) with primed velocities, one then obtains

$$\frac{1}{9}\int_0^T \int \left\langle\!\!\left\langle \frac{\delta_\epsilon \sigma' |g'_{\epsilon 1}||g'_\epsilon|}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} |q_\epsilon| \right\rangle\!\!\right\rangle dx\, dt \leq \frac{3^5}{2^7 \alpha} C_b \int_0^T \int \left\langle \frac{\sigma_1'^{\beta} g_{\epsilon 1}'^2}{N'_{\epsilon 1}} \right\rangle dx\, dt \left\langle \sigma'^{\beta} r^*\left(\frac{\sigma'}{3}\right)\right\rangle + 4\alpha C^{\mathrm{in}} . \tag{9.23}$$

Because (9.21) holds in $L^\infty(dt)$ while $\langle \sigma^\beta r^*(\frac{1}{3}\sigma)\rangle < \infty$, Lemma 9.3 therefore follows from (9.23) by optimizing over α and multiplying the result by δ_ϵ. □

Finally, Lemma 9.4 is an analogue for the scaled collision integrands q_ϵ of the nonlinear entropy estimate (4.20) for the fluctuations g_ϵ. It arises from the dissipation rate bound (9.13) exactly as (4.20) arises from the relative entropy bound (4.11).

LEMMA 9.4 *Let β, δ_ϵ, η_ϵ, g_ϵ, q_ϵ, and N_ϵ be as in Proposition* 9.1. *Then*

$$(\sigma + \sigma_1) \frac{q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = O(|\log(\eta_\epsilon \delta_\epsilon)|) \quad \text{in } L^1_{\mathrm{loc}}(dt; L^1(d\mu\, dx)) \text{ as } \epsilon \to 0 .$$

PROOF: The idea will be to exploit estimates that were used in the proof of proposition 3.4 of [3] to establish the nonlinear estimate (4.20) from the relative entropy bound. Specifically, that proof uses a Young-type inequality for the convex function h defined by

$$h(z) = (1+z)\log(1+z) - z \quad \text{for every } z > -1 .$$

Because h and r satisfy the elementary inequality

$$h(z) \leq r(z) \quad \text{for every } z > -1 , \tag{9.24}$$

the dissipation control (9.13) implies that

$$\frac{1}{\eta_\epsilon^2 \delta_\epsilon^2} \int_0^\infty \int \left\langle\!\!\left\langle h\left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon \right\rangle\!\!\right\rangle dx\, dt \leq 4C^{\mathrm{in}} . \tag{9.25}$$

Upon applying the argument in the proof of Proposition 3.3 of [3], one obtains the Young-type inequality

$$\Lambda(\eta_\epsilon \delta_\epsilon) \frac{1}{8} (\sigma + \sigma_1) \frac{1}{\eta_\epsilon^2 \delta_\epsilon^2} s\left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) \leq \frac{1}{\eta_\epsilon^2 \delta_\epsilon^2} h\left(\frac{\eta_\epsilon \delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) + C \exp\left(\frac{7}{16}(\sigma + \sigma_1)\right) , \tag{9.26}$$

where C is a positive constant, $s(z)$ is defined by

$$s(z) = \frac{\frac{1}{2} z^2}{1 + \frac{1}{3} z} ,$$

and $0 < \Lambda(y) < 1$ is defined implicitly for every $y \in (0, 1)$ by

$$1 - \Lambda \log(\Lambda) - (1 - \Lambda)\log(1 - \Lambda) + \Lambda \log(y) = 0 .$$

It follows from this definition that

$$\frac{1}{\Lambda(y)} = O(|\log(y)|) \quad \text{as } y \to 0\,. \tag{9.27}$$

Let $T \in [0, \infty)$ and integrate both sides of the inequality (9.26) over the set $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$ with respect to the measure

$$\frac{G_{\epsilon 1}}{N_{\epsilon 1}} \frac{G_\epsilon}{N_\epsilon}\, d\mu\, dx\, dt\,.$$

By using the bound (9.19), the fact $\langle \sigma^\beta \exp(\frac{7}{16}\sigma)\rangle < \infty$, and the asymptotics (9.27), one then obtains

$$\int_0^T \int \left\langle\!\!\left\langle (\sigma + \sigma_1) \frac{q_\epsilon^2}{G_{\epsilon 1}G_\epsilon + G'_{\epsilon 1}G'_\epsilon} \frac{1}{N_{\epsilon 1}N_\epsilon} \right\rangle\!\!\right\rangle dx\, dt = O(|\log(\eta_\epsilon \delta_\epsilon)|) \tag{9.28}$$

as $\epsilon \to 0$. Upon using the $d\mu$-symmetries of the collision integrand and the elementary inequality

$$G_{\epsilon 1}G_\epsilon + G'_{\epsilon 1}G'_\epsilon \le 3^2(N_{\epsilon 1}N_\epsilon + N'_{\epsilon 1}N'_\epsilon)\,,$$

we see that the left-hand side of (9.28) satisfies

$$\begin{aligned}
2\int_0^T &\int \left\langle\!\!\left\langle (\sigma + \sigma_1) \frac{q_\epsilon^2}{G_{\epsilon 1}G_\epsilon + G'_{\epsilon 1}G'_\epsilon} \frac{1}{N_{\epsilon 1}N_\epsilon} \right\rangle\!\!\right\rangle dx\, dt \\
&= \int_0^T \int \left\langle\!\!\left\langle (\sigma + \sigma_1) \frac{q_\epsilon^2}{G_{\epsilon 1}G_\epsilon + G'_{\epsilon 1}G'_\epsilon} \left(\frac{1}{N_{\epsilon 1}N_\epsilon} + \frac{1}{N'_{\epsilon 1}N'_\epsilon} \right) \right\rangle\!\!\right\rangle dx\, dt \\
&= \int_0^T \int \left\langle\!\!\left\langle (\sigma + \sigma_1) \frac{q_\epsilon^2}{G_{\epsilon 1}G_\epsilon + G'_{\epsilon 1}G'_\epsilon} \frac{N'_{\epsilon 1}N'_\epsilon + N_{\epsilon 1}N_\epsilon}{N_{\epsilon 1}N_\epsilon N'_{\epsilon 1}N'_\epsilon} \right\rangle\!\!\right\rangle dx\, dt \\
&\ge \frac{1}{3^2} \int_0^T \int \left\langle\!\!\left\langle (\sigma + \sigma_1) \frac{q_\epsilon^2}{N_{\epsilon 1}N_\epsilon N'_{\epsilon 1}N'_\epsilon} \right\rangle\!\!\right\rangle dx\, dt\,.
\end{aligned}$$

The announced estimate therefore follows from (9.28). □

10 Control of the Stokes Fluxes

In order to control the renormalized momentum and heat fluxes, the proof of the weak Stokes limit given in Section 8.2 asserted the limits (8.15) and (8.13). Moreover, it asserted we can control every moment of the form

$$\frac{1}{\epsilon}\left\langle (\mathcal{L}\xi) \frac{g_\epsilon}{N_\epsilon} \right\rangle$$

for any $\xi \in \mathrm{Dom}(\mathcal{L})$. This assertion was crucial in the proof of (5.5). The proof of these assertions was deferred to this section. They follow from the following:

THEOREM 10.1 (Moment Theorem) *Let the collision kernel b satisfy the bound* (2.6) *for some $\beta \in [0, 1]$ and let* $\mathrm{Dom}(\mathcal{L})$ *satisfy condition* (5.1). *Consider the Stokes scaling $\tau_\epsilon = \epsilon$, $\eta_\epsilon = \epsilon$, and $\delta_\epsilon > 0$ satisfying* (5.2) *as $\epsilon \to 0$.*

Let $G_\epsilon \geq 0$ be a family of functions in $C([0,\infty); w\text{-}L^1(M\,dv\,dx))$ that satisfies the entropy bound (4.11). *Let g_ϵ, q_ϵ, and N_ϵ be given by* (4.13), (4.14), *and* (4.15). *Then for every $\xi \in \mathrm{Dom}(\mathcal{L})$ one has that*

$$\frac{1}{\epsilon}\left\langle (\mathcal{L}\xi)\frac{g_\epsilon}{N_\epsilon}\right\rangle \quad \textit{is relatively compact in } w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(dx))\,. \tag{10.1}$$

If in addition there exists $q \in L^2(d\mu\,dx\,dt)$ such that

$$\frac{q_\epsilon}{N_\epsilon} \to q \quad \textit{in } w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma\,d\mu\,dx)) \textit{ as } \epsilon \to 0\,, \tag{10.2}$$

then for every $\xi \in \mathrm{Dom}(\mathcal{L})$ one has the limit

$$\frac{1}{\epsilon}\left\langle (\mathcal{L}\xi)\frac{g_\epsilon}{N_\epsilon}\right\rangle \to -\langle\!\langle \xi q\rangle\!\rangle \quad \textit{in } w\text{-}L^1_{\mathrm{loc}}(dt; L^1(dx)) \textit{ as } \epsilon \to 0\,. \tag{10.3}$$

The proof of this theorem is given in Section 10.2 after establishing two preliminary lemmas.

10.1 More Fluctuation Lemmas

The proof of Proposition 10.1 rests on two lemmas. The first gives an elementary L^2-like bound on the scaled collision integrands q_ϵ. The second is the key estimate. It enables one to control the linear part of the scaled collision integrands. Both these lemmas can be viewed as extensions of the theory of fluctuations developed in [3]. We again work in the general setting of Proposition 9.1; namely, δ_ϵ is only required to satisfy (3.2), $\delta_\epsilon \to 0$ as $\epsilon \to 0$, while G_ϵ is only required to satisfy the entropy inequality (4.11).

We first give the analogue for the scaled collision integrands q_ϵ of the nonlinear bound (4.19) for the fluctuations g_ϵ.

LEMMA 10.2 *Let β, δ_ϵ, η_ϵ, g_ϵ, q_ϵ, and N_ϵ be as in Proposition* 9.1. *Then*

$$\int_0^\infty \int \left\langle\!\!\left\langle \frac{q_\epsilon^2}{N_\epsilon N_{\epsilon 1} N'_\epsilon N'_{\epsilon 1}} \right\rangle\!\!\right\rangle dx\,dt \leq \frac{3^4}{2} C^{\mathrm{in}}\,. \tag{10.4}$$

PROOF: Start from the dissipation rate bound (9.13) and use the elementary inequality

$$\frac{z^2}{1+\frac{1}{2}z} \leq r(z) \quad \text{for every } z \in (-1,\infty) \tag{10.5}$$

to obtain the bound

$$\frac{1}{2}\int_0^\infty \int \left\langle\!\!\left\langle \frac{q_\epsilon^2}{G'_{\epsilon 1}G'_\epsilon + G_{\epsilon 1}G_\epsilon} \right\rangle\!\!\right\rangle dx\,dt \leq C^{\mathrm{in}}\,.$$

When this is combined with the elementary arithmetic inequality

$$G'_{\epsilon 1}G'_\epsilon + G_{\epsilon 1}G_\epsilon \leq \frac{3^4}{2^2} N_\epsilon N_{\epsilon 1} N'_\epsilon N'_{\epsilon 1}\,,$$

it leads to the announced bound. □

We now give the key new estimate. It gives an L^2 control that is much better than the L^1 control of estimates in earlier works. It is essential in controlling the heat flux in the Stokes limit. Indeed, it supplants all the Stokes limit estimates in section 5 of [3]. It can also be used to give a much more elegant proof of assertion (vi) of the fluctuation lemma (Lemma 4.2) than is found in [3].

LEMMA 10.3 *Let β, δ_ϵ, η_ϵ, g_ϵ, q_ϵ, and N_ϵ be as in Theorem* 9.1. *One has the bound*

$$\frac{q_\epsilon}{N'_{\epsilon1}N'_\epsilon N_{\epsilon1}N_\epsilon} - \frac{1}{\eta_\epsilon}\left(\frac{g'_{\epsilon1}}{N'_{\epsilon1}} + \frac{g'_\epsilon}{N'_\epsilon} - \frac{g_{\epsilon1}}{N_{\epsilon1}} - \frac{g_\epsilon}{N_\epsilon}\right) = O\left(\frac{\delta_\epsilon|\log(\delta_\epsilon)|^\beta}{\eta_\epsilon}\right)$$

in $L^\infty(dt; L^1(dx; L^2(d\mu)))$ as $\epsilon \to 0$.

PROOF: The key to the argument is the computation

$$\begin{aligned}
&\frac{1}{\eta_\epsilon\delta_\epsilon}\frac{G'_{\epsilon1}G'_\epsilon - G_{\epsilon1}G_\epsilon}{N'_{\epsilon1}N'_\epsilon N_{\epsilon1}N_\epsilon} - \frac{1}{\eta_\epsilon}\left(\frac{g'_{\epsilon1}}{N'_{\epsilon1}} + \frac{g'_\epsilon}{N'_\epsilon} - \frac{g_{\epsilon1}}{N_{\epsilon1}} - \frac{g_\epsilon}{N_\epsilon}\right)\\
&\quad= \frac{1}{\eta_\epsilon}\left(\frac{g'_{\epsilon1} + g'_\epsilon - g_{\epsilon1} - g_\epsilon}{N'_{\epsilon1}N'_\epsilon N_{\epsilon1}N_\epsilon} - \frac{g'_{\epsilon1} + g'_\epsilon}{N'_{\epsilon1}N'_\epsilon} + \frac{g_{\epsilon1} + g_\epsilon}{N_{\epsilon1}N_\epsilon}\right)\\
&\qquad+ \frac{1}{\eta_\epsilon}\left(\frac{g'_{\epsilon1} + g'_\epsilon}{N'_{\epsilon1}N'_\epsilon} - \frac{g'_{\epsilon1}}{N'_{\epsilon1}} - \frac{g'_\epsilon}{N'_\epsilon}\right) - \frac{1}{\eta_\epsilon}\left(\frac{g_{\epsilon1} + g_\epsilon}{N_{\epsilon1}N_\epsilon} - \frac{g_{\epsilon1}}{N_{\epsilon1}} - \frac{g_\epsilon}{N_\epsilon}\right)\\
&\qquad+ \frac{\delta_\epsilon}{\eta_\epsilon}\frac{g'_{\epsilon1}g'_\epsilon - g_{\epsilon1}g_\epsilon}{N'_{\epsilon1}N'_\epsilon N_{\epsilon1}N_\epsilon}\\
&\quad= \left(\frac{1}{9}\frac{\delta_\epsilon(g_{\epsilon1} + g_\epsilon)}{N_{\epsilon1}N_\epsilon}\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g'_{\epsilon1}g'_\epsilon}{N'_{\epsilon1}N'_\epsilon} - \frac{1}{9}\frac{\delta_\epsilon(g'_{\epsilon1} + g'_\epsilon)}{N'_{\epsilon1}N'_\epsilon}\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g_{\epsilon1}g_\epsilon}{N_{\epsilon1}N_\epsilon}\right)\\
&\qquad- \frac{2}{3}\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g'_{\epsilon1}g'_\epsilon}{N'_{\epsilon1}N'_\epsilon} + \frac{2}{3}\frac{\delta_\epsilon}{\eta_\epsilon} + \frac{1}{N_{\epsilon1}N_\epsilon}\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g'_{\epsilon1}g'_\epsilon}{N'_{\epsilon1}N'_\epsilon} - \frac{1}{N'_{\epsilon1}N'_\epsilon}\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g_{\epsilon1}g_\epsilon}{N_{\epsilon1}N_\epsilon}\,,
\end{aligned}$$

which results in the identity

$$\begin{aligned}
&\frac{q_\epsilon}{N'_{\epsilon1}N'_\epsilon N_{\epsilon1}N_\epsilon} - \frac{1}{\eta_\epsilon}\left(\frac{g'_{\epsilon1}}{N'_{\epsilon1}} + \frac{g'_\epsilon}{N'_\epsilon} - \frac{g_{\epsilon1}}{N_{\epsilon1}} - \frac{g_\epsilon}{N_\epsilon}\right)\\
&\quad= \frac{1}{3}\left(\frac{1}{N_{\epsilon1}N_\epsilon} + \frac{1}{N_{\epsilon1}} + \frac{1}{N_\epsilon} - 2\right)\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g'_{\epsilon1}g'_\epsilon}{N'_{\epsilon1}N'_\epsilon}\\
&\qquad- \frac{1}{3}\left(\frac{1}{N'_{\epsilon1}N'_\epsilon} + \frac{1}{N'_{\epsilon1}} + \frac{1}{N'_\epsilon} - 2\right)\frac{\delta_\epsilon}{\eta_\epsilon}\frac{g_{\epsilon1}g_\epsilon}{N_{\epsilon1}N_\epsilon}\,.
\end{aligned}\tag{10.6}$$

However, the bound (9.19) on the collision kernel b yields

$$\int \left\langle\!\!\left\langle \left(\frac{g'_{\epsilon1}g'_\epsilon}{N'_{\epsilon1}N'_\epsilon}\right)^2 \right\rangle\!\!\right\rangle^{\frac{1}{2}} dx = \int \left\langle\!\!\left\langle \left(\frac{g_{\epsilon1}g_\epsilon}{N_{\epsilon1}N_\epsilon}\right)^2 \right\rangle\!\!\right\rangle^{\frac{1}{2}} dx \le C_b \int \left\langle \frac{\sigma^\beta g_\epsilon^2}{N_\epsilon^2} \right\rangle dx\,.$$

This and (9.21) imply that

$$\frac{g'_{\epsilon1}g'_\epsilon}{N'_{\epsilon1}N'_\epsilon} = O\big(|\log(\delta_\epsilon)|^\beta\big)\,, \qquad \frac{g_{\epsilon1}g_\epsilon}{N_{\epsilon1}N_\epsilon} = O\big(|\log(\delta_\epsilon)|^\beta\big)\,,$$

in $L^\infty(dt; L^1(dx; L^2(d\mu)))$ as $\epsilon \to 0$. On the other hand,

$$\left|\frac{1}{N_{\epsilon 1}N_\epsilon} + \frac{1}{N_{\epsilon 1}} + \frac{1}{N_\epsilon} - 2\right| \le \frac{13}{4}\,, \qquad \left|\frac{1}{N'_{\epsilon 1}N'_\epsilon} + \frac{1}{N'_{\epsilon 1}} + \frac{1}{N'_\epsilon} - 2\right| \le \frac{13}{4}\,.$$

Bringing these last two estimates in (10.6) concludes the proof of Lemma 10.3. □

10.2 Proof of the Moment Theorem

With the above preliminary results at our disposal, we give the proof of Theorem 10.1.

PROOF OF THEOREM 10.1: Let $\xi \in \mathrm{Dom}(\mathcal{L})$. The domain condition (5.1) then implies $\xi \in L^2(d\mu)$. One also has $g_\epsilon/N_\epsilon \in L^\infty(d\mu\, dx\, dt)$ for every ϵ because $|g_\epsilon/N_\epsilon| < 3/\epsilon$. Thus, by applying the $d\mu$-symmetries (2.15), one obtains the identity

$$\left\langle\!\!\left\langle \xi\left(\frac{g'_{\epsilon 1}}{N'_{\epsilon 1}} + \frac{g'_\epsilon}{N'_\epsilon} - \frac{g_{\epsilon 1}}{N_{\epsilon 1}} - \frac{g_\epsilon}{N_\epsilon}\right)\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle (\xi'_1 + \xi' - \xi_1 - \xi)\frac{g_\epsilon}{N_\epsilon}\right\rangle\!\!\right\rangle = -\left\langle (\mathcal{L}\xi)\frac{g_\epsilon}{N_\epsilon}\right\rangle.$$

This identity and Lemma 10.3 applied with the Stokes scaling (i.e., with $\eta_\epsilon = \epsilon$ and δ_ϵ satisfying (5.2)) imply that

$$\frac{1}{\epsilon}\left\langle (\mathcal{L}\xi)\frac{g_\epsilon}{N_\epsilon}\right\rangle + \left\langle\!\!\left\langle \xi\frac{q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\right\rangle\!\!\right\rangle \to 0 \tag{10.7}$$

in $L^\infty(dt; L^1(dx))$ as $\epsilon \to 0$. On the other hand, Lemma 10.2 implies that the family

$$\frac{q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} \quad \text{is relatively compact in } w\text{-}L^2(d\mu\, dx\, dt)\,, \tag{10.8}$$

whereby the family

$$\left\langle\!\!\left\langle \xi\frac{q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\right\rangle\!\!\right\rangle \quad \text{is relatively compact in } w\text{-}L^2(dx\, dt)\,. \tag{10.9}$$

This fact combined with (10.7) yields (10.1).

Finally, to prove (10.3), first observe that

$$0 \le \frac{1}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}} \le \frac{3^3}{2^3} \quad \text{and} \quad \frac{1}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}} \to 1 \quad d\mu\text{-almost everywhere}\,.$$

With the weak L^1 convergence (10.2) of q_ϵ/N_ϵ, these last two properties and the product limit theorem of [3] imply that

$$\frac{q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} \to q \quad \text{in } w\text{-}L^1_{\mathrm{loc}}(dt; w\text{-}L^1(\sigma\, d\mu\, dx))\,. \tag{10.10}$$

By (10.8), the convergence (10.10) holds in w-$L^2(d\mu\, dx\, dt)$. In particular, this implies that

$$\left\langle\!\!\left\langle \xi\frac{q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\right\rangle\!\!\right\rangle \to \langle\!\langle \xi q\rangle\!\rangle \quad \text{in } w\text{-}L^2(dx\, dt) \text{ as } \epsilon \to 0\,.$$

When this limit is combined with (10.7), it proves (10.3). □

11 Concluding Remarks

Perhaps the hardest problem left open in this paper is the mathematical justification for the acoustic limit for all scalings for which its formal derivation holds, that is to say, whenever the size of the fluctuations of number density δ_ϵ vanish in the limit as $\epsilon \to 0$. Although the role of the entropy dissipation is by now fairly well understood in the derivation of hydrodynamic equations of parabolic type (as shown by the essentially complete derivation of the Stokes limit), this role remains to be fully assessed in hydrodynamic limits leading to hyperbolic equations. The acoustic limit might be a possible starting point in this direction.

A somewhat less difficult open problem is the full mathematical justification for the incompressible Navier-Stokes limit. This is the major goal of the program proposed in [3] to tie the global theory of DiPerna-Lions for the Boltzmann equation to the global theory of Leray for the Navier-Stokes system. One step in that direction would be to remove the logarithmic term that appears in the Stokes scaling condition (5.2), thereby justifying the Stokes limit for all scalings for which its formal derivation holds, namely, whenever δ_ϵ satisfies (3.11). Of course, this gap will certainly be bridged by any full justification of the incompressible Navier-Stokes limit.

Otherwise, it would be interesting to know how the acoustic and Stokes limits can be unified in the domain in which they are both known to be valid, that is, when δ_ϵ satisfies the Stokes scaling condition (5.2). Based on formal arguments, one expects the fluid fluctuations to be governed by what might be called the *compressible Stokes system*, which is the linearization about a homogeneous state of the compressible Navier-Stokes system. After a suitable choice of units, in this model the fluid fluctuations $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ satisfy

$$
\begin{aligned}
&\partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon = 0\,,\\
&\partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) = \epsilon\nu\nabla_x \cdot \left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^{\mathsf{T}} - \frac{2}{D}\nabla_x \cdot u_\epsilon I\right],\\
&\frac{D}{2}\partial_t \theta_\epsilon + \nabla_x \cdot u_\epsilon = \epsilon\kappa\,\Delta_x \theta_\epsilon\,,
\end{aligned}
\tag{11.1}
$$

with initial data $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$. Notice that unlike the Stokes and acoustic systems, in this system the Knudsen number ϵ appears explicitly, whereby the solutions also depend on ϵ even though the initial data does not. It is a relatively easy exercise to show that solutions of this system converge to those of the acoustic system (1.3) with the same initial data as $\epsilon \to 0$. It is only a bit harder to show that on time scales of order $1/\epsilon$, solutions of this system converge (generally weakly) to those of the Stokes system (1.1)–(1.2) with initial data

$$\left(\Pi u^{\mathrm{in}}, \frac{D}{D+2}\theta^{\mathrm{in}} - \frac{2}{D+2}\rho^{\mathrm{in}}\right) \quad \text{as } \epsilon \to 0\,.$$

It is therefore natural to ask whether this system governs the asymptotics of solutions of the Boltzmann equation (3.4) uniformly over time scales of $o(\epsilon^{-2})$ or longer.

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