The Lorentz gas

François Golse
Ecole polytechnique, CMLS
golse@math.polytechnique.fr

Work with J. Bourgain, E. Caglioti, B. Wennberg

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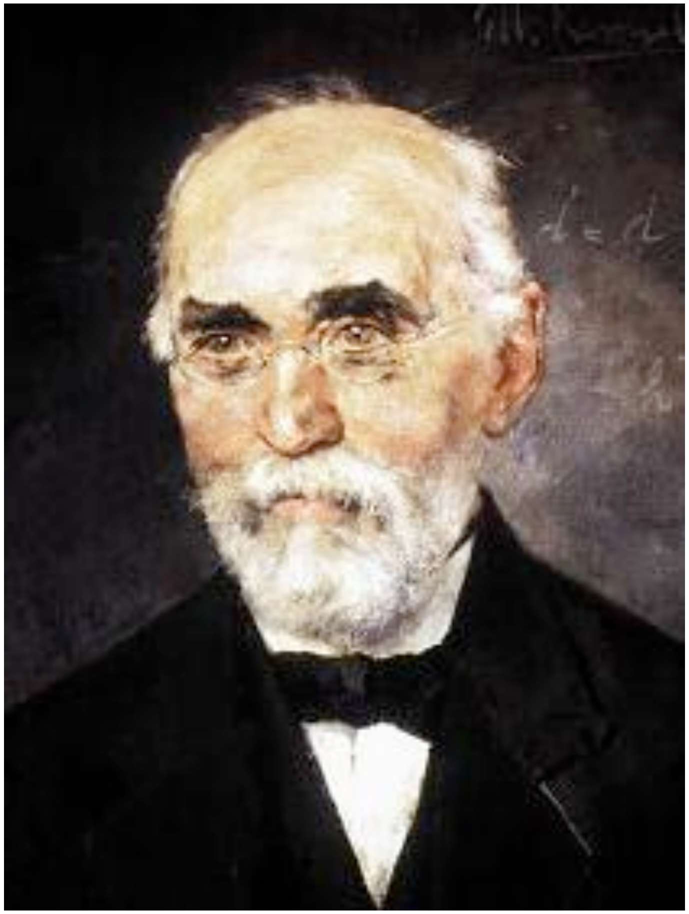
$\boxed{\text{Lecture 1}}$

- Gallavotti’s thm (1972) for Poisson distribution of obstacles

- Santalò’s formula (1942) for the mean free path



In 1905, H. Lorentz proposed to describe the motion of electrons in metals by the methods of kinetic theory

Introduction: the Lorentz kinetic model

•Gas of electrons described by its phase-space density $f \equiv f(t, x, v)$ (density of electrons at the position x with velocity v at time t)

•Electron-electron collisions neglected (unlike in the kinetic theory of gases)

•Only the collisions between electrons and metallic atoms are considered

$\Rightarrow$ LINEAR KINETIC EQUATION

unlike Boltzmann's equation in the kinetic theory of gases

•Equation for the phase-space density of electrons $f \equiv f(t,x,v)$:

$$\boxed{(\partial_t + v \cdot \nabla_x + \tfrac{1}{m} F(t,x) \cdot \nabla_v) f(t,x,v) = N_{at} r_{at}^2 |v| \mathcal{C}(f(t,x,\cdot))(v)}$$

where $\mathcal{C}$ is the Lorentz collision integral

$$\boxed{\mathcal{C}(\phi)(v) = \int_{\substack{|\omega|=1 \\ \omega\cdot v>0}} \Big(\phi(v - 2(v\cdot\omega)\omega) - \phi(v)\Big) \cos(v,\omega) d\omega}$$

Notation: m =mass of the electron; N_{at}, r_{at} density, radius of metallic atoms; $F \equiv F(t,x)$ electric force (given).

•Can one derive the Lorentz kinetic equation from a microscopic, purely mechanical particle model?

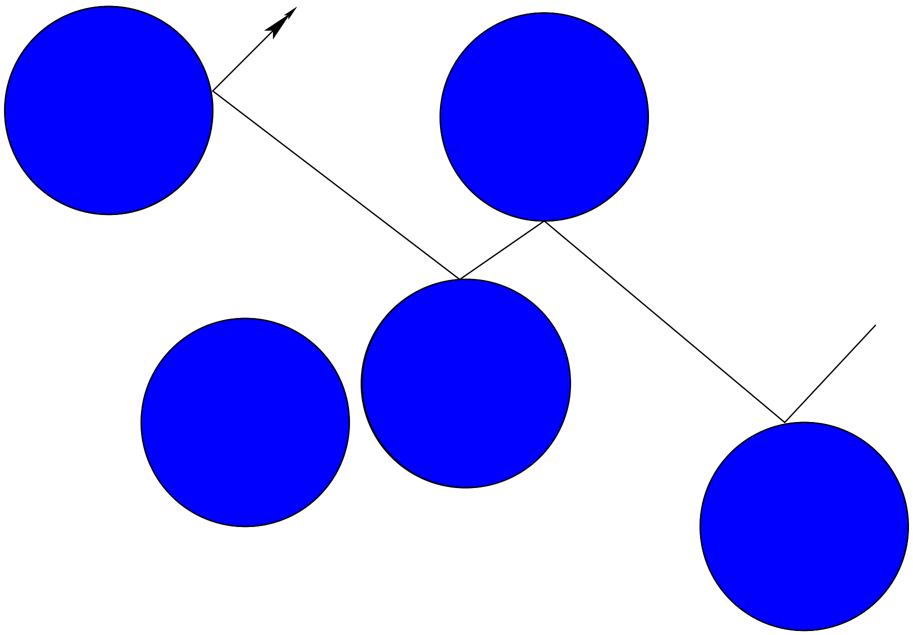
•We consider a gas of point particles (the electrons) moving in a system fixed spherical obstacles (the metallic atoms).

•We assume elastic collisions between the electrons and the metallic atoms: in other words, the point particles are specularly reflected upon colliding with the obstacles.

•The most interesting part of the Lorentz kinetic equation is the collision integral $\Rightarrow$ we assume for simplicity that there is no applied electric field

$$\boxed{F \equiv 0}$$

•Microscopic model= billiard system (=gas of point particles moving at a constant speed in a configuration of fixed spherical obstacles, and specularly reflected at the surface of the obstacles).



- Gallavotti (1969) derived the Lorentz kinetic equation from a billiard system consisting of randomly distributed obstacles (Poisson, possibly overlapping) in some limit — the Boltzmann-Grad limit, whose definition is given below

- Slightly more general, random distributions of scatterers by Spohn (1978)

- Almost sure convergence by Boldrighini-Bunimovich-Sinai (1983)

- What about the case of a periodic configuration of obstacles?

The Lorentz gas with Poisson distribution of obstacles

•Assume that the obstacles are disks of radius r in the Euclidian plane $\mathbf{R}^2$, centered at $c^1, c^2, \ldots, c^j, \ldots \in \mathbf{R}^2$. Henceforth, we denote

$$\{c\} = \{c^1, c^2, \ldots, c^j, \ldots\} = \text{ a configuration of obstacle centers}$$

•Assume further that $\{c\}$ is distributed under Poisson's law with parameter n, meaning that

$$\boxed{\text{Prob}(\{\{c\} \,|\, \#(A \cap \{c\}) = p\}) = e^{-n|A|}\frac{(n|A|)^p}{p!}}$$

•Obstacles may overlap: in other words, configurations $\{c\}$ such that

$$\text{for some } j \neq k \in \{1, 2, \ldots\}, \text{ one has } |c^i - c^j| < 2r$$

are not excluded

•<u>Billiard flow</u> (defined a.e.) in the obstacle configuration $\{c\}$:

$$\dot{X}(t;x,v,\{c\}) = V(t;x,v,\{c\}),$$
$$\dot{V}(t;x,v,\{c\}) = 0, \qquad \text{whenever } |X(t;x,v,\{c\}) - c^i| > r \text{ for all } i$$

with specular reflection in case of a collision with the i-th obstacle:

$$X(t+0;x,v,\{c\}) = X(t-0;x,v,\{c\})$$
$$V(t+0;x,v,\{c\}) = \mathcal{R}\left[\frac{X(t;x,v,\{c\}) - c^i}{r}\right] V(t-0;x,v,\{c\})$$

where $\mathcal{R}[\omega]$ denotes the reflection w.r.t. the line$(\mathbf{R}\omega)^{\perp}$:

$$\mathcal{R}[\omega]v = v - 2(\omega\cdot v)\omega\,,\quad |\omega| = 1$$

•Given a probability density $f^{in}_{\{c\}} \equiv f^{in}_{\{c\}}(x,v)$ on $\mathbf{R}^2 \times \mathbf{S}^1$ (with support outside the obstacles) define

$$\boxed{f(t,x,v,\{c\}) = f^{in}_{\{c\}}(X(-t;x,v,\{c\}),V(-t;x,v,\{c\}))}$$

•Let $\tau_1(x,v,\{c\}),\tau_2(x,v,\{c\}),\dots,\tau_j(x,v,\{c\}),\dots$ be the sequence of collision times for a particle starting from x in the direction $-v$ at $t=0$:

$$\tau_j(x,v,\{c\}) =$$
$$\sup\{t \,|\, \#\{s\in[0,t] \,|\, \mathrm{dist}(X(-s,x,v,\{c\});\{c\})=r\}=j-1\}$$

•Denoting $\tau_0=0$ and $\Delta\tau_k=\tau_k-\tau_{k-1}$, the 1-particle density f is (a.e.)

$$\boxed{\begin{aligned} f(t,x,v,\{c\}) &= f^{in}(x-tv,v)\mathbf{1}_{t<\tau_1} \\ &+\sum_{j\geq 1} f^{in}\left(x-\sum_{k=1}^{j}\Delta\tau_k V(-\tau_k^-)-(t-\tau_j)V(-\tau_j^+),V(-\tau_j^+)\right)\mathbf{1}_{\tau_j<t<\tau_{j+1}}\end{aligned}}$$

•**Remark:** in the case of physically admissible initial data, there should be no particle whose initial position is inside an obstacle: hence we assumed that $f^{in}_{\{c\}} = 0$ in the union of all the disks of radius r centered at the $c^j \in \{c\}$.

•When dealing with bounded initial data, this constraint disappears in the BG limit, since the volume fraction occupied by the obstacles $\to 0$ in that limit.

$\Rightarrow$ henceforth we neglect this difficulty and proceed as if f^{in} were any bounded probability density on $\mathbf{R}^2 \times \mathbf{S}^1$

•Goal: average the summation above in the obstacle configuration $\{c\}$ under the Poisson distribution, and identify a scaling on the obstacle radius r and the parameter of the Poisson distribution n leading to a nontrivial limit.

•Interpretation of the parameter n: the expected number of obstacle centers in a domain $\Omega \subset \mathbf{R}^2$ is

$$\sum_{p\geq 0} p\mathrm{Prob}(\{\{c\} \,|\, \#(\Omega \cap \{c\}) = p\}) = \sum_{p\geq 0} p e^{-n|\Omega|}\frac{(n|\Omega|)^p}{p!} = n|\Omega|$$

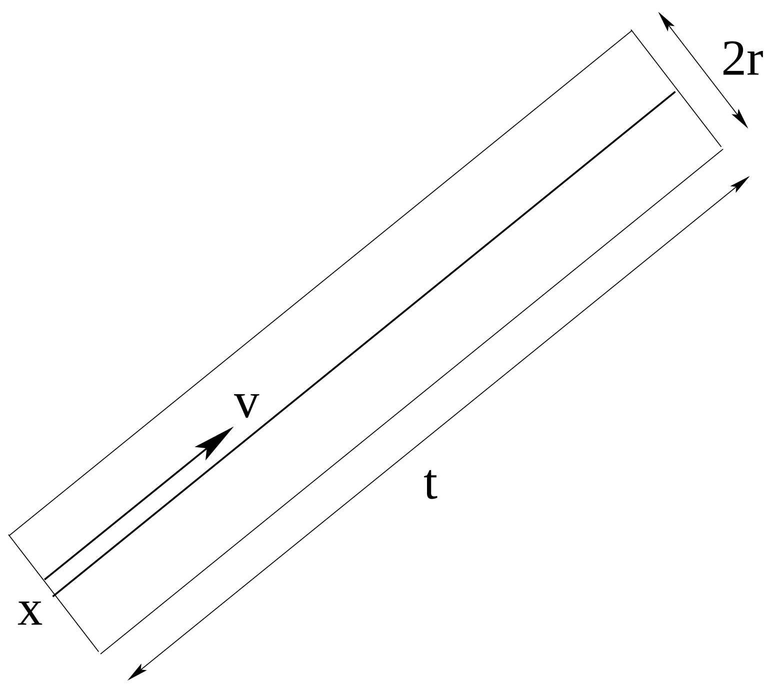
so that

$$\boxed{n = \#\text{ obstacles per unit volume}}$$

• The average of the first term in $f(t, x, v, \{c\})$ is

$$f^{in}(x - tv, v)\langle \mathbf{1}_{t<\tau_1} \rangle = f^{in}(x - tv, v)e^{-n2rt}$$

since $t < \tau_1$ means that the tube of width $2r$ and length t contains 0 obstacle centers.



•Henceforth, we seek a scaling limit corresponding to small obstacles, i.e. $r \to 0$, and a large number of obstacles per unit volume, i.e. $n \to \infty$.

• BOLTZMANN-GRAD SCALING IN DIM. 2 In order for the average of the first term above to have a nontrivial limit

$$\boxed{2nr \to \sigma > 0}$$

so that

$$\langle f^{in}(x - tv, v)\mathbf{1}_{t<\tau_1}\rangle \to f^{in}(x - tv, v)e^{-\sigma t}$$

•Strategy: this first term corresponds with an exponential damping effect.

The subsequent terms in the summation should converge to the Duhamel formula for the Lorentz kinetic equation

Theorem. (Gallavotti 1972) *Let f^{in} be a continuous, bounded probability density on $\mathbf{R}^2\times\mathbf{S}^1$, and let $f_r(t,x,v,\{c\}) = f^{in}((X^r,V^r)(-t,x,v,\{c\}))$, where $(t,x,v)\mapsto (X^r,V^r)(t,x,v,\{c\})$ is the billiard flow in the system of disks of radius r centered at the elements of $\{c\}$. Assuming that the obstacle centers are distributed under the Poisson law of parameter $n=\sigma/2r$ with $\sigma>0$, the expected single particle density*

$\langle f_r(t,x,v,\cdot)\rangle \to f(t,x,v)$ *in $L^1(\mathbf{R}^2\times\mathbf{S}^1)$ uniformly on compact t-sets*

where f is the solution of the Lorentz kinetic equation

$$\boxed{\begin{aligned}&\frac{\partial f}{\partial t}+v\cdot\nabla_x f+\sigma f=\sigma\int_0^{2\pi} f(t,x,R[\beta]v)\sin\tfrac{\beta}{2}\tfrac{d\beta}{4}\\ &f\Big|_{t=0}=f^{in}\end{aligned}}$$

End of proof: The general term in the summation giving $f(t, x, v, \{c\})$ is

$$\boxed{f^{in}\left(x - \sum_{k=1}^{j} \Delta\tau_k V^r(-\tau_k^-) - (t-\tau_j)V^r(-\tau_j^+), V^r(-\tau_j^+)\right)\mathbf{1}_{\tau_j<t<\tau_{j+1}}}$$

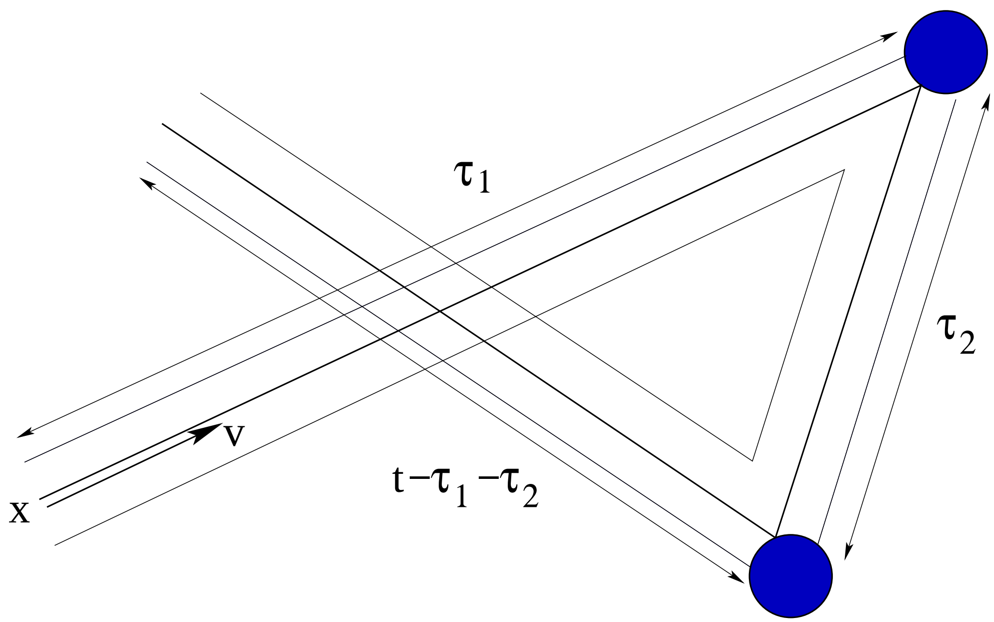
and its average under the Poisson distribution on $\{c\}$ is

$$\boxed{\int f^{in}\left(x - \sum_{k=1}^{j} \Delta\tau_k V^r(-\tau_k^-) - (t-\tau_j)V^r(-\tau_j^+), V^r(-\tau_j^-)\right) e^{-n|T(t;c_1,\ldots,c_j)|}\frac{n^j dc_1 \ldots dc_j}{j!}}$$

where $T(t; c_1, \ldots, c_j)$ is the tube of width $2r$ around the particle trajectory colliding first with the obstacle centered at c_1, . . . , and whose j-th collision is with the obstacle centered at c_j.

•As before, the surface of that tube is

$$|T(t; c_1, \ldots, c_j)| = 2rt + O(r^2)$$



The tube $T(t, c_1, c_2)$

•In the j-th term, change variables by expressing the positions of the j encountered obstacles in terms of free flight times and deflection angles:

$$(c_1, \ldots, c_j) \mapsto (\tau_1, \ldots, \tau_j; \beta_1, \ldots, \beta_j)$$

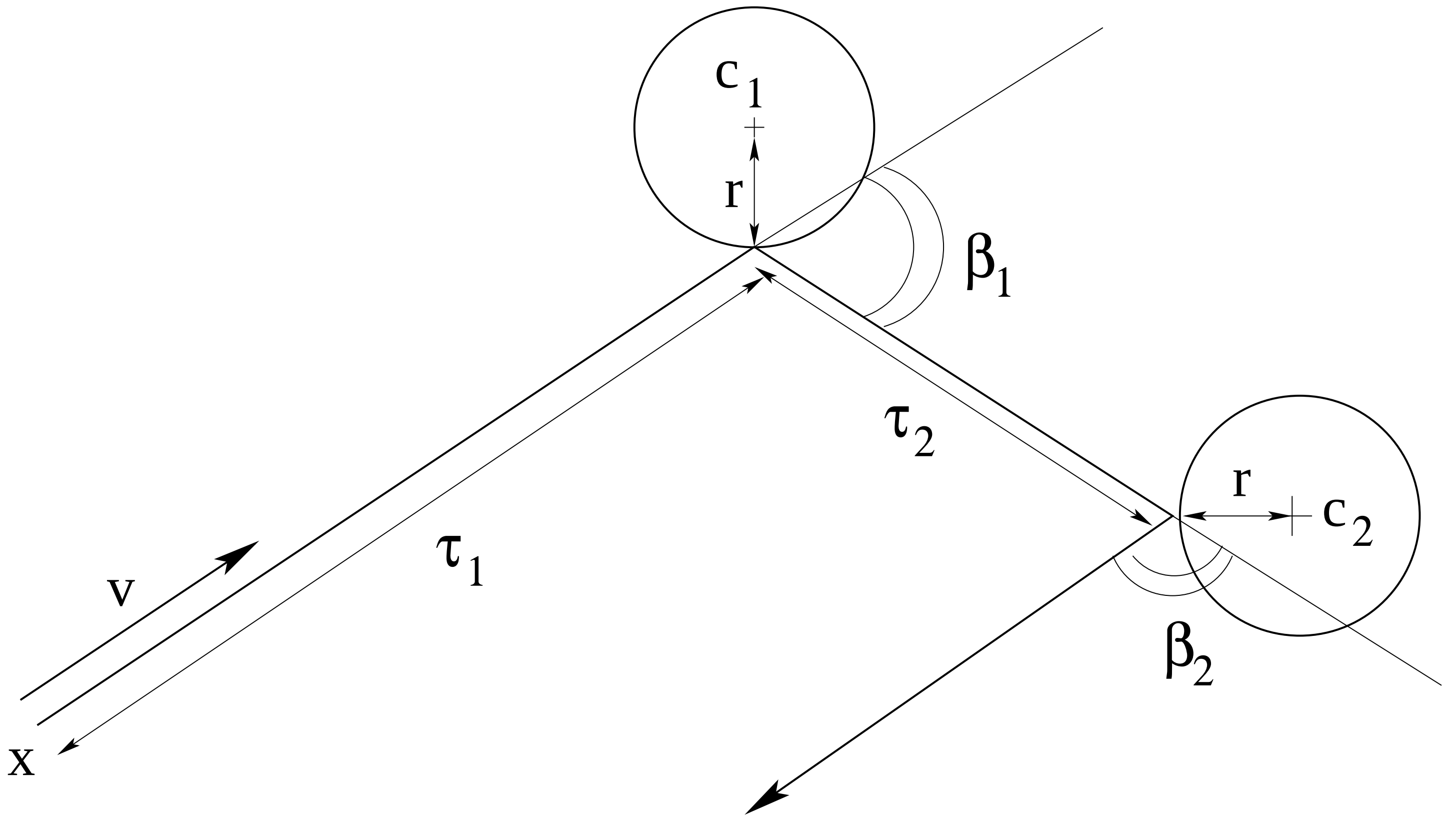
The volume element in the j-th integral is changed into

$$\boxed{\frac{dc_1 \ldots dc_j}{j!} = r^j \sin \frac{\beta_1}{2} \ldots \sin \frac{\beta_j}{2} \, \frac{d\beta_1}{2} \ldots \frac{d\beta_j}{2} d\tau_1 \ldots d\tau_j}$$

•The measure in the left-hand side is invariant by permutations of $c_1, \ldots, c_j$; on the right-hand side, we assume that

$$\tau_1 < \tau_2 < \ldots < \tau_j$$

so that the $1/j!$ factor disappears.



The substitution $(c_1, c_2) \mapsto (\tau_1, \tau_2, \beta_1, \beta_2)$

•The substitution above is possible only if the particle does not hit twice the SAME obstacle. Define

$$A_r(T,x,v) = \bigcup_{j\geq 1} \{\{c\} \,|\, \text{dist}(X^r(t,x,v,\{c\}),c^j) = r \text{ for some } 0 < t_1 < t_2 < T\}$$

and set

$$\boxed{\begin{aligned} f_r^M(t,x,v,\{c\}) &= f_r(t,x,v,\{c\}) - f_r^R(t,x,v,\{c\}) \\ f_r^R(t,x,v,\{c\}) &= f_r(t,x,v,\{c\})\mathbf{1}_{A_r(T,x,v)}(\{c\}) \end{aligned}}$$

(respectively the Markovian part and the recollision part in f_r.)

•After averaging over the obstacle configuration $\{c\}$, the contribution of the j-th term in f_r^M is, to leading order in r:

$$\boxed{\begin{aligned}&(2nr)^j e^{-2nrt}\int_{0<\tau_1<\ldots<\tau_j<t}\int_{[0,2\pi]^j}\sin\frac{\beta_1}{2}\ldots\sin\frac{\beta_j}{2}\frac{d\beta_1}{4}\ldots\frac{d\beta_j}{4}d\tau_1\ldots d\tau_j\\ &\quad\times f^{in}\left(x-\sum_{k=1}^{j}\Delta\tau_k R\left[\sum_{l=1}^{k-1}\beta_l\right]v-(t-\tau_j)R\left[\sum_{l=1}^{j-1}\beta_l\right]v, R\left[\sum_{l=1}^{j}\beta_l\right]v\right)\end{aligned}}$$

•It is dominated by

$$\|f^{in}\|_{L^\infty}O(\sigma)^j e^{-O(\sigma)t}\frac{t^j}{j!}$$

which is the general term of a converging series.

•Passing to the limit as $n \to +\infty$, $r \to 0$ so that $2rn \to \sigma$, one finds (by dominated convergence in the series) that

$$\boxed{\begin{aligned}&\langle f_r^M(t,x,v,\{c\})\rangle \to e^{-\sigma t} f^{in}(x-tv,v)\\ &+\sigma e^{-\sigma t}\int_0^t\int_0^{2\pi} f^{in}(x-\tau_1 v-(t-\tau_1)R[\beta_1]v, R[\beta_1]v)\sin\frac{\beta_1}{2}\frac{d\beta_1}{4}d\tau_1\\ &+\sum_{j\geq 2}\sigma^j e^{-\sigma t}\int_{0<\tau_j<\ldots<\tau_1<t}\int_{[0,2\pi]^j}\sin\frac{\beta_1}{2}\ldots\sin\frac{\beta_j}{2}\\ &\times f^{in}\left(x-\sum_{k=1}^{j}\Delta\tau_k R\left[\sum_{l=1}^{k-1}\beta_l\right]v-(t-\tau_j)R\left[\sum_{l=1}^{j-1}\beta_l\right]v, R\left[\sum_{l=1}^{j}\beta_l\right]v\right)\\ &\times\frac{d\beta_1}{4}\ldots\frac{d\beta_j}{4}d\tau_1\ldots d\tau_j\end{aligned}}$$

which is the Duhamel series giving the solution of the Lorentz kinetic equation

•We have proved that

$$\langle f_r^M(t,x,v,\cdot)\rangle \to f(t,x,v) \text{ uniformly on bounded sets as } r \to 0$$

where f is the solution of the Lorentz kinetic equation. Since

$$\iint_{\mathbf{R}^2\times\mathbf{S}^1} f(t,x,v)dxdv = \iint_{\mathbf{R}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv$$

and

$$\iint_{\mathbf{R}^2\times\mathbf{S}^1} f_r(t,x,v,\{c\})dxdv = \iint_{\mathbf{R}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv$$

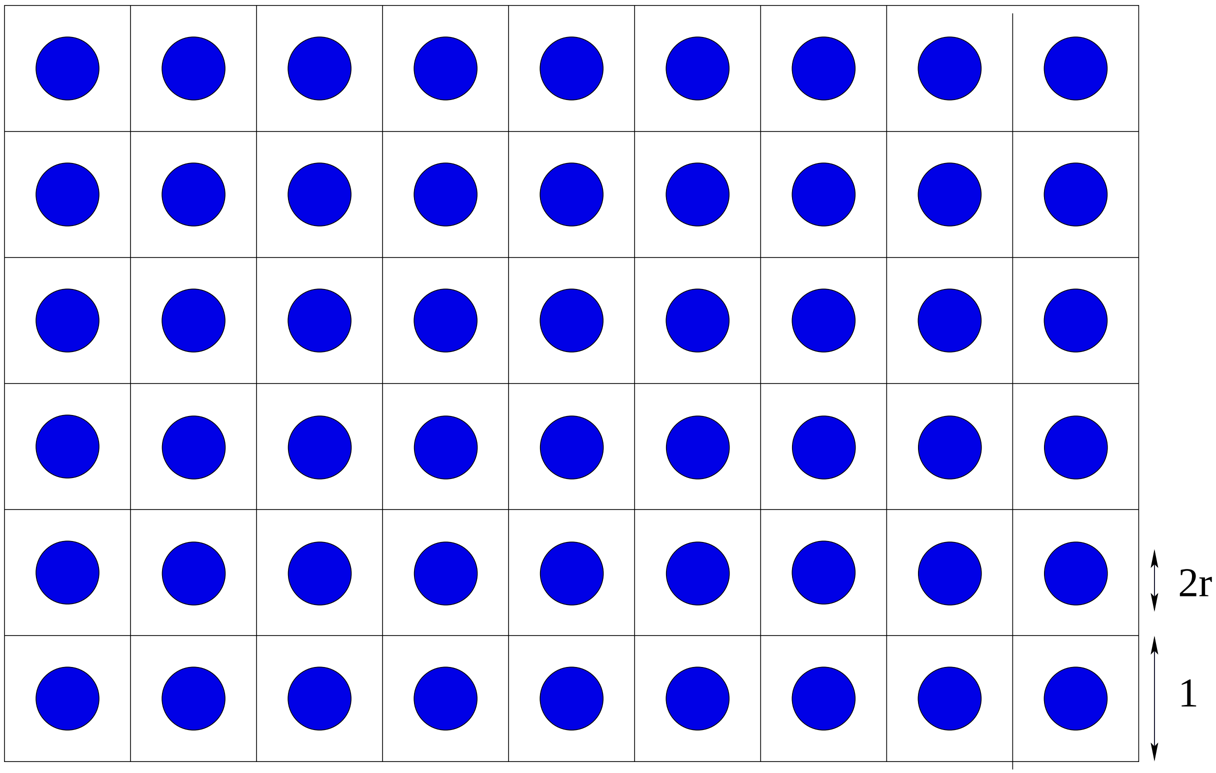
we conclude by Fatou's lemma that

$$\boxed{\begin{array}{c}\langle f_r^A\rangle \to 0 \text{ in } L^1(\mathbf{R}^2\times\mathbf{S}^1) \text{ uniformly on bounded } t\text{-sets}\\ \langle f_r^M\rangle \to f \text{ in } L^1(\mathbf{R}^2\times\mathbf{S}^1) \text{ uniformly on bounded } t\text{-sets}\end{array}}$$

This conclude the proof of Gallavotti's theorem. □

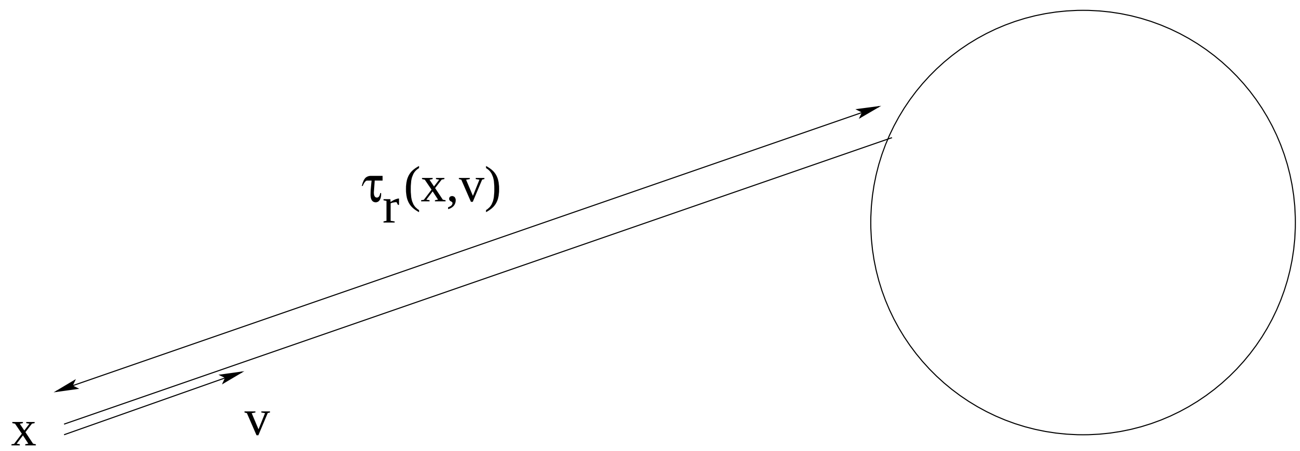
The periodic case: the mean free path

•For $r \in (0, \frac{1}{2})$, define the billiard table $Z_r = \{x \in \mathbf{R}^D \,|\, \text{dist}(x, \mathbf{Z}^D) > r\}$;



•The free path length starting from $x \in Z_r$ in the direction $v \in \mathbf{S}^{D-1}$ is

$$\tau_r(x,v) = \min\{t > 0 \,|\, x + tv \in \partial Z_r\}$$



$$\text{Mean Free Path} = \langle \tau_r \rangle$$

where $\langle \cdot \rangle$ is the average under some appropriate probability measure

Two natural probability measures for the Lorentz gas:

a) the uniform probability measure in $Z_r/\mathbf{Z}^D \times \mathbf{S}^{D-1}$

$$\boxed{d\mu_r(x,v) = \frac{dxdv}{|Z_r/\mathbf{Z}^D|\,|\mathbf{S}^{D-1}|}}$$

invariant under the billiard flow

b) the invariant measure of the billiard map

$$\boxed{d\nu_r(x,v) = \frac{v\cdot n_x dxdv}{v\cdot n_x dxdv\text{-meas}(\Gamma_+^r/\mathbf{Z}^D)}}$$

where n_x is the unit inward normal at $x \in \partial Z_r$ and

$$\Gamma_+^r := \{(x,v) \in \partial Z_r \times \mathbf{S}^{D-1} \,|\, v\cdot n_x > 0\}$$

•Billiard flow:

$$\begin{cases} \dot{X}_r = V_r \\ \dot{V}_r = 0 \end{cases} \quad \text{whenever } X(t) \notin \partial Z_r$$

$$\begin{cases} X_r(t^+) = X_r(t^-) =: X_r(t) \text{ if } X(t^\pm) \in \partial Z_r \\ V_r(t^+) = \mathcal{R}[n_{X_r(t)}]V_r(t^-) \end{cases}$$

with $\mathcal{R}[n]v = v - 2v\cdot nn$ denoting the reflection w.r.t. $(\mathbf{R}n)^\perp$

•Billiard map:

$$\Gamma_+^r \ni (x,v) \mapsto \mathcal{B}_r(x,v) := (X_r, V_r)(\tau_r(x,v); x, v) \in \Gamma_+^r$$

•A first notion of mean free path: if the billiard map $\mathcal{B}_r$ is ergodic for the measure ν_r, then for ν_r-a.e. $(x,v) \in \Gamma_r^+$,

$$\boxed{\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \tau_r(\mathcal{B}_r^k(x,v)) = \int_{\Gamma_+^r/\mathbf{Z}^D} \tau_r d\nu_r}$$

SANTALÒ'S FORMULA FOR THE MEAN FREE PATH (1942)

$$\ell_r := \int_{\Gamma^r_+/\mathbf{Z}^D} \tau_r(x,v)\,d\nu_r(x,v) = \frac{1-|\mathbf{B}^D|r^D}{|\mathbf{B}^{D-1}|r^{D-1}}$$

In fact, one has the following slightly more general

Lemma. (H.S. Dumas, L. Dumas, F.G. 1996) *For $f \in C^1(\mathbf{R}_+)$ such that $f(0)=0$, one has*

$$\int_{\Gamma^r_+/\mathbf{Z}^D} f(\tau_r(x,v))v\cdot n_x\,dx\,dv = \int_{(Z_r/\mathbf{Z}^D)\times\mathbf{S}^{D-1}} f'(\tau_r(x,v))\,dx\,dv$$

The Santalò formula corresponds with $f(z)=z$ in the identity above.

Proof: for each $(x,v)\in Z_r\times\mathbf{S}^{D-1}$ one has

$$\tau_r(x+tv,v)=\tau_r(x,v)-t\,,\quad \text{so that } \frac{d}{dt}\tau_r(x+tv,v)=-1$$

Hence

$$\begin{cases} v\cdot\nabla_x\tau_r(x,v)=-1\,, & x\in Z_r\,, \quad v\in\mathbf{S}^{D-1}\\ \tau_r(x,v)=0\,, & x\in\partial Z_r\,, \quad v\cdot n_x<0\end{cases}$$

Since $f\in C^1(\mathbf{R}_+)$ and $f(0)=0$, one has

$$\begin{cases} v\cdot\nabla_x f(\tau_r(x,v))=-f'(\tau_r(x,v))\,, & x\in Z_r\,, \quad v\in\mathbf{S}^{D-1}\\ f(\tau_r(x,v))=0\,, & x\in\partial Z_r\,, \quad v\cdot n_x<0\end{cases}$$

Integrating both sides of the equality above, and applying Green's formula establishes the identity in the lemma. □

Boltzmann-Grad limit

Small obstacles: $r \to 0^+$; long time scale so that

$$\# \text{ collisions per unit of time } = O(1)$$

Given $f^{in} \in C_c(\mathbf{R}^D \times \mathbf{S}^{D-1})$, define f_r to be

$$f_r(t,x,v) = f^{in}\left(r^{D-1} X_r\left(-\frac{t}{r^{D-1}}; \frac{x}{r^{D-1}}, v\right), V_r\left(-\frac{t}{r^{D-1}}; \frac{x}{r^{D-1}}, v\right)\right)$$

where (X_r, V_r) is the billiard flow in Z_r with specular reflection on ∂Z_r.

Problem: to find an equation governing the L^∞ weak-* limit points of the scaled number density f_r as $r \to 0^+$.

Final remarks:

a) the proof of Santalò's thm shows that it holds in greater generality (smooth obstacles other than circles, trajectories of particles subject to some external force field...)

b) assuming no external force field is not as inocuous as it may seem. For instance, in the case of Poisson distributed holes (so that particles falling into the holes disappear from the system), the presence of an external force may introduce memory effects in the BG limit (Desvillettes-Ricci, 2004)