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Velocity averaging in L^1 for the transport equation

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Abstract A new result of L^1-compactness for velocity averages of solutions to the transport equation is stated and proved in this Note. This result, proved by a new interpolation argument, extends to the case of any space dimension Lemma 8 of Golse–Lions–Perthame–Sentis [J. Funct. Anal. 76 (1988) 110–125], proved there in space dimension 1 only. This is a key argument in the proof of the hydrodynamic limits of the Boltzmann or BGK equations to the incompressible Euler or Navier–Stokes equations. ***To cite this article: F. Golse, L. Saint-Raymond, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 557–562.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Moyennisation en vitesse dans L^1 pour l'équation de transport

Résumé On énonce et démontre dans cette Note un nouveau résultat de compacité dans L^1 pour les moyennes en vitesse des solutions de l'équation de transport. Ce résultat, établi par un nouvel argument d'interpolation, généralise à toute dimension d'espace le Lemme 8 de Golse–Lions–Perthame–Sentis [J. Funct. Anal. 76 (1988) 110–125], qui n'était jusqu'ici connu qu'en dimension 1 d'espace. C'est un point crucial dans les preuves des limites hydrodynamiques des équations de Boltzmann ou de BGK vers les équations de Navier–Stokes. ***Pour citer cet article : F. Golse, L. Saint-Raymond, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 557–562.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

La compacité par moyennisation en vitesse des solutions d'équations de transport a été établie dans [8] — des résultats partiels de même nature ayant été obtenus indépendamment dans [1]. Pour une fonction $f \equiv f(x,v)$ telle que f et $v \cdot \mathrm{grad}_x f$ appartiennent à $L^p(\mathbf{R}^D \times \mathbf{R}^D)$ pour $1 < p < +\infty$, la régularité (au sens de l'appartenance à un espace de Sobolev ou de Besov correspondant à des dérivées fractionnaires d'ordre strictement positif) des moyennes en vitesse de la forme

$$\rho(x) := \int f(x,v)\psi(v)\,\mathrm{d}v \quad \text{pour } \psi \in L^\infty_{\mathrm{comp}}\big(\mathbf{R}^D\big) \tag{1}$$

a été étudiée dans [7] avec des résultats plus précis obtenus dans [6] et [5].

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Lorsque $p = 1$, un contre-exemple de [7] (pp. 123–124) montre que, pour une suite $f_n \equiv f_n(x, v)$, l'hypothèse « f_n et $v \cdot \mathrm{grad}_x f_n$ bornées dans $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$ » n'implique même pas en général que la suite des moments ρ_n lui étant associée par la formule (1) est relativement faiblement compacte dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D)$. Le Théorème 3 de [7] montre que l'hypothèse « f_n et $v \cdot \mathrm{grad}_x f_n$ relativement faiblement compactes dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$ » implique que la suite ρ_n est relativement compacte dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D)$.

Le résultat principal de cette Note montre que les hypothèses de ce théorème peuvent être considérablement affaiblies, et en particulier qu'il suffit d'éliminer la possibilité de concentrations pour la suite f_n *en la variable v seulement.*

THÉORÈME 0.1. – *Soit $f_\varepsilon \equiv f_\varepsilon(x, v)$ une famille bornée de $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$ localement équiintégrable en v et telle que la famille $v \cdot \mathrm{grad}_x f_\varepsilon$ est bornée dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$. Alors*

- *la famille f_ε est localement équiintégrable (en les deux variables x et v) sur $\mathbf{R}^D \times \mathbf{R}^D$;*
- *pour tout $\psi \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbf{R}^D)$ la famille ρ_ε associée à la famille f_ε par la formule (1) est relativement compacte dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D)$.*

La notion d'« équiintégrabilité en v » fait l'objet de la Définition 2.1 de la version anglaise. Par exemple, si la famille f_ε est bornée dans $\mathrm{L}^1_x(\mathrm{L}^p_v)$ avec $p > 1$, elle est localement équiintégrable en v.

Bien que de nature purement locale, ce résultat repose (a) sur la propriété de dispersion de l'opérateur de transport énoncée dans la formule (6) ci-dessous et (b) sur une formule d'interpolation – voir (7) dans la version anglaise – qui n'est pas sans évoquer la définition d'« espaces de traces » comme interpolés d'un espace de Hilbert H et du domaine $D(A)$ du générateur non borné A d'un semi-groupe sur H : voir [11].

Une version partielle du Théorème 0.1, à savoir la relative compacité faible de ρ_ε dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D)$ sous l'hypothèse « f_ε bornée dans $\mathrm{L}^1_x(\mathrm{L}^\infty_v)$ et $v \cdot \mathrm{grad}_x f_\varepsilon$ bornée dans $\mathrm{L}^1_{x,v}$ » avait été établie dans [14]. Un analogue de ce théorème pour le cas d'évolution est un point crucial dans l'obtention de la limite hydrodynamique des équations de BGK [15] ou de Boltzmann [9] vers les équations de Navier–Stokes.

1. A counterexample

Regularization by velocity averaging is a property of the transport equation discovered in [8] – partial results of a similar nature had been obtained independently in [1]. Theorem 1 of [8] implies in particular that, for any $p \in]1, +\infty[$ and any bounded family $f_\varepsilon \equiv f_\varepsilon(x, v)$ of $\mathrm{L}^p(\mathbf{R}^D \times \mathbf{R}^D)$ such that the family $v \cdot \mathrm{grad}_x f_\varepsilon$ is also bounded in $\mathrm{L}^p(\mathbf{R}^D \times \mathbf{R}^D)$, the family of velocity averages

$$\rho_\varepsilon(x) := \int f_\varepsilon(x, v)\psi(v)\,\mathrm{d}v \text{ is relatively compact in } \mathrm{L}^p_{\mathrm{loc}}\big(\mathbf{R}^D\big) \tag{2}$$

for any $\psi \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbf{R}^D)$. It was later proved in [7,6,5] that the family ρ_ε is in fact bounded in some Sobolev or Besov space (corresponding to derivatives of positive fractional order). The optimality of these regularity results is established in [12].

However, the compactness result in (2) fails if $p = 1$. Pick any bounded sequence $g_n \equiv g_n(x, v)$ in $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$ that converges weakly to $\delta_0(x) \otimes \delta_{v^*}(v)$, where $v^* \neq 0$. Let f_n be the unique L^1 solution of the equation $f_n + v \cdot \mathrm{grad}_x f_n = g_n$. Both f_n and $v \cdot \mathrm{grad}_x f_n$ are bounded sequences of $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$, but an elementary computation shows that the sequence of velocity averages satisfies

$$\int \chi(x)\left(\int f_n(x, v)\psi(v)\,\mathrm{d}v\right)\mathrm{d}x \to \psi\big(v^*\big)\int_0^{+\infty} \mathrm{e}^{-t}\chi\big(tv^*\big)\,\mathrm{d}t \tag{3}$$

for each test function $\chi \in \mathrm{C}_c(\mathbf{R}^D)$. In particular the sequence of velocity averages is not even *weakly* relatively compact in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D)$ since it converges in the sense of distributions to a density carried by the half-line $\mathbf{R}_+ \cdot v^*$. (This example was given in [7], pp. 123–124.)

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2. Equiintegrability in v

The main result of this Note is that the lack of compactness in the example (3) can be eliminated if one assumes moreover that the family f_ε has "no concentrations in the variable v". The appropriate notion for our purpose is as follows

DEFINITION 2.1. – Let $f_\varepsilon \equiv f_\varepsilon(x,v)$ be a bounded family of $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$. It is said to be locally equiintegrable in v if and only if, for each $\eta > 0$ and each compact $K \subset \mathbf{R}^D \times \mathbf{R}^D$, there exists $\alpha > 0$ such that, for each measurable family $(A_x)_{x\in\mathbf{R}^D}$ of measurable subsets of $\mathbf{R}^D$ satisfying $\sup_{x\in\mathbf{R}^D} |A_x| < \alpha$, one has

$$\int \left(\int_{A_x} \mathbf{1}_K(x,v) \left| f_\varepsilon(x,v) \right| \mathrm{d}v \right) \mathrm{d}x < \eta$$

for each ε.

The following variant of the de la Vallée–Poussin criterion (*see* [13], p. 38) leads to a slightly more concrete formulation of the property of equiintegrability in v.

PROPOSITION 2.2. – *A bounded family $f_\varepsilon \equiv f_\varepsilon(x,v)$ of $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$ is locally equiintegrable in v if and only if, for each compact $K \subset \mathbf{R}^D \times \mathbf{R}^D$, there exists a positive increasing convex function Φ defined on $\mathbf{R}_+$ that satisfies $\Phi(0) = 0$ and $\Phi(z)/z \to +\infty$ as $z \to +\infty$, and is such that the family $\mathbf{1}_K f_\varepsilon$ is bounded in* (*We denote by* $\mathrm{L}^\Phi(\mathbf{R}^D)$ *the Orlicz space of measurable functions f such that $\Phi(|f|)$ belongs to* $\mathrm{L}^1(\mathbf{R}^D)$.) $\mathrm{L}^1(\mathbf{R}^D_x; \mathrm{L}^\Phi(\mathbf{R}^D_v))$.

Since this proposition is not necessary for our main result (Theorem 3.1 below) we choose to give its proof elsewhere (*see* [10]).

3. The compactness results

The main results in this Note are summarized in the following theorem.

THEOREM 3.1. – *Let $f_\varepsilon \equiv f_\varepsilon(x,v)$ be a bounded family of $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$ that is locally equiintegrable in v and such that the family $v \cdot \mathrm{grad}_x f_\varepsilon$ is also bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$. Then*

- *the family f_ε is locally equiintegrable* (*in both variables x and v*) *in $\mathbf{R}^D \times \mathbf{R}^D$;*
- *for each $\psi \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbf{R}^D)$, the family of moments*

$$\rho_\varepsilon(x) := \int f_\varepsilon(x,v)\psi(v)\,\mathrm{d}v \text{ is relatively compact in } \mathrm{L}^1_{\mathrm{loc}}\big(\mathbf{R}^D\big). \tag{4}$$

4. Interpolation based on dispersion

Dispersion effects of the transport operator are well known: see [2] or [4]. The following statement is Proposition 1.11 in [3].

LEMMA 4.1. – *Let $\phi^0 \equiv \phi^0(x,v) \in \mathrm{L}^p_x(\mathrm{L}^q_v)$ for some $1 \leqslant p < q \leqslant +\infty$, and let $\phi \equiv \phi(t,x,v)$ be the solution of the Cauchy problem*

$$\partial_t \phi + v \cdot \mathrm{grad}_x \phi = 0, \quad \phi(0,x,v) = \phi^0(x,v), \quad x, v \in \mathbf{R}^D. \tag{5}$$

Then, for all $t \in \mathbf{R}^*$,

$$\left\|\phi(t,\cdot,\cdot)\right\|_{\mathrm{L}^q_x(\mathrm{L}^p_v)} \leqslant |t|^{-D(1/p-1/q)} \left\|\phi^0\right\|_{\mathrm{L}^p_x(\mathrm{L}^q_v)}. \tag{6}$$

The proof of the first part of Theorem 3.1 uses the following interpolation formula.

LEMMA 4.2. – *For each* $f \equiv f(x,v) \in \mathrm{L}^1_{\mathrm{comp}}(\mathbf{R}^D \times \mathbf{R}^D)$ *such that* $v \cdot \mathrm{grad}_x f$ *belongs to* $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$ *and each* $\phi^0 \in \mathrm{L}^\infty(\mathbf{R}^D \times \mathbf{R}^D)$, *one has*

$$\iint f(x,v)\phi^0(x,v)\,\mathrm{d}x\,\mathrm{d}v = \iint f(x,v)\phi(t,x,v)\,\mathrm{d}x\,\mathrm{d}v - \int_0^t \iint \phi(s,x,v)v \cdot \mathrm{grad}_x f(x,v)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \tag{7}$$

for all $t \in \mathbf{R}^*$, *where* ϕ *is the solution of* (5).

Proof. – Apply Green's formula to the integral

$$\int_\Omega f(x,v)(\partial_t + v \cdot \mathrm{grad}_x)\phi(s,x,v)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v = 0,$$

where $\Omega =]0,t[\times \mathbf{R}^D \times \mathbf{R}^D$. □

The proof of Theorem 3.1 uses Lemma 4.1 to estimate the right-hand side of (7); then one optimises in $t \in \mathbf{R}_+^*$.

5. Proof of the equiintegrability statement in Theorem 3.1

A first step in the proof of Theorem 3.1 is

LEMMA 5.1. – *Under the same assumptions as in Theorem* 3.1, *the family* ρ_ε *in* (4) *is locally equiintegrable in* $\mathbf{R}^D$ *for each* $\psi \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbf{R}^D)$.

Proof. – Without loss of generality, assume that all the f_ε's are supported in the same compact $K \subset \mathbf{R}^D \times \mathbf{R}^D$, that $f_\varepsilon \geqslant 0$ a.e. and that $\|\psi\|_{\mathrm{L}^\infty} = 1$. Let A be a measurable subset of $\mathbf{R}^D$; let $\phi^0(x,v) = \mathbf{1}_A(x)$ for each $x, v \in \mathbf{R}^D$ and let ϕ be the solution of (5). Pick $\eta > 0$; Definition 2.1 associates $\alpha > 0$ to this η and the compact K. Pick then t and α' such that $0 < t < \eta / \sup_\varepsilon \|v \cdot \mathrm{grad}_x f_\varepsilon\|_{\mathrm{L}^1}$ and $0 < \alpha' < t^D \alpha$.

First, $\|\phi^0\|_{\mathrm{L}^1_x(\mathrm{L}^\infty_v)} = |A|$. Secondly ϕ takes its values in $\{0,1\}$ as does ϕ^0. For each $s > 0$ and $x \in \mathbf{R}^D$, define $A(s)_x = \{v \in \mathbf{R}^D \mid \phi(s,x,v) = 1\}$. With the choice of t as above, (6) implies that

$$\left|A(t)_x\right| = \|\phi(t,0,0)\|_{\mathrm{L}^\infty_x(\mathrm{L}^1_v)} \leqslant |A|/t^D < \alpha \quad \text{provided that } |A| < \alpha'. \tag{8}$$

Applying (7) with $f_\varepsilon(x,v)\psi(v)$ in the place of f shows that, whenever $|A| < \alpha'$,

$$\begin{aligned}\int \mathbf{1}_A(x)\rho_\varepsilon(x)\,\mathrm{d}x &= \int \left(\int_{A(t)_x} f(x,v)\psi(v)\,\mathrm{d}v\right)\mathrm{d}x - \int_0^t \iint \phi(s,x,v)v \cdot \mathrm{grad}_x f(x,v)\psi(v)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \\ &\leqslant \eta + t\|\phi\psi\|_{\mathrm{L}^\infty}\|v \cdot \mathrm{grad}_x f\|_{\mathrm{L}^1} \leqslant 2\eta.\end{aligned}$$

for each ε, which implies the equiintegrability of ρ_ε. □

The result in Lemma 5.1 was discovered in [14] under the assumption that f_ε is bounded in $\mathrm{L}^1_x(\mathrm{L}^\infty_v)$.

LEMMA 5.2. – *Let* $g_\varepsilon \equiv g_\varepsilon(x,v)$ *be a bounded family of* $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^D \times \mathbf{R}^D)$ *that is locally equiintegrable in* v. *If for each* $R > 0$ *the family*

$$\int_{|v| \leqslant R} \left|g_\varepsilon(x,v)\right|\mathrm{d}v \text{ is locally equiintegrable in } \mathbf{R}^D_x \tag{9}$$

then the family g_ε *is locally equiintegrable in* $\mathbf{R}^D \times \mathbf{R}^D$ (*in all variables* x *and* v).

Proof. – Without loss of generality assume that all the g_ε's are supported in the same compact $K \subset [-R,R]^D \times [-R,R]^D$. Let B be a measurable subset of $\mathbf{R}^D \times \mathbf{R}^D$. For each $x \in \mathbf{R}^D$ let $B_x = \{v \in \mathbf{R}^D \mid (x,v) \in B\}$. Consider the set $E = \{x \in \mathbf{R}^D \mid |B_x| \leqslant |B|^{1/2}\}$. Pick $\eta > 0$; Definition 2.1 associates some $\alpha_1 > 0$ to this η and the compact K. By assumption (9), there exists $\alpha_2 > 0$ such that, for each measurable

$A \subset \mathbf{R}^D$ of measure $|A| < \alpha_2$,

$$\int_A \left(\int_{|v| \leqslant R} \left| g_\varepsilon(x,v) \right| \mathrm{d}v \right) \mathrm{d}x < \eta. \tag{10}$$

Assume that $|B| < \inf(\alpha_1^2, \alpha_2^2)$; since one has clearly $|E^c| \leqslant |B|^{1/2} < \alpha_2$ by the Bienaymé–Chebychev inequality,

$$\iint \mathbf{1}_B |g_\varepsilon| \,\mathrm{d}x \,\mathrm{d}v \leqslant \int_E \left(\int_{B_x} \left| g_\varepsilon(x,v) \right| \mathrm{d}v \right) \mathrm{d}x + \int_{E^c} \left(\int_{|v| \leqslant R} \left| g_\varepsilon(x,v) \right| \mathrm{d}v \right) \mathrm{d}x \leqslant \eta + \eta$$

for each ε, by using Definition 2.1 and estimate (10). Therefore the family g_ε is equiintegrable in both variables x and v. □

The first statement in Theorem 3.1 easily follows from both Lemmas 5.1 and 5.2. The classical formula $v \cdot \mathrm{grad}_x |f_\varepsilon| = \mathrm{sgn}(f_\varepsilon)(v \cdot \mathrm{grad}_x f_\varepsilon)$ shows that $|f_\varepsilon|$ satisfies the assumptions of Theorem 3.1 if f_ε does. Further, $v \cdot \mathrm{grad}_x(\chi f_\varepsilon) = \chi(v \cdot \mathrm{grad}_x f_\varepsilon) + f_\varepsilon(v \cdot \mathrm{grad}_x \chi)$ is bounded in $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$ if f_ε satisfies the assumptions of Theorem 3.1 and if χ is a smooth compactly supported bump function. Therefore one can assume that the f_ε's are nonnegative and supported in the same compact $K \subset \mathbf{R}^D \times \mathbf{R}^D$ without loss of generality.

By Lemma 5.1 applied in the case where $\psi(v) = \mathbf{1}_{|v| \leqslant R}$, the family of moments

$$\int_{|v| \leqslant R} f_\varepsilon(x,v) \,\mathrm{d}v \text{ is equiintegrable in } \mathbf{R}_x^D.$$

By Lemma 5.2, this and the fact that the family f_ε is assumed to be locally equiintegrable in v implies the first statement of Theorem 3.1.

6. Proof of the strong compactness statement in Theorem 3.1

For each $\lambda > 0$, define $R_\lambda = (\lambda I + v \cdot \mathrm{grad}_x)^{-1}$ the resolvent of the transport operator $v \cdot \mathrm{grad}_x$ on $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$. One easily see that R_λ is given by the formula

$$(R_\lambda g)(x,v) = \int_0^{+\infty} \mathrm{e}^{-\lambda t} g(x - tv, v) \,\mathrm{d}t$$

for all $g \in \mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$. In particular

$$\text{for each } \lambda > 0, \quad \|R_\lambda\|_{\mathcal{L}(\mathrm{L}^1)} = \frac{1}{\lambda}. \tag{11}$$

We begin with an amplification of Theorem 3 of [7].

PROPOSITION 6.1. – *Let $f_\varepsilon \equiv f_\varepsilon(x,v)$ be a locally equiintegrable family of* $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$ *such that* $v \cdot \mathrm{grad}_x f_\varepsilon$ *is a bounded family of* $\mathrm{L}^1(\mathbf{R}^D \times \mathbf{R}^D)$*. Then for each* $\psi \in \mathrm{L}^\infty_{\mathrm{comp}}(\mathbf{R}^D)$*, the family of moments*

$$\rho_\varepsilon(x) := \int f_\varepsilon(x,v)\psi(v) \,\mathrm{d}v \text{ is relatively compact in } \mathrm{L}^1_{\mathrm{loc}}\big(\mathbf{R}^D\big).$$

Theorem 3 of [7] reached the same conclusion under the assumption that both f_ε and $v \cdot \mathrm{grad}_x f_\varepsilon$ are locally equiintegrable.

The second statement of Theorem 3.1 is a direct consequence of the first statement and Proposition 6.1.

Proof. – Without loss of generality, assume that the f_ε's are supported in the same compact $K \subset \mathbf{R}^D \times \mathbf{R}^D$. Then, for each ε and each $\lambda > 0$, one has

$$\int f_\varepsilon(x,v)\psi(v) \,\mathrm{d}v = \lambda \int (R_\lambda f_\varepsilon)(x,v)\psi(v) \,\mathrm{d}v + \int \big[R_\lambda(v \cdot \mathrm{grad}_x f_\varepsilon)\big](x,v)\psi(v) \,\mathrm{d}v. \tag{12}$$

Pick $\eta > 0$ and set $\lambda = \sup_\varepsilon \|v \cdot \mathrm{grad}_x f_\varepsilon\|_{\mathrm{L}^1} \|\psi\|_{\mathrm{L}^\infty}/\eta$. In the right-hand side of (12), the second term has norm no greater than η in $\mathrm{L}^1(\mathbf{R}^D)$ by (11) and the choice of λ above. For this λ, the family $\lambda \int (R_\lambda f_\varepsilon)(x, v)\psi(v)\,\mathrm{d}v$ is relatively compact in $\mathrm{L}^1(\mathbf{R}^D)$ since f_ε is equiintegrable on $\mathbf{R}^D \times \mathbf{R}^D$, by Theorem 3 of [8]. Hence, for each $\eta > 0$, there exists a compact set $\mathcal{K}_\eta \subset \mathrm{L}^1(\mathbf{R}^D)$ such that the family ρ_ε lies in $\mathcal{K}_\eta + B(0, \eta)$. Therefore this family is precompact. □

7. Extensions and applications

Although the results in Theorem 3.1 use the dispersion properties of the transport operator that are of a global nature, all these results are local. Indeed, if f_ε satisfies the assumptions in Theorem 3.1, so does χf_ε, for any bump function $\chi \equiv \chi(x) \in \mathrm{C}_c^\infty(\mathbf{R}^D)$. Thus, Theorem 3.1 also holds when the variable x lies in either a regular open set Ω of $\mathbf{R}^D$ or the D-torus $\mathbf{T}^D$.

The interpolation mechanism used in the proof of Proposition 6.1 can also be used to study the regularity (in the sense of Besov spaces) of ρ defined as in (1) under the assumption that $f \in \mathrm{L}^p_{x,v}$ and $v \cdot \mathrm{grad}_x f \in \mathrm{L}^q_{x,v}$ with different p's and q's. We shall return to these questions in [10].

An analogue of Theorem 3.1 for some scaled evolution transport operator is a key ingredient in the proof of the incompressible Navier–Stokes limit for the BGK (*see* [15]) or the Boltzmann equations (*see* [9]).

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