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ON THE DISTRIBUTION OF FREE PATH LENGTHS FOR THE PERIODIC LORENTZ GAS II [*,**]

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Abstract. Consider the domain $Z_\epsilon = \{x \in \mathbb{R}^n \,;\, \mathrm{dist}(x, \epsilon\mathbb{Z}^n) > \epsilon^\gamma\}$ and let the free path length be defined as $\tau_\epsilon(x,v) = \inf\{t > 0\,;\, x - tv \in \partial Z_\epsilon\}$. In the Boltzmann-Grad scaling corresponding to $\gamma = \frac{n}{n-1}$, it is shown that the limiting distribution ϕ_ϵ of τ_ϵ is bounded from below by an expression of the form C/t, for some $C > 0$. A numerical study seems to indicate that asymptotically for large t, $\phi_\epsilon \sim C/t$. This is an extension of a previous work [J. Bourgain *et al.*, *Comm. Math. Phys.* **190** (1998) 491–508]. As a consequence, it is proved that the linear Boltzmann type transport equation is inappropriate to describe the Boltzmann-Grad limit of the periodic Lorentz gas, at variance with the usual case of a Poisson distribution of scatterers treated in [G. Gallavotti (1972)].

Résumé. Considérons le domaine $Z_\epsilon = \{x \in \mathbb{R}^n \,;\, \mathrm{dist}(x, \epsilon\mathbb{Z}^n) > \epsilon^\gamma\}$ et définissons le temps de sortie par la formule $\tau_\epsilon(x,v) = \inf\{t > 0\,;\, x - tv \in \partial Z_\epsilon\}$. Sous l'hypothèse de la loi d'échelle de Boltzmann-Grad, qui correspond au cas où $\gamma = \frac{n}{n-1}$, on montre que la fonction de répartition ϕ_ϵ des valeurs de τ_ϵ est minorée asymptotiquement lorsque $\epsilon \to 0$ par une expression de la forme C/t, avec $C > 0$. Des simulations numériques semblent indiquer que, pour $\epsilon \to 0$ et $t \to +\infty$, $\phi_\epsilon \sim C/t$. Ce travail généralise ce qui a été montré précédemment [J. Bourgain *et al.*, *Comm. Math. Phys.* **190** (1998) 491–508]. On en déduit l'impossibilité de décrire la limite de Boltzmann-Grad d'un gaz de Lorentz par une équation de Boltzmann linéaire dans le cas d'une configuration périodique des obstacles, contrairement au cas d'une distribution poissonienne d'obstacles traité dans [G. Gallavotti (1972)].

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1. The distribution of free path lengths

The Lorentz gas is a classical model of non-equilibrium statistical mechanics. It consists of a cloud of point particles interacting by collisions with a prescribed distribution of obstacles. Collisions between particles are neglected. One of the main issues concerning this model is the large scale dynamics, which is expected to be described by either kinetic or diffusion equations, depending on the various scaling assumptions made. While

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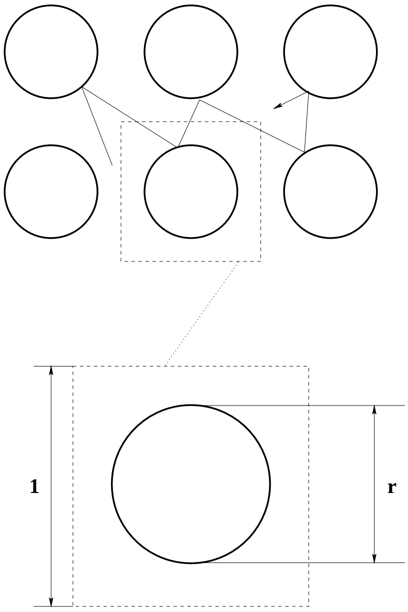

FIGURE 1. The billiard table.

satisfying results on this model have been obtained in the case of a random distribution of obstacles (see for example [1], [8] or [10]), the case of periodic distributions of obstacles has been at the origin of highly nontrivial techniques in ergodic theory (see [3–6]).

If one has in mind to describe the large scale dynamics of the periodic Lorentz gas by a kinetic (linear Boltzmann) equation, a very natural first ingredient to consider is the notion of mean free path, or, more generally, the distribution of free path lengths. The latter notion was studied in [2], with fairly complete success in the two dimensional case, and only partial success in all dimensions higher than two. The present paper extends the two dimensional result of [2] to any space dimension, and discusses applications thereof to the derivation of the linear Boltzmann equation.

1.1. **The periodic billiard table**

The Lorentz gas model considered in this paper is defined by a periodic distribution of spherical obstacles centered at the nodes of a cubic lattice in the Euclidean space.

More precisely, let $n \geq 2$ denote the space dimension. Choose the minimal distance between distinct lattice points as unit of length, and pick the radius of the spherical obstacles to be $r \in]0, \frac{1}{2}[$. The spatial domain where particle motion takes place is therefore

$$Z[r] = \{x \in \mathbb{R}^n \,;\, \mathrm{dist}(x, \mathbb{Z}^n) > r\}.$$

For simplicity, we restrict our attention to a one-speed gas (as shall be seen, the dynamics of the system under consideration here does not involve changes in the kinetic energy of particles induced by collisions with the obstacles). By an appropriate choice of a macroscopic time scale, particles are assumed to move at (uniform) speed 1. In other words, the velocity variable v is assumed to be a unit vector $v \in S^{n-1}$.

The free path length (or exit time) is defined as follows for all points $x \in \overline{Z}[r]$ and $v \in S^{n-1}$

$$\tau(x, v; r) = \inf\{t > 0 \,;\, x - tv \in \partial Z[r]\}.$$

Clearly $\tau(\cdot,\cdot;r)$ is a Borelian function for all $r \in]0,\frac{1}{2}[$. The notion of mean free path, which is central in the classical kinetic theory of gases, is defined as some appropriate average of $\tau(\cdot,\cdot;r)$.

Therefore, the next logical step is to define a measure on $\overline{Z}[r] \times S^{n-1}$ so as to have a precise definition of the notion of mean free path. There are two natural choices of such a measure, whose respective merits are discussed in [7]. The only one considered in the present work has the advantage of taking into proper account the considerable oscillations of the free path length $\tau(x,v;r)$ as the velocity variable v runs through the unit sphere.

Let $Y[r] = \overline{Z}[r]/\mathbb{Z}^n$; topologically $Y[r]$ is a punctured torus. Let $Q[r] = \mathrm{d}x\mathrm{d}v - \mathrm{meas}(Y[r] \times S^{n-1})$ and let $\mu[r]$ be the Borelian measure defined on $Y[r] \times S^{n-1}$ by the formula

$$\mathrm{d}\mu[r](x,v) = \frac{1}{Q[r]}\mathrm{d}x\mathrm{d}v.$$

Since the free path length satisfies the relation

$$\tau(x+k,v;r) = \tau(x,v;r) \text{ for all } x \in \overline{Z}[r]\,,\ k \in \mathbb{Z}^n,\ v \in S^{n-1},$$

the function $\tau(\cdot,\cdot;r)$ induces a Borelian function on the quotient phase space $Y[r] \times S^{n-1}$, still, somewhat abusively, denoted by τ_ϵ, which is the main object of study in the present paper.

1.2. **Estimates on the distribution of τ**

As will be seen below, if one proceeds to defining the mean free path as the average of $\tau(\cdot,\cdot;r)$ under the measure defined above, one finds

$$\text{mean free path } = \int_{Y[r]\times S^{n-1}} \tau(x,v;r)\mathrm{d}\mu[r](x,v) = +\infty. \tag{1}$$

This already casts doubts on the validity of a linear Boltzmann equation as the limiting governing equation for the dynamics of the Lorentz gas in the so-called Boltzmann-Grad scaling. We discuss this issue in the next section.

However, more information is to be found in the distribution $\phi[r]$ of the free path length $\tau(\cdot,\cdot;r)$ under $\mu[r]$, which is the push-forward under the map $\tau(\cdot,\cdot;r) : Y[r] \times S^{n-1} \to \mathbb{R}_+$ of the measure μ. We recall that this phrase designates the Borelian probability measure on $\mathbb{R}_+$ such that, for all $a \le b \in \mathbb{R}_+$,

$$\phi[r]([a,b]) = \mu[r](\{(x,v) \in Y[r] \times S^{n-1}\,;\, \tau(x,v;r) \in [a,b]\}). \tag{2}$$

Our main result is:

Theorem 1.1. *For any integer $n > 1$, there exist two positive constants $C(n) < C'(n)$ such that, for all $r \in]0,\frac{1}{2}[$ and all $t > \frac{1}{r^{n-1}}$*

$$\frac{C(n)}{r^{n-1}t} \le \phi[r]([t,+\infty[) \le \frac{C'(n)}{r^{n-1}t}\,\cdot \tag{3}$$

In [2], the upper bound in (3) was established for all n, while the lower bound was proved only in the case where $n = 2$. Here we complete the result of [2] by extending the validity of the proof for the lower bound there to arbitrary space dimension. Also, in [2] (see Rem. 1, p. 495), we announced a weaker lower bound of the form

$$\phi[r]([t,+\infty[) \ge \frac{C''(n)}{rt^{n-1}}, \tag{4}$$

for all $t > \frac{1}{r^{\frac{1}{n-1}}}$. While such an estimate would have the same effect regarding the application of kinetic theory to the periodic Lorentz gas as does (3), it is a weaker estimate for large t's, in particular one that does not entail that the mean free path is infinite (see (1)).

On the contrary, estimate (3), which implies (4), is clearly optimal as regards the decay rate in t. Thus, the present paper improves on the lower bound announced in [2] in a significant way.

The remaining part of the paper is organized as follows: Section 3 contains the proof of Theorem 1.1 above, while applications to kinetic theory are given in Section 2 below.

2. Applications to kinetic theory

2.1. **The Boltzmann-Grad scaling**

So far, we have only considered the geometry of the billiard table. Now we consider the dynamics of a gas of point particles on the billiard table, undergoing elastic collisions with the spherical obstacles (with inter-particle collisions neglected). The question is whether one can model the large scale effect of the obstacles by an equivalent absorption/scattering mechanism, as can be done in the case of a random (Poisson) distribution of obstacles (see [1], [8] or [10]).

This suggests to pick a macroscopic length scale as unit of length instead of the minimal distance between lattice points, which is now considered as a microscopic length scale. In other words, one chooses as macroscopic length scale the typical length scale on which the initial density of particles varies significantly: this length scale henceforth defines the unit of length. One calls ϵ the ratio of the minimal distance between lattice points to this unit of length; in this scaling, the radius of the obstacles becomes ϵr so as to keep the ratio of this radius to the lattice length scale equal to r as in Section 1. In this process, the speed of the particles is unchanged (equal to 1); thus the free path length of this new billiard table, henceforth denoted by $\tau_\epsilon(\cdot,\cdot;\epsilon r)$ scales exactly as the lattice, *i.e.*

$$\tau_\epsilon(\epsilon x, v; \epsilon r) = \epsilon \tau(x, v; r).$$

Thus, if one defines $\phi_\epsilon[\epsilon r]$ to be the distribution of free path length $\tau_\epsilon(\cdot,\cdot;\epsilon r)$, one sees that (3) transforms into

$$\frac{C(n)}{r^{n-1}\frac{t}{\epsilon}} \le \phi_\epsilon[\epsilon r]([t, +\infty[) \le \frac{C'(n)}{r^{n-1}\frac{t}{\epsilon}} \cdot \tag{5}$$

Now, the only case where the number of collisions per unit of time is not either 0 or $+\infty$ as $\epsilon \to 0$ is the case where r^{n-1}/ϵ converges to some $r_* \in]0, +\infty[$ as $\epsilon \to 0$. Up to a trivial change of time scale, we may pick $r_* = 1$, which suggests that the value

$$r = \epsilon^{\frac{1}{n-1}} \tag{6}$$

plays a special role.

Indeed, if $r_\epsilon >> \epsilon^{\frac{1}{n-1}}$, the upper bound in (5) shows that $\phi_\epsilon[\epsilon r]([t, +\infty[) \to 0$ for all $t > 0$: thus $\phi_\epsilon[\epsilon r]$ converges weakly to the Dirac measure at $t = 0$, which is another way of saying the free path length is statistically small compared to the macroscopic length scale (or that there are infinitely many collisions per unit of time).

On the other hand, if $r_\epsilon << \epsilon^{\frac{1}{n-1}}$, then, for all $t > t_\epsilon$ with $t_\epsilon = \epsilon/r^{n-1}$, then the lower estimate in (3) shows that $\phi_\epsilon[\epsilon r]([t, +\infty[) \ge C(n)t_\epsilon/t$. In particular does one have $\phi_\epsilon[\epsilon r]([t_\epsilon, +\infty[) \ge C(n)$, which indicate that, in the limit as $\epsilon \to 0$ the free path length statistically becomes large compared to the macroscopic length scale (in other words, there are almost no collision per unit of time).

But, if $r = \epsilon^{\frac{1}{n-1}}$, the number of collision per unit of time takes all positive values with nontrivial probability, at least when averaged over a sufficiently large interval of time. This is why the scaling (6) is, for the present model, the closest possible approximation of a Boltzmann-Grad scaling.

2.2. **Why a kinetic description is impossible**

We assume the Boltzmann-Grad scaling (6) henceforth and use the following abbreviations:

$$Z_\epsilon = \epsilon Z[\epsilon^{1/(n-1)}], \qquad \tau_\epsilon(x,v) = \tau_\epsilon(x,v;\epsilon^{n/(n-1)}).$$

A kinetic description of the periodic Lorentz gas in the billiard table defined by Z_ϵ would consist in replacing the effect of collisions on the periodic distribution of obstacles by an absorption and scattering mechanism.

As a first step, one could consider the case of fully absorbing obstacles (*i.e.* think of these obstacles as holes into which particles would fall and disappear); the corresponding question for this system is whether one can model the effect of the periodic distribution of traps by an absorption cross-section in the limit as $\epsilon \to 0$. On the basis of estimate (3), we show below that this is not possible.

The mathematical formulation of the problem is as follows. Consider the free transport of the gas of point particles moving at unit speed; the gas is described by its number density $f_\epsilon(t,x,v)$ which is the density of particles which, at time t are in position $x \in Z_\epsilon$ and move in the direction $v \in S^{n-1}$. Before falling in the traps, particles are transported with no acceleration, so that

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = 0, \quad x \in Z_\epsilon,\ v \in S^{n-1},\ t > 0.$$

No particle can leave a trap: hence

$$f_\epsilon(\epsilon k + \epsilon^{n/(n-1)}\omega, v) = 0, \quad k \in \mathbb{Z}^n,\ v,\ \omega \in S^{n-1}.$$

Finally, the initial number density is prescribed

$$f_\epsilon(0,x,v) = f_\epsilon^{\mathrm{in}}(x,v), \quad (x,v) \in Z_\epsilon \times S^{n-1}.$$

The main result in this section, answering the question raised above concerning the validity of a kinetic limit for the periodic, fully absorbing Lorentz gas is provided by the following theorem.

Theorem 2.1. *Assume that there exists $M > 0$ such that*

$$0 \le f_\epsilon^{\mathrm{in}} \le M\,, \qquad a.e.\ on\ Z_\epsilon \times S^{n-1}.$$

For each $\epsilon > 0$, the number density f_ϵ is extended by by 0 in $\mathbb{R} \times Z_\epsilon^c \times S^{n-1}$ (the resulting extension being still denoted by f_ϵ). Then the family f_ϵ is relatively compact in $L^\infty(\mathbb{R} \times \mathbb{R}^n \times S^{n-1})$ equipped with the weak- topology, and, for any limit point f of f_ϵ as $\epsilon \to 0$, there exists no absorption cross-section $\kappa \equiv \kappa(x,v)$ such that*

$$\kappa(x,v) \ge \sigma > 0, \qquad \text{for all } (x,v) \in \mathbb{R}^n \times S^{n-1} \tag{7}$$

and

$$\partial_t f + v \cdot \nabla_x f + \kappa f = 0. \tag{8}$$

Proof. Suppose, on the contrary, that there exists a subsequence of f_ϵ (still denoted f_ϵ for simplicity) converging to f in $L^\infty(\mathbb{R} \times \mathbb{R}^n \times S^{n-1})$ weak-*, and a constant $\sigma > 0$ such that

$$\partial_t f + v \cdot \nabla_x f + \sigma f \le 0. \tag{9}$$

Solving the transport equation by the method of characteristics shows that

$$f_\epsilon(t,x,v) = f_\epsilon^{\mathrm{in}}(x - tv, v)\mathbb{1}_{t \le \tau_\epsilon(x,v)}.$$

Without loss of generality, we can consider the case where f_ϵ^{in} is $\mathbb{1}_{Z_\epsilon}$. Since $f_\epsilon \to f$ as $\epsilon \to 0$ in $L^\infty(\mathbb{R}\times\mathbb{R}^n\times S^{n-1})$ weak-*, for any test function $\phi \in C_c^1(\mathbb{R}^n \times S^{n-1})$ and any $0 < t_1 < t_2$, one has

$$\int_{t_1}^{t_2} \iint f_\epsilon(t,x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}t = \iint \phi(x,v)\int_{t_1}^{t_2} \mathbb{1}_{t\le\tau_\epsilon(x,v)}\mathrm{d}t\mathrm{d}x\mathrm{d}v$$
$$\to \int_{t_1}^{t_2} \iint f(t,x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}t$$

as $\epsilon \to 0$. Pick ϕ so that $\phi \ge 0$ on $\mathbb{R}^n \times S^{n-1}$ and $\phi \ge 1$ on $[-R,R]^n \times S^{n-1}$; thus

$$\liminf_{\epsilon\to0} \iint \phi(x,v)\int_{t_1}^{t_2} \mathbb{1}_{t\le\tau_\epsilon(x,v)}\mathrm{d}t\mathrm{d}x\mathrm{d}v \ge (2R)^n \int_{t_1}^{t_2} \phi_\epsilon([t,+\infty[)\mathrm{d}t \ge (2R)^n C(n)\log\left(\frac{t_2}{t_1}\right), \tag{10}$$

by (3)–(5), provided that $t_1 > 1$. On the other hand, if f satisfies (8),

$$f(t,x+tv,v) \le f(0,x,v)\mathrm{e}^{-\sigma t} \le \mathrm{e}^{-\sigma t},$$

so that

$$\int_{t_1}^{t_2} \iint f(t,x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}t \le \frac{\mathrm{e}^{-\sigma t_1}}{\sigma}\iint \phi(x,v)\mathrm{d}x\mathrm{d}v. \tag{11}$$

Keeping t_1 fixed and letting $t_2 \to +\infty$, one sees that, for t_2 large enough,

$$(2R)^n C(n)\log\left(\frac{t_2}{t_1}\right) > 2\frac{\mathrm{e}^{-\sigma t_1}}{\sigma}\iint \phi(x,v)\mathrm{d}x\mathrm{d}v;$$

thus, in view of (10) and (11) one has

$$\liminf_{\epsilon\to0}\int_{t_1}^{t_2} \iint f_\epsilon(t,x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}t > \int_{t_1}^{t_2} \iint f(t,x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}t.$$

This contradiction shows that the weak-* limit f of f_ϵ as $\epsilon \to 0$ in $L^\infty(\mathbb{R}\times\mathbb{R}^n\times S^{n-1})$ cannot satisfy (8), and this holds for any positive σ.

Further, any nonnegative f that satisfies (8) with an absorption cross-section κ as in (7) must satisfy (9); by the previous argument, it is therefore impossible that f satisfy (8) with κ as in (7). □

2.3. An open problem

Nevertheless, the maximum principle for the transport equation implies that, under the same assumptions as in the previous subsection, the family f_ϵ is bounded (and therefore relatively compact for the weak-* topology) in $L^\infty(\mathbb{R}\times\mathbb{R}^n\times S^{n-1})$. A natural problem is to describe the weak-* limit points of f_ϵ as $\epsilon \to 0$.

Another formulation of the same problem is as follows: the bounds (5) imply that, for all $t > 1$ and all $\epsilon > 0$,

$$\frac{C(n)}{t} \le \phi_\epsilon([t,+\infty[) \le \frac{C'(n)}{t}\,.$$

It would then be natural to investigate the possible limit points of ϕ_ϵ as $\epsilon \to 0$ in the set of probability measures on $\mathbb{R}_+$ equipped with the vague topology. For example, if ϕ is such a limit point, does there exists a constant $A(n)$ such that

$$\phi([t,+\infty[) = \frac{A(n)}{t}? \tag{12}$$

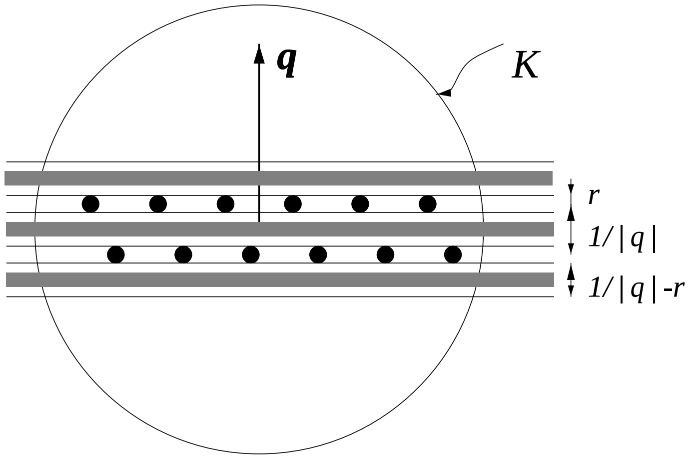

FIGURE 2. Sandwich-layers (channels) in $Z[r]$.

3. Sandwiches and long trajectories in n-dimensional regular lattices

This section is devoted to the proof of the main theorem, *i.e.* of the estimate (3). The upper bound is established in [2], and also the lower one in the two-dimensional case. Here we give a proof, valid in an arbitrary dimension, for the lower bound. The idea is very much the same as for two dimensions: one considers a line segment of length L, starting at $x_0 \in Z[r]$, and in the direction $v \in S^{n-1}$. If x_0 and v are taken at random, uniformly distributed, can one find an estimate of the probability that the entire line segment lies in $Z[r]$? The estimate (3) is an answer to this, and the construction below gives its proof.

In the two-dimensional case, it is clear that only the line segments with a rational direction can be arbitrarily extended without intersecting with the holes. In [2] this fact was used as follows. A rational vector $q = (q^1, q^2)$ defines a set of channels which are orthogonal to q and which extend to infinity inside $Z[r]$, and which do not contain any of the points in $\mathbb{Z}^n$. Assuming that the greatest common divisor of q^1 and q^2 is one ($g.c.d.(q^1, q^2) = 1$), it is easy to compute the distance separating the channels: Any lattice vector $p = (p^1, p^2) \in \mathbb{Z}^2$ satisfies $q^1p^1 + q^2p^2 = k$, where $k = 0, \pm1, \pm2, \pm3\dots$ This defines a set of lines that includes all lattice points, and the distance between these lines is $d = 1/|q|$, or, without the assumption that q^1 q^2 are coprime, $d = g.c.d.(p^1, q^2)/|q|$. The width of an actual channel is $|q|^{-1} - r$, because of the holes that are blocking the path. This means that the number of rational directions for which infinitely long channels exist is finite. The estimate from below in (3) is then obtained by considering the a middle third of a channel: any line-segment starting in the middle third with a direction v such that $|v \cdot q|/|q| < d/3L$ lies entirely in that same channel. And the estimate can be concluded by summing over the finite number of open channels that exist for a given r.

One can do very much the same operation in any space dimension. In $\mathbb{R}^3$, the channels are replaced by a layered structure, and similarly in higher dimensions. We shall call the n-dimensional analogue of a channel a *sandwich layer*.

Take an integer vector q such that $g.c.d.(q) \equiv g.c.d.(q^1, \dots, q^n) = 1$. The sandwich layers corresponding to the direction q are separated by planes,

$$P_{q,z} = \{x \in \mathbb{R}^n;\, (x - z) \cdot q = 0\}, \quad z \in \mathbb{Z}^n$$

which also in this case are separated by a distance $d = 1/|q|$, and if $r < |q|^{-1}$, there are hole-free layers with thickness $|q|^{-1} - r$. And the "middle third" of a hole-free layer has thickness

$$a_{q,r} = \frac{1}{3}\Big(\frac{1}{|q|} - r\Big) \cdot$$

From now on, a typical hole free sandwich layer will be denoted by $\tilde{\Lambda}_{q,r}$, and the corresponding middle third by $\tilde{\Lambda}_{\frac{1}{3},q,r}$. Moreover, the union of all middle thirds is denoted

$$\Lambda_{\frac{1}{3},q,r} = \bigcup \tilde{\Lambda}_{\frac{1}{3},q,r}.$$

Like in the two-dimensional case, any line segment of length L which begins at a point in one middle third, $x_0 \in \tilde{\Lambda}_{\frac{1}{3},q,r}$, and has a direction v belonging to the set

$$A_{q,r,L} = \Big\{v \in S^{n-1};\, |v \cdot q|/|q| \leq \frac{a_{q,r}}{3L}\Big\}$$

lies entirely inside the layer $\Lambda_{q,r}$. If only directions q for which

$$|q| \leq q_{\max} \;\equiv\; (2r)^{-1},$$

are considered, then for any large ball $K \in \mathbb{R}^n$,

$$\frac{\operatorname{meas}\big(\{x\,;\, x \text{ belongs to some middle third}\} \bigcap K\big)}{\operatorname{meas}(K)} \sim \frac{1}{3}\big(\frac{1}{|q|} - r\big) > 1/6.$$

This is the same as saying that if $|q|$ is not to large (depending on the diameter r), then the density of the middle third layers is larger than $1/6$. This means that in considering only those line segments that have an endpoint in a middle third, one does not loose too much:

$$\frac{\mathrm{d}x\mathrm{d}v\text{-meas}\big\{(x,v)\;\;;\;\; v \in A_{q,r,L},\; x \in \Lambda_{\frac{1}{3},q,r} \cap K\big\}}{\mathrm{d}x\text{-meas}(K)} \geq \frac{1}{6}\mathrm{d}v\text{-meas}(A_{q,r,L}).$$

The total $\mathrm{d}x\mathrm{d}v$-measure of the set of $x_0 \in K, v \in S^{n-1}$ for which the corresponding line segment of length L remains in $Z[r]$ is consequently bounded from below by

$$\frac{1}{6}\mathrm{d}v\text{-meas}\Big(\bigcup_{|q|\leq q_{\max}} A_{q,r,L}\Big).$$

The measure of a set $A_{q,r,L}$ is approximately $\mathrm{d}v\text{-meas}(S^{n-2})a_{q,r}$, but when summing of the different q, one must take into account that any two sandwich layers of different directions intersect. Let the number of directions q with $|q| \leq q_{\max}$ be J. Enumerate these directions so that

$$j \leq k \implies |q_j| \leq |q_k|, \qquad k \leq J.$$

By the inclusion-exclusion principle,

$$\mathrm{d}v\text{-meas}(A_{q_1,r,L} \cup A_{q_2,r,L}) \geq \mathrm{d}v\text{-meas}(A_{q_1,r,L}) + \mathrm{d}v\text{-meas}(A_{q_2,r,L}) - \mathrm{d}v\text{-meas}(A_{q_1,r,L} \cap A_{q_2,r,L})$$

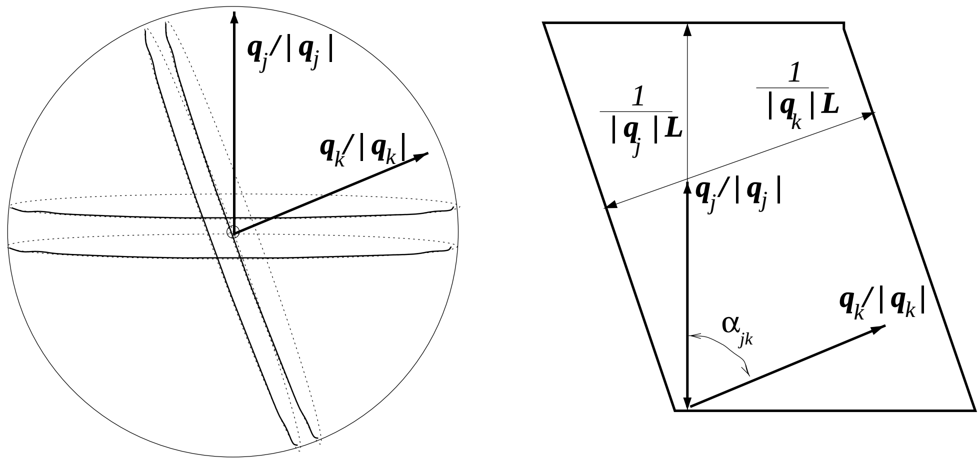

FIGURE 3. Intersection of a sandwich-layer and a sphere.

and then adding more sandwich layers increases the measure; the kth sandwich layer contributes to the total measure by at least

$$\mathrm{d}v\text{-meas}(A_{q_k,r,L}) - \mathrm{d}v\text{-meas}\Big(A_{q_k,r,L} \cap \big(\cup_{j=1}^{k-1} A_{q_j,r,L}\big)\Big) \geq \mathrm{d}v\text{-meas}(A_{q_k,r,L}) - \sum_{j=1}^{k-1} \mathrm{d}v\text{-meas}\big(A_{q_k,r,L} \cap A_{q_j,r,L}\big).$$

Consider now the unit sphere in $\mathbb{R}^n$. We shall be interested in the intersection of this sphere and a sandwich layer. The picture below illustrates the situation for $n = 3$, and for simplicity we begin by carrying out the calculations for this case. The calculations in higher dimension are similar.

The two layers intersect with the sphere along two bands with width $a_{q_j,r}/L$ and $a_{q_k,r}/L$ respectively, and these two bands intersect at an angle $\alpha_{j,k}$, which can be computed from q_k and q_j. The area of the intersection is then

$$\frac{2a_{q_k,r}a_{q_j,r}}{L^2 \sin\alpha_{j,k}} \leq \frac{2}{36|q_j||q_k|L^2 \sin\alpha_{j,k}} \qquad \text{for} \qquad |q_j|\,,\ |q_k| \leq q_{\max}.$$

The factor 2 comes from the fact that there are two intersections, and actually the factor should have been slightly larger, because in this way we are not taking into account that the bands of intersection are spherical. Hence

$$\begin{aligned}\sum_{j=1}^{k-1} \mathrm{d}v\text{-meas}\big(A_{q_k,r,L} \cap A_{q_j,r,L}\big) &\leq \sum_{j=1}^{k-1} \frac{2}{36|q_j||q_k|L^2 \sin\alpha_{j,k}} \\ &= \frac{1}{18L^2|q_k|} \sum_{j=1}^{k-1} \frac{1}{|q_j|\sin\alpha_{j,k}}\,. \end{aligned} \tag{13}$$

Recall that the sum from $j = 1$ to $j = k - 1$ really means that the sum is taken over the set

$$\big\{q \in \mathbb{Z}^n;\ |q| \leq |q_k|,\ g.c.d.(q) = 1,\ q \neq q_k\big\}.$$

If the restriction that $g.c.d.(q) = 1$ is removed, then each rational direction q in the sum is represented by several colinear vectors; the multiplicity of a given direction is

$$\#\{q' \in \mathbb{Z}^n\,;\, q' \parallel q\,,\ |q'| \le |q_k|\} = \left[|q_k| \frac{g.c.d.(q)}{|q|}\right]$$

where $\#$ denotes the number of elements in a set, and $[z]$ denotes the largest integer smaller than or equal to z. The sum in (13) is then not larger than

$$\frac{1}{18L^2|q_k|} \sum_{|q|\le|q_k|, q\ne q_k} \frac{g.c.d.(q)}{|q|\left[|q_k|\frac{g.c.d.(q)}{|q|}\right] \sin\alpha_{j,k}} \le \frac{C}{L^2|q_k|} \sum_{|q|\le|q_k|, q\ne q_k} \frac{1}{|q_k| \sin\alpha_{j,k}}\,\cdot \tag{14}$$

This sum may now be estimated by an integral:

$$\frac{C}{L^2|q_k|^2} \int_0^{|q_k|} \int_0^{2\pi} \int_0^{\pi} \frac{\mathbb{1}_{\{|\sin\alpha|>\alpha_{\min}\}}}{\sin\alpha} q^2\, \mathrm{d}q\, \sin\alpha\, \mathrm{d}\alpha\, \mathrm{d}\phi \le C\frac{|q_k|}{L^2} \tag{15}$$

and hence the fraction lost is

$$\frac{\sum_{j=1}^{k-1} \mathrm{d}v\text{-meas}\big(A_{q_k,r,L} \cap A_{q_j,r,L}\big)}{\mathrm{d}v\text{-meas}\big(A_{q_k,r,L}\big)} \le C\frac{|q_k|}{L^2}\Big(\frac{1}{3L}\Big(\frac{1}{|q_k|} - r\Big)\Big)^{-1} \ = \ C\frac{|q_k|^2}{L}\,\cdot \tag{16}$$

Recall that only $|q_k| < q_{\max} = 1/2r$ are considered, and hence the last quantity is smaller than

$$\frac{C_n}{r^2L}$$

where C_n is an absolute constant (possibly depending on the dimension). Therefore, if $r^2L > 2C_n$

$$\mathrm{d}v\text{-meas}\Big(\bigcup_{|q|\le q_{\max}} A_{q,r,L}\Big) \ge \frac{1}{2}C'_n \sum_{|q|\le q_{\max}, g.c.d.(q)=1} \frac{1}{L|q|}$$

and this can be estimated just as the union of all intersections above, and it is bounded from below by

$$C'_n \frac{{q_{\max}}^2}{2L} = C'_n \frac{1}{Lr^2}\,\cdot$$

These estimates can be carried out in exactly the same way in the general n-dimensional case. The dv-measure of

$$A_{q,r,L} = \Big\{v \in S^{n-1}; |v\cdot q|/|q| \le \frac{a_{q,r}}{3L}\Big\}$$

is proportional to $1/|q|L$, and $q_{\max} = 1/2r$ gives hole-free layers just as in the three-dimensional case. Only the constants involved depend on the dimension. To estimate the measure of the intersection of $A_{q,r,L} \cap A_{p,r,L}$ with the $n-1$-dimensional unit sphere, note that in a suitable coordinate system, $q/|q| = (1, 0, \cdots, 0)$ and $p/|p| = (\cos\alpha, \sin\alpha, 0, \cdots, 0)$, and hence

$$A_{q,r,L} \cap A_{p,r,L} \cap S^n = \big\{\omega = (\omega^1, \cdots, \omega^n) \in S^n;\ \ |\omega^1| < 1/|q|L, \quad |\omega^1\cos\alpha + \omega^2 \sin\alpha| < 1/|p|L\big\}.$$

Therefore

$$
\begin{aligned}
\mathrm{d}v\text{-meas}(A_{q,r,L} \cap A_{p,r,L} \cap S^n) &= \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \int_{|\bar{x}|^2 = 1-x_1^2-x_2^2} \mathbb{1}_{A_{q,r,L} \cap A_{p,r,L} \cap S^n} \, \mathrm{d}\bar{x} \mathrm{d}x_2 \mathrm{d}x_1 \\
&= \mathrm{Vol}(S^{n-3}) \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \mathbb{1}_{A_{q,r,L} \cap A_{p,r,L} \cap S^n} (1 - x_1^2 - x_2^2)^{(n-3)/2} \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\leq C_n \frac{1}{\sin(\alpha)} \frac{1}{|p|L} \frac{1}{|q|L}
\end{aligned}
$$

just as in the three-dimensional case: (13) and (14) are unchanged, whereas the integral in (15) is replaced by the corresponding n-dimensional one, and as a consequence, the last member of (16) is replaced by

$$
C_n \frac{q_k^{n-1}}{L} \cdot
$$

All together, this proves the main result of this section:

Proposition 3.1. *Let $M_{r,L} \subset S^{n-1} \times \mathbb{R}^n$ denote the set of (v, x_0) such that the line segment of length L, starting from x_0 in the direction v lies entirely in $Z[r]$. There is a constant $C_n > 0$ depending only on the dimension n, such that for any set $K \subset \mathbb{R}^n$,*

$$
\frac{\mathrm{d}x \, \mathrm{d}v\text{-meas}\big(M_{r,L} \cap S^{n-1} \times K\big)}{\mathrm{d}x \, \mathrm{d}v\text{-meas}\big(S^{n-1} \times K\big)} \geq C_n \frac{1}{r^{n-1}L} \qquad \textit{if} \qquad r^{n-1}L \geq 2C_n.
$$

A final remark: In estimating the sum in (3.2) by an integral, it is tacitly assumed that this poses no problem. It is not quite an innocent assumption, because of the factor $1/\alpha$. However, in the sum, α is never zero, and a more careful calculation estimating separately the contribution from each cylinder

$$
\{|q| < q_{\max};\, 2^j \alpha_{\min} \leq \alpha < 2^{j+1} \alpha_{\min}\}, \qquad j \geq 0
$$

would give the same result. After all, here only an upper bound is needed.

4. Numerical experiments

In this section we go back to the inequalities (3)

$$
\frac{C(n)}{r^{n-1}t} \;\leq\; \phi[r]([t, +\infty[) \;\leq\; \frac{C'(n)}{r^{n-1}t},
$$

and to the question posed at the end of Section 2.2: could it be that, asymptotically for large $r^{n-1}t$ one could take $C(n)$ and $C'(n)$ arbitrarily close to each other? We have not been able to prove this, but the numerical simulations described in this section indicate that this is actually the case. The results that we report here are from two-dimensional simulations, and for moderately small r. For $D \geq 3$, and for very small r, the method that we have used is too time consuming to give good results (a small number of particles give noisy data).

The method used is the simplest one conceivable:

1. An initial point is chosen at random in the unit square.
2. Then the particle is advanced (using the periodicity of the cell) until the trajectory passes through a ball of radius R in the centre of the cell.
3. The exact exit-time can then be calculated.

This is repeated a large number of times (in this case 5×10^6 times).

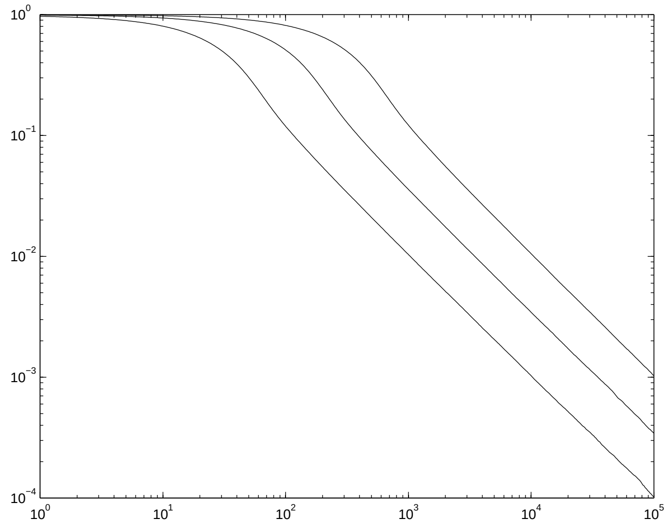

FIGURE 4. Log-log-plot of $\phi[r]([t,\infty])$ *vs.* t for $r = 0.01$, $r = 0.03$ and $r = 0.001$.

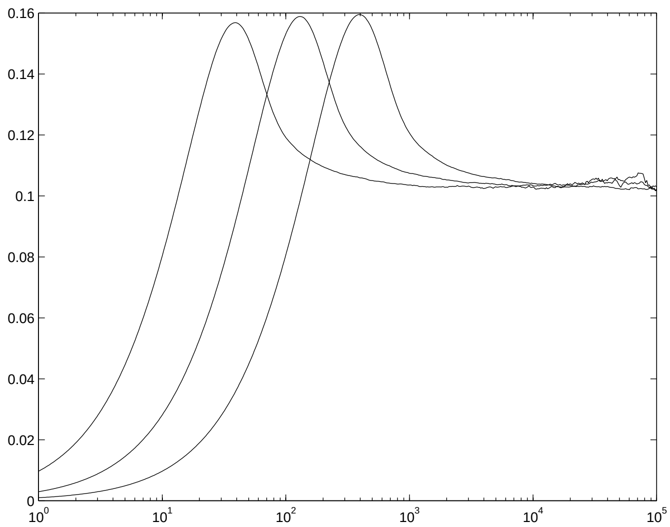

FIGURE 5. Plot of $tr\phi[r]([t,\infty[)$ *vs.* t.

The plots above show the results from three runs, with $r = 0.01$, $r = 0.03$ and $r = 0.001$. In Figure 4, $\phi[r]([t,\infty[)$ is plotted as a function of t in a log-log-diagram, and Figure 5 shows $tr\phi[r]([t,\infty[)$ as a function of t. Obviously this type of experiment does not prove anything, but at least indicates that $C(2)$ and $C'(2)$ in equation (3) should not be too different. There is one thing to remember, though: for a fixed r, the picture would be rather different if t was allowed to increase arbitrarily. The asymptotic result expected (namely (12)) can only hold when $r \to 0$ and $rt \to \infty$ simultaneously.

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