Available online at www.sciencedirect.com
ScienceDirect
J. Math. Pures Appl. 91 (2009) 508–552
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

The incompressible Navier–Stokes limit of the Boltzmann equation for hard cutoff potentials

François Golse [a,b,*], Laure Saint-Raymond [c]

[a] *Ecole polytechnique, Centre de mathématiques L. Schwartz, F91128 Palaiseau cedex, France*
[b] *Université Paris Diderot - Paris 7, Laboratoire J.-L. Lions, 4 place Jussieu, Boîte courrier 187, F75252 Paris cedex 05, France*
[c] *Ecole Normale Supérieure, Département de Mathématiques et Applications, 45 rue d'Ulm, F75230 Paris cedex 05, France*

Received 2 October 2008

Available online 10 April 2009

Abstract

The present paper proves that all limit points of sequences of renormalized solutions of the Boltzmann equation in the limit of small, asymptotically equivalent Mach and Knudsen numbers are governed by Leray solutions of the Navier–Stokes equations. This convergence result holds for hard cutoff potentials in the sense of H. Grad, and therefore completes earlier results by the same authors [Invent. Math. 155 (2004) 81–161] for Maxwell molecules.

Résumé

On montre dans cet article qu'à extraction de sous-suites près, les suites de solutions renormalisées de l'équation de Boltzmann convergent vers des limites décrites par les solutions de Leray des équations de Navier–Stokes dans la limite où les nombres de Mach et de Knudsen sont petits et asymptotiquement équivalents. Cette convergence est établie ici dans le cas de potentiels durs avec troncature angulaire au sens de H. Grad, ce qui complète les résultats antérieurs des mêmes auteurs [Invent. Math. 155 (2004) 81–161] pour le cas des molécules Maxwelliennes.

MSC: 35Q35; 35Q30; 82C40

Keywords: Hydrodynamic limit; Boltzmann equation; Incompressible Navier–Stokes equations; Renormalized solutions; Leray solutions

1. Introduction

The subject matter of this paper is the derivation of the Navier–Stokes equations for incompressible fluids from the Boltzmann equation, which is the governing equation in the kinetic theory of rarefied, monatomic gases.

In the kinetic theory of gases founded by Maxwell and Boltzmann, the state of a monatomic gas is described by the molecular number density in the single-body phase space, $f \equiv f(t,x,v) \geqslant 0$ that is the density with respect to the Lebesgue measure $dx\,dv$ of molecules with velocity $v \in \mathbf{R}^3$ and position $x \in \mathbf{R}^3$ at time $t \geqslant 0$. Henceforth, we restrict

* Corresponding author at: Ecole polytechnique, Centre de mathématiques L. Schwartz, F91128 Palaiseau cedex, France.
E-mail addresses: golse@math.polytechnique.fr (F. Golse), saintray@dma.ens.fr (L. Saint-Raymond).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2009.01.013

our attention to the case where the gas fills the Euclidean space $\mathbf{R}^3$. For a perfect gas, the number density f satisfies the Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = \mathcal{B}(f, f), \quad x, v \in \mathbf{R}^3, \tag{1.1}$$

where $\mathcal{B}(f, f)$ is the Boltzmann collision integral.

The Boltzmann collision integral acts only on the v variable in the number density f. In other words, $\mathcal{B}$ is a bilinear operator defined on functions of the single variable v, and it is understood that the notation

$$\mathcal{B}(f, f)(t, x, v) \text{ designates } \mathcal{B}\big(f(t, x, \cdot), f(t, x, \cdot)\big)(v). \tag{1.2}$$

For each continuous $f \equiv f(v)$ rapidly decaying at infinity, the collision integral is given by:

$$\mathcal{B}(f, f)(v) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} \big(f(v')f(v_1') - f(v)f(v_1)\big) b(v - v_1, \omega)\, dv_1\, d\omega, \tag{1.3}$$

where

$$\begin{aligned} v' &\equiv v'(v, v_1, \omega) = v - (v - v_1)\cdot\omega\omega, \\ v_1' &\equiv v_1'(v, v_1, \omega) = v_1 + (v - v_1)\cdot\omega\omega. \end{aligned} \tag{1.4}$$

The collision integral is then extended by continuity to wider classes of densities f, depending on the specifics of the function b.

The function $b \equiv b(v - v_1, \omega)$, called the collision kernel, is measurable, a.e. positive, and satisfies the symmetry:

$$b(v - v_1, \omega) = b(v_1 - v, \omega) = b(v' - v_1', \omega) \quad \text{a.e. in } (v, v_1, \omega). \tag{1.5}$$

Throughout the present paper, we assume that b satisfies:

$$\begin{gathered} 0 < b(z, \omega) \leqslant C_b\big(1 + |z|\big)^{\beta}\big|\cos(\widehat{z, \omega})\big| \quad \text{a.e. on } \mathbf{R}^3 \times \mathbf{S}^2, \\ \int_{\mathbf{S}^2} b(z, \omega)\, d\omega \geqslant \frac{1}{C_b}\frac{|z|}{1 + |z|} \quad \text{a.e. on } \mathbf{R}^3, \end{gathered} \tag{1.6}$$

for some $C_b > 0$ and $\beta \in [0, 1]$. The bounds (1.6) are verified by all collision kernels coming from a repulsive, binary intermolecular potential of the form $U(r) = U_0/r^s$ with Grad's angular cutoff (see [15]) and $s \geqslant 4$. Such power-law potentials are said to be "hard" if $s \geqslant 4$ and "soft" otherwise: in other words, we shall be dealing with hard cutoff potentials. The case of a hard-sphere interaction (binary elastic collisions between spherical particles) corresponds with

$$b(z, \omega) = |z \cdot \omega|; \tag{1.7}$$

it is a limiting case of hard potentials that obviously satisfies (1.6), even without Grad's cutoff. At the time of this writing, the Boltzmann equation has been derived from molecular dynamics — i.e. Newton's equations of classical mechanics applied to a large number of spherical particles — in the case of hard sphere collisions, by O.E. Lanford [16], see also [9] for the case of compactly supported potentials. Thus the collision kernel b given by (1.7) plays an important role in the mathematical theory of the Boltzmann equation.

The only nonnegative, measurable number densities f such that $\mathcal{B}(f, f) = 0$ are Maxwellian densities, i.e. densities of the form:

$$f(v) = \frac{R}{(2\pi\Theta)^{3/2}} e^{-\frac{|v-U|^2}{2\Theta}} =: \mathcal{M}_{R,U,\Theta}(v) \tag{1.8}$$

for some $R \geqslant 0$, $\Theta > 0$ and $U \in \mathbf{R}^3$. Maxwellian densities whose parameters R, U, Θ are constants are called "uniform Maxwellians", whereas Maxwellian densities whose parameters R, U, Θ are functions of t and x are referred to as "local Maxwellians". Uniform Maxwellians are solutions of (1.1); however, local Maxwellians are not solutions of (1.1) in general.

The incompressible Navier–Stokes limit of the Boltzmann equation can be stated as follows.

Navier–Stokes limit of the Boltzmann equation. Let $u^{\mathrm{in}} \equiv u^{\mathrm{in}}(x) \in \mathbf{R}^3$ be a divergence-free vector field on $\mathbf{R}^3$. For each $\epsilon > 0$, consider the initial number density:

$$f_\epsilon^{\mathrm{in}}(x, v) = \mathcal{M}_{1,\epsilon u^{\mathrm{in}}(\epsilon x),1}(v). \tag{1.9}$$

Notice that the number density f_ϵ^{in} is a slowly varying perturbation of order ϵ of the uniform Maxwellian $\mathcal{M}_{1,0,1}$. Let f_ϵ solve the Boltzmann equation (1.1) with initial data (1.9), and define:

$$u_\epsilon(t, x) := \frac{1}{\epsilon} \int\limits_{\mathbf{R}^3} v f_\epsilon\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}, v\right) dv. \tag{1.10}$$

Then, in the limit as $\epsilon \to 0^+$ (and possibly after extracting a converging subsequence), the velocity field u_ϵ satisfies

$$u_\epsilon \to u \quad \text{in } \mathcal{D}'\big(\mathbf{R}_+ \times \mathbf{R}^3\big),$$

where u is a solution of the incompressible Navier–Stokes equations:

$$\begin{aligned} \partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \quad x \in \mathbf{R}^3,\ t > 0, \\ \operatorname{div}_x u = 0, \end{aligned} \tag{1.11}$$

with initial data

$$u|_{t=0} = u^{\mathrm{in}}. \tag{1.12}$$

The viscosity ν is defined in terms of the collision kernel b, by some implicit formula, that will be given below.

(More general initial data than (1.9) can actually be handled with our method: see below for a precise statement of the Navier–Stokes limit theorem.)

Hydrodynamic limits of the Boltzmann equation leading to incompressible fluid equations have been extensively studied by many authors. See in particular [2] for formal computations, and [1,3] for a general program of deriving global solutions of incompressible fluid models from global solutions of the Boltzmann equation. The derivation of global weak (Leray) solutions of the Navier–Stokes equations from global weak (renormalized à la DiPerna–Lions) solutions of the Boltzmann equation is presented in [3], under additional assumptions on the Boltzmann solutions which remained unverified. In a series of later publications [20,22,4,10] some of these assumptions have been removed, except one that involved controlling the build-up of particles with large kinetic energy and possible concentrations in the x-variable. This last assumption was removed by the second author in the case of the model BGK equation [23,24], by a kind of dispersion argument based on the fact that relaxation to local equilibrium improves the regularity in v of number density fluctuations. Finally, a complete proof of the Navier–Stokes limit of the Boltzmann equation was proposed in [13]. In this paper, the regularization in v was obtained by a rather different argument — specifically, by the smoothing properties of the gain part of Boltzmann's collision integral — since not much is known about relaxation to local equilibrium for weak solutions of the Boltzmann equation.

While the results above holds for global solutions of the Boltzmann equation without restriction on the size (or symmetries) of its initial data, earlier results had been obtained in the regime of smooth solutions [7,5]. Since the regularity of Leray solutions of the Navier–Stokes equations in 3 space dimensions is not known at the time of this writing, such results are limited to either local (in time) solutions, or to solutions with initial data that are small in some appropriate norm.

The present paper extends the result of [13] to the case of hard cutoff potentials in the sense of Grad — i.e. assuming that the collision kernel satisfies (1.6). Indeed, [13] only treated the case of Maxwell molecules, for which the collision kernel is of the form:

$$b(z, \omega) = \big|\cos(z, \omega)\big| b^*\big(\big|\cos(z, \omega)\big|\big) \quad \text{with } \frac{1}{C_*} \leqslant b_* \leqslant C_*.$$

The method used in the present paper also significantly simplifies the original proof in [13] in the case of Maxwell molecules.

Independently, C.D. Levermore and N. Masmoudi have extended the analysis of [13] to a wider class of collision kernels that includes soft potentials with a weak angular cutoff in the sense of DiPerna–Lions: see [17]. Their proof is written in the case where the spatial domain is the 3-torus $\mathbf{R}^3/\mathbf{Z}^3$.

In the present paper, we handle the case of the Euclidean space $\mathbf{R}^3$, which involves additional technical difficulties concerning truncations at infinity and the Leray projection on divergence-free vector fields — see Appendix C below.

2. Formulation of the problem and main results

2.1. Global solutions of the Boltzmann equation

The only global existence theory for the Boltzmann equation without extra smallness assumption on the size of the initial data known to this date is the DiPerna–Lions theory of renormalized solutions [8,18]. We shall present their theory in the setting best adapted to the hydrodynamic limit considered in the present paper.

All incompressible hydrodynamic limits of the Boltzmann equation involve some background, uniform Maxwellian equilibrium state — whose role from a physical viewpoint is to set the scale of the speed of sound. Without loss of generality, we assume this uniform equilibrium state to be the centered, reduced Gaussian density:

$$M(v) := \mathcal{M}_{1,0,1}(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}. \tag{2.1}$$

Our statement of the Navier–Stokes limit of the Boltzmann equation given above suggests that one has to handle the scaled number density:

$$F_\epsilon(t,x,v) = f_\epsilon\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}, v\right), \tag{2.2}$$

where f_ϵ is a solution of the Boltzmann equation (1.1). This scaled number density is a solution of the scaled Boltzmann equation:

$$\epsilon^2 \partial_t F_\epsilon + \epsilon v \cdot \nabla_x F_\epsilon = \mathcal{B}(F_\epsilon, F_\epsilon), \quad x, v \in \mathbf{R}^3,\ t > 0. \tag{2.3}$$

Throughout the present section, ϵ is any fixed, positive number.

Definition 2.1. A renormalized solution of the scaled Boltzmann equation (2.3) relatively to the global equilibrium M is a function,

$$F \in C\big(\mathbf{R}^+, L^1_{\mathrm{loc}}\big(\mathbf{R}^3 \times \mathbf{R}^3\big)\big)$$

such that

$$\Gamma'\left(\frac{F}{M}\right)\mathcal{B}(F,F) \in L^1_{\mathrm{loc}}\big(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3\big),$$

and which satisfies,

$$M\big(\epsilon^2 \partial_t + \epsilon v \cdot \nabla_x\big)\Gamma\left(\frac{F}{M}\right) = \Gamma'\left(\frac{F}{M}\right)\mathcal{B}(F,F), \tag{2.4}$$

for each normalizing nonlinearity:

$$\Gamma \in C^1\big(\mathbf{R}^+\big) \quad \text{such that } \big|\Gamma'(z)\big| \leqslant \frac{C}{\sqrt{1+z}}, \quad z \geqslant 0.$$

The DiPerna–Lions theory is based on the only a priori estimates that have natural physical interpretation. In particular, the distance between any number density $F \equiv F(x,v)$ and the uniform equilibrium M is measured in terms of the relative entropy:

$$H(F|M) := \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left(F \ln\left(\frac{F}{M}\right) - F + M\right) dx\, dv. \tag{2.5}$$

Introducing

$$h(z) = (1+z)\ln(1+z) - z \geqslant 0, \quad z > -1, \tag{2.6}$$

we see that

$$H(F|M) = \iint\limits_{\mathbf{R}^3\times\mathbf{R}^3} h\left(\frac{F}{M} - 1\right) M\,dv\,dx \geqslant 0,$$

with equality if and only if $F = M$ a.e. in x, v.

While the relative entropy measures the distance of a number density F to the particular equilibrium M, the local entropy production rate "measures the distance" of F to the set of all Maxwellian densities. Its expression is as follows:

$$\mathcal{E}(F) = \frac{1}{4} \iiint\limits_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (F'F_1' - FF_1) \ln\left(\frac{F'F_1'}{FF_1}\right) b(v - v_1, \omega)\,dv\,dv_1\,d\omega. \tag{2.7}$$

The DiPerna–Lions existence theorem is the following statement [8,18].

Theorem 2.2. *Assume that the collision kernel b satisfies Grad's cutoff assumption* (1.6) *for some $\beta \in [0, 1]$. Let $F^{\text{in}} \equiv F^{\text{in}}(x, v)$ be any measurable, a.e. nonnegative function on $\mathbf{R}^3 \times \mathbf{R}^3$ such that*

$$H\big(F^{\text{in}}|M\big) < +\infty. \tag{2.8}$$

Then, for each $\epsilon > 0$, there exists a renormalized solution,

$$F_\epsilon \in C\big(\mathbf{R}^+, L^1_{\text{loc}}\big(\mathbf{R} \times \mathbf{R}^3\big)\big),$$

relatively to M of the scaled Boltzmann equation (2.3) *such that*

$$F_\epsilon|_{t=0} = F^{\text{in}}.$$

Moreover, F_ϵ satisfies:

(a) *the continuity equation*

$$\epsilon\partial_t \int\limits_{\mathbf{R}^3} F_\epsilon\,dv + \operatorname{div}_x \int\limits_{\mathbf{R}^3} vF_\epsilon\,dv = 0, \tag{2.9}$$

and

(b) *the entropy inequality*

$$H(F_\epsilon|M)(t) + \frac{1}{\epsilon^2}\int\limits_0^t \int\limits_{\mathbf{R}^3} \mathcal{E}(F_\epsilon)(s, x)\,ds\,dx \leqslant H\big(F^{\text{in}}|M\big), \quad t > 0. \tag{2.10}$$

Besides the continuity equation (2.9), classical solutions of the scaled Boltzmann equation (2.3) with fast enough decay as $|v| \to \infty$ would satisfy the local conservation of momentum,

$$\epsilon\partial_t \int\limits_{\mathbf{R}^3} vF_\epsilon\,dv + \operatorname{div}_x \int\limits_{\mathbf{R}^3} v \otimes vF_\epsilon\,dv = 0, \tag{2.11}$$

as well as the local conservation of energy,

$$\epsilon\partial_t \int\limits_{\mathbf{R}^3} \frac{1}{2}|v|^2 F_\epsilon\,dv + \operatorname{div}_x \int\limits_{\mathbf{R}^3} v\frac{1}{2}|v|^2 F_\epsilon\,dv = 0. \tag{2.12}$$

Renormalized solutions of the Boltzmann equation (2.3) are not known to satisfy any of these conservation laws except that of mass — i.e. the continuity equation (2.9). Since these local conservation laws are the fundamental objects in every fluid theory, we expect to recover them somehow in the hydrodynamic limit $\epsilon \to 0^+$.

2.2. The convergence theorem

It will be more convenient to replace the number density F_ϵ by its ratio to the uniform Maxwellian equilibrium M; also we shall be dealing mostly with perturbations of order ϵ of the uniform Maxwellian state M. Thus we define:

$$G_\epsilon = \frac{F_\epsilon}{M}, \qquad g_\epsilon = \frac{G_\epsilon - 1}{\epsilon}. \tag{2.13}$$

Likewise, the Lebesgue measure dv will be replaced with the unit measure $M\,dv$, and we shall systematically use the notation:

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M(v)\,dv, \quad \text{for each } \phi \in L^1(M\,dv). \tag{2.14}$$

For the same reason, quantities like the local entropy production rate involve the measure:

$$d\mu(v, v_1, \omega) = b(v - v_1, \omega) M_1\,dv_1 M\,dv\,d\omega, \qquad \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d\mu(v, v_1, \omega) = 1, \tag{2.15}$$

whose normalization can be assumed without loss of generality, by some appropriate choice of physical units for the collision kernel b. We shall also use the notation:

$$\langle\!\langle \psi \rangle\!\rangle = \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \psi(v, v_1, \omega)\,d\mu(v, v_1, \omega) \quad \text{for } \psi \in L^1\big(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2, d\mu\big). \tag{2.16}$$

From now on, we consider solutions of the scaled Boltzmann equation (2.3) that are perturbations of order ϵ about the uniform Maxwellian M. This is conveniently expressed in terms of the relative entropy.

Proposition 2.3 *(Uniform a priori estimates). Let $F_\epsilon^{\mathrm{in}} \equiv F_\epsilon^{\mathrm{in}}(x, v)$ be a family of measurable, a.e. nonnegative functions such that*

$$\sup_{\epsilon>0} \frac{1}{\epsilon^2} H\big(F_\epsilon^{\mathrm{in}} | M\big) = C^{\mathrm{in}} < +\infty. \tag{2.17}$$

Consider a family (F_ϵ) of renormalized solutions of the scaled Boltzmann equation (2.3) *with initial data,*

$$F_\epsilon|_{t=0} = F_\epsilon^{\mathrm{in}}. \tag{2.18}$$

Then

(a) *the family of relative number density fluctuations g_ϵ satisfies*

$$\frac{1}{\epsilon^2} \int_{\mathbf{R}^3} \big\langle h\big(\epsilon g_\epsilon(t, x, \cdot)\big)\big\rangle\,dx \leqslant C^{\mathrm{in}}, \tag{2.19}$$

where h is the function defined in (2.6);

(b) *the family $\frac{1}{\epsilon}(\sqrt{G_\epsilon} - 1)$ is bounded in $L^\infty(\mathbf{R}_+; L^2(M\,dv\,dx))$:*

$$\int_{\mathbf{R}^3} \left\langle \left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right)^2 \right\rangle dx \leqslant C^{\mathrm{in}}; \tag{2.20}$$

(c) *hence the family g_ϵ is relatively compact in $L^1_{\mathrm{loc}}(dt\,dx; L^1(M\,dv))$;*

(d) *the family of relative number densities G_ϵ satisfies the entropy production — or dissipation estimate*:

$$\int_0^\infty \int_{\mathbf{R}^3} \left\langle\!\!\left\langle \left(\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} \right)^2 \right\rangle\!\!\right\rangle dx\,dt \leqslant C^{\mathrm{in}}. \tag{2.21}$$

Proof. The entropy inequality implies that

$$H(F_\epsilon|M)(t) = \int_{\mathbf{R}^3} \langle h(G_\epsilon - 1)\rangle(t,x)\,dx \leqslant H\big(F_\epsilon^{\mathrm{in}}|M\big) \leqslant C^{\mathrm{in}}\epsilon^2,$$

which is the estimate (a).

The estimate (b) follows from (a) and the elementary identity:

$$\begin{aligned} h(z-1) - (\sqrt{z}-1)^2 &= z\ln z - (\sqrt{z}-1)(\sqrt{z}+1) - (\sqrt{z}-1)^2 \\ &= 2z\ln\sqrt{z} - 2(\sqrt{z}-1)\sqrt{z} \\ &= 2\sqrt{z}(\sqrt{z}\ln\sqrt{z} - \sqrt{z}+1) \geqslant 0. \end{aligned}$$

From the identity,

$$g_\epsilon = 2\frac{\sqrt{G_\epsilon}-1}{\epsilon} + \epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2, \tag{2.22}$$

and the bound (b), we deduce the weak compactness statement (c).

Finally, the entropy inequality implies that

$$\int_0^\infty \int_{\mathbf{R}^3} \mathcal{E}(F_\epsilon)(s,x)\,dx\,ds \leqslant C^{\mathrm{in}}\epsilon^4.$$

Observing that

$$\begin{aligned} \mathcal{E}(F_\epsilon) &= \frac{1}{4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (F_\epsilon' F_{\epsilon 1}' - F_\epsilon F_{\epsilon 1}) \ln\left(\frac{F_\epsilon' F_{\epsilon 1}'}{F_\epsilon F_{\epsilon 1}}\right) b(v-v_1,\omega)\,dv\,dv_1\,d\omega \\ &= \frac{1}{4}\left\langle\!\!\left\langle (G_\epsilon' G_{\epsilon 1}' - G_\epsilon G_{\epsilon 1}) \ln\left(\frac{G_\epsilon' G_{\epsilon 1}'}{G_\epsilon G_{\epsilon 1}}\right)\right\rangle\!\!\right\rangle, \end{aligned}$$

and using the elementary inequality,

$$\frac{1}{4}(X-Y)\ln\frac{X}{Y} \geqslant (\sqrt{X}-\sqrt{Y})^2, \quad X,Y>0,$$

leads to the dissipation estimate (d). □

Our main result in the present paper is a description of all limit points of the family of number density fluctuations g_ϵ.

Theorem 2.4. *Let F_ϵ^{in} be a family of measurable, a.e. nonnegative functions defined on $\mathbf{R}^3\times\mathbf{R}^3$ satisfying the scaling condition* (2.17). *Let F_ϵ be a family of renormalized solutions relative to M of the scaled Boltzmann equation* (2.3) *with initial data* (2.18)*, for a hard cutoff collision kernel b that satisfies* (1.6) *with $\beta \in [0,1]$. Define the relative number density G_ϵ and the number density fluctuation g_ϵ by the formulas* (2.13).

Then, any limit point g in $L^1_{\mathrm{loc}}(dt\,dx; L^1(M\,dv))$ of the family of number density fluctuations g_ϵ is an infinitesimal Maxwellian of the form,

$$g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2-5\big),$$

where the vector field u and the function θ are solutions of the Navier–Stokes–Fourier system:

$$\begin{gathered} \partial_t u + \operatorname{div}_x(u\otimes u) + \nabla_x p = \nu\Delta_x u, \quad \operatorname{div}_x u = 0, \\ \partial_t\theta + \operatorname{div}_x(u\theta) = \kappa\Delta_x\theta, \end{gathered} \tag{2.23}$$

with initial data

$$u^{\mathrm{in}} = w - \lim_{\epsilon \to 0} P\left(\frac{1}{\epsilon}\int vF_\epsilon^{\mathrm{in}}\,dv\right),$$
$$\theta^{\mathrm{in}} = w - \lim_{\epsilon \to 0} \frac{1}{\epsilon}\int \left(\frac{1}{5}|v|^2 - 1\right)\left(F_\epsilon^{\mathrm{in}} - M\right)dv, \tag{2.24}$$

where P is the Leray orthogonal projection in $L^2(\mathbf{R}^3)$ on the space of divergence-free vector fields and the weak limits above are taken along converging subsequences. Finally, the weak solution (u,θ) of (2.23) *so obtained satisfies the energy inequality*:

$$\int_{\mathbf{R}^3} \left(\frac{1}{2}\left|u(t,x)\right|^2 + \frac{5}{4}\left|\theta(t,x)\right|^2\right)dx + \int_0^t\int_{\mathbf{R}^3}\left(\nu|\nabla_x u|^2 + \frac{5}{2}\kappa|\nabla_x\theta|^2\right)dx$$
$$\leqslant \varliminf_{\epsilon \to 0^+} \frac{1}{\epsilon^2} H\left(F_\epsilon^{\mathrm{in}}|M\right). \tag{2.25}$$

The viscosity ν and thermal conductivity κ are defined implicitly in terms of the collision kernel b by the formulas (2.27) *below.*

There are several ways of stating the formulas giving ν and κ. Perhaps the quickest route to arrive at these formulas is as follows.

Consider the Dirichlet form associated to the Boltzmann collision integral linearized at the uniform equilibrium M:

$$\mathcal{D}_M(\Phi) := \frac{1}{8}\left\langle\!\left\langle |\Phi' + \Phi_1' - \Phi - \Phi_1|^2 \right\rangle\!\right\rangle. \tag{2.26}$$

The notation $|\cdot|^2$ designates the Euclidean norm on $\mathbf{R}^3$ when Φ is vector-valued, or the Frobenius norm on $M_3(\mathbf{R})$ (defined by $|A| = \mathrm{trace}(A^*A)^{1/2}$) when Φ is matrix-valued. Let $\mathcal{D}^*$ be the Legendre dual of $\mathcal{D}$, defined by the formula

$$\mathcal{D}^*(\Psi) := \sup_\Phi\left(\langle \Psi \cdot \Phi\rangle - \mathcal{D}(\Phi)\right),$$

where the notation $\Phi(v)\cdot\Psi(v)$ designates the Euclidean inner product in $\mathbf{R}^3$ whenever Φ, Ψ are vector valued, or the Frobenius inner product in $M_3(\mathbf{R})$ whenever Φ, Ψ are matrix-valued (the Frobenius inner product being defined by $A \cdot B = \mathrm{trace}(A^*B)$).

With these notations, one has:

$$\nu := \frac{1}{5}\mathcal{D}^*\left(v \otimes v - \frac{1}{3}|v|^2 I\right), \qquad \kappa := \frac{4}{15}\mathcal{D}^*\left(\frac{1}{2}v\left(|v|^2 - 5\right)\right). \tag{2.27}$$

The weak solutions of the Navier–Stokes–Fourier system obtained in Theorem 2.4 satisfy the energy inequality (2.25) and thus are strikingly similar to Leray solutions of the Navier–Stokes equations in 3 space dimensions — of which they are a generalization. The reader is invited to check that, whenever the initial data F_ϵ^{in} is chosen so that

$$\frac{1}{\epsilon^2}H\left(F_\epsilon^{\mathrm{in}}|M\right) \to \frac{1}{2}\int_{\mathbf{R}^3}\left|u^{\mathrm{in}}(x)\right|^2 dx \quad \text{as } \epsilon \to 0^+,$$

then the vector field u obtained in Theorem 2.4 is indeed a Leray solution of the Navier–Stokes equations. More information on this kind of issues can be found in [13]. See in particular the statements of Corollary 1.8 and Theorem 1.9 in [13], which hold verbatim in the case of hard cutoff potentials considered in the present paper, and which are deduced from Theorem 2.4 as explained in [13].

2.3. Mathematical tools and notations for the hydrodynamic limit

An important feature of the Boltzmann collision integral is the following symmetry relations (the collision symmetries). These collision symmetries are straightforward, but fundamental consequences of the identities (1.5) verified by the collision kernel, and can be formulated in the following manner. Let $\Phi \equiv \Phi(v, v_1)$ be such that $\Phi \in L^1(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2, d\mu)$. Then

$$\begin{aligned}\iiint\limits_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \Phi(v, v_1)\, d\mu(v, v_1, \omega) &= \iiint\limits_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \Phi(v_1, v)\, d\mu(v, v_1, \omega)\\ &= \iiint\limits_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \Phi\big(v'(v, v_1, \omega), v_1'(v, v_1, \omega)\big)\, d\mu(v, v_1, \omega),\end{aligned} \tag{2.28}$$

where v' and v_1' are defined in terms of v, v_1, ω by the formulas (1.4).

Since the Navier–Stokes limit of the Boltzmann equation is a statement on number density fluctuations about the uniform Maxwellian M, it is fairly natural to consider the linearization at M of the collision integral.

First, the quadratic collision integral is polarized into a symmetric bilinear operator, by the formula

$$\mathcal{B}(F, G) := \frac{1}{2}\big(\mathcal{B}(F+G, F+G) - \mathcal{B}(F, F) - \mathcal{B}(G, G)\big).$$

The linearized collision integral is defined as

$$\mathcal{L} f = -2M^{-1}\mathcal{B}(M, Mf). \tag{2.29}$$

Assuming that the collision kernel b comes from a hard cutoff potential in the sense of Grad (1.6), one can show (see [15] for instance) that $\mathcal{L}$ is a possibly unbounded, self-adjoint, nonnegative Fredholm operator on the Hilbert space $L^2(\mathbf{R}^3, M\, dv)$ with domain,

$$D(\mathcal{L}) = L^2\big(\mathbf{R}^3, a\big(|v|\big)^2 M\, dv\big),$$

and nullspace,

$$\operatorname{Ker}\mathcal{L} = \operatorname{span}\big\{1, v_1, v_2, v_3, |v|^2\big\}, \tag{2.30}$$

and that $\mathcal{L}$ can be decomposed as

$$\mathcal{L} g(v) = a\big(|v|\big) g(v) - \mathcal{K} g(v),$$

where $\mathcal{K}$ is a compact integral operator on $L^2(M\, dv)$ and $a = a(|v|)$ is a scalar function called the collision frequency that satisfies, for some $C > 1$,

$$0 < a_- \leqslant a\big(|v|\big) \leqslant a_+\big(1 + |v|\big)^{\beta}.$$

In particular, $\mathcal{L}$ has a spectral gap, meaning that there exists $C > 0$ such that

$$\langle f \mathcal{L} f\rangle \geqslant C\|f - \Pi f\|^2_{L^2(Ma\, dv)}, \tag{2.31}$$

for each $f \in D(\mathcal{L})$, where Π is the orthogonal projection on $\operatorname{Ker}\mathcal{L}$ in $L^2(\mathbf{R}^3, M\, dv)$, i.e.

$$\Pi f = \langle f\rangle + \langle v f\rangle \cdot v + \left\langle\left(\frac{1}{3}|v|^2 - 1\right) f\right\rangle \frac{1}{2}\big(|v|^2 - 3\big). \tag{2.32}$$

The bilinear collision integral intertwined with the multiplication by M is defined by:

$$\mathcal{Q}(f, g) = M^{-1}\mathcal{B}(Mf, Mg). \tag{2.33}$$

Under the only assumption that the collision kernel satisfies (1.5) together with the bound,

$$\int\limits_{\mathbf{S}^2} b(z, \omega)\, d\omega \leqslant a_+\big(1 + |z|\big)^{\beta}, \tag{2.34}$$

$\mathcal{Q}$ maps continuously $L^2(\mathbf{R}^3, M(1 + |v|)^{\beta}\, dv)$ into $L^2(\mathbf{R}^3, a^{-1}M\, dv)$. Indeed, by using the Cauchy–Schwarz inequality and the collision symmetries (2.28) entailed by (1.5):

$$\begin{aligned}
\|\mathcal{Q}(g,h)\|^2_{L^2(a^{-1}M\,dv)} &= \int_{\mathbf{R}^3} a\big(|v|\big)^{-1}\bigg(\frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{S}^2}(g'h'_1+g'_1h'-gh_1-g_1h)b(v-v_1,\omega)M_1\,dv_1\,d\omega\bigg)^2 M\,dv\\
&\leqslant \frac{1}{4}\int_{\mathbf{R}^3} a\big(|v|\big)^{-1}\bigg(\iint_{\mathbf{R}^3\times\mathbf{S}^2} b(v-v_1,\omega)M_1\,dv_1\,d\omega\bigg)\\
&\quad\times\bigg(\iint_{\mathbf{R}^3\times\mathbf{S}^2}(g'h'_1+g'_1h'-gh_1-g_1h)^2 b(v-v_1,\omega)M_1\,dv_1\,d\omega\bigg)M\,dv\\
&\leqslant \sup_{v\in\mathbf{R}^3} a\big(|v|\big)^{-1}\iint_{\mathbf{R}^3\times\mathbf{S}^2} b(v-v_1,\omega)M_1\,dv_1\,d\omega\\
&\quad\times\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\big((g'h'_1)^2+(g'_1h')^2+(gh_1)^2+(g_1h)^2\big)\,d\mu(v,v_1,\omega)\\
&\leqslant 2C\iint_{\mathbf{R}^3\times\mathbf{R}^3}\big((gh_1)^2+(g_1h)^2\big)\bigg(\int_{\mathbf{S}^2} b(v-v_1,\omega)\,d\omega\bigg)MM_1\,dv\,dv_1\\
&\leqslant 4C^2\|g\|^2_{L^2((1+|v|)^\beta M\,dv)}\|h\|^2_{L^2((1+|v|)^\beta M\,dv)}.
\end{aligned}\tag{2.35}$$

Another important property of the bilinear operator $\mathcal{Q}$ is the following relation:

$$\mathcal{Q}(f,f)=\frac{1}{2}\mathcal{L}\big(f^2\big)\quad\text{for each } f\in\operatorname{Ker}\mathcal{L},\tag{2.36}$$

which follows from differentiating twice both sides of the equality,

$$\mathcal{B}(\mathcal{M}_{R,U,\Theta},\mathcal{M}_{R,U,\Theta})=0,$$

with respect to $R\geqslant 0$, $\Theta>0$ and $U\in\mathbf{R}^3$ — see for instance [2], formulas (59)–(60) for a quick proof of this identity.

Young's inequality. Since the family of number density fluctuations g_ϵ satisfies the uniform bound (a) in Proposition 2.3 and the measure $M\,dv$ has total mass 1, the fluctuation g_ϵ can be integrated against functions of v with at most quadratic growth at infinity, by an argument analogous to the Hölder inequality. This argument will be used in various places in the proof, and we present it here for the reader's convenience. To the function h in (2.6), we associate its Legendre dual h^* defined by:

$$h^*(\zeta):=\sup_{z>-1}\big(\zeta z-h(z)\big)=e^\zeta-\zeta-1.$$

Thus, for each $\zeta>0$ and each $z>-1$, one has:

$$\zeta|z|\leqslant h\big(|z|\big)+h^*(\zeta)\leqslant h(z)+h^*(\zeta),\tag{2.37}$$

since

$$h\big(|z|\big)\leqslant h(z),\qquad z>-1.$$

The inequality (2.37) is referred to as the Young inequality (by analogy with the classical Young inequality:

$$\zeta z\leqslant\frac{z^p}{p}+\frac{\zeta^q}{q},\qquad z,\zeta>0,$$

which holds whenever $1<p,q<\infty$ satisfy $\frac{1}{p}+\frac{1}{q}=1$).

Notations regarding functional spaces. Finally, we shall systematically use the following notations. First, Lebesgue spaces without mention of the domain of integration always designate that Lebesgue space on the largest domain of integration on which the measure is defined. For instance:

$$L^p(M\,dv) \quad \text{designates } L^p\big(\mathbf{R}^3; M\,dv\big),$$
$$L^p(M\,dv\,dx) \quad \text{designates } L^p\big(\mathbf{R}^3\times\mathbf{R}^3; M\,dv\,dx\big),$$
$$L^p(d\mu) \quad \text{designates } L^p\big(\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2; d\mu\big).$$

When the measure is the Lebesgue measure, we shall simply denote:

$$L^p_x := L^p\big(\mathbf{R}^3; dx\big), \qquad L^p_{t,x} := L^p\big(\mathbf{R}_+\times\mathbf{R}^3;\, dt\,dx\big).$$

Whenever E is a normed space, the notations $\mathrm{O}(\delta)_E$ and $\mathrm{o}(\delta)_E$ designate a family of elements of E whose norms are $\mathrm{O}(\delta)$ or $\mathrm{o}(\delta)$. (For instance $\mathrm{O}(1)_E$ designates a bounded family in E, while $\mathrm{o}(1)_E$ designates a sequence that converges to 0 in E.)

Although L^p_{loc} spaces are not normed spaces, we designate by the notation $\mathrm{O}(\delta)_{L^p_{\mathrm{loc}}(\Omega)}$ a family $f_\epsilon \in L^p_{\mathrm{loc}}(\Omega)$ such that, for each compact $K\subset\Omega$,

$$\|f_\epsilon\|_{L^p(K)} = \mathrm{O}(\delta).$$

The notation $\mathrm{o}(\delta)_{L^p_{\mathrm{loc}}(\Omega)}$ is defined similarly.

2.4. *Outline of the proof of Theorem 2.4*

In terms of the fluctuation g_ϵ, the scaled Boltzmann equation (2.3) with initial condition (2.18) can be put in the form:

$$\begin{gathered}\epsilon\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon = -\frac{1}{\epsilon}\mathcal{L}(g_\epsilon) + \mathcal{Q}(g_\epsilon, g_\epsilon),\\ g_{\epsilon|t=0} = g_\epsilon^{\mathrm{in}}.\end{gathered} \tag{2.38}$$

Step 1: We first prove that any limit point g of the family of fluctuations g_ϵ as $\epsilon\to 0^+$ satisfies,

$$g = \Pi g,$$

where Π is the orthogonal projection on the nullspace of $\mathcal{L}$ defined in (2.32).

Hence, the limiting fluctuation g is an infinitesimal Maxwellian, i.e. of the form:

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2-3\big). \tag{2.39}$$

The limiting form of the continuity equation (2.9) is equivalent to the incompressibility condition on u:

$$\operatorname{div}_x u = 0.$$

Step 2: In order to obtain equations for the moments,

$$\rho = \langle g\rangle, \qquad u = \langle vg\rangle, \quad \text{and} \quad \theta = \left\langle\left(\frac{1}{3}|v|^2-1\right)g\right\rangle,$$

we pass to the limit in approximate local conservation laws deduced from the Boltzmann equation in the following manner.

Besides the square-root renormalization, we use a renormalization of the scaled Boltzmann equation (2.3) based on a smooth truncation γ such that

$$\gamma\in C^\infty\big(\mathbf{R}^+, [0,1]\big), \quad \gamma|_{[0,\frac{3}{2}]}\equiv 1, \quad \gamma|_{[2,+\infty)}\equiv 0. \tag{2.40}$$

Define:

$$\hat\gamma(z) = \frac{d}{dz}\big((z-1)\gamma(z)\big). \tag{2.41}$$

Notice that

$$\operatorname{supp}(\hat\gamma)\subset[0,2], \quad \hat\gamma|_{[0,\frac{3}{2}]}\equiv 1, \quad \text{and} \quad \|1-\hat\gamma\|_{L^\infty}\leqslant 1+\|\gamma'\|_{L^\infty}. \tag{2.42}$$

We use below the notation γ_ϵ and $\hat\gamma_\epsilon$ to denote respectively $\gamma(G_\epsilon)$ and $\hat\gamma(G_\epsilon)$.

We also use a truncation of high velocities, defined as follows: given $\mathbf{k} > 6$, we set:

$$K_\epsilon = \mathbf{k}|\ln \epsilon|. \tag{2.43}$$

For each continuous scalar function, or vector- or tensor-field $\xi \equiv \xi(v)$, we denote by ξ_{K_ϵ} the following truncation of ξ:

$$\xi_{K_\epsilon}(v) = \xi(v)\mathbf{1}_{|v|^2 \leqslant K_\epsilon}. \tag{2.44}$$

Renormalizing the scaled Boltzmann equation (2.3) with the nonlinearity $\Gamma(Z) = (Z-1)\gamma(Z)$, we arrive at the following form of (2.38):

$$\partial_t(g_\epsilon\gamma_\epsilon) + \frac{1}{\epsilon} v \cdot \nabla_x(g_\epsilon\gamma_\epsilon) = \frac{1}{\epsilon^3}\hat{\gamma}_\epsilon \mathcal{Q}(G_\epsilon, G_\epsilon).$$

Multiplying each side of the equation above by ξ_{K_ϵ}, and averaging in the variable v leads to

$$\partial_t\langle \xi_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle + \operatorname{div}_x \frac{1}{\epsilon}\langle v\xi_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle = \frac{1}{\epsilon^3}\langle\!\langle \xi_{K_\epsilon}\hat{\gamma}_\epsilon\left(G'_\epsilon G'_{\epsilon 1} - G_\epsilon G_{\epsilon 1}\right)\rangle\!\rangle. \tag{2.45}$$

Henceforth we use the following notations for the fluxes of momentum or energy:

$$\mathbf{F}_\epsilon(\zeta) = \frac{1}{\epsilon}\langle \zeta_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle, \tag{2.46}$$

with

$$\zeta(v) = A(v) := v^{\otimes 2} - \frac{1}{3}|v|^2 I, \quad \text{or} \quad \zeta(v) = B(v) := \frac{1}{2} v\big(|v|^2 - 5\big).$$

Likewise, we use the notation,

$$\mathbf{D}_\epsilon(\xi) = \frac{1}{\epsilon^3}\langle\!\langle \xi_{K_\epsilon}\hat{\gamma}_\epsilon\left(G'_\epsilon G'_{\epsilon 1} - G_\epsilon G_{\epsilon 1}\right)\rangle\!\rangle, \tag{2.47}$$

for the conservation defect corresponding with the (truncated) quantity $\xi \equiv \xi(v)$, where $\xi \in \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$.

The Navier–Stokes motion equation is obtained by passing to the limit as $\epsilon \to 0$ modulo gradient fields in Eq. (2.45) for $\xi(v) = v_j$, $j = 1, 2, 3$, recast as

$$\partial_t\langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle + \operatorname{div}_x \mathbf{F}_\epsilon(A) + \nabla_x\left\langle \frac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\right\rangle = \mathbf{D}_\epsilon(v), \tag{2.48}$$

while the temperature equation is obtained by passing to the limit in that same equation with $\xi(v) = \frac{1}{2}(|v|^2 - 5)$, i.e. in

$$\partial_t\left\langle \frac{1}{2}\big(|v|^2 - 5\big)_{K_\epsilon} g_\epsilon\gamma_\epsilon\right\rangle + \operatorname{div}_x \mathbf{F}_\epsilon(B) = \mathbf{D}_\epsilon\left(\frac{1}{2}\big(|v|^2 - 5\big)\right). \tag{2.49}$$

For the mathematical study of that limiting process, the uniform a priori estimates obtained from the scaled entropy inequality are not sufficient. Our first task is therefore to improve these estimates using both:

(a) the properties of the collision operator (see Section 3), namely a suitable control on the relaxation based on the coercivity estimate (2.31):

$$\langle \phi\mathcal{L}\phi\rangle \geqslant C\|\phi - \Pi\phi\|^2_{L^2(M a\, dv)},$$

(b) and the properties of the free transport operator (see Section 4), namely dispersion and velocity averaging.

With the estimates obtained in Sections 3–4, we first prove (in Section 5) that the conservation defects vanish asymptotically:

$$\mathbf{D}_\epsilon(\xi) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\, dx), \quad \xi \in \operatorname{span}\big\{v_1, v_2, v_3, |v|^2\big\}.$$

Next we analyze the asymptotic behavior of the flux terms. This requires splitting these flux terms into a convection and a diffusion part (Section 6),

$$\mathbf{F}_\epsilon(\zeta) - 2\left\langle \zeta\left(\Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2\right\rangle + \frac{2}{\epsilon^2}\big\langle\hat{\zeta}\,\mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\big\rangle \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\, dx),$$

where $\hat{\zeta}$ is the unique solution in $(\operatorname{Ker}\mathcal{L})^\perp$ of the Fredholm integral equation,

$$\mathcal{L}\hat{\zeta} = \zeta.$$

For instance, the tensor field A and the vector field B defined by,

$$A(v) := v \otimes v - \frac{1}{3}|v|^2 I, \qquad B(v) := \frac{1}{2}\big(|v|^2 - 5\big)v \tag{2.50}$$

satisfy

$$A \perp \operatorname{Ker}\mathcal{L}, \qquad B \perp \operatorname{Ker}\mathcal{L} \tag{2.51}$$

componentwise, so that there exists a unique tensor field $\hat{A}$ and a unique vector field $\hat{B}$ such that

$$\mathcal{L}\hat{A} = A, \qquad \mathcal{L}\hat{B} = B, \quad \hat{A} \text{ and } \hat{B} \perp \operatorname{Ker}\mathcal{L}, \tag{2.52}$$

The diffusion terms are easily proved to converge towards the dissipation terms in the Navier–Stokes–Fourier system:

$$\frac{2}{\epsilon^2}\big\langle \hat{\zeta}\,\mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon})\big\rangle \to \big\langle \hat{\zeta}\,(v \cdot \nabla_x g)\big\rangle \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx).$$

The formulas (2.27) for the viscosity ν and heat conduction κ are easily shown to be equivalent to

$$\nu = \frac{1}{10}\langle \hat{A} : A\rangle, \qquad \kappa = \frac{2}{15}\langle \hat{B} \cdot B\rangle. \tag{2.53}$$

The (nonlinear) convection terms require a more careful treatment, involving in particular some spatial regularity argument and the filtering of acoustic waves (see Section 7).

3. Controls on the velocity dependence of the number density fluctuations

The goal of this section is to prove that the square number density fluctuation — or more precisely the following variant thereof,

$$\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2,$$

is uniformly integrable in v with the weight $(1 + |v|)^p$ for each $p < 2$.

In our previous work [13], we obtained this type of control for $p = 0$ only, by a fairly technical argument (see Section 6 of [13]). Basically, we used the entropy production bound to estimate some notion of distance between the number density and the gain part of a fictitious collision integral. The conclusion followed from earlier results by Grad and Caflisch on the v-regularity of the gain term in Boltzmann's collision integral linearized at some uniform Maxwellian state.

Unfortunately, this method seems to provide only estimates without the weight $(1 + |v|)^\beta$ (with β as in (1.6)) that is crucial for treating hard potentials other than the case of Maxwell molecules. Obtaining the weighted estimates requires some new ideas presented in this section.

The first such idea is to use the spectral gap estimate (2.31) for the linearized collision integral. Instead of comparing the number density to (some variant of) the local Maxwellian equilibrium — as in the case of the BGK model equation, treated in [23,24], or in the case of the Boltzmann equation with Maxwell molecules as in [13] — we directly compare the number density fluctuation to the infinitesimal Maxwellian that is its projection on hydrodynamic modes.

The lemma below provides the basic argument for arriving at such estimates.

Lemma 3.1. *Under the assumptions of Theorem* 2.4*, one has*:

$$\left\| \frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi \frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right\|_{L^2(M\,dv)} \leqslant \mathrm{O}(\epsilon)_{L^2_{t,x}} + \mathrm{O}(\epsilon) \left\| \frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right\|^2_{L^2(M\,dv)}. \tag{3.1}$$

Proof. In order to simplify the presentation we first define some fictitious collision integrals $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{Q}}$,

$$\tilde{\mathcal{L}}g = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (g + g_1 - g' - g_1')M_1\tilde{b}(v - v_1, \omega)\, dv_1\, d\omega,$$

$$\tilde{\mathcal{Q}}(g, h) = \frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{S}^2} (g'h_1' + g_1'h' - gh_1 - g_1h)M_1\tilde{b}(v - v_1, \omega)\, dv_1\, d\omega,$$

obtained from $\mathcal{L}$ and $\mathcal{Q}$ by replacing the original collision kernel b with

$$\tilde{b}(z, \omega) = \frac{b(z, \omega)}{1 + \int_{\mathbf{S}^2} b(z, \omega_1)\, d\omega_1}.$$

Start from the elementary formula:

$$\tilde{\mathcal{L}}\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon^2}\right) = \tilde{\mathcal{Q}}\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon}, \frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right) - \frac{1}{\epsilon^2}\tilde{\mathcal{Q}}\big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\big). \tag{3.2}$$

Multiplying both sides of this equation by $(I - \Pi)(\sqrt{G_\epsilon} - 1)$ and using the spectral gap estimate (2.31) leads to

$$\begin{aligned}&\left\|\frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right\|_{L^2(M\,dv)} \\ &\quad\leqslant \epsilon\left\|\tilde{\mathcal{Q}}\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon}, \frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)\right\|_{L^2(M\,dv)} + \epsilon\left\|\frac{1}{\epsilon^2}\tilde{\mathcal{Q}}\big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\big)\right\|_{L^2(M\,dv)}.\end{aligned} \tag{3.3}$$

Denote

$$d\tilde{\mu}(v, v_1, \omega) = MM_1\tilde{b}(v - v_1, \omega)\, d\omega\, dv\, dv_1.$$

By definition of $\tilde{b}$, one has:

$$\int_{\mathbf{S}^2} \tilde{b}(v - v_1, \omega)\, d\omega \leqslant 1.$$

Hence $\mathcal{Q}$ is continuous on $L^2(M\,dv)$: by (2.35)

$$\big\|\tilde{\mathcal{Q}}(g, h)\big\|_{L^2(M\,dv)} \leqslant 2\|g\|_{L^2(M\,dv)}\|h\|_{L^2(M\,dv)}.$$

(Notice that $\tilde{b}$ verifies (1.5) as does b.)

Plugging this estimate in (3.3) leads to

$$\left\|\frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right\|_{L^2(M\,dv)} \leqslant C\epsilon\left\|\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right\|^2_{L^2(M\,dv)} + \epsilon\left\|\frac{1}{\epsilon^2}\mathcal{Q}\big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\big)\right\|_{L^2(M\,dv)} \tag{3.4}$$

Finally, applying the Cauchy–Schwarz inequality as in the proof of (2.35), one finds that

$$\begin{aligned}\left\|\frac{1}{\epsilon^2}\tilde{\mathcal{Q}}\big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\big)\right\|^2_{L^2(M\,dv)} &\leqslant \left(\sup_{v\in\mathbf{R}^3}\iint_{\mathbf{R}^3\times\mathbf{S}^2} \tilde{b}(v - v_1, \omega)M_1\, dv_1\, d\omega\right)\frac{1}{\epsilon^4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \Big(\sqrt{G_\epsilon' G_{\epsilon 1}'} - \sqrt{G_\epsilon G_{\epsilon 1}}\Big)^2\, d\tilde{\mu}(v, v_1, \omega) \\ &\leqslant \frac{1}{\epsilon^4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \Big(\sqrt{G_\epsilon' G_{\epsilon 1}'} - \sqrt{G_\epsilon G_{\epsilon 1}}\Big)^2\, d\mu(v, v_1, \omega),\end{aligned}$$

since $0 \leqslant \tilde{b} \leqslant b$. By the entropy production estimate (d) in Proposition 2.3, the inequality above implies that

$$\left\|\frac{1}{\epsilon^2}\tilde{\mathcal{Q}}\big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\big)\right\|_{L^2(M\,dv)} = \mathrm{O}(1)_{L^2_{t,x}}.$$

This estimate and (3.4) entail the inequality (3.1). □

Notice that we could have used directly $\mathcal{L}$ and $\mathcal{Q}$ instead of their truncated analogues $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{Q}}$, obtaining bounds in weighted L^2 spaces by some loop argument, unfortunately much more technical than the proof above.

The main result in this section — and one of the key new estimate in this paper is:

Proposition 3.2. *Under the assumptions of Theorem* 2.4*, for each $T > 0$, each compact $K \subset \mathbf{R}^3$, and each $p < 2$, the family*

$$\left(1+|v|\right)^p\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2$$

is uniformly integrable in v on $[0,T]\times K\times\mathbf{R}^3$ with respect to the measure $dt\,dx\,M\,dv$. (*This means that, for each $\eta > 0$, there exists $\alpha > 0$ such that, for each measurable $\varphi \equiv \varphi(x,v)$ verifying*:

$$\|\varphi\|_{L^\infty_{x,v}} \leqslant 1 \quad \textit{and} \quad \|\varphi\|_{L^\infty_x(L^1_v)} \leqslant \alpha,$$

one has:

$$\int_0^T\int_K\int_{\mathbf{R}^3} \varphi\left(1+|v|\right)^p\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 M\,dv\,dx\,dt \leqslant \eta.)$$

Proof. Start from the decomposition:

$$\begin{aligned} J &:= \left(1+|v|\right)^p\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 \\ &= \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\left(1+|v|\right)^p \Pi\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) + \left(1+|v|\right)^{\frac{p}{2}}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\left(1+|v|\right)^{\frac{p}{2}}\left(\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) - \Pi\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right). \end{aligned} \tag{3.5}$$

We recall from the entropy bound (b) in Proposition 2.3 that

$$\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) = \mathrm{O}(1)_{L^\infty_t(L^2(dx\,M\,dv))}$$

so that, by definition (2.32) of the hydrodynamic projection Π

$$\Pi\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) = \mathrm{O}(1)_{L^\infty_t(L^2_x(L^q(M\,dv)))}, \tag{3.6}$$

for all $q < +\infty$. Therefore the first term in the right-hand side of (3.5) satisfies,

$$I = \left|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right|\left(1+|v|\right)^p\left|\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right| = \mathrm{O}(1)_{L^\infty_t(L^1_x(L^r(M\,dv)))}, \tag{3.7}$$

for all $0 \leqslant p < +\infty$ and $1 \leqslant r < 2$.

In order to estimate the second term in the right-hand side of (3.5), we first remark that, for each $\delta > 0$, each $p < 2$ and each $q < +\infty$, there exists some $C = C(p,q,\delta)$ such that

$$\left(1+|v|\right)^{p/2}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) = \mathrm{O}(\delta)_{L^\infty_t(L^2(dx\,M\,dv))} + \mathrm{O}\left(\frac{C(p,q,\delta)}{\epsilon}\right)_{L^\infty_{t,x}(L^q(M\,dv))}. \tag{3.8}$$

Indeed, by Young's inequality and Proposition 2.3(a),

$$\begin{aligned} \left(1+|v|\right)^p\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 &\leqslant \frac{\delta^2}{\epsilon^2}|G_\epsilon - 1|\left(\frac{(1+|v|)^p}{\delta^2}\right) \\ &\leqslant \frac{\delta^2}{\epsilon^2}h(G_\epsilon-1) + \frac{\delta^2}{\epsilon^2}h^*\left(\frac{(1+|v|)^p}{\delta^2}\right) \\ &= \mathrm{O}\left(\delta^2\right)_{L^\infty_t(L^1(dx\,M\,dv))} + \frac{\delta^2}{\epsilon^2}\exp\left(\frac{(1+|v|)^p}{\delta^2}\right). \end{aligned}$$

We next use (3.8) with the two following observations: first, the obvious continuity statement (3.6). Also, because of (3.1) and the entropy bound (b) in Proposition 2.3, one has:

$$\left\| \frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi \frac{\sqrt{G_\epsilon}-1}{\epsilon} \right\|_{L^2(M\,dv)} = \mathrm{O}(\epsilon)_{L^1_{\mathrm{loc}}(dt\,dx)}. \tag{3.9}$$

Hence

$$\begin{aligned}
&\bigl(1+|v|\bigr)^{\frac{p}{2}} \left| \frac{\sqrt{G_\epsilon}-1}{\epsilon} \right| \bigl(1+|v|\bigr)^{\frac{p}{2}} \left| \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) - \Pi \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) \right| \\
&\quad \leqslant \frac{\delta}{\epsilon} h(G_\epsilon - 1)^{1/2} \bigl(1+|v|\bigr)^{p/2} \left| \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) - \Pi \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) \right| \\
&\qquad + \frac{\delta}{\epsilon} \bigl(1+|v|\bigr)^{p/2} \exp\left(\frac{(1+|v|)^p}{2\delta^2} \right) \left| \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) - \Pi \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) \right| \\
&\quad =: \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Now

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{\delta}{2\epsilon^2} h(G_\epsilon - 1) + \delta \bigl(1+|v|\bigr)^p \left| \Pi \frac{\sqrt{G_\epsilon}-1}{\epsilon} \right|^2 + \delta \bigl(1+|v|\bigr)^p \left| \frac{\sqrt{G_\epsilon}-1}{\epsilon} \right|^2 \\
&= \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))} + \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))} + \delta J.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|\mathrm{III}\|_{L^1_{\mathrm{loc}}(dt\,dx;L^r(M\,dv))} &\leqslant \delta \left\| \bigl(1+|v|\bigr)^{p/2} \exp\left(\frac{(1+|v|)^p}{2\delta^2} \right) \right\|_{L^q(M\,dv)} \\
&\quad \times \left\| \frac{1}{\epsilon} \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) - \Pi \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right) \right\|_{L^1_{\mathrm{loc}}(dt\,dx;L^2(M\,dv))} \\
&= \mathrm{O}\bigl(\delta C(p,q,\delta)\bigr),
\end{aligned}$$

with $r = \frac{2q}{q+2}$.

Putting all these controls together shows that

$$\begin{aligned}
J \leqslant \mathrm{I} + \mathrm{II} + \mathrm{III} &= \mathrm{O}(1)_{L^\infty_t(L^1_x(L^r(M\,dv)))} + \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))} + \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))} + \delta J \\
&\quad + \mathrm{O}\bigl(\delta C(p,q,\delta)\bigr)_{L^1_{\mathrm{loc}}(dt\,dx;L^r(M\,dv))},
\end{aligned} \tag{3.10}$$

i.e.

$$\begin{aligned}
&(1-\delta)\bigl(1+|v|\bigr)^p \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right)^2 \\
&\quad \leqslant \mathrm{O}(1)_{L^\infty_t(L^1_x(L^r(M\,dv)))} + \mathrm{O}\bigl(\delta C(p,q,\delta)\bigr)_{L^1_{\mathrm{loc}}(dt\,dx;L^r(M\,dv))} + \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))},
\end{aligned}$$

which entails the uniform integrability in v stated in Proposition 3.2. □

Remark. Replacing the estimate for II above with

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{8\delta}{2\epsilon^2} h(G_\epsilon - 1) + \frac{\delta}{8} \bigl(1+|v|\bigr)^p \left| \Pi \frac{\sqrt{G_\epsilon}-1}{\epsilon} \right|^2 + \frac{\delta}{8} \bigl(1+|v|\bigr)^p \left| \frac{\sqrt{G_\epsilon}-1}{\epsilon} \right|^2 \\
&= \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))} + \mathrm{O}(\delta)_{L^\infty_t(L^1(M\,dv\,dx))} + \frac{\delta}{8} J,
\end{aligned}$$

and choosing $\delta = 4$ in (3.10) shows that

$$\bigl(1+|v|\bigr)^2 \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} \right)^2 \text{ is bounded in } L^1_{\mathrm{loc}}\bigl(dt\,dx; L^1(M\,dv)\bigr).$$

In [3], the Navier–Stokes limit of the Boltzmann equation is established assuming the uniform integrability in $[0,T]\times K\times\mathbf{R}^3$ for the measure $dt\,dx\,M\,dv$ of a quantity analogous to the one considered in this bound. As we shall see, the Navier–Stokes–Fourier limit of the Boltzmann equation is derived in the present paper by using only the weaker information in Proposition 3.2.

4. Compactness results for the number density fluctuations

The following result is the main technical step in the present paper.

Proposition 4.1. *Under the assumptions in Theorem* 2.4*, for each $T>0$, each compact $K\subset\mathbf{R}^3$ and each $p<2$, the family of functions,*

$$\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\bigl(1+|v|\bigr)^p,$$

is uniformly integrable on $[0,T]\times K\times\mathbf{R}^3$ for the measure $dt\,dx\,M\,dv$.

This proposition is based on the uniform integrability in v of that same quantity, established in Proposition 3.2, together with a bound on the streaming operator applied to (a variant of) the number density fluctuation (stated in Lemma 4.2). Except for some additional truncations, the basic principle of the proof is essentially the same as explained in Lemma 3.6 of [13] (which is recalled in Appendix B). In other words, while the result of Proposition 3.2 provides some kind of regularity in v only for the number density fluctuation, the bound on the free transport part of the Boltzmann equation gives the missing regularity (in the x-variable).

The technical difficulty comes from the fact that the square-root renormalization $\Gamma(Z)=\sqrt{Z}$ is not admissible for the Boltzmann equation due to the singularity at $Z=0$. We will therefore use an approximation of the square-root, namely $z\mapsto\sqrt{z+\epsilon^\alpha}$ for some $\alpha\in\,]1,2[$.

Lemma 4.2. *Under the assumptions in Theorem* 2.4*, for each $\alpha>0$, one has*:

$$(\epsilon\partial_t+v\cdot\nabla_x)\frac{\sqrt{\epsilon^\alpha+G_\epsilon}-1}{\epsilon}=\mathrm{O}\bigl(\epsilon^{2-\alpha/2}\bigr)_{L^1(M\,dv\,dx\,dt)}+\mathrm{O}(1)_{L^2((1+|v|)^{-\beta}M\,dv\,dx\,dt)}+\mathrm{O}(\epsilon)_{L^1_{\mathrm{loc}}(dt\,dx;L^2((1+|v|)^{-\beta}M\,dv))}.$$

Proof. Start from the renormalized form of the scaled Boltzmann equation (2.3), with normalizing function:

$$\Gamma_\epsilon(Z)=\frac{\sqrt{\epsilon^\alpha+Z}-1}{\epsilon}.$$

This equation can be written as

$$(\epsilon\partial_t+v\cdot\nabla_x)\frac{\sqrt{\epsilon^\alpha+G_\epsilon}-1}{\epsilon}=\frac{1}{\epsilon^2}\frac{1}{2\sqrt{\epsilon^\alpha+G_\epsilon}}\mathcal{Q}(G_\epsilon,G_\epsilon)=Q^1_\epsilon+Q^2_\epsilon, \tag{4.1}$$

with

$$\begin{aligned}Q^1_\epsilon&=\frac{1}{\epsilon^2}\frac{1}{2\sqrt{\epsilon^\alpha+G_\epsilon}}\iint\Bigl(\sqrt{G'_\epsilon G'_{\epsilon1}}-\sqrt{G_\epsilon G_{\epsilon1}}\Bigr)^2 b(v-v_1,\omega)\,d\omega\,M_1\,dv_1,\\ Q^2_\epsilon&=\frac{1}{\epsilon^2}\frac{1}{\sqrt{\epsilon^\alpha+G_\epsilon}}\iint\sqrt{G_\epsilon G_{\epsilon1}}\Bigl(\sqrt{G'_\epsilon G'_{\epsilon1}}-\sqrt{G_\epsilon G_{\epsilon1}}\Bigr)b(v-v_1,\omega)\,d\omega\,M_1\,dv_1.\end{aligned} \tag{4.2}$$

The entropy production estimate (d) in Proposition 2.3 and the obvious inequality

$$\sqrt{\epsilon^\alpha+G_\epsilon}\geqslant\epsilon^{\alpha/2}$$

imply that

$$\|Q^1_\epsilon\|_{L^1(M\,dv\,dx\,dt)}\leqslant\frac{1}{2}C^{\mathrm{in}}\epsilon^{2-\alpha/2}. \tag{4.3}$$

On the other hand

$$Q_\epsilon^2 = \frac{\sqrt{G_\epsilon}}{\sqrt{\epsilon^\alpha + G_\epsilon}} \iint \sqrt{G_{\epsilon 1}} \frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} b(v - v_1, \omega)\, d\omega\, M_1\, dv_1.$$

Write

$$\sqrt{G_{\epsilon 1}} = 1 + \epsilon \frac{\sqrt{G_{\epsilon 1}} - 1}{\epsilon}.$$

Apply the Cauchy–Schwarz inequality as in the proof of (2.35), then

$$\begin{aligned}
&\left\| \iint \sqrt{G_{\epsilon 1}} \frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} b(v - v_1, \omega) M_1\, dv_1\, d\omega \right\|_{L^2((1+|v|)^{-\beta} M\, dv)} \\
&\quad \leqslant \sup_{v \in \mathbf{R}^3} \left(\left(1 + |v|\right)^{-\beta} \iint b(v - v_1, \omega) M_1\, dv_1\, d\omega \right)^{1/2} \left\langle\!\!\left\langle \left(\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} \right)^2 \right\rangle\!\!\right\rangle^{1/2} \\
&\qquad + \epsilon \sup_{v \in \mathbf{R}^3} \left(\left(1 + |v|\right)^{-\beta} \iint M_1 \left(\frac{\sqrt{G_{\epsilon 1}} - 1}{\epsilon} \right)^2 b(v - v_1, \omega)\, dv_1\, d\omega \right)^{1/2} \left\langle\!\!\left\langle \left(\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} \right)^2 \right\rangle\!\!\right\rangle^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\| \iint \sqrt{G_{\epsilon 1}} \frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} b(v - v_1, \omega) M_1\, dv_1 \right\|_{L^2((1+|v|)^{-\beta} M\, dv)} \\
&\quad \leqslant C \left(1 + \epsilon \left\| \frac{\sqrt{G_{\epsilon 1}} - 1}{\epsilon} \right\|_{L^2(M_1(1+|v_1|^\beta)\, dv_1)} \right) \left\langle\!\!\left\langle \left(\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} \right)^2 \right\rangle\!\!\right\rangle^{1/2},
\end{aligned}$$

because of the upper bound in Grad's cut-off assumption (1.6).

Hence, on account of Proposition 3.2 and the entropy production estimate (d) in Proposition 2.3

$$Q_\epsilon^2 = \mathrm{O}(1)_{L^2((1+|v|)^{-\beta} M\, dv\, dx\, dt)} + \mathrm{O}(\epsilon)_{L^1_{\mathrm{loc}}(dt\, dx; L^2((1+|v|)^{-\beta} M\, dv))}. \tag{4.4}$$

Both estimates (4.3) and (4.4) together with (4.1) entail the control in Lemma 4.2. □

With Lemma 4.2 at our disposal, we next proceed to the:

Proof of Proposition 4.1. *Step* 1. We claim that, for $\alpha > 1$,

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon} \right)^2 - \left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right)^2 = \mathrm{O}\left(\epsilon^{\alpha - 1}\right)_{L^\infty_t(L^2_{\mathrm{loc}}(dx; L^2(M\, dv)))} + \mathrm{O}\left(\epsilon^{\alpha/2}\right)_{L^\infty_t(L^1(M\, dv\, dx))}. \tag{4.5}$$

Indeed,

$$\begin{aligned}
\left| \frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon} - \frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right| &\leqslant \frac{\epsilon^\alpha \mathbf{1}_{G_\epsilon > 1/2}}{\epsilon(\sqrt{\epsilon^\alpha + G_\epsilon} + \sqrt{G_\epsilon})} + \epsilon^{\alpha/2 - 1} \mathbf{1}_{G_\epsilon \leqslant 1/2} \\
&\leqslant \mathrm{O}\left(\epsilon^{\alpha - 1}\right)_{L^\infty_{t,x,v}} + \epsilon^{\alpha/2} \frac{\sqrt{2}}{\sqrt{2} - 1} \left| \frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right| \\
&= \mathrm{O}\left(\epsilon^{\alpha - 1}\right)_{L^\infty_{t,x,v}} + \mathrm{O}\left(\epsilon^{\alpha/2}\right)_{L^\infty_t(L^2(M\, dv\, dx))},
\end{aligned} \tag{4.6}$$

and we conclude with the decomposition,

$$\begin{aligned}
\left| \left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon} \right)^2 - \left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right)^2 \right| &= \left(\mathrm{O}\left(\epsilon^{\alpha - 1}\right)_{L^\infty_{t,x,v}} + \mathrm{O}\left(\epsilon^{\alpha/2}\right)_{L^\infty_t(L^2(M\, dv\, dx))} \right) \\
&\quad \times \left(\mathrm{O}\left(\epsilon^{\alpha - 1}\right)_{L^\infty_{t,x,v}} + \mathrm{O}\left(\epsilon^{\alpha/2}\right)_{L^\infty_t(L^2(M\, dv\, dx))} + 2 \left| \frac{\sqrt{G_\epsilon} - 1}{\epsilon} \right| \right),
\end{aligned}$$

together with the fluctuation control (b) in Proposition 2.3.

Step 2. Let γ be a smooth truncation as in (2.40), and set

$$\phi_\epsilon^\delta = \left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 \gamma\left(\epsilon\delta\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)\right).$$

We claim that, for each fixed $\delta > 0$,

$$(\epsilon\partial_t + v\cdot\nabla_x)\phi_\epsilon^\delta = \mathrm{O}\left(\frac{1}{\delta}\right)_{L^1_{\mathrm{loc}}(M\,dv\,dx\,dt)}. \tag{4.7}$$

Indeed,

$$(\epsilon\partial_t + v\cdot\nabla_x)\phi_\epsilon^\delta = \tilde\gamma\left(\epsilon\delta\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)\right)\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)\left(Q_\epsilon^1 + Q_\epsilon^2\right),$$

where $\tilde\gamma(Z) = 2\gamma(Z) + Z\gamma'(Z)$, while Q_ϵ^1 and Q_ϵ^2 are defined in (4.2).

Clearly, $\tilde\gamma$ has support in $[0, 2]$, so that

$$\tilde\gamma\left(\epsilon\delta\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)\right)\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right) = \mathrm{O}\left(\frac{1}{\epsilon\delta}\right)_{L^\infty_{t,x,v}}.$$

On the other hand, the fluctuation control (b) in Proposition 2.3 and the estimate (4.6) imply that

$$\tilde\gamma\left(\epsilon\delta\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)\right)\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right) = \mathrm{O}(1)_{L^\infty_t(L^2_{\mathrm{loc}}(dx; L^2(M\,dv)))}.$$

Together with Lemma 4.2, these last two estimates lead to the following bound:

$$\begin{aligned}(\epsilon\partial_t + v\cdot\nabla_x)\phi_\epsilon^\delta = {}& \mathrm{O}\left(\frac{\epsilon^{1-\alpha/2}}{\delta}\right)_{L^1(M\,dv\,dx\,dt)} + \mathrm{O}(1)_{L^2_t(L^1_{\mathrm{loc}}(dx; L^1((1+|v|)^{-\beta/2}M\,dv)))} \\ & + \mathrm{O}\left(\frac{1}{\delta}\right)_{L^1_{\mathrm{loc}}(dt\,dx; L^2((1+|v|)^{-\beta}M\,dv))}.\end{aligned}$$

Pick then $\alpha \in (1, 2)$; the last estimate implies that (4.7) holds for each $\delta > 0$, as announced.

Step 3. On the other hand, we already know from the fluctuation control (b) in Proposition 2.3 and (4.5) that

$$\phi_\epsilon^\delta = \mathrm{O}(1)_{L^\infty_t(L^1_{\mathrm{loc}}(M\,dv\,dx))}. \tag{4.8}$$

Moreover

$$\phi_\epsilon^\delta \text{ is locally uniformly integrable in the } v\text{-variable.} \tag{4.9}$$

Indeed, for each $\varphi \in L^\infty_{x,v} \cap L^\infty_x(L^1_v)$, one has:

$$\begin{aligned}\left|\iint_K \phi_\epsilon^\delta \varphi M\,dv\,dx\right| \leqslant {}& \iint \left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2 |\varphi| M\,dx\,dv \\ & + \iint_K \left|\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 - \left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2\right| |\varphi| M\,dv\,dx.\end{aligned}$$

The second term is $\mathrm{O}(\epsilon^{\alpha-1})\|\phi\|_{L^\infty}$. Hence this term can be made smaller than any given η whenever $\epsilon < \epsilon_0(\eta)$. Since ϵ denotes an extracted subsequence converging to 0, there remain only finitely many terms, say $N \equiv N(\eta)$ that can also be made smaller that η, this time by choosing $\|\phi\|_{L^\infty_x(L^1_v)}$ smaller than $c \equiv c(N(\eta), \eta)$. As for the first term, it can be made less than η whenever $\|\phi\|_{L^\infty_x(L^1_v)} \leqslant c'(\eta)$, by Proposition 3.2. Therefore

$$\left|\iint_K \phi_\epsilon^\delta \varphi M\,dv\,dx\right| \leqslant 2\eta \quad \text{for each } \epsilon \text{ and } \delta > 0,$$

whenever $\|\phi\|_{L^\infty_x(L^1_v)} \leqslant \min(c(N(\eta), \eta), c'(\eta))$, which establishes (4.9).

Applying Theorem B.1 (taken from [13]) in Appendix B, we conclude from (4.8), (4.9) and (4.7) that

$$\text{for each } \delta > 0, \quad \phi_\epsilon^\delta \text{ is locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3, \tag{4.10}$$

for the measure $M\,dv\,dx\,dt$.

Step 4. But, for each $\epsilon, \delta \in (0,1)$, one has:

$$\begin{aligned}\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 - \phi_\epsilon^\delta &= \left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 \left(1 - \gamma\left(\epsilon\delta\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)\right)\right) \\ &\leqslant \left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 \mathbf{1}_{G_\epsilon > 1/\delta^2} \\ &\leqslant \frac{1}{\epsilon^2} G_\epsilon \mathbf{1}_{G_\epsilon > 1/\delta^2} \leqslant \frac{C}{|\ln\delta|}\frac{1}{\epsilon^2} h(G_\epsilon - 1)\mathbf{1}_{G_\epsilon > 1/\delta^2},\end{aligned}$$

so that

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 - \phi_\epsilon^\delta = O\left(\frac{1}{|\ln\delta|}\right)_{L_t^\infty(L^1(M\,dv\,dx))},$$

by the fluctuation control (a) in Proposition 2.3. This and (4.10) imply that

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 \text{ is also locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3, \tag{4.11}$$

for the measure $M\,dv\,dx\,dt$.

Because of the estimate (4.5) in Step 1, we finally conclude that

$$\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2 \text{ is locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3, \tag{4.12}$$

for the measure $M\,dv\,dx\,dt$.

Together with the control of large velocities in Proposition 3.2, the statement (4.12) entails Proposition 4.1. □

Here is a first consequence of Proposition 4.1, bearing on the relaxation to infinitesimal Maxwellians.

Proposition 4.3. *Under the assumptions of Theorem* 2.4*, one has*:

$$\frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon} \to 0 \quad \text{in } L^2_{\text{loc}}\big(dt\,dx; L^2\big((1+|v|)^p M\,dv\big)\big),$$

for each $p < 2$ *as* $\epsilon \to 0$.

Proof. By Proposition 4.1, the family,

$$\big(1+|v|\big)^p\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2,$$

is uniformly integrable on $[0,T] \times K \times \mathbf{R}^3$ for the measure $M\,dv\,dx\,dt$, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$.

On the other hand, (3.1) and the fluctuation control (b) in Proposition 2.3 imply that

$$\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right) \to 0 \quad \text{in } L^1_{\text{loc}}(M\,dv\,dx\,dt),$$

and therefore in $M\,dv\,dx\,dt$-measure locally on $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$.

Therefore

$$\big(1+|v|\big)^p\left(\frac{\sqrt{G_\epsilon} - 1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon} - 1}{\epsilon}\right)^2 \to 0 \quad \text{in } L^1_{\text{loc}}\big(dt\,dx; L^1(M\,dv)\big),$$

which implies the convergence stated above. □

We conclude this section with the following variant of the classical velocity averaging theorem [11,12], stated as Theorem B.2 in [13]. This result is needed in order to handle the nonlinear terms appearing in the hydrodynamic limit.

Proposition 4.4. *Under the assumptions of Theorem* 2.4*, for each* $\xi \in L^2(M\,dv)$*, each* $T > 0$ *and each compact* $K \subset \mathbf{R}^3$*,*

$$\int_0^T \int_K \left| \langle \xi g_\epsilon \gamma_\epsilon \rangle (t, x+y) - \langle \xi g_\epsilon \gamma_\epsilon \rangle (t,x) \right|^2 dx\,dt \to 0,$$

as $|y| \to 0^+$*, uniformly in* $\epsilon > 0$*.*

Proof. Observe that

$$g_\epsilon \gamma_\epsilon - 2\frac{\sqrt{G_\epsilon}-1}{\epsilon} = \frac{\sqrt{G_\epsilon}-1}{\epsilon}\big((\sqrt{G_\epsilon}+1)\gamma_\epsilon - 2\big),$$

since, up to extraction,

$$(\sqrt{G_\epsilon}+1)\gamma_\epsilon - 2 \to 0 \text{ a.e.} \quad \text{and} \quad \left|(\sqrt{G_\epsilon}+1)\gamma_\epsilon - 2\right| \leqslant 3 + \sqrt{2},$$

it follows from Proposition 4.1 and Theorem A.1 in Appendix A, referred to as the Product Limit Theorem, that

$$g_\epsilon \gamma_\epsilon - 2\frac{\sqrt{G_\epsilon}-1}{\epsilon} \to 0 \quad \text{in } L^2_{\mathrm{loc}}\big(dt\,dx; L^2(M\,dv)\big), \tag{4.13}$$

as $\epsilon \to 0$.

This estimate, and Step 1 in the proof of Proposition 4.1 (and especially the estimate (4.5) there) shows that one can replace $g_\epsilon \gamma_\epsilon$ with $\frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon}$ with $\alpha > 1$ in the equicontinuity statement of Proposition 4.4.

Using (4.11) shows that, for each $\alpha \in (1,2)$, the family,

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon}\right)^2 \text{ is locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3,$$

for the measure $M\,dv\,dx\,dt$. In view of the estimate (4.5) and Proposition 3.2, we also control the contribution of large velocities in the above term, so that, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon}\right)^2 \text{ is uniformly integrable on } [0,T] \times K \times \mathbf{R}^3,$$

for the measure $M\,dv\,dx\,dt$.

On the other hand, Lemma 4.2 shows that the family,

$$(\epsilon \partial_t + v \cdot \nabla_x)\frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon}$$

is bounded in $L^1_{\mathrm{loc}}(M\,dv\,dx\,dt)$.

Applying then Theorem B.2 (taken from [13]) in Appendix B shows that, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$, one has:

$$\int_0^T \int_K \left| \left\langle \xi \frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon} \right\rangle (t, x+y) - \left\langle \xi \frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon} \right\rangle (t,x) \right|^2 dx\,dt \to 0,$$

as $|y| \to 0$ uniformly in ϵ, which concludes the proof of Proposition 4.4. □

5. Vanishing of conservation defects

Conservation defects appear in the renormalized form of the Boltzmann equation precisely because the natural symmetries of the collision integral are broken by the renormalization procedure. However, these conservation defects vanish in the hydrodynamic limit, as shown by the following:

Proposition 5.1. *Under the same assumptions as in Theorem* 2.4, *for each* $\xi \in \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$, *one has the following convergence for the conservation defects* $\mathbf{D}_\epsilon(\xi)$ *defined by* (2.47):

$$\mathbf{D}_\epsilon(\xi) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx) \text{ as } \epsilon \to 0.$$

Proof. For $\xi \in \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$, the associated defect $\mathbf{D}_\epsilon(\xi)$ is split as follows:

$$\mathbf{D}_\epsilon(\xi) = \mathbf{D}^1_\epsilon(\xi) + \mathbf{D}^2_\epsilon(\xi), \tag{5.1}$$

with

$$\mathbf{D}^1_\epsilon(\xi) = \frac{1}{\epsilon^3}\langle\!\langle \xi_{K_\epsilon}\hat{\gamma}_\epsilon\big(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}\big)^2\rangle\!\rangle,$$

and

$$\mathbf{D}^2_\epsilon(\xi) = \frac{2}{\epsilon^3}\langle\!\langle \xi_{K_\epsilon}\hat{\gamma}_\epsilon\sqrt{G_\epsilon G_{\epsilon 1}}\big(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}\big)\rangle\!\rangle,$$

with the notation (2.15) and (2.16).

That the term $\mathbf{D}^1_\epsilon(\xi)$ vanishes for $\xi(v) = \mathrm{O}(|v|^2)$ as $|v| \to +\infty$ is easily seen as follows:

$$\big\|\mathbf{D}^1_\epsilon(\xi)\big\|_{L^1_{t,x}} \leqslant \epsilon\|\xi_{K_\epsilon}\hat{\gamma}_\epsilon\|_{L^\infty_{t,x,v}}\left\|\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\right\|^2_{L^2_{t,x,\mu}} \leqslant \epsilon\mathrm{O}(K_\epsilon)\mathrm{O}(1) = \mathrm{O}\big(\epsilon|\ln\epsilon|\big), \tag{5.2}$$

because of the entropy production estimate in Proposition 2.3(d) and the choice of K_ϵ in (2.43).

We further decompose $\mathbf{D}^2_\epsilon(\xi)$ in the following manner:

$$\mathbf{D}^2_\epsilon(\xi) = \mathbf{D}^{21}_\epsilon(\xi) + \mathbf{D}^{22}_\epsilon(\xi) + \mathbf{D}^{23}_\epsilon(\xi), \tag{5.3}$$

with

$$\mathbf{D}^{21}_\epsilon(\xi) = -\frac{2}{\epsilon}\left\langle\!\!\left\langle \xi\mathbf{1}_{|v|^2>K_\epsilon}\hat{\gamma}_\epsilon\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\sqrt{G_\epsilon G_{\epsilon 1}}\right\rangle\!\!\right\rangle,$$

$$\mathbf{D}^{22}_\epsilon(\xi) = \frac{2}{\epsilon}\left\langle\!\!\left\langle \xi\hat{\gamma}_\epsilon(1 - \hat{\gamma}_{\epsilon 1}\hat{\gamma}'_\epsilon\hat{\gamma}'_{\epsilon 1})\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\sqrt{G_\epsilon G_{\epsilon 1}}\right\rangle\!\!\right\rangle,$$

and, by symmetry in the v and v_1 variables,

$$\mathbf{D}^{23}_\epsilon(\xi) = \frac{1}{\epsilon}\left\langle\!\!\left\langle (\xi + \xi_1)\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon 1}\hat{\gamma}'_\epsilon\hat{\gamma}'_{\epsilon 1}\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\sqrt{G_\epsilon G_{\epsilon 1}}\right\rangle\!\!\right\rangle.$$

The terms $\mathbf{D}^{21}_\epsilon(\xi)$ and $\mathbf{D}^{23}_\epsilon(\xi)$ are easily mastered by the following classical estimate on the tail of Gaussian distributions (see for instance [13] on p. 103 for a proof).

Lemma 5.2. *Let* $\mathbf{G}_N(z)$ *be the centered, reduced Gaussian density in* $\mathbf{R}^N$, *i.e.*

$$\mathbf{G}_N(z) = \frac{1}{(2\pi)^{N/2}}e^{-\frac{1}{2}|z|^2}.$$

Then

$$\int\limits_{|z|^2>R} |z|^p\mathbf{G}_N(z)\,dz \sim (2\pi)^{-N/2}\big|\mathbf{S}^{N-1}\big|R^{\frac{p+N}{2}-1}e^{-\frac{1}{2}R},$$

as $R \to +\infty$.

Indeed, because of the upper bound on the collision cross-section in (1.6), for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\begin{aligned}\left\|\mathbf{D}_\epsilon^{21}(\xi)\right\|_{L^1([0,T]\times K)} &\leqslant \frac{2}{\epsilon}\|\xi\mathbf{1}_{|v|^2>K_\epsilon}\hat{\gamma}_\epsilon\sqrt{G_\epsilon G_{\epsilon 1}}\|_{L^2([0,T]\times K,L^2_\mu)}\left\|\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\right\|_{L^2_{t,x,\mu}}\\ &\leqslant \frac{C_b^{1/2}}{\epsilon}\left\|\xi^2\mathbf{1}_{|v|^2>K_\epsilon}\left(1+|v|\right)^\beta\right\|^{1/2}_{L^1(M\,dv)}\|\hat{\gamma}_\epsilon\sqrt{G_\epsilon}\|_{L^\infty_{t,x,v}}\\ &\quad\times\left\|G_{\epsilon 1}\left(1+|v_1|\right)^\beta\right\|^{1/2}_{L^1([0,T]\times K,L^1(M_1\,dv_1))}\left\|\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\right\|_{L^2_{t,x,\mu}}.\end{aligned}$$

In the last right-hand side of the above chain of inequalities, one has obviously:

$$\|\hat{\gamma}_\epsilon\sqrt{G_\epsilon}\|_{L^\infty_{t,x,v}} = \mathrm{O}(1).$$

From Young's inequality and the entropy bound (2.19), we deduce that

$$G_\epsilon\left(1+|v|\right)^2 \leqslant \left(1+|v|^2\right)+4\left(h(G_\epsilon-1)+h^*\left(\frac{1+|v|^2}{4}\right)\right) = \mathrm{O}(1)_{L^1([0,T]\times K,L^1(M\,dv))}.$$

Lemma 5.2 and the condition $\xi(v) = \mathrm{O}(|v|^2)$ as $|v| \to +\infty$ imply that

$$\left\|\xi^2\mathbf{1}_{|v|^2>K_\epsilon}\left(1+|v|\right)^\beta\right\|^{1/2}_{L^1(M\,dv)} = \mathrm{O}\left(K_\epsilon^{\frac{\beta+5}{2}}e^{-\frac{1}{2}K_\epsilon}\right) = \mathrm{O}\left(\epsilon^{\mathbf{k}/2}|\ln\epsilon|^{\frac{\beta+5}{2}}\right),$$

on account of (2.43). Thus

$$\left\|\mathbf{D}_\epsilon^{21}(\xi)\right\|_{L^1([0,T]\times K)} = \mathrm{O}\left(\epsilon^{\mathbf{k}/2-1}|\ln\epsilon|^{\frac{\beta+5}{2}}\right) \to 0, \tag{5.4}$$

for all $\xi(v) = \mathrm{O}(|v|^2)$ as $|v| \to +\infty$ as soon as $\mathbf{k} > 2$.

Next we handle $\mathbf{D}_\epsilon^{23}(\xi)$. Whenever ξ is a collision invariant (i.e. whenever ξ belongs to the linear span of $\{1, v_1, v_2, v_3, |v|^2\}$) then $\xi + \xi_1 = \xi' + \xi'_1$, and using the $(v, v_1) - (v', v'_1)$ symmetry (2.28) in the integral defining $\mathbf{D}_\epsilon^{23}(\xi)$ leads to

$$\mathbf{D}_\epsilon^{23}(\xi) = -\frac{1}{\epsilon}\left\langle\!\!\left\langle(\xi+\xi_1)\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon 1}\hat{\gamma}'_\epsilon\hat{\gamma}'_{\epsilon 1}\frac{\left(\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}\right)^2}{2\epsilon^2}\right\rangle\!\!\right\rangle = -\mathbf{D}_\epsilon^{231}(\xi) - \mathbf{D}_\epsilon^{232}(\xi),$$

where

$$\mathbf{D}_\epsilon^{231}(\xi) = \frac{1}{2}\epsilon\left\langle\!\!\left\langle(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1^2|\leqslant K_\epsilon}\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon 1}\hat{\gamma}'_\epsilon\hat{\gamma}'_{\epsilon 1}\frac{\left(\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}\right)^2}{\epsilon^4}\right\rangle\!\!\right\rangle,$$

and

$$\mathbf{D}_\epsilon^{232}(\xi) = \frac{1}{2}\epsilon\left\langle\!\!\left\langle(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1^2|>K_\epsilon}\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon 1}\hat{\gamma}'_\epsilon\hat{\gamma}'_{\epsilon 1}\frac{\left(\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}\right)^2}{\epsilon^4}\right\rangle\!\!\right\rangle.$$

Then

$$\begin{aligned}\left\|\mathbf{D}_\epsilon^{231}(\xi)\right\|_{L^1_{t,x}} &\leqslant \epsilon\left\|\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\right\|^2_{L^2_{t,x,\mu}}\frac{1}{2}\left\|(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2\leqslant K_\epsilon}\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon 1}\hat{\gamma}'_\epsilon\hat{\gamma}'_{\epsilon 1}\right\|_{L^\infty_{t,x,v,v_1,\omega}}\\ &= \epsilon\cdot\mathrm{O}(1)\cdot\mathrm{O}(K_\epsilon)\|\hat{\gamma}\|^4_{L^\infty},\end{aligned}$$

so that

$$\left\|\mathbf{D}_\epsilon^{231}(\xi)\right\|_{L^1_{t,x}} = \mathrm{O}(\epsilon K_\epsilon) \to 0 \quad \text{as } \epsilon \to 0. \tag{5.5}$$

On the other hand, since $G_\epsilon \in [0,2]$ whenever $\hat{\gamma}(G_\epsilon) \neq 0$,

$$\begin{aligned}
\left\|\mathbf{D}_\epsilon^{232}(\xi)\right\|_{L^\infty_{t,x}} &\leqslant 16\|\hat{\gamma}\|^4_{L^\infty}\frac{1}{\epsilon^3}\left\|\frac{1}{2}(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2>K_\epsilon}\right\|_{L^1_\mu} \\
&\leqslant \mathrm{O}\left(\frac{1}{\epsilon^3}\right)\left\|\left(1+|v|^2+|v_1|^2\right)\left(1+|v-v_1|\right)^\beta \mathbf{1}_{|v|^2+|v_1|^2>K_\epsilon}\right\|_{L^1(MM_1\,dv\,dv_1)} \\
&= \mathrm{O}\left(\frac{1}{\epsilon^3}\right)\left\|\left(1+|v|^2+|v_1|^2\right)^{1+\beta/2}\mathbf{1}_{|v|^2+|v_1|^2>K_\epsilon}\right\|_{L^1(MM_1\,dv\,dv_1)} \\
&= \mathrm{O}\left(\frac{1}{\epsilon^3}\right)\mathrm{O}\left(e^{-K_\epsilon/2}K_\epsilon^{\frac{\beta+6}{2}}\right),
\end{aligned}$$

so that

$$\left\|\mathbf{D}_\epsilon^{232}(\xi)\right\|_{L^\infty_{t,x}} = \mathrm{O}\left(\epsilon^{\mathbf{k}/2-3}|\ln\epsilon|^{\frac{\beta+6}{2}}\right) \to 0 \quad \text{as } \epsilon \to 0, \tag{5.6}$$

for $\mathbf{k} > 6$, by a direct application of Lemma 5.2 in $\mathbf{R}^3_v \times \mathbf{R}^3_{v_1}$ — i.e. with $N = 6$.

Whereas the terms $\mathbf{D}^1_\epsilon(\xi)$, $\mathbf{D}^{21}_\epsilon(\xi)$, and $\mathbf{D}^{23}_\epsilon(\xi)$ are shown to vanish by means of only the entropy and entropy production bounds in Proposition 2.3(a)–(d) and Lemma 5.2, the term $\mathbf{D}^{22}_\epsilon(\xi)$ is much less elementary to handle.

First, we split $\mathbf{D}^{22}_\epsilon(\xi)$ as

$$\begin{aligned}
\mathbf{D}^{22}_\epsilon(\xi) &= \frac{2}{\epsilon}\left\langle\!\!\left\langle \xi\hat{\gamma}_\epsilon(1-\hat{\gamma}_{\epsilon1})\sqrt{G_\epsilon G_{\epsilon1}}\frac{\sqrt{G'_\epsilon G'_{\epsilon1}}-\sqrt{G_\epsilon G_{\epsilon1}}}{\epsilon^2}\right\rangle\!\!\right\rangle \\
&\quad + \frac{2}{\epsilon}\left\langle\!\!\left\langle \xi\left(\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon1}(1-\hat{\gamma}'_\epsilon)+\hat{\gamma}_\epsilon\hat{\gamma}_{\epsilon1}\hat{\gamma}'_\epsilon(1-\hat{\gamma}'_{\epsilon1})\right)\frac{\sqrt{G'_\epsilon G'_{\epsilon1}}-\sqrt{G_\epsilon G_{\epsilon1}}}{\epsilon^2}\sqrt{G_\epsilon G_{\epsilon1}}\right\rangle\!\!\right\rangle \\
&= \mathbf{D}^{221}_\epsilon(\xi) + \mathbf{D}^{222}_\epsilon(\xi).
\end{aligned}$$

For each $T > 0$ and each compact $K \subset \mathbf{R}^3$, the first term satisfies,

$$\begin{aligned}
\left\|\mathbf{D}^{221}_\epsilon(\xi)\right\|_{L^1([0,T]\times K)} &\leqslant 2C_b\left\|\frac{1}{\epsilon}(1-\hat{\gamma}_{\epsilon1})\sqrt{G_{\epsilon1}}\left(1+|v_1|\right)^{\beta/2}\right\|_{L^2([0,T]\times K;L^2(M_1\,dv_1))}\left\|\hat{\gamma}_\epsilon\sqrt{G_\epsilon}\right\|_{L^\infty_{t,x,v}} \\
&\quad \times \left\||\xi|\left(1+|v|\right)^{\beta/2}\right\|_{L^2(M\,dv)}\left\|\frac{\sqrt{G'_\epsilon G'_{\epsilon1}}-\sqrt{G_\epsilon G_{\epsilon1}}}{\epsilon^2}\right\|_{L^2_{t,x,\mu}} \\
&= \mathrm{O}(1)\left\|\frac{1}{\epsilon}(1-\hat{\gamma}_{\epsilon1})\sqrt{G_{\epsilon1}}\left(1+|v_1|\right)^{\beta/2}\right\|_{L^2([0,T]\times K;L^2(M_1\,dv_1))},
\end{aligned}$$

provided that $\xi(v) = \mathrm{O}(|v|^m)$ for some $m \in \mathbf{N}$.

Since $\operatorname{supp}(1-\hat{\gamma}) \subset [\frac{3}{2}, +\infty)$, then $\frac{\sqrt{G_\epsilon}}{\sqrt{G_\epsilon}-1} \leqslant \frac{\sqrt{3/2}}{\sqrt{3/2}-1}$ whenever $\hat{\gamma}_\epsilon \neq 1$, and one has:

$$\frac{1}{\epsilon}|1-\hat{\gamma}_\epsilon|\sqrt{G_\epsilon} \leqslant \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}|1-\hat{\gamma}_\epsilon|\frac{|\sqrt{G_\epsilon}-1|}{\epsilon}.$$

Furthermore, as

$$|1-\hat{\gamma}_\epsilon| \leqslant 1 + \|\gamma'\|_{L^\infty} \quad \text{and} \quad 1-\hat{\gamma}_\epsilon \to 0 \text{ a.e.}, \tag{5.7}$$

the uniform integrability stated in Proposition 4.1 and the Product Limit Theorem (see Appendix A) imply that

$$|1-\hat{\gamma}_\epsilon|\frac{|\sqrt{G_\epsilon}-1|}{\epsilon} \to 0 \quad \text{in } L^2\left([0,T]\times K, L^2\left(M\left(1+|v|\right)^\beta dv\right)\right). \tag{5.8}$$

Thus

$$\left\|\mathbf{D}^{221}_\epsilon(\xi)\right\|_{L^1([0,T]\times K)} \to 0 \quad \text{as } \epsilon \to 0. \tag{5.9}$$

Finally, we consider the term $\mathbf{D}_\epsilon^{222}(\xi)$, one has:

$$\begin{aligned}\big\|\mathbf{D}_\epsilon^{222}(\xi)\big\|_{L^1([0,T]\times K)} &\leqslant \frac{2}{\epsilon}\big(\big\|(1-\hat{\gamma}'_\epsilon)\xi\big\|_{L^2([0,T]\times K;L^2_\mu)}+\big\|(1-\hat{\gamma}'_{\epsilon 1})\xi\big\|_{L^2([0,T]\times K;L^2_\mu)}\big)\\ &\quad\times\big\|\hat{\gamma}_\epsilon\sqrt{G_\epsilon}\big\|^2_{L^\infty_{t,x,v}}\left\|\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\right\|_{L^2_{t,x,\mu}}\\ &= \mathrm{O}(1)\left\|\frac{1-\hat{\gamma}_\epsilon}{\epsilon}\big(1+|v|^2+|v_1|^2\big)\right\|_{L^2([0,T]\times K;L^2_\mu)}\\ &= \mathrm{O}(1)\left\|\frac{1-\hat{\gamma}_\epsilon}{\epsilon}\big(1+|v|\big)^{2+\beta/2}\right\|_{L^2([0,T]\times K;L^2(M\,dv))},\end{aligned}$$

where the first equality uses the $(vv_1)-(v'v'_1)$ symmetry in (2.28).

Since $\operatorname{supp}(1-\hat{\gamma})\subset[\frac{3}{2},+\infty)$, $\frac{1}{\sqrt{G_\epsilon}-1}\leqslant\frac{1}{\sqrt{3/2}-1}$ whenever $\hat{\gamma}_\epsilon\neq 1$, one has:

$$\begin{aligned}\frac{|1-\hat{\gamma}_\epsilon|^2}{\epsilon^2} &\leqslant \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}\frac{|1-\hat{\gamma}_\epsilon|}{\epsilon}\frac{\sqrt{G_\epsilon}-1}{\epsilon}\\ &\leqslant \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}\frac{|1-\hat{\gamma}_\epsilon|}{\epsilon}\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}+\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right).\end{aligned}$$

By (5.7) and (5.8),

$$\frac{|1-\hat{\gamma}_\epsilon|}{\epsilon}\leqslant\frac{1+\|\gamma'\|_{L^\infty}}{\epsilon}\quad\text{and}\quad\frac{|1-\hat{\gamma}_\epsilon|}{\epsilon}\to 0\quad\text{in } L^2_{\mathrm{loc}}\big(dt\,dx,L^2(M\,dv)\big),\tag{5.10}$$

since $\sqrt{G_\epsilon}-1>\sqrt{3/2}-1$ whenever $\hat{\gamma}_\epsilon\neq 1$, whereas by Proposition 2.3(b) and Lemma 3.1,

$$\begin{aligned}\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}&=\mathrm{O}(1)_{L^\infty_t(L^2_x(L^q(M\,dv)))},\\ \frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}&=\mathrm{O}(\epsilon)_{L^1_{\mathrm{loc}}(dt\,dx,L^2(M\,dv))},\end{aligned}$$

for all $q>+\infty$. Then,

$$\frac{|1-\hat{\gamma}_\epsilon|^2}{\epsilon^2}=\mathrm{O}(1)_{L^1_{\mathrm{loc}}(dt\,dx,L^q(M\,dv))},$$

for all $q<2$. In particular, for each $r<+\infty$, $(\frac{1}{\epsilon}(1-\hat{\gamma}_\epsilon)(1+|v|)^r)$ is uniformly bounded in $L^2_{\mathrm{loc}}(dt\,dx,L^2(M\,dv))$. By interpolation with (5.10) we conclude that

$$\left\|\frac{1-\hat{\gamma}_\epsilon}{\epsilon}\big(1+|v|\big)^r\right\|^2_{L^2([0,T]\times K;L^2(M\,dv))}\to 0\quad\text{as }\epsilon\to 0,\tag{5.11}$$

and consequently

$$\mathbf{D}_\epsilon^{222}(\xi)\to 0\quad\text{in } L^1_{\mathrm{loc}}(dt\,dx)\text{ as }\epsilon\to 0.\tag{5.12}$$

The convergences (5.2), (5.4), (5.9), (5.12), (5.5) and (5.6) eventually imply Proposition 5.1. □

Remark. The same arguments leading to (5.8) and to (5.11) imply that, for all $r\in\mathbf{R}$,

$$\left\|\frac{1-\gamma_\epsilon}{\epsilon}\big(1+|v|\big)^r\right\|^2_{L^2([0,T]\times K;L^2(M\,dv))}\to 0\quad\text{as }\epsilon\to 0.\tag{5.13}$$

6. Asymptotic behavior of the flux terms

The purpose of the present section is to establish the following:

Proposition 6.1. *Under the same assumptions as in Theorem* 2.4, *one has*:

$$\mathbf{F}_\epsilon(\zeta) - 2\left\langle \zeta\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle + \frac{2}{\epsilon^2}\left\langle \hat{\zeta}\,\mathcal{Q}\big(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\big)\right\rangle \to 0 \quad \textit{in } L^1_{\mathrm{loc}}(dt\,dx),$$

as $\epsilon \to 0$, *where* ζ *and* $\hat{\zeta}$ *designate respectively either* A *and* $\hat{A}$ *or* B *and* $\hat{B}$ *defined by* (2.50) *and* (2.52).

Proof. First, we decompose the flux term $\mathbf{F}_\epsilon(\zeta)$ as follows:

$$\begin{aligned}\mathbf{F}_\epsilon(\zeta) &= \frac{1}{\epsilon}\langle \zeta_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle = \left\langle \zeta_{K_\epsilon}\frac{G_\epsilon-1}{\epsilon^2}\gamma_\epsilon\right\rangle \\ &= \left\langle \zeta_{K_\epsilon}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\gamma_\epsilon\right\rangle + \frac{2}{\epsilon}\left\langle \zeta_{K_\epsilon}\frac{\sqrt{G_\epsilon}-1}{\epsilon}\gamma_\epsilon\right\rangle \\ &= \mathbf{F}^1_\epsilon(\zeta) + \mathbf{F}^2_\epsilon(\zeta).\end{aligned}$$

We further split the term $\mathbf{F}^1_\epsilon(\zeta)$ as

$$\mathbf{F}^1_\epsilon(\zeta) = \mathbf{F}^{11}_\epsilon(\zeta) + \mathbf{F}^{12}_\epsilon(\zeta) + \mathbf{F}^{13}_\epsilon(\zeta)$$

with

$$\begin{aligned}\mathbf{F}^{11}_\epsilon(\zeta) &= \left\langle \zeta_{K_\epsilon}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} + \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\gamma_\epsilon\right\rangle, \\ \mathbf{F}^{12}_\epsilon(\zeta) &= \left\langle \zeta(\mathbf{1}_{|v|^2\leqslant K_\epsilon}\gamma_\epsilon - 1)\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle, \\ \mathbf{F}^{13}_\epsilon(\zeta) &= \left\langle \zeta\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle.\end{aligned} \tag{6.1}$$

The term $\mathbf{F}^{12}_\epsilon(\zeta)$ is easily disposed of. Indeed, the definition (2.32) of the hydrodynamic projection Π implies that, $(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon})^2(1+|v|)^p$ is, for each $p \geqslant 0$, a (finite) linear combination of functions of v of order $O(|v|^{p+4})$ as $|v| \to +\infty$, with coefficients that are quadratic in $\langle \xi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\rangle$ for $\xi \in \{1, v_1, v_2, v_3, |v|^2\}$. Together with Proposition 4.1, this implies that, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\big(1+|v|\big)^p \text{ is uniformly integrable on } [0,T]\times K\times\mathbf{R}^3, \tag{6.2}$$

for the measure $M\,dv\,dx\,dt$. On the other hand,

$$\mathbf{1}_{|v|^2\leqslant K_\epsilon}\gamma_\epsilon - 1 \to 0 \quad\text{and}\quad |\mathbf{1}_{|v|^2\leqslant K_\epsilon}\gamma_\epsilon - 1| \leqslant 1 \text{ a.e.}$$

Since $\zeta(v) = O(|v|^3)$ as $|v| \to +\infty$, this and the Product Limit Theorem imply that

$$\mathbf{F}^{12}_\epsilon(\zeta) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx). \tag{6.3}$$

The term $\mathbf{F}^{11}_\epsilon(\zeta)$ requires a slightly more involved treatment. We start with the following decomposition: for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\begin{aligned}\big\|\mathbf{F}^{11}_\epsilon(\zeta)\big\|_{L^1([0,T]\times K)} \leqslant{}& \left\|\zeta_{K_\epsilon}\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} + \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right\|_{L^2([0,T]\times K;L^2(M\,dv))} \\ &\times \left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2([0,T]\times K;L^2(M\,dv))}.\end{aligned} \tag{6.4}$$

Since $\gamma_\epsilon = \gamma(G_\epsilon) = 0$ whenever $G_\epsilon > 2$, one has for each $q < +\infty$,

$$\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 = \gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon} + \left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right)$$
$$= \mathrm{O}(1)_{L_t^\infty(L^2(dx\,M\,dv))}\mathrm{O}(1)_{L_t^\infty(L_x^2(L^q(M\,dv)))} + \mathrm{O}\left(\frac{1}{\epsilon}\right)\mathrm{O}(\epsilon)_{L^1_{\mathrm{loc}}(dt\,dx,L^2(M\,dv))}.$$

In particular

$$\left\|\zeta_{K_\epsilon}\gamma_\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2([0,T]\times K;L^2(M\,dv))} = \mathrm{O}(1),$$

since $\zeta(v) = \mathrm{O}(|v|^3)$ as $|v| \to +\infty$. This and (6.2) imply that

$$\left\|\zeta_{K_\epsilon}\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} + \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right\|_{L^2_{\mathrm{loc}}(dt\,dx;L^2(M\,dv))} = \mathrm{O}(1). \tag{6.5}$$

Using (6.4), (6.5) and Proposition 4.3 show that

$$\mathbf{F}_\epsilon^{11}(\zeta) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx). \tag{6.6}$$

This and (6.3) imply that

$$\mathbf{F}_\epsilon^{1}(\zeta) - \left\langle\zeta\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx), \tag{6.7}$$

as $\epsilon \to 0$.

Next we handle the term $\mathbf{F}_\epsilon^2(\zeta)$. We first decompose it as follows:

$$\mathbf{F}_\epsilon^2(\zeta) = -\frac{2}{\epsilon}\left\langle\zeta\mathbf{1}_{|v|^2>K_\epsilon}\gamma_\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle + 2\left\langle\zeta\frac{\gamma_\epsilon-1}{\epsilon}\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle + \frac{2}{\epsilon}\left\langle\zeta\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle$$
$$= \mathbf{F}_\epsilon^{21}(\zeta) + \mathbf{F}_\epsilon^{22}(\zeta) + \mathbf{F}_\epsilon^{23}(\zeta). \tag{6.8}$$

Then, by (2.20) and Lemma 5.2, one has:

$$\left\|\mathbf{F}_\epsilon^{21}(\zeta)\right\|_{L_t^\infty(L_x^2)} \leqslant \frac{2}{\epsilon}\|\gamma\|_{L^\infty}\|\zeta\mathbf{1}_{|v|^2>K_\epsilon}\|_{L^2(M\,dv)}\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L_t^\infty(L^2(M\,dv\,dx))}$$
$$\leqslant \frac{2}{\epsilon}\mathrm{O}\big(e^{-K_\epsilon/2}K_\epsilon^2\big) = \mathrm{O}\big(\epsilon^{\mathbf{k}/2-1}|\ln\epsilon|^2\big). \tag{6.9}$$

On the other hand, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\left\|\mathbf{F}_\epsilon^{22}(\zeta)\right\|_{L^1([0,T]\times K)} \leqslant 2T^{1/2}\left\|\zeta\frac{\gamma_\epsilon-1}{\epsilon}\right\|_{L^2([0,T]\times K;L^2(M\,dv))}\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L_t^\infty(L^2(M\,dv\,dx))} \to 0 \quad \text{as } \epsilon \to 0, \tag{6.10}$$

because of (2.20) and of (5.13), since $\zeta(v) = \mathrm{O}(|v|^3)$ as $|v| \to +\infty$.

Finally, we transform $\mathbf{F}_\epsilon^{23}(\zeta)$ as follows:

$$\mathbf{F}_\epsilon^{23}(\zeta) = 2\left\langle\hat{\zeta}\frac{1}{\epsilon}\mathcal{L}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right\rangle = 2\left\langle\hat{\zeta}\left(\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) - \frac{1}{\epsilon^2}\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right)\right\rangle.$$

Writing

$$\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) = \mathcal{Q}\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon},\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)$$
$$+ \mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon} + \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right),$$

and using the classical relation (see [2] for instance),

$$\mathcal{Q}(\phi,\phi)=\frac{1}{2}\mathcal{L}\big(\phi^2\big)\quad\text{for each }\phi\in\operatorname{Ker}\mathcal{L},$$

we arrive at

$$\begin{aligned}\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)=&\ \frac{1}{2}\mathcal{L}\left(\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right)\\&+\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}+\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right).\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{F}^{23}_\epsilon(\zeta)=&\left\langle\zeta\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle-\frac{2}{\epsilon^2}\big\langle\hat\zeta\,\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\big\rangle\\&+2\left\langle\hat\zeta\,\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}+\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right\rangle.\end{aligned}\tag{6.11}$$

By continuity of $\mathcal{Q}$ (see (2.35)),

$$\begin{aligned}&\left\|\left\langle\hat\zeta\,\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}+\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)\right\rangle\right\|_{L^1([0,T]\times K)}\\&\quad\leqslant C\|\hat\zeta\|_{L^2(aM\,dv)}\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2([0,T]\times K;L^2((1+|v|)^\beta M\,dv))}\\&\quad\times\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}+\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2([0,T]\times K;L^2((1+|v|)^\beta M\,dv))}\to 0\end{aligned}$$

as $\epsilon\to 0$, for each $T>0$ and each compact $K\subset\mathbf{R}^3$, because of (6.2) and Proposition 4.3.

Thus, by (6.9), (6.10) and (6.11)

$$\mathbf{F}^2_\epsilon(\zeta)-\left\langle\zeta\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle+\frac{2}{\epsilon^2}\left\langle\hat\zeta\,\mathcal{Q}\big(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\big)\right\rangle\to 0\tag{6.12}$$

in $L^1_{\mathrm{loc}}(dt\,dx)$ as $\epsilon\to 0$.

The convergences (6.7) and (6.12) eventually imply Proposition 6.1. □

7. Proof of Theorem 2.4

Throughout this section $U\equiv U(x)$ designates an arbitrary compactly supported, C^∞, divergence-free vector field on $\mathbf{R}^3$. Taking the inner product with U of both sides of (2.48) gives

$$\partial_t\int\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle\cdot U\,dx-\int\mathbf{F}_\epsilon(A):\nabla_x U\,dx=\int\mathbf{D}_\epsilon(v)\cdot U\,dx\to 0\quad\text{in }L^1_{\mathrm{loc}}(dt),\tag{7.1}$$

by Proposition 5.1. Likewise, the energy equation (2.49) and Proposition 5.1 lead to

$$\partial_t\left\langle\frac{1}{2}\big(|v|^2-5\big)_{K_\epsilon}g_\epsilon\gamma_\epsilon\right\rangle+\operatorname{div}_x\mathbf{F}_\epsilon(B)=\mathbf{D}_\epsilon\left(\frac{1}{2}\big(|v|^2-5\big)\right)\to 0\quad\text{in }L^1_{\mathrm{loc}}(dt\,dx).\tag{7.2}$$

By Proposition 6.1, one can decompose the fluxes as

$$\begin{aligned}\mathbf{F}_\epsilon(A)&=\mathbf{F}^{\mathrm{conv}}_\epsilon(A)+\mathbf{F}^{\mathrm{diff}}_\epsilon(A)+\mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)},\\\mathbf{F}_\epsilon(B)&=\mathbf{F}^{\mathrm{conv}}_\epsilon(B)+\mathbf{F}^{\mathrm{diff}}_\epsilon(B)+\mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)},\end{aligned}\tag{7.3}$$

where

$$\begin{aligned}\mathbf{F}_\epsilon^{\mathrm{conv}}(A)&=2\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle,\\ \mathbf{F}_\epsilon^{\mathrm{diff}}(A)&=-2\left\langle\hat{A}\frac{1}{\epsilon^2}\mathcal{Q}\big(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\big)\right\rangle,\end{aligned}\tag{7.4}$$

while

$$\begin{aligned}\mathbf{F}_\epsilon^{\mathrm{conv}}(B)&=2\left\langle B\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle,\\ \mathbf{F}_\epsilon^{\mathrm{diff}}(B)&=-2\left\langle\hat{B}\frac{1}{\epsilon^2}\mathcal{Q}\big(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\big)\right\rangle.\end{aligned}\tag{7.5}$$

Classical computations (that can be found for instance in [3]) using the fact that A is orthogonal in $L^2(M\,dv)$ to $\operatorname{Ker}\mathcal{L}$ as well as to odd functions of v and functions of $|v|^2$ show that

$$\mathbf{F}_\epsilon^{\mathrm{conv}}(A)=2\langle A\otimes A\rangle:\left\langle v\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle^{\otimes 2}.$$

In a similar way, B is orthogonal in $L^2(M\,dv)$ to $\operatorname{Ker}\mathcal{L}$ and to even functions of v, so that

$$\mathbf{F}_\epsilon^{\mathrm{conv}}(B)=2\langle B\otimes B\rangle\cdot\left\langle v\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle\left\langle\left(\frac{1}{3}|v|^2-1\right)\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle.$$

7.1. Convergence of the diffusion terms

The convergence of $\mathbf{F}_\epsilon^{\mathrm{diff}}(A)$ and $\mathbf{F}_\epsilon^{\mathrm{diff}}(B)$ comes only from weak compactness results, and from the following characterization of the weak limits.

Proposition 7.1. *Under the same assumptions as in Theorem* 2.4, *one has, up to extraction of a subsequence* $\epsilon_n\to 0$,

$$g_{\epsilon_n}\to g,\quad\textit{and}\quad\frac{\sqrt{G'_{\epsilon_n}G'_{\epsilon_n 1}}-\sqrt{G_{\epsilon_n}G_{\epsilon_n 1}}}{\epsilon_n^2}\to\tilde{q},\tag{7.6}$$

in $w-L^1_{\mathrm{loc}}(dt\,dx;L^1(M\,dv))$ *and in* $w-L^2(dt\,dx\,d\mu)$, *respectively.*

Furthermore $g\in L^\infty_t(L^2(dx\,M\,dv))$ *is an infinitesimal Maxwellian of the form,*

$$g(t,x,v)=u(t,x)\cdot v+\theta(t,x)\frac{1}{2}\big(|v|^2-5\big),\quad \operatorname{div}_x u=0,\tag{7.7}$$

and $\tilde{q}\in L^2(dt\,dx\,d\mu)$ *satisfies*:

$$\iint\tilde{q}b(v-v_1,\omega)\,d\omega M_1\,dv_1=\frac{1}{2}v\cdot\nabla_x g=\frac{1}{2}(A:\nabla_x u+B\cdot\nabla_x\theta).\tag{7.8}$$

Proof. Proposition 2.3(c) shows that

$$(g_\epsilon)\text{ is relatively compact in }w-L^1_{\mathrm{loc}}\big(dt\,dx;L^1(M\,dv)\big)$$

while (2.21) implies that

$$\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2}\text{ is relatively compact in }w-L^2(dt\,dx\,d\mu).$$

Pick then any sequence $\epsilon_n\to 0$ such that

$$g_{\epsilon_n}\to g,\quad\text{and}\quad\frac{\sqrt{G'_{\epsilon_n}G'_{\epsilon_n 1}}-\sqrt{G_{\epsilon_n}G_{\epsilon_n 1}}}{\epsilon_n^2}\to\tilde{q},$$

in $w-L^1_{\mathrm{loc}}(dt\,dx;L^1(M\,dv))$ and in $w-L^2(dt\,dx\,d\mu)$ respectively.

Step 1: From (2.22) we deduce that

$$\frac{1}{\epsilon_n}(\sqrt{G_{\epsilon_n}}-1) \to g \quad \text{in } w-L^2_{\mathrm{loc}}\big(dt, L^2(dx\, M\, dv)\big).$$

In particular, by Proposition 4.3, g is an infinitesimal Maxwellian, i.e. of the form:

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{1}{2}\big(|v|^2-3\big).$$

Taking limits in the local conservation of mass leads then to

$$\mathrm{div}_x\langle vg\rangle = 0,$$

or in other words

$$\mathrm{div}_x\, u = 0,$$

which is the incompressibility constraint.

Multiplying the approximate momentum equation (2.48) by ϵ,

$$\epsilon\partial_t\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle + \epsilon\,\mathrm{div}_x\,\mathbf{F}_\epsilon(A) + \frac{1}{3}\nabla_x\left\langle\frac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\right\rangle = \epsilon\mathbf{D}_\epsilon(v),$$

using Propositions 5.1 and 6.1 to control $\mathbf{D}_\epsilon(v)$ and the remainder term in $\mathbf{F}_\epsilon(A)$,

$$\mathbf{F}_\epsilon(A) - 2\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle + 2\left\langle\!\!\left\langle \hat{A}\frac{1}{\epsilon^2}(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}})\right\rangle\!\!\right\rangle \to 0,$$

and estimating $\mathbf{F}^{\mathrm{conv}}_\epsilon(A)$ and $\mathbf{F}^{\mathrm{diff}}_\epsilon(A)$ by the entropy and entropy production bounds (2.20)–(2.21),

$$\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle = \mathrm{O}(1) \quad \text{in } L^\infty_t\big(L^1_x\big),$$
$$\left\langle\!\!\left\langle \hat{A}\frac{1}{\epsilon^2}(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}})\right\rangle\!\!\right\rangle = \mathrm{O}(1)_{L^2_{t,x}},$$

we also obtain:

$$\nabla_x\big\langle|v|^2 g\big\rangle = 0,$$

or equivalently, since $\langle|v|^2 g\rangle = 3(p+\theta) \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^3))$,

$$\rho+\theta = 0,$$

which is the Boussinesq relation. One therefore has (7.7).

Step 2: Start from (4.1) in the proof of Lemma 4.2:

$$(\epsilon\partial_t + v\cdot\nabla_x)\frac{\sqrt{\epsilon^\alpha + G_\epsilon}-1}{\epsilon} = \frac{1}{\epsilon^2}\frac{1}{2\sqrt{\epsilon^\alpha+G_\epsilon}}\mathcal{Q}(G_\epsilon, G_\epsilon) = Q^1_\epsilon + Q^2_\epsilon.$$

Recall that

$$Q^1_\epsilon \to 0 \quad \text{in } L^1(M\, dv\, dx\, dt). \tag{7.9}$$

Next observe that

$$Q^2_\epsilon = \frac{\sqrt{G_\epsilon}}{\sqrt{\epsilon^\alpha + G_\epsilon}}\iint \sqrt{G_{\epsilon 1}}\frac{\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}}{\epsilon^2} b(v-v_1,\omega)\, d\omega\, M_1\, dv_1.$$

Proposition 4.1 implies that

$$\sqrt{G_\epsilon} \to 1 \quad \text{in } L^2_{\mathrm{loc}}\big(dt\, dx; L^2\big((1+|v|)^\beta M\, dv\big)\big) \text{ as } \epsilon\to 0;$$

this and the second limit in (7.6) imply that

$$\iint \sqrt{G_{\epsilon_n 1}}\frac{\sqrt{G'_{\epsilon_n}G'_{\epsilon_n 1}}-\sqrt{G_{\epsilon_n}G_{\epsilon_n 1}}}{\epsilon_n^2}b(v-v_1,\omega)\,d\omega\,M_1\,dv_1 \to \iint \tilde{q}b(v-v_1,\omega)\,d\omega\,M_1\,dv_1,$$

in $w-L^1_{\mathrm{loc}}(dt\,dx;L^1(M\,dv))$ as $n\to+\infty$. Since on the other hand,

$$\frac{\sqrt{G_\epsilon}}{\sqrt{\epsilon^\alpha+G_\epsilon}}\to 1 \text{ a.e.}\quad \text{as } \epsilon\to 0 \text{ with } 0\leqslant \frac{\sqrt{G_\epsilon}}{\sqrt{\epsilon^\alpha+G_\epsilon}}\leqslant 1,$$

we conclude from the Product Limit Theorem that

$$Q^2_{\epsilon_n}\to \iint \tilde{q}b(v-v_1,\omega)\,d\omega\,M_1\,dv_1, \tag{7.10}$$

in $w-L^1_{\mathrm{loc}}(dt\,dx;L^1(M\,dv))$ as $n\to+\infty$.

By (4.6), (4.13) and (7.6),

$$\frac{\sqrt{\epsilon_n^\alpha+G_\epsilon}-1}{\epsilon_n}\to\frac{1}{2}g,$$

in $w-L^1_{\mathrm{loc}}(dt\,dx;L^1((1+|v|^2)M\,dv))$ whenever $\alpha\in\,]1,2[$. Using (7.9), (7.10) and the convergence above, and passing to the limit in (4.1) as $\epsilon_n\to 0$ leads to

$$\iint \tilde{q}b(v-v_1,\omega)\,d\omega\,M_1\,dv_1=\frac{1}{2}v\cdot\nabla_x g,$$

which is precisely the first equality in (7.8). Finally, replacing g by its expression (7.7) in the formula above leads to the second equality in (7.8). □

Since $\hat{A}$ and $\hat{B}\in L^2(aM\,dv)$, the second limit in (7.6) and identity (7.8) show that

$$\begin{aligned}\mathbf{F}^{\mathrm{diff}}_{\epsilon_n}(A)&=-2\left\langle \hat{A}\frac{1}{\epsilon^2}\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right\rangle\to-\langle\hat{A}\otimes A\rangle:\nabla_x u=-\nu\left(\nabla_x u+(\nabla_x u)^T\right),\\ \mathbf{F}^{\mathrm{diff}}_{\epsilon_n}(B)&=-2\left\langle \hat{B}\frac{1}{\epsilon^2}\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right\rangle\to-\langle\hat{B}\otimes B\rangle:\nabla_x\theta=-\kappa\nabla_x\theta,\end{aligned}\tag{7.11}$$

in $w-L^2(dt\,dx)$ as $\epsilon\to 0$, because of the divergence-free condition in (7.7).

7.2. *Convergence of the convection terms*

The goal of this section is to establish that

$$\int \mathbf{F}^{\mathrm{conv}}_\epsilon(A):\nabla_x U\,dx\to\int u\otimes u:\nabla_x U\,dx\quad\text{and}\quad \mathrm{div}_x\,\mathbf{F}^{\mathrm{conv}}_\epsilon(B)\to\frac{5}{2}\,\mathrm{div}_x(u\theta)$$

in the sense of distributions on $\mathbf{R}^*_+$ and on $\mathbf{R}^*_+\times\mathbf{R}^3$ respectively.

First, we replace $\mathbf{F}^{\mathrm{conv}}_\epsilon(A)$ and $\mathbf{F}^{\mathrm{conv}}_\epsilon(B)$ by asymptotically equivalent expressions.

Indeed, because of (4.13),

$$\langle vg_\epsilon\gamma_\epsilon\rangle-2\left\langle v\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle\to 0\quad\text{in } L^2_{\mathrm{loc}}(dt\,dx),$$

and

$$\left\langle\left(\frac{1}{3}|v|^2-1\right)g_\epsilon\gamma_\epsilon\right\rangle-2\left\langle\left(\frac{1}{3}|v|^2-1\right)\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle\to 0\quad\text{in } L^2_{\mathrm{loc}}(dt\,dx).$$

On the other hand, $g_\epsilon\gamma_\epsilon$ is bounded in $L^\infty_t(L^2(M\,dv\,dx))$ while $v\mathbf{1}_{|v|^2>K_\epsilon}\to 0$ and $(\frac{1}{3}|v|^2-1)\mathbf{1}_{|v|^2>K_\epsilon}\to 0$ in $L^2(M\,dv)$; therefore

$$\langle vg_\epsilon\gamma_\epsilon\rangle-\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle\to 0\quad\text{and}\quad\left\langle\left(\frac{1}{3}|v|^2-1\right)g_\epsilon\gamma_\epsilon\right\rangle-\left\langle\left(\frac{1}{3}|v|^2-1\right)_{K_\epsilon}g_\epsilon\gamma_\epsilon\right\rangle\to 0,$$

in $L^2_{\mathrm{loc}}(dt\,dx)$. Therefore

$$\begin{aligned}\mathbf{F}^{\mathrm{conv}}_{\epsilon}(A) &= \frac{1}{2}\langle A\otimes A\rangle : \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle^{\otimes 2} + \mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)}\\ &= \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle^{\otimes 2} - \frac{1}{3}\left|\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle\right|^2 I + \mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)},\end{aligned} \tag{7.12}$$

while

$$\begin{aligned}\mathbf{F}^{\mathrm{conv}}_{\epsilon}(B) &= \langle B\otimes B\rangle \cdot \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \left\langle \left(\frac{1}{3}|v|^2 - 1\right)_{K_\epsilon} g_\epsilon\gamma_\epsilon\right\rangle + \mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)}\\ &= \frac{5}{2}\left\langle \left(\frac{1}{3}|v|^2 - 1\right)_{K_\epsilon} g_\epsilon\gamma_\epsilon\right\rangle \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle + \mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)}.\end{aligned} \tag{7.13}$$

Furthermore, since $g_{\epsilon_n} \to g$ weakly in $L^1_{\mathrm{loc}}(dt\,dx; L^1((1+|v|^2)M\,dv))$ while

$$v_{K_\epsilon}\gamma_\epsilon \to v \quad \text{and} \quad \left(\frac{1}{3}|v|^2 - 1\right)_{K_\epsilon}\gamma_\epsilon \to \left(\frac{1}{3}|v|^2 - 1\right) \text{ a.e.,} \quad \text{and}$$
$$|v_{K_\epsilon}\gamma_\epsilon| + \left|\left(\frac{1}{3}|v|^2 - 1\right)_{K_\epsilon}\gamma_\epsilon\right| \leqslant C\left(1+|v|^2\right)$$

one has by the Product Limit Theorem:

$$\begin{aligned}&\langle v_{K_{\epsilon_n}}\gamma_{\epsilon_n} g_{\epsilon_n}\rangle \to \langle vg\rangle = u,\\ &\left\langle \left(\frac{1}{3}|v|^2 - 1\right)_{K_{\epsilon_n}} \gamma_{\epsilon_n} g_{\epsilon_n}\right\rangle \to \left\langle \left(\frac{1}{3}|v|^2 - 1\right) g\right\rangle = \theta,\end{aligned} \tag{7.14}$$

in $w-L^1_{\mathrm{loc}}(dt\,dx)$. In fact, these limits also hold in $w-L^2_{\mathrm{loc}}(dt\,dx)$ since the family $g_\epsilon\gamma_\epsilon$ is bounded in $L^\infty_t(L^2(M\,dv\,dx))$.

Taking limits in (7.12) and (7.13), which are quadratic functions of the moments, requires then to establish some strong compactness on $(\langle \zeta_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle)$.

7.2.1. Strong compactness in the x-variable

Applying Proposition 4.4 with $\xi = v$ and $\xi = \frac{1}{2}(|v|^2 - 5)$ shows that, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\begin{aligned}&\int_0^T\!\!\int_K \left|\left\langle \frac{1}{2}\left(|v|^2-5\right) g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle(t,x+y) - \left\langle \frac{1}{2}\left(|v|^2-5\right) g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle(t,x)\right|^2 dx\,dt\\ &\quad + \int_0^T\!\!\int_K \left|\langle v g_{\epsilon_n}\gamma_{\epsilon_n}\rangle(t,x+y) - \langle v g_{\epsilon_n}\gamma_{\epsilon_n}\rangle(t,x)\right|^2 dx\,dt \to 0,\end{aligned}$$

as $|y| \to 0$ uniformly in n. An easy consequence of the above convergence properties is that

$$\begin{aligned}&\int_0^T\!\!\int_K \left|\left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle(t,x+y) - \left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle(t,x)\right|^2 dx\,dt\\ &\quad + \int_0^T\!\!\int_K \left|\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle(t,x+y) - \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle(t,x)\right|^2 dx\,dt \to 0,\end{aligned} \tag{7.15}$$

as $|y| \to 0$ uniformly in n.

In order to study the convergence of $\mathbf{F}_\epsilon(A)$, we need some similar statements for the solenoidal and gradient parts of $\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle$, since the first one is expected to converge strongly in $L^2_{\mathrm{loc}}(dt\,dx)$.

The difficulty comes then from the fact that the Leray projection is a nonlocal pseudodifferential operator, in particular it is not continuous on $L^2_{\mathrm{loc}}(dx)$.

Introducing some convenient truncation χ in x and using the properties of the commutator $[\chi, P]$, one can nevertheless prove the following equicontinuity statement (see Lemma C.1): for each compact $K \subset \mathbf{R}^3$ and each $T > 0$, one has:

$$\int_0^T\int_K \left|P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle(t, x+y) - P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle(t, x)\right|^2 dx\,dt \to 0, \tag{7.16}$$

as $|y| \to 0$, uniformly in n.

7.2.2. Strong compactness in the t-variable

As we shall see below, the temperature fluctuation $\langle \frac{1}{2}(|v|^2-5)_{K_\epsilon} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle$ and the solenoidal part $P\langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle$ of $\langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle$ are strongly compact in the t-variable. However the orthogonal complement of $P\langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle$ — which is a gradient field — is not in general because of high frequency oscillations in t.

Proposition 7.2. *Under the assumptions of Theorem* 2.4, *one has*:

$$P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \to \langle vg\rangle = u,$$
$$\left\langle \frac{1}{2}\big(|v|^2-5\big)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \to \left\langle \frac{1}{2}\big(|v|^2-5\big)g\right\rangle = \frac{5}{2}\theta,$$

in $C(\mathbf{R}_+; w - L^2_x)$ and in $L^2_{\mathrm{loc}}(dt\,dx)$ as $n \to +\infty$.

Proof. The conservation law (7.1) implies that

$$\partial_t \int_{\mathbf{R}^3} \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \cdot U\,dx = \mathrm{O}(1) \quad \text{in } L^1_{\mathrm{loc}}(dt), \tag{7.17}$$

for each compactly supported, solenoidal vector field $U \in C^\infty(\mathbf{R}^3)$, since we know from Proposition 6.1 together with the bounds (2.21) and (2.20) that $\mathbf{F}_{\epsilon_n}(A)$ is bounded in $L^1_{\mathrm{loc}}(dt\,dx)$.

In the same way, the conservation law (7.2) implies that

$$\partial_t\left\langle \frac{1}{2}\big(|v|^2-5\big)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle = \mathrm{O}(1) \quad \text{in } L^1_{\mathrm{loc}}\big(dt; W^{-1,1}_{\mathrm{loc}}\big(\mathbf{R}^3\big)\big). \tag{7.18}$$

Also, we recall that $g_\epsilon\gamma_\epsilon$ is bounded in $B(\mathbf{R}_+; L^2(M\,dv\,dx))$ — where $B(X, Y)$ denotes the class of bounded maps from X to Y — because of the entropy bound (2.20). Indeed, since $\gamma_\epsilon = 0$ whenever $G_\epsilon > 2$, one has:

$$|g_{\epsilon_n}\gamma_{\epsilon_n}| \leqslant \mathbf{1}_{G_\epsilon\leqslant 2}\frac{|G_\epsilon - 1|}{\epsilon} \leqslant (1+\sqrt{2})\frac{|\sqrt{G_{\epsilon_n}}-1|}{\epsilon_n}. \tag{7.19}$$

In particular, one has:

$$\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle = \mathrm{O}(1) \quad \text{in } B\big(\mathbf{R}_+; L^2_x\big),$$
$$\left\langle \frac{1}{2}\big(|v|^2-5\big)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle = \mathrm{O}(1) \quad \text{in } B\big(\mathbf{R}_+; L^2_x\big). \tag{7.20}$$

Since the class of C^∞, compactly supported solenoidal vector fields is dense in that of all L^2 solenoidal vector fields (see Appendix A of [19]), (7.20) and (7.17) imply that

$$P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \text{ is relatively compact in } C\big(\mathbf{R}_+; w - L^2\big(\mathbf{R}^3\big)\big), \tag{7.21}$$

by a variant of Ascoli's theorem that can be found in Appendix C of [19].

The same argument shows that

$$\left\langle \frac{1}{2}\big(|v|^2-5\big)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \text{ is also relatively compact in } C\big(\mathbf{R}_+; w - L^2_x\big). \tag{7.22}$$

As for the $L^2_{\mathrm{loc}}(dt\,dx)$ compactness, notice (7.21)–(7.22) imply that

$$P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \star \chi_\delta \text{ is relatively compact in } L^2_{\mathrm{loc}}(dt\,dx),$$
$$\left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \star \chi_\delta \text{ is relatively compact in } L^2_{\mathrm{loc}}(dt\,dx),$$

where χ_δ designates any mollifying sequence and $\star$ is the convolution in the x-variable only. Hence

$$P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \cdot P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \star \chi_\delta \to Pu\cdot Pu\star\chi_\delta,$$
$$\left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle\left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \star \chi_\delta \to \left(\frac{5}{2}\theta\right)\left(\frac{5}{2}\theta\star\chi_\delta\right),$$

in $w-L^1_{\mathrm{loc}}(dt\,dx)$ as $n\to\infty$. By (7.15)–(7.16),

$$P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \star \chi_\delta \to P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle,$$
$$\left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \star \chi_\delta \to \left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle, \tag{7.23}$$

in $L^2_{\mathrm{loc}}(dt\,dx)$ uniformly in n as $\delta\to 0$. With this, we conclude that

$$\left|P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\right|^2 \to |Pu|^2 \quad \text{in } w-L^1_{\mathrm{loc}}(dt\,dx),$$
$$\left|\left\langle \frac{1}{2}\left(|v|^2-5\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle\right|^2 \to \left(\frac{5}{2}\theta\right)^2 \quad \text{in } w-L^1_{\mathrm{loc}}(dt\,dx),$$

which implies the expected strong compactness in $L^2_{\mathrm{loc}}(dt\,dx)$. □

7.2.3. Passing to the limit in the convection terms

As explained above, $P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle$ is strongly relatively compact in $L^2_{\mathrm{loc}}(dt\,dx)$; however, the term $\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle$ itself may not be strongly relatively compact in $L^2_{\mathrm{loc}}(dt\,dx)$ — at least in general. For that reason, on account of (7.12), it is not clear that

$$\mathbf{F}^{\mathrm{conv}}_\epsilon(A) \to u\otimes u - \frac{1}{3}|u|^2 I.$$

Likewise $\langle (|v|^2-5)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle$ is strongly relatively compact in $L^2_{\mathrm{loc}}(dt\,dx)$, and, on account of (7.13), it is not clear that

$$\mathbf{F}^{\mathrm{conv}}_\epsilon(B) \to \frac{5}{2}u\theta,$$

as one would expect.

What we shall prove in this section is

Proposition 7.3. *Under the assumptions of Theorem* 2.4*, one has,*

$$\int_{\mathbf{R}^3} \nabla_x U : \mathbf{F}^{\mathrm{conv}}_{\epsilon_n}(A)\,dx \to \int_{\mathbf{R}^3} \nabla_x U : u\otimes u\,dx,$$

in the sense of distributions on $\mathbf{R}^*_+$ for each solenoidal vector field $U\in C^\infty_c(\mathbf{R}^3;\mathbf{R}^3)$, and

$$\operatorname{div}_x \mathbf{F}^{\mathrm{conv}}_{\epsilon_n}(B) \to \frac{5}{2}\operatorname{div}_x(u\theta),$$

in the sense of distributions on $\mathbf{R}^*_+\times\mathbf{R}^3$.

The proof of this result relies on a compensated compactness argument due to P.-L. Lions and N. Masmoudi [21] and recalled in Appendix A (Theorem A.2), and on the following observation:

Lemma 7.4. *Let $\delta > 0$, and $\xi \in C_c^\infty(\mathbf{R}^3)$ be a bump function such that*

$$\operatorname{supp}\xi \subset B(0,1), \quad \xi \geqslant 0, \quad \textit{and} \quad \int \xi\,dx = 1;$$

let $\xi_\delta(x) = \delta^{-3}\chi(x/\delta)$ and $\lambda_\delta = \xi_\delta \star \xi_\delta \star \xi_\delta$. Denote by $Q = I - P$ the orthogonal projection on gradient fields in $L^2(\mathbf{R}^3;\mathbf{R}^3)$. Under the assumptions of Theorem 2.4, one has:

$$\epsilon\partial_t Q\big(\lambda_\delta \star \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big) + \nabla_x\lambda_\delta \star \left\langle \frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \to 0,$$
$$\epsilon\partial_t\lambda_\delta \star \left\langle \frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle + \frac{5}{3}\operatorname{div}_x Q\big(\lambda_\delta \star \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big) \to 0,$$

in $L^1_{\mathrm{loc}}(dt; H^s_{\mathrm{loc}}(\mathbf{R}^3))$ for each $s > 0$.

Proof. The second convergence statement above is obvious: indeed, considering the truncated, renormalized energy equation (2.45) with $\xi = \frac{1}{3}|v|^2$, and applying the mollifier λ_δ leads to

$$\begin{aligned}\epsilon\partial_t\lambda_\delta \star \left\langle \frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle + \frac{5}{3}\operatorname{div}_x Q\big(\lambda_\delta \star \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)\\ = -\frac{2}{3}\epsilon\operatorname{div}_x\lambda_\delta \star \mathbf{F}_\epsilon(B) + \frac{1}{3}\epsilon\lambda_\delta \star \mathbf{D}_\epsilon\big(|v|^2\big).\end{aligned}$$

It follows from Proposition 6.1, the entropy bound (2.20) and the entropy production estimate (2.21) that $\mathbf{F}_\epsilon(B)$ is bounded in $L^1_{\mathrm{loc}}(dt\,dx)$; this and Proposition 5.1 eventually entail that the right-hand side of the above equality vanishes in $L^1_{\mathrm{loc}}(dt; H^s_{\mathrm{loc}}(\mathbf{R}^3))$.

The first convergence statement above is much trickier. Start from the analogous truncated, renormalized momentum equation (2.45) with $\xi = v$:

$$\epsilon\partial_t\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle + \nabla_x\frac{1}{3\epsilon}\big\langle |v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\big\rangle = -\epsilon\operatorname{div}_x\mathbf{F}_\epsilon(A) + \epsilon\mathbf{D}_\epsilon(v). \tag{7.24}$$

Applying Q to both sides of the equality above is not obvious, because we only know that the right-hand side vanishes in $L^1_{\mathrm{loc}}(dt; W^{-1,1}_{\mathrm{loc}}(\mathbf{R}^3))$, while Q is known to be continuous on global Sobolev spaces only.

However, $Q = \nabla_x\Delta_x^{-1}\operatorname{div}_x$ is a singular integral operator whose integral kernel decays at infinity. More precisely, we shall use Lemma C.2 together with the following estimates on the right-hand side of (7.24):

Lemma 7.5. *One has:*

$$\epsilon\mathbf{F}_\epsilon(A) \to 0 \quad \textit{and} \quad \epsilon\mathbf{D}_\epsilon(v) \to 0 \quad \textit{in } L^1_{\mathrm{loc}}(dt\,dx),$$

as $\epsilon \to 0$. Furthermore,

$$\epsilon\mathbf{F}_\epsilon(A) = \mathrm{O}(1)_{L^\infty_t(L^2_x)},$$
$$\epsilon\mathbf{D}_\epsilon(v) = \mathrm{O}\big(\epsilon^2 K_\epsilon^{1/2}\big)_{L^1_{t,x}} + \mathrm{O}(\sqrt{\epsilon})_{L^2_t(L^1_x)} + \mathrm{O}(1)_{L^2_{t,x}}.$$

Note that the convergence statement in Lemma 7.5 is a simple consequence of Propositions 5.1 and 6.1 (already used in the derivation of the Boussinesq relation in Section 7.1).

Then let us postpone the proof of the global estimates, which is based on the entropy and entropy production bounds (2.20)–(2.21), and conclude the proof of Lemma 7.4.

Define $\zeta_\delta = \xi_\delta \star \xi_\delta$. One has then

$$\epsilon\partial_t Q\big(\zeta_\delta \star \langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle\big) + \nabla_x\zeta_\delta \star \left\langle \frac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\right\rangle = -Q\big(\xi_\delta \star \big(\epsilon\mathbf{F}_\epsilon(A) \star \nabla\xi_\delta\big)\big) + Q\big(\zeta_\delta \star \big(\epsilon\mathbf{D}_\epsilon(v)\big)\big).$$

For each $\delta > 0$ fixed,

$$Q\big(\xi_\delta \star \big(\epsilon\mathbf{F}_\epsilon(A) \star \nabla\xi_\delta\big)\big) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx), \text{ as } \epsilon \to 0,$$

by the first convergence result in Lemma 7.5 and Lemma C.2.

Next decompose

$$\epsilon \mathbf{D}_\epsilon(v) = \mathbf{D}^0_\epsilon(v) + \mathbf{D}'_\epsilon(v)$$

with

$$\mathbf{D}^0_\epsilon(v) = \mathrm{O}(1)_{L^2_{t,x}} \quad \text{and} \quad \mathbf{D}'_\epsilon(v) = \mathrm{O}\big(\epsilon^2 K_\epsilon^{1/2}\big)_{L^1_{t,x}} + \mathrm{O}(\sqrt{\epsilon})_{L^2_t(L^1_x)}.$$

Thus, one has:

$$\zeta_\delta \star \mathbf{D}'_\epsilon(v) \to 0 \quad \text{in } L^1_{\mathrm{loc}}\big(dt; L^2_x\big),$$

so that

$$Q\big(\zeta_\delta \star \mathbf{D}'_\epsilon(v)\big) \to 0 \quad \text{in } L^1_{\mathrm{loc}}\big(dt; L^2_x\big),$$

as $\epsilon \to 0$, by the L^2-continuity of pseudo-differential operators of order 0. Finally, since $\mathbf{D}^0_\epsilon(v) \to 0$ in $L^1_{\mathrm{loc}}(dt\,dx)$ and is bounded in $L^2_{t,x}$, it follows from Lemma C.2 that

$$Q\big(\zeta_\delta \star \mathbf{D}^0_\epsilon(v)\big) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx).$$

Eventually, we have proved that

$$\epsilon \partial_t Q\big(\zeta_\delta \star \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle\big) + \nabla_x \zeta_\delta \star \left\langle \frac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \right\rangle \to 0,$$

in $L^1_{\mathrm{loc}}(dt\,dx)$ as $\epsilon \to 0$. Therefore, denoting $\lambda_\delta = \xi_\delta \star \xi_\delta \star \xi_\delta$, one has,

$$\epsilon \partial_t Q\big(\lambda_\delta \star \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle\big) + \nabla_x \lambda_\delta \star \left\langle \frac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \right\rangle \to 0,$$

in $L^1_{\mathrm{loc}}(dt; H^s_{\mathrm{loc}}(\mathbf{R}^3))$ for each $s > 0$ as $\epsilon \to 0$. □

Let us now turn to the:

Proof of Lemma 7.5. First, $g_\epsilon \gamma_\epsilon = \mathrm{O}(1)$ in $L^\infty_t(L^2(dx\,M\,dv))$, while $A \in L^2(M\,dv)$: hence

$$\langle A_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle = \mathrm{O}(1)_{L^\infty_t(L^2_x)}.$$

Next decompose $\epsilon \mathbf{D}_\epsilon(v)$ as

$$\epsilon \mathbf{D}_\epsilon(v) = T_1 + T_2 + T_3,$$

where

$$T_1 = \left\langle\!\!\left\langle v_{K_\epsilon} \hat{\gamma}_\epsilon \frac{1}{\epsilon^2}\big(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}\big)^2 \right\rangle\!\!\right\rangle,$$

$$T_2 = 2\left\langle\!\!\left\langle v_{K_\epsilon} \hat{\gamma}_\epsilon \sqrt{G_\epsilon} \frac{1}{\epsilon^2}\big(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}\big) \right\rangle\!\!\right\rangle,$$

$$T_3 = 2\left\langle\!\!\left\langle v_{K_\epsilon} \hat{\gamma}_\epsilon \sqrt{G_\epsilon}\big(\sqrt{G_{\epsilon 1}} - 1\big) \frac{1}{\epsilon^2}\big(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}}\big) \right\rangle\!\!\right\rangle,$$

Since $\frac{1}{\epsilon^4}(\sqrt{G'_\epsilon G'_{\epsilon 1}} - \sqrt{G_\epsilon G_{\epsilon 1}})^2$ is bounded in $L^1_{t,x,\mu}$ (see (2.21)), one has:

$$T_1 = \mathrm{O}\big(\epsilon^2 K_\epsilon^{1/2}\big)_{L^1_{t,x}}.$$

Likewise, $\hat{\gamma}_\epsilon \sqrt{G_\epsilon} = \mathrm{O}(1)$ in $L^\infty_{t,x,v}$ and $v \in L^2(d\mu)$, so that

$$T_2 = \mathrm{O}(1)_{L^2_{t,x}}.$$

The same argument is used for T_3, except that one has to control the terms $v(\sqrt{G_{\epsilon 1}} - 1)$ instead of v in L^2_μ. By Young's inequality,

$$\begin{aligned}\big(1+|v_1|\big)(\sqrt{G_{\epsilon 1}}-1)^2 &\leqslant \big(1+|v_1|\big)|G_{\epsilon 1}-1|\\ &\leqslant \frac{1}{\epsilon}\Big(h(G_{\epsilon 1}-1)+h^*\big(\epsilon\big(1+|v_1|\big)\big)\Big)\\ &\leqslant \frac{1}{\epsilon}h(\epsilon g_{\epsilon 1})+\epsilon h^*\big(1+|v_1|\big)\\ &= \mathrm{O}(\epsilon)_{L^\infty_t(L^1(M_1\,dv_1dx))}+\mathrm{O}(\epsilon)_{L^\infty_{t,x}(L^1(M_1\,dv_1))}.\end{aligned}$$

The 3rd inequality above comes from the superquadratic nature of h^*. Indeed

$$h^*(p)=e^p-p-1=\sum_{n\geqslant 2}\frac{p^n}{n!}$$

so that

$$h^*(\lambda p)\leqslant \lambda^2 h^*(p),\quad \text{for each } p\geqslant 0 \quad\text{and}\quad \lambda\in[0,1].$$

With the upper bound on $\int b(v-v_1,\omega)\,d\omega$, this shows that

$$\begin{aligned}|T_3| &\leqslant \|v_{K_\epsilon}\|_{L^2((1+|v|)^\beta M\,dv)}\|\hat{\gamma}_\epsilon G_\epsilon\|_{L^\infty_v}\|\sqrt{G_{\epsilon 1}}-1\|_{L^2((1+|v_1|)^\beta M_1\,dv_1)}\left\|\frac{1}{\epsilon^2}\big(\sqrt{G'_\epsilon G'_{\epsilon 1}}-\sqrt{G_\epsilon G_{\epsilon 1}}\big)\right\|_{L^2_\mu}\\ &= \mathrm{O}(\sqrt{\epsilon})_{L^2_t(L^1_x)}+\mathrm{O}(\sqrt{\epsilon})_{L^2_{t,x}}.\end{aligned}$$

Combining the previous results leads to the expected estimate for $\mathbf{D}_\epsilon(v)$. □

At this point, we conclude this section with the

Proof of Proposition 7.3. First, we apply the compensated compactness argument for the acoustic system in [21] — see also Theorem A.2 — to conclude from the statement in Lemma 7.4 that

$$\int \nabla_x U : Q\big(\lambda_\delta \star \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)^{\otimes 2}\,dx \to 0,$$
$$\operatorname{div}_x\left(\lambda_\delta\star\left\langle\frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle Q\big(\lambda_\delta\star\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)\right)\to 0,$$

in the sense of distributions on $\mathbf{R}^*_+$ and $\mathbf{R}^*_+\times\mathbf{R}^3$ respectively, for each divergence-free vector field $U\in C^\infty_c(\mathbf{R}^3;\mathbf{R}^3)$.

On the other hand, the compactness property in the x-variable stated in Proposition 4.4 and (7.23) implies that

$$Q\big(\lambda_\delta\star\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)-Q\big(\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)\to 0,$$
$$\lambda_\delta\star\left\langle\frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle-\left\langle\frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle\to 0,$$

in $L^2_{\mathrm{loc}}(dt\,dx)$ as $\delta\to 0$, uniformly in n. Therefore, one has:

$$\int \nabla_x U : Q\big(\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)^{\otimes 2}\,dx \to 0,$$
$$\operatorname{div}_x\left(\left\langle\frac{1}{3}|v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle Q\big(\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)\right)\to 0, \tag{7.25}$$

in the sense of distributions on $\mathbf{R}^*_+$ and $\mathbf{R}^*_+\times\mathbf{R}^3$ respectively, for each divergence-free vector field $U\in C^\infty_c(\mathbf{R}^3;\mathbf{R}^3)$.

Also, we recall from Proposition 7.2 and (7.14) that

$$P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\to u\quad\text{strongly in } L^2_{\mathrm{loc}}(dt\,dx),$$
$$Q\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\to 0\quad\text{weakly in } L^2_{\mathrm{loc}}(dt\,dx).$$

Therefore, for each compactly supported, C^∞ solenoidal vector field U, one has:

$$\begin{aligned}\int_{\mathbf{R}^3} \nabla_x U : \langle v_{K_{\epsilon_n}} g_{\epsilon_n} \gamma_{\epsilon_n}\rangle^{\otimes 2}\, dx &= \int_{\mathbf{R}^3} \nabla_x U : \big(P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)^{\otimes 2}\, dx + \int_{\mathbf{R}^3} \nabla_x U : \big(Q\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)^{\otimes 2}\, dx \\ &\quad + \int_{\mathbf{R}^3} \nabla_x U : \big(P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \otimes Q\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)\, dx \\ &\quad + \int_{\mathbf{R}^3} \nabla_x U : \big(Q\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle \otimes P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big)\, dx \\ &\to \int_{\mathbf{R}^3} \nabla_x U : u\otimes u\, dx,\end{aligned}$$

in the sense of distributions on $\mathbf{R}_+^*$. Together with (7.12), this implies the first limit in Proposition 7.3.

On the other hand, Proposition 7.2 and (7.14) imply that

$$\begin{aligned}&\left\langle \left(\frac{1}{5}|v|^2 - 1\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \to \theta \quad \text{strongly in } L^2_{\mathrm{loc}}(dt\,dx), \\ &\big\langle |v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\big\rangle \to 0 \quad \text{weakly in } L^2_{\mathrm{loc}}(dt\,dx).\end{aligned}$$

Hence

$$\begin{aligned}&\operatorname{div}_x\left(\left\langle \left(\frac{1}{3}|v|^2 - 1\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle \langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\right) \\ &\quad = \operatorname{div}_x\left(\left\langle \left(\frac{1}{5}|v|^2 - 1\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\right) + \frac{2}{15}\operatorname{div}_x\big(\big\langle |v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\big\rangle Q\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big) \\ &\qquad + \frac{2}{15}\operatorname{div}_x\big(\big\langle |v|^2_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\big\rangle P\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\big) + \operatorname{div}_x\left(\left\langle \left(\frac{1}{5}|v|^2 - 1\right)_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\right\rangle Q\langle v_{K_{\epsilon_n}} g_{\epsilon_n}\gamma_{\epsilon_n}\rangle\right) \\ &\quad \to \operatorname{div}_x(u\theta),\end{aligned}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$. With (7.13), this entails the second statement in Proposition 7.3. □

7.3. End of the proof of Theorem 2.4

At this point we return to the renormalized, truncated momentum and energy conservations in the form (7.1) and (7.2).

Asymptotic conservation of momentum. By using the convergence properties in (7.11) and Proposition 7.3 with the decomposition (7.3), one sees that, for each C^∞, compactly supported, solenoidal vector field U,

$$\int_{\mathbf{R}^3} \nabla_x U : \mathbf{F}_{\epsilon_n}(A)\, dx \to \int_{\mathbf{R}^3} \nabla_x U : u\otimes u\, dx - \nu \int_{\mathbf{R}^3} \nabla_x U : \big(\nabla_x u + (\nabla_x u)^T\big)\, dx,$$

in the sense of distributions on $\mathbf{R}_+^*$, while

$$\operatorname{div}_x \mathbf{F}_{\epsilon_n}(B) \to \operatorname{div}_x(u\theta) - \kappa\Delta_x\theta,$$

in the sense of distributions in $\mathbf{R}_+^* \times \mathbf{R}^3$. Furthermore, since $\operatorname{div}_x u = 0$, one has:

$$\int_{\mathbf{R}^3} \nabla_x U : (\nabla_x u)^T\, dx = \int_{\mathbf{R}^3} \nabla_x(\operatorname{div}_x U)\cdot u\, dx = 0,$$

for each solenoidal test vector field U, so that

$$\int_{\mathbf{R}^3} \nabla_x U : \mathbf{F}_{\epsilon_n}(A)\,dx \to \int_{\mathbf{R}^3} \nabla_x U : u \otimes u\,dx - \nu \int_{\mathbf{R}^3} \nabla_x U : \nabla_x u\,dx,$$

in the sense of distributions on $\mathbf{R}_+^*$.

On the other hand, by Proposition 7.2,

$$\int_{\mathbf{R}^3} U \cdot \langle v_{K_{\epsilon_n}} g_{\epsilon_n} \gamma_{\epsilon_n} \rangle\,dx \to \int_{\mathbf{R}^3} U \cdot u\,dx,$$

uniformly on $[0, T]$ for each $T > 0$. In particular, for $t = 0$, one has:

$$\int_{\mathbf{R}^3} U \cdot u|_{t=0}\,dx = \lim_{\epsilon \to 0} \int_{\mathbf{R}^3} U \cdot P\left(\frac{1}{\epsilon} \int_{\mathbf{R}^3} v F_\epsilon^{\mathrm{in}}\,dv\right) dx = \int_{\mathbf{R}^3} U \cdot u^{\mathrm{in}}\,dx.$$

Therefore, u satisfies:

$$\partial_t \int_{\mathbf{R}^3} U \cdot u\,dx - \int_{\mathbf{R}^3} \nabla_x U : u \otimes u\,dx + \nu \int_{\mathbf{R}^3} \nabla_x U : \nabla_x u\,dx = 0, \quad t > 0,$$
$$u|_{t=0} = u^{\mathrm{in}}.$$

Asymptotic conservation of energy. Likewise,

$$\left\langle \left(\frac{1}{5}|v|^2 - 1\right)_{K_{\epsilon_n}} g_{\epsilon_n} \gamma_{\epsilon_n} \right\rangle \to \theta,$$

in $C(\mathbf{R}_+; w - L_x^2)$. In particular, for $t = 0$, one has:

$$\theta|_{t=0} = w - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbf{R}^3} \left(\frac{1}{5}|v|^2 - 1\right) F_\epsilon^{\mathrm{in}}\,dv = \theta^{\mathrm{in}}.$$

Therefore, θ satisfies:

$$\partial_t \theta + \operatorname{div}_x(u\theta) - \kappa \Delta_x \theta = 0, \quad x \in \mathbf{R}^3,\ t > 0,$$
$$\theta|_{t=0} = \theta^{\mathrm{in}}.$$

Notice that one has also

$$\frac{1}{\epsilon_n} \int_{\mathbf{R}^3} v F_{\epsilon_n}\,dv = \langle v g_{\epsilon_n} \rangle \to u,$$
$$\frac{1}{\epsilon_n} \int_{\mathbf{R}^3} \left(\frac{1}{5}|v|^2 - 1\right)(F_{\epsilon_n} - M)\,dv = \left\langle \left(\frac{1}{5}|v|^2 - 1\right) g_{\epsilon_n} \right\rangle \to \theta,$$

weakly in $L^1_{\mathrm{loc}}(dt\,dx)$, because of (7.6) and (7.7).

Asymptotic energy inequality. By Proposition 7.1 and (2.22), one has

$$\frac{2}{\epsilon_n}\left(\sqrt{G_{\epsilon_n}} - 1\right) \to g \quad \text{in } w - L^2_{\mathrm{loc}}\big(dt, L^2(dx\,M\,dv)\big)$$

and

$$\frac{1}{\epsilon_n^2}\left(\sqrt{G'_{\epsilon_n 1} G'_{\epsilon_n}} - \sqrt{G_{\epsilon_n 1} G_{\epsilon_n}}\right) \to \tilde{q} \quad \text{in } w - L^2(dt\,dx\,d\mu).$$

Then, by (2.20) and (2.21),

$$\iint Mg^2(t,x,v)\,dx\,dv \leqslant \varliminf_{n\to\infty} 4\iint M\left(\frac{\sqrt{G_{\epsilon_n}}-1}{\epsilon_n}\right)^2(t,x,v)\,dv\,dx \leqslant \varliminf_{n\to\infty}\frac{2}{\epsilon_n^2}H(F_{\epsilon_n}|M)(t),$$

and

$$\int_0^t\iint \tilde q^2\,ds\,dx\,d\mu \leqslant \varliminf_{n\to\infty}\int_0^t\iint\left(\frac{\sqrt{G'_{\epsilon_n 1}G'_{\epsilon_n}}-\sqrt{G_{\epsilon_n 1}G_{\epsilon_n}}}{\epsilon_n^2}\right)^2 ds\,dx\,d\mu$$

$$\leqslant \varliminf_{n\to\infty}\frac{1}{\epsilon_n^4}\int_0^t\iint \mathcal{E}(F_{\epsilon_n})\,ds\,dx.$$

Explicit computations based on the limiting forms (7.7) and (7.8) of

$$g \quad \text{and} \quad \iint \tilde q b(v-v_1,\omega)\,d\omega\,M_1\,dv_1,$$

and using the symmetries of $\tilde q$ under the $d\mu$-symmetries imply that

$$\iint Mg^2(t,x,v)\,dx\,dv = \int\left(|u|^2(t,x)+\frac{5}{2}|\theta|^2(t,x)\right)dx,$$

while

$$\int \tilde q^2 d\mu \geqslant \frac{1}{2}\nu\left|\nabla_x u+(\nabla_x u)^T\right|^2+\frac{5}{2}\kappa|\nabla_x\theta|^2$$

(see Lemma 4.7 in [3] for a detailed proof of these statements).

Taking limits in the scaled entropy inequality,

$$\frac{1}{\epsilon^2}H(F_\epsilon|M)(t)+\frac{1}{\epsilon^4}\int_0^t\iint \mathcal{E}(F_\epsilon)(s,x)\,dx\,ds \leqslant \frac{1}{\epsilon^2}H\big(F_\epsilon^{\mathrm{in}}|M\big),$$

entails the expected energy inequality:

$$\int_{\mathbf{R}^3}\left(\frac{1}{2}\left|u(t,x)\right|^2+\frac{5}{4}\left|\theta(t,x)\right|^2\right)dx+\int_0^t\int_{\mathbf{R}^3}\left(\nu|\nabla_x u|^2+\frac{5}{2}\kappa|\nabla_x\theta|^2\right)dx\,ds \leqslant \varliminf\frac{1}{\epsilon^2}H\big(F_\epsilon^{\mathrm{in}}|M\big).$$

With this last observation, the proof of Theorem 2.4 is complete.

Appendix A. Some results about the limits of products

For the sake of completeness, we recall here without proof some classical results used in the present paper to pass to the limit in nonlinear terms.

The first one is due to DiPerna and Lions [8], and is referred to as the Product Limit Theorem in [3]:

Theorem A.1. *Let μ be a finite, positive Borel measure on a Borel subset X of $\mathbf{R}^N$. Consider two sequences of real-valued measurable functions defined on X denoted φ_n and ψ_n.*

Assume that (ψ_n) is bounded in $L^\infty(d\mu)$ and such that $\psi_n\to\psi$ a.e. on X while $\varphi_n\to\varphi$ in $w-L^1(d\mu)$. Then

$$\varphi_n\psi_n\to\varphi\psi \quad \text{in } L^1(d\mu) \text{ weak}.$$

The second one is due to Lions and Masmoudi [21], and can be viewed as a compensated compactness result. It states that (fast oscillating) acoustic waves do not contribute to the macroscopic dynamics in the incompressible limit:

Theorem A.2. *Let $c \neq 0$. Consider two families (φ_ϵ) and $(\nabla_x \psi_\epsilon)$ bounded in $L^\infty_{\mathrm{loc}}(dt, L^2_{\mathrm{loc}}(dx))$, such that*

$$\partial_t \varphi_\epsilon + \frac{1}{\epsilon}\Delta_x \psi_\epsilon = \frac{1}{\epsilon} F_\epsilon,$$
$$\partial_t \nabla \psi_\epsilon + \frac{c^2}{\epsilon}\nabla_x \varphi_\epsilon = \frac{1}{\epsilon} G_\epsilon,$$

for some F_ϵ, G_ϵ converging to 0 in $L^1_{\mathrm{loc}}(dt, L^2_{\mathrm{loc}}(dx))$.

Then the quadratic quantities,

$$P\nabla_x \cdot \big((\nabla_x \psi_\epsilon)^{\otimes 2}\big) \quad \text{and} \quad \nabla_x \cdot (\varphi_\epsilon \nabla_x \psi_\epsilon),$$

converge to 0 in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$.

Appendix B. Some regularity results for the free transport operator

The main new idea in our previous work on the Navier–Stokes limit of the Boltzmann equation [13] was to improve integrability and regularity with respect to the x variables using the integrability with respect to the v variables.

This property is obtained by combining the velocity averaging lemma [11,12] with dispersive properties of the free transport operator [6].

We state here two results of this kind used in the present paper, whose proof can be found in [14] or [13].

The first such result, based on the dispersive properties of free transport, explains how the streaming operator transfers uniform integrability from the v variables to the x variables.

Theorem B.1. *Consider a bounded family (ψ_ϵ) of $L^\infty_{\mathrm{loc}}(dt, L^1_{\mathrm{loc}}(dx\,dv))$ such that $(\epsilon\partial_t + v \cdot \nabla_x)\psi_\epsilon$ is bounded in $L^1_{\mathrm{loc}}(dt\,dx\,dv)$. Assume that (ψ_ϵ) is locally uniformly integrable in the v-variable — see Proposition 3.2 for a definition of this notion. Then (ψ_ϵ) is locally uniformly integrable (in all variables t, x and v).*

The second one, which is based on the classical velocity averaging theorem in [11,12], explains how this additional integrability is used to prove a L^1 averaging lemma.

Theorem B.2. *Consider a bounded family (φ_ϵ) of $L^2_{\mathrm{loc}}(dt\,dx, L^2(M\,dv))$ such that $(\epsilon\partial_t + v \cdot \nabla_x)\varphi_\epsilon$ is weakly relatively compact in $L^1_{\mathrm{loc}}(dt\,dx\,M\,dv)$. Assume that $(|\varphi_\epsilon|^2)$ is locally uniformly integrable with respect to the measure $dt\,dx\,M\,dv$.*

Then, for each function $\xi \equiv \xi(v)$ in $L^2(M\,dv)$, each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\left\| \int \varphi_\epsilon(t, x+y, v) M\xi(v)\,dv - \int \varphi_\epsilon(t, x, v) M\xi(v)\,dv \right\|_{L^2([0,T]\times K)} \to 0,$$

as $|y| \to 0$ uniformly in ϵ.

Appendix C. Some regularity results for the Leray projection

One annoying difficulty in handling incompressible or weakly compressible models is the nonlocal nature of the Leray projection P — defined on the space $L^2(\mathbf{R}^3;\mathbf{R}^3)$ of square integrable vector fields, on the closed subspace of divergence-free vector fields. By definition, P is continuous on $L^2(\mathbf{R}^3;\mathbf{R}^3)$, as well as on $H^s(\mathbf{R}^3;\mathbf{R}^3)$ — since P is a classical pseudo-differential operator of order 0. However, P is not continuous on local spaces of the type $L^p_{\mathrm{loc}}(dx)$. Here is how we make up for this lack of continuity.

A first observation leads to a *local* L^2-equicontinuity statement provided that some *global bound* is known to hold.

Lemma C.1. *Consider a sequence of vector fields (ψ_n) uniformly bounded in $L^\infty_t(L^2(dx))$. Assume that, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,*

$$\int_0^T \int_K \big|\psi_n(t, x+y) - \psi_n(t,x)\big|^2\,dx\,dt \to 0 \quad \text{as } |y| \to 0,$$

uniformly in n.

Then, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\int_0^T \int_K \left| P\psi_n(t, x+y) - P\psi_n(t,x) \right|^2 dx\, dt \to 0 \quad \text{as } |y| \to 0,$$

uniformly in n.

Proof. For each $\delta \in (0,1)$ and $R > 0$, let $\chi \equiv \chi(x)$ be a C^∞ bump function satisfying:

$$\chi(x) = 1 \quad \text{for } |x| \leqslant R, \qquad \chi(x) = 0 \quad \text{for } |x| \geqslant R + \delta,$$
$$0 \leqslant \chi \leqslant 1, \quad |\chi'| \leqslant 2/\delta.$$

Obviously, for $|y| \leqslant 1$, one has:

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^3} \left| \chi(x+y)\psi_n(t,x+y) - \chi(x)\psi_n(t,x) \right|^2 dx\, dt \\
&\quad \leqslant 2 \int_0^T \int_{\mathbf{R}^3} \chi(x+y)^2 \left| \psi_n(t,x+y) - \psi_n(t,x) \right|^2 dx\, dt + 2 \int_0^T \int_{\mathbf{R}^3} \left| \chi(x+y) - \chi(x) \right|^2 \left| \psi_n(t,x) \right|^2 dx\, dt \\
&\quad \leqslant 2 \int_0^T \int_{|x| \leqslant R+2} \left| \psi_n(t,x+y) - \psi_n(t,x) \right|^2 dx\, dt + 2 \left(\frac{2}{\delta} \right)^2 |y|^2 T \|\psi_n\|_{L^\infty_t(L^2_x)}
\end{aligned}$$

so that

$$\int_0^T \int_{\mathbf{R}^3} \left| \chi(x+y)\psi_n(t,x+y) - \chi(x)\psi_n(t,x) \right|^2 dx\, dt \to 0$$

as $|y| \to 0$ uniformly in n, since ψ_n is bounded in $L^\infty_t(L^2(M\, dv\, dx))$.

Consider next the decomposition:

$$\chi P = P\chi + [\chi, P],$$

where χ denotes the pointwise multiplication by the function χ, which is another pseudo-differential operator of order 0 on $\mathbf{R}^3$. In particular, $[\chi, P]$ is a classical pseudo-differential operator of order -1 on $\mathbf{R}^3$.

With this decomposition, we consider the expression:

$$\begin{aligned}
&\int_0^T \int_{|x| \leqslant R} \left| \chi(x+y) P\psi_n(t,x+y) - \chi(x) P\psi_n(t,x) \right|^2 dx\, dt \\
&\quad \leqslant 2 \int_0^T \int_{|x| \leqslant R} \left| P(\chi\psi_n)(t,x+y) - P(\chi\psi_n)(t,x) \right|^2 dx\, dt \\
&\qquad + 2 \int_0^T \int_{|x| \leqslant R} \left| [\chi, P]\psi_n(t,x+y) - [\chi, P]\psi_n(t,x) \right|^2 dx\, dt.
\end{aligned}$$

Because P is an $L^2(dx)$-orthogonal projection, the first integral on the right-hand side of the inequality above satisfies:

$$\begin{aligned}
&\int_0^T \int_{|x|\leqslant R} \left|P(\chi\psi_n)(t,x+y) - P(\chi\psi_n)(t,x)\right|^2 dx\,dt \\
&\quad \leqslant \int_0^T \int_{\mathbf{R}^3} \left|P(\chi\psi_n)(t,x+y) - P(\chi\psi_n)(t,x)\right|^2 dx\,dt \\
&\quad \leqslant \int_0^T \int_{\mathbf{R}^3} \left|\chi(x+y)\psi_n(t,x+y) - \chi(x)\psi_n(t,x)\right|^2 dx\,dt \to 0,
\end{aligned}$$

as $|y| \to 0$, uniformly in n. On the other hand, because $[\chi, P]$ is a classical pseudo-differential operator of order -1 on $\mathbf{R}^3$ (see [25], §7.16, on p. 268): therefore $[\chi, P]$ maps $L^2(\mathbf{R}^3)$ continuously into $H^1(\mathbf{R}^3)$. This implies in particular that $[\chi, P]\psi_n$ is bounded in $L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^3))$ so that, for each $R > 0$,

$$\int_0^T \int_{|x|\leqslant R} \left|[\chi, P]\psi_n(t,x+y) - [\chi, P]\psi_n(t,x)\right|^2 dx\,dt \to 0,$$

as $|y| \to 0$, uniformly in n. Hence

$$\int_0^T \int_{|x|\leqslant R} \left|\chi(x+y)P\psi_n(t,x+y) - \chi(x)P\psi_n(t,x)\right|^2 dx\,dt \to 0,$$

as $|y| \to 0$, uniformly in n. Assuming that $R > 2$, that the parameter δ in the definition of χ satisfies $\delta \in (0,1)$ and that $|y| \leqslant 1$, we conclude that

$$\int_0^T \int_{|x|\leqslant R-2} \left|P\psi_n(t,x+y) - P\psi_n(t,x)\right|^2 dx\,dt \to 0,$$

as $|y| \to 0$, uniformly in n, for each $R > 0$ sufficiently large. □

A second observation provides continuity of P in L^1_{loc} under some appropriate global bounds.

Lemma C.2. *Let $\psi_\epsilon \equiv \psi_\epsilon(t,x) \in \mathbf{R}^3$ be a family of vector fields such that $\psi_\epsilon \to 0$ in $L^1_{\mathrm{loc}}(dt\,dx)$ and $\psi_\epsilon = \mathrm{O}(1)$ in $L^1_{\mathrm{loc}}(dt; L^2_x)$. Let ξ_δ be a mollifying sequence on $\mathbf{R}^3$ defined by $\xi_\delta(x) = \delta^{-3}\xi(x/\delta)$ where $\xi \in C_c^\infty(\mathbf{R}^3)$ is a bump function such that*

$$\operatorname{supp}\xi \subset B(0,1), \quad \xi \geqslant 0, \quad \text{and} \quad \int \xi\,dx = 1.$$

Then, for each $\delta > 0$,

$$Q(\xi_\delta \star \psi_\epsilon) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(dt\,dx) \text{ as } \epsilon \to 0.$$

Proof. Let $\chi \in C_c^\infty(\mathbf{R}^3)$. Then

$$\int_0^T \int_{\mathbf{R}^3} \chi(x)\left|Q(\xi_\delta \star \psi_\epsilon)(t,x)\right| dx\,dt = \int_0^T \int_{\mathbf{R}^3} \chi(x)\Omega(t,x) \cdot Q(\xi_\delta \star \psi_\epsilon)(t,x)\,dx\,dt,$$

where Ω is any measurable unit vector field such that

$$\Omega(t,x) = \frac{Q(\xi_\delta \star \psi_\epsilon)}{|Q(\xi_\delta \star \psi_\epsilon)|}(t,x) \quad \text{whenever } Q(\xi_\delta \star \psi_\epsilon)(t,x) \neq 0.$$

Hence

$$\int_0^T\int_{\mathbf{R}^3} \chi(x)\left|Q(\xi_\delta \star \psi_\epsilon)(t,x)\right| dx\,dt = -\int_0^T\int_{\mathbf{R}^3} \Delta_x^{-1}\operatorname{div}_x(\chi\Omega)\operatorname{div}_x(\xi_\delta\star\psi_\epsilon)(t,x)\,dx\,dt.$$

Let $\mathcal{G}(x)=\frac{x}{4\pi|x|^3}$ be the convolution kernel corresponding to $-\nabla_x\Delta_x^{-1}$; for $R>0$, denote $\mathcal{G}_R(x)=\mathcal{G}(x)\mathbf{1}_{|x|<R}$ and $\mathcal{G}^R(x)=\mathcal{G}(x)\mathbf{1}_{|x|\geqslant R}$. Thus

$$\begin{aligned}&\int_0^T\int_{\mathbf{R}^3} \chi(x)\left|Q(\xi_\delta \star \psi_\epsilon)(t,x)\right| dx\,dt\\ &\quad=\int_0^T\int_{\mathbf{R}^3} \mathcal{G}_R\star(\chi\Omega)(\nabla\xi_\delta)\star\psi_\epsilon(t,x)\,dx\,dt+\int_0^T\int_{\mathbf{R}^3} \mathcal{G}^R\star(\chi\Omega)(\nabla\xi_\delta)\star\psi_\epsilon(t,x)\,dx\,dt.\end{aligned}$$

We have used here the following simplifying notation: if a and b are two vector fields on $\mathbf{R}^3$, we denote:

$$a\star b(x)=\int_{\mathbf{R}^3} a(x-y)\cdot b(y)\,dy,$$

where $\cdot$ is the canonical inner product on $\mathbf{R}^3$.

Observe that $\mathcal{G}^R=\mathrm{O}(1/\sqrt{R})$ in L^2_x, while $\chi\Omega\in L^\infty_t(L^1_x)$ (since $|\Omega|=1$ and $\operatorname{supp}(\chi)$ is compact). Hence

$$\mathcal{G}^R\star(\chi\Omega)=\mathrm{O}(1/\sqrt{R})\quad\text{in } L^1_{\mathrm{loc}}\big(dt;L^2_x\big),$$

and $(\nabla\xi_\delta)\star\psi_\epsilon=\mathrm{O}(1)$ in $L^1_{\mathrm{loc}}(dt;L^2_x)$ for each $\delta>0$ since $\psi_\epsilon=\mathrm{O}(1)$ in $L^1_{\mathrm{loc}}(dt;L^2_x)$. Hence the second integral is $\mathrm{O}(1/\sqrt{R})$ for each $\delta>0$.

Next $\mathcal{G}_R=\mathrm{O}(R)$ in L^1_x and thus $\mathcal{G}_R\star(\chi\Omega)=\mathrm{O}(R)$ in L^∞_x since $|\Omega|=1$; moreover,

$$\operatorname{supp}_x\big(\mathcal{G}_R\star(\chi\Omega)\big)\subset\operatorname{supp}(\chi)+B(0,R),$$

is bounded for each $R>0$. On the other hand $\nabla\xi_\delta\star\psi_\epsilon\to 0$ in $L^1_{\mathrm{loc}}(dt\,dx)$, so that the first integral vanishes as $\epsilon\to 0$ for each $\delta>0$ and each $R>0$. Passing to the limsup as $\epsilon\to 0^+$, then letting $R\to 0^+$ leads to the announced result. $\square$

References

[1] C. Bardos, F. Golse, C.D. Levermore, Macroscopic limits of kinetic equations, in: Multidimensional Hyperbolic Problems and Computations, Minneapolis, MN, 1989, in: IMA Vol. Math. Appl., vol. 29, Springer, New York, 1991, pp. 1–12.

[2] C. Bardos, F. Golse, C.D. Levermore, Fluid dynamic limits of kinetic equations I: Formal derivations, J. Stat. Phys. 63 (1991) 323–344.

[3] C. Bardos, F. Golse, C.D. Levermore, Fluid dynamic limits of the Boltzmann equation II: Convergence proofs, Comm. Pure Appl. Math. 46 (1993) 667–753.

[4] C. Bardos, F. Golse, C.D. Levermore, The acoustic limit for the Boltzmann equation, Arch. Rational Mech. Anal. 153 (2000) 177–204.

[5] C. Bardos, S. Ukai, The classical incompressible Navier–Stokes limit of the Boltzmann equation, Math. Models Methods Appl. Sci. 1 (1991) 235–257.

[6] F. Castella, B. Perthame, Estimations de Strichartz pour les équations de transport cinétique, C. R. Acad. Sci. Paris Sér. I Math. 322 (1996) 535–540.

[7] A. DeMasi, R. Esposito, J. Lebowitz, Incompressible Navier–Stokes and Euler limits of the Boltzmann equation, Comm. Pure Appl. Math. 42 (1990) 1189–1214.

[8] R.J. DiPerna, P.-L. Lions, On the Cauchy problem for the Boltzmann equation: Global existence and weak stability results, Ann. of Math. 130 (1989) 321–366.

[9] R. Esposito, M. Pulvirenti, From Particles to Fluids, in: Handbook of Mathematical Fluid Dynamics, vol. 3, North-Holland, Amsterdam, 2004, pp. 1–82.

[10] F. Golse, C.D. Levermore, The Stokes–Fourier and acoustic limits for the Boltzmann equation, Comm. Pure Appl. Math. 55 (2002) 336–393.

[11] F. Golse, P.-L. Lions, B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation, J. Funct. Anal. 76 (1988) 110–125.

[12] F. Golse, B. Perthame, R. Sentis, Un résultat de compacité pour les équations de transport et application au calcul de la valeur propre principale de l'opérateur de transport, C. R. Acad. Sci. Paris 301 (1985) 341–344.

[13] F. Golse, L. Saint-Raymond, The Navier–Stokes limit of the Boltzmann equation for bounded collision kernels, Invent. Math. 155 (2004) 81–161.

[14] F. Golse, L. Saint-Raymond, Velocity averaging in L^1 for the transport equation, C. R. Math. Acad. Sci. Paris 334 (2002) 557–562.

[15] H. Grad, Asymptotic theory of the Boltzmann equation II, in: Rarefied Gas Dynamics, Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962, vol. I, 1963, pp. 26–59.

[16] O.E. Lanford, Time Evolution of Large Classical Systems, in: J. Moser (Ed.), Lecture Notes in Phys., vol. 38, Springer-Verlag, 1975, pp. 1–111.

[17] C.D. Levermore, N. Masmoudi, From the Boltzmann equation to an incompressible Navier–Stokes–Fourier system, Preprint, 2008.

[18] P.-L. Lions, Conditions at infinity for Boltzmann's equation, Comm. Partial Differential Equations 19 (1994) 335–367.

[19] P.-L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1: Incompressible Models, Oxford University Press, 1996.

[20] P.-L. Lions, N. Masmoudi, Une approche locale de la limite incompressible, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999) 387–392.

[21] P.-L. Lions, N. Masmoudi, From Boltzmann equation to Navier–Stokes equations I, Arch. Rational Mech. Anal. 158 (2001) 173–193.

[22] P.-L. Lions, N. Masmoudi, From Boltzmann equation to the Navier–Stokes and Euler equations II, Arch. Rational Mech. Anal. 158 (2001) 195–211.

[23] L. Saint-Raymond, Discrete time Navier–Stokes limit for the BGK Boltzmann equation, Comm. Partial Differential Equations 27 (2002) 149–184.

[24] L. Saint-Raymond, From the BGK model to the Navier–Stokes equations, Ann. Sci. École Norm. Sup. 36 (2003) 271–317.

[25] E. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.